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ERROR ANALYSIS FOR A FINITE DIFFERENCE SCHEME FOR AXISYMMETRIC MEAN CURVATURE FLOW OF GENUS-0 SURFACES*

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Abstract. We consider a finite difference approximation of mean curvature flow for axisymmetric surfaces of genus zero. A careful treatment of the degeneracy at the axis of rotation for the one-dimensional partial differential equation for a parameterization of the generating curve allows us to prove error bounds with respect to discrete L^2- and H^1-norms for a fully discrete approximation. The theoretical results are confirmed with the help of numerical convergence experiments. We also present numerical simulations for some genus-0 surfaces, including for a nonembedded self-shrinker for mean curvature flow.

Key words. mean curvature flow, axisymmetry, finite differences, error analysis, self-shrinker

AMS subject classifications. 65M60, 65M12, 65M15, 53C44, 35K55

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1. Introduction.

Consider a family of surfaces $(\mathcal{S}(t))_{t\in[0,T]} \subset \mathbb{R}^3$ evolving by mean curvature flow, i.e.,

$$\mathcal{V}_{\mathcal{S}} = k_m \qquad \text{on } \mathcal{S}(t). \tag{1.1}$$

Here, $\mathcal{V}_{\mathcal{S}}$ denotes the normal velocity of $\mathcal{S}(t)$ in the direction of the normal $\vec{\nu}_{\mathcal{S}(t)}$, and k_m is the mean curvature of $\mathcal{S}(t)$, i.e., the sum of its principal curvatures. As the L^2-gradient flow for the area functional, (1.1) is one of the most important geometric evolution equations with applications in materials science and image processing. We refer the reader to [12, 17] for an introduction and important results of mean curvature flow.

In this paper we are concerned with the numerical approximation of solutions of (1.1) using a parametric approach. If $\vec{X} : \mathcal{M} \times [0,T) \to \mathbb{R}^3$ is a family of embeddings such that $\mathcal{S}(t) = \vec{X}(\mathcal{M}, t)$, then (1.1) is satisfied if $\vec{X}_t \circ \vec{X}^{-1} = k_m \vec{\nu}_{\mathcal{S}(t)}$ on $\mathcal{S}(t)$. Making use of the fact that the mean curvature vector $k_m \vec{\nu}_{\mathcal{S}(t)}$ can be written as $\Delta_{\mathcal{S}(t)} \vec{\mathrm{id}}$, where $\Delta_{\mathcal{S}(t)}$ denotes the Laplace–Beltrami operator on $\mathcal{S}(t)$, Dziuk [10] suggested a finite element method in order to approximate solutions of (1.1). While this approach has been widely used in the following years, the numerical analysis of the method remained open. Only recently, Kovács, Li, and Lubich [16] obtained error estimates for a parametric approach that uses not only the position $\vec{X}$ but also the mean curvature and the normal as variables. Both approaches are based on evolution equations in which the velocity vector points purely in normal direction, which may lead to degenerate meshes at the discrete level. A way to tackle this issue is to introduce a suitable additional tangential motion in such a way that mesh points are better distributed on the approximate surface. Corresponding schemes have been suggested by Barrett,

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Garcke, and Nürnberg [4], as well as by Elliott and Fritz [13], using DeTurck's trick. For the approach from [13], error bounds for a finite difference scheme in the case of surfaces of torus type have recently been obtained in [18]. For more details on the numerical approximation of geometric evolution equations we refer to the review articles [8, 6].

In what follows, we are interested in the case that the evolving surfaces are axisymmetric with respect to the x_2-axis; i.e., we assume that there exists a mapping $\vec{x}(\cdot,t) : [0,1] \to \mathbb{R}_{\geq 0} \times \mathbb{R}$ such that

$$\mathcal{S}(t) = \left\{ (\vec{x}(\rho,t) \cdot \vec{e}_1 \cos\theta, \vec{x}(\rho,t) \cdot \vec{e}_2, \vec{x}(\rho,t) \cdot \vec{e}_1 \sin\theta)^T : \rho \in [0,1], \theta \in [0,2\pi] \right\}.$$

As shown in [5, 2], the law (1.1) translates into the following evolution equation for the curves $(\Gamma(t))_{t\in[0,T]}$ parameterized by $\vec{x}(\cdot,t)$:

$$\vec{x}_t \cdot \vec{\nu} = \varkappa - \frac{\vec{\nu} \cdot \vec{e}_1}{\vec{x} \cdot \vec{e}_1}, \tag{1.2}$$

where $\vec{\nu}$ is a unit normal to $\Gamma(t)$ and $\varkappa = \vec{\varkappa} \cdot \vec{\nu}$ denotes curvature with $\vec{\varkappa} = \frac{1}{|\vec{x}_\rho|}(\frac{\vec{x}_\rho}{|\vec{x}_\rho|})_\rho$ the curvature vector. We note that without the last term on the right-hand side of (1.2), the problem collapses to the curve shortening flow,

$$\vec{x}_t \cdot \vec{\nu} = \varkappa, \tag{1.3}$$

which is the analog of (1.1) for curves. Since the relations (1.2) and (1.3) only prescribe the normal velocity, there is a certain freedom in choosing the tangential part of the velocity vector. Setting the tangential velocity to zero for (1.3) leads to the formulation $\vec{x}_t = \frac{1}{|\vec{x}_\rho|}(\frac{\vec{x}_\rho}{|\vec{x}_\rho|})_\rho$, and optimal error bounds for a semidiscrete continuous-in-time finite element approximation of it have been obtained by Dziuk [11]. On the other hand, an application of DeTurck's trick gives rise to the formulation $\vec{x}_t = \frac{\vec{x}_{\rho\rho}}{|\vec{x}_\rho|^2}$ for the classical curve shortening flow. An error analysis for a corresponding semidiscrete finite element scheme has been first presented in [7], and this was later extended in [13] to the family of problems $\alpha\vec{x}_t + (1-\alpha)(\vec{x}_t \cdot \vec{\nu})\vec{\nu} = \frac{\vec{x}_{\rho\rho}}{|\vec{x}_\rho|^2}$, $\alpha \in (0,1]$. Inspired by the ideas in [7], the present authors in [2] applied DeTurck's trick to the flow (1.2) to obtain the system

$$\vec{x}_t = \frac{\vec{x}_{\rho\rho}}{|\vec{x}_\rho|^2} - \frac{\vec{\nu} \cdot \vec{e}_1}{\vec{x} \cdot \vec{e}_1}\vec{\nu}; \tag{1.4}$$

cf. [2, equation (1.7)]. Note that (1.4) is strictly parabolic and that a solution of (1.4) satisfies (1.2). The difference to the curve shortening flow consists in the presence of the term $\frac{\vec{\nu}\cdot\vec{e}_1}{\vec{x}\cdot\vec{e}_1}$, which is the principal curvature related to the parallels of $\mathcal{S}(t)$. It is possible to rewrite (1.4) in the following divergence form,

$$\vec{x} \cdot \vec{e}_1 \, |\vec{x}_\rho|^2 \vec{x}_t = \big((\vec{x} \cdot \vec{e}_1)\vec{x}_\rho\big)_\rho - |\vec{x}_\rho|^2 \vec{e}_1, \tag{1.5}$$

giving rise to a natural variational formulation. On the basis of this weak formulation, a semi-implicit scheme using piecewise linear finite elements in space and a backward Euler method in time was suggested by the authors in [2]. In particular, in [2, Theorem 2.2] optimal error bounds both in H^1 and L^2 are obtained in the case of genus-1 surfaces. While the numerical method still performs well also for genus-0 surfaces, it is, however, not possible to apply the employed analysis to genus-0 surfaces.

The reason for the additional difficulties in the genus-0 case comes from the different properties of the curves $\Gamma(t)$: for genus-1 surfaces, $\Gamma(t)$ is a closed curve satisfying $\vec{x} \cdot \vec{e}_1 > 0$ on $[0,1]$ so that this term is bounded strictly from below on compact time intervals, thus simplifying the analysis. In contrast, a description of a genus-0 surface in our setting requires $\Gamma(t)$ to be open with its endpoints lying on the x_2-axis, which means that $\vec{x} \cdot \vec{e}_1 = 0$ at the endpoints of the interval $[0,1]$. Furthermore, in order to guarantee smoothness of the surface $\mathcal{S}(t)$, the curve $\Gamma(t)$ has to meet the x_2-axis at a right angle. In order to formulate the resulting initial boundary problem, it is convenient to rewrite (1.4). To do so, we choose $\vec{\nu} = \vec{\tau}^{\perp}$ with the unit tangent $\vec{\tau} = \frac{\vec{x}_\rho}{|\vec{x}_\rho|}$ and $\cdot^{\perp}$ denoting a clockwise rotation by $\frac{\pi}{2}$. Observing that

$$(\vec{\nu} \cdot \vec{e}_1)\, \vec{\nu} = \frac{1}{|\vec{x}_\rho|^2} (\vec{x}_\rho^{\perp} \cdot \vec{e}_1)\, \vec{x}_\rho^{\perp} = \frac{1}{|\vec{x}_\rho|^2} (\vec{x}_\rho \cdot \vec{e}_2)\, \vec{x}_\rho^{\perp},$$

we are led to the following system:

$$\vec{x}_t = \frac{\vec{x}_{\rho\rho}}{|\vec{x}_\rho|^2} - \frac{1}{|\vec{x}_\rho|^2} \frac{\vec{x}_\rho \cdot \vec{e}_2}{\vec{x} \cdot \vec{e}_1}\, \vec{x}_\rho^{\perp} \quad \text{in } (0,1) \times (0,T], \tag{1.6a}$$

$$\vec{x} \cdot \vec{e}_1 = 0,\ \vec{x}_\rho \cdot \vec{e}_2 = 0 \quad \text{on } \{0,1\} \times [0,T]. \tag{1.6b}$$

Since $\vec{x}(\rho,t) \cdot \vec{e}_1 \to 0$, as $\rho \to \rho_0 \in \{0,1\}$, the last term in (1.6a) needs to be treated with care. Using the boundary conditions (1.6b), it is shown in (A.3) in Appendix A, with the help of L'Hospital's rule, that

$$\lim_{\rho \searrow 0} \left[-\frac{1}{|\vec{x}_\rho(\rho,t)|^2} \frac{\vec{x}_\rho(\rho,t) \cdot \vec{e}_2}{\vec{x}(\rho,t) \cdot \vec{e}_1}\, \vec{x}_\rho^{\perp}(\rho,t) \right] = \frac{\vec{x}_{\rho\rho}(0,t) \cdot \vec{e}_2}{|\vec{x}_\rho(0,t)|^2}\, \vec{e}_2,$$

so that the expression acts like a second order operator close to the boundary without affecting the parabolicity of the problem. Nevertheless, the different behavior of $\frac{\vec{\nu} \cdot \vec{e}_1}{\vec{x} \cdot \vec{e}_1}$ in the interior and close to the boundary is a major problem for the analysis of a numerical scheme. Rather than using the variational form (1.5) that worked well for genus-1 surfaces, we shall introduce a scheme which directly discretizes (1.6a) with the help of finite differences. Our main result are optimal error bounds measuring the error in discrete versions of the usual integral norms.

The paper is organized as follows. In section 2, we formulate our assumptions on the solution of (1.6) and derive a number of properties that will be used in the error analysis. In the second part, we introduce our numerical scheme and provide an estimate for the consistency error. Section 3 is devoted to the proof of our main error estimates, which include an $\mathcal{O}(h^2 + \Delta t)$ bound for a discrete H^1-norm. Finally, in section 4 we present the results of several numerical simulations.

We end this section with a few comments about notation. Throughout this paper, C denotes a generic positive constant independent of the mesh parameter h and the time step size Δt. At times ε will play the role of a (small) positive parameter with $C_\varepsilon > 0$ depending on ε but independent of h and Δt.

2. Finite difference discretization.

Assumption 2.1. Let $\vec{x} : [0,1] \times [0,T] \to \mathbb{R}_{\geq 0} \times \mathbb{R}$ be a solution of (1.6) such that $\partial_t^i \partial_\rho^j \vec{x}$ exists and is continuous on $[0,1] \times [0,T]$ for all $i, j \in \mathbb{N}_0$ with $2i + j \leq 4$. Furthermore, we assume that $\vec{x}_\rho(\rho,t) \neq 0$ for all $(\rho,t) \in [0,1] \times [0,T]$, as well as

$$\vec{x} \cdot \vec{e}_1 > 0 \quad \text{in } (0,1) \times [0,T]. \tag{2.1}$$

It is beyond the scope of this paper to prove the existence of a solution to (1.6) with the above regularity. We note, however, that the well-posedness of the corresponding problem, in the case that the curves $\Gamma(t)$ can be written as a graph, was recently studied in [14].

Let us collect a few properties of the solution which will be used in the error analysis. To begin, there exist constants $0 < c_0 \leq C_0$ such that

$$c_0 \leq |\vec{x}_\rho| \leq C_0 \quad \text{in } [0,1] \times [0,T]. \tag{2.2}$$

Recalling (1.6b), we infer that $\vec{x}_\rho(0,t) \cdot \vec{e}_1 \geq c_0, \vec{x}_\rho(1,t) \cdot \vec{e}_1 \leq -c_0$, which together with (2.1) implies that there exist $c_1 > 0, \delta > 0$ with

$$\vec{x}_\rho \cdot \vec{e}_1 \geq \tfrac{1}{2} c_0 \quad \text{in } [0,\delta] \times [0,T], \tag{2.3a}$$

$$\vec{x}_\rho \cdot \vec{e}_1 \leq -\tfrac{1}{2} c_0 \quad \text{in } [1-\delta,1] \times [0,T], \tag{2.3b}$$

$$\vec{x} \cdot \vec{e}_1 \geq c_1 \quad \text{in } [\tfrac{1}{2}\delta, 1-\tfrac{1}{2}\delta] \times [0,T]. \tag{2.3c}$$

Let us formally describe how this observation can be translated into an estimate on the solution. If we multiply (1.6a) by $-\vec{x}_{\rho\rho}$ and integrate over $[0,1]$, we find, upon integration by parts and observing from (1.6b) that $\vec{x}_t \cdot \vec{x}_\rho = 0$ on $\{0,1\} \times [0,T]$, that

$$\tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 |\vec{x}_\rho|^2 \,\mathrm{d}\rho + \int_0^1 \frac{|\vec{x}_{\rho\rho}|^2}{|\vec{x}_\rho|^2} \,\mathrm{d}\rho - \int_0^1 \frac{1}{|\vec{x}_\rho|^2} \frac{\vec{x}_\rho \cdot \vec{e}_2}{\vec{x} \cdot \vec{e}_1} \vec{x}_\rho^{\perp} \cdot \vec{x}_{\rho\rho} \,\mathrm{d}\rho = 0. \tag{2.4}$$

Since (1.6b) implies $\frac{\vec{x}_\rho^{\perp}(0,t)}{|\vec{x}_\rho(0,t)|} \approx -\vec{e}_2$ on $[0,\delta]$, we can rewrite the third term on $[0,\delta]$, while noting (2.2) and (2.3a), as

$$\begin{aligned}
&-\int_0^\delta \frac{1}{|\vec{x}_\rho|^2} \frac{\vec{x}_\rho \cdot \vec{e}_2}{\vec{x} \cdot \vec{e}_1} \vec{x}_\rho^{\perp} \cdot \vec{x}_{\rho\rho} \,\mathrm{d}\rho \\
&\approx \int_0^\delta \frac{1}{|\vec{x}_\rho|} \frac{\vec{x}_\rho \cdot \vec{e}_2}{\vec{x} \cdot \vec{e}_1} \vec{x}_{\rho\rho} \cdot \vec{e}_2 \,\mathrm{d}\rho = \tfrac{1}{2} \int_0^\delta \frac{1}{|\vec{x}_\rho|} \frac{1}{\vec{x} \cdot \vec{e}_1} \left[(\vec{x}_\rho \cdot \vec{e}_2)^2 \right]_\rho \,\mathrm{d}\rho \\
&= \tfrac{1}{2} \left[\frac{1}{|\vec{x}_\rho|} \frac{1}{\vec{x} \cdot \vec{e}_1} (\vec{x}_\rho \cdot \vec{e}_2)^2 \right]_0^\delta + \tfrac{1}{2} \int_0^\delta \frac{\vec{x}_\rho \cdot \vec{e}_1}{|\vec{x}_\rho|} \frac{(\vec{x}_\rho \cdot \vec{e}_2)^2}{(\vec{x} \cdot \vec{e}_1)^2} \,\mathrm{d}\rho + \tfrac{1}{2} \int_0^\delta \frac{\vec{x}_\rho \cdot \vec{x}_{\rho\rho}}{|\vec{x}_\rho|^3} \frac{(\vec{x}_\rho \cdot \vec{e}_2)^2}{\vec{x} \cdot \vec{e}_1} \,\mathrm{d}\rho \\
&\geq \tfrac{1}{4} \frac{c_0}{C_0} \int_0^\delta \frac{(\vec{x}_\rho \cdot \vec{e}_2)^2}{(\vec{x} \cdot \vec{e}_1)^2} \,\mathrm{d}\rho + \tfrac{1}{2} \int_0^\delta \frac{\vec{x}_\rho \cdot \vec{x}_{\rho\rho}}{|\vec{x}_\rho|^3} \frac{(\vec{x}_\rho \cdot \vec{e}_2)^2}{\vec{x} \cdot \vec{e}_1} \,\mathrm{d}\rho,
\end{aligned} \tag{2.5}$$

so that we obtain L^2–control of $\frac{\vec{x}_\rho \cdot \vec{e}_2}{\vec{x} \cdot \vec{e}_1}$ close to 0. A similar calculation applies close to 1, while the denominator $\vec{x} \cdot \vec{e}_1$ is bounded away from 0 on $[\delta, 1-\delta]$ in view of (2.3c). Our aim is to mimic this argument within the error analysis (cf. Lemma 3.4). To do so, we will directly discretize (1.6a) using a finite difference scheme, and the discrete analog of the above estimate is then obtained by multiplying with a suitable second order finite difference.

In order to define our finite difference scheme, let us introduce the set of grid points $\mathcal{G}_h := \{q_0, q_1, \ldots, q_J\}$, where $q_j = jh$ and $h = \frac{1}{J}, j = 0, \ldots, J$. For a grid function $\vec{v} : \mathcal{G}_h \to \mathbb{R}^2$ we write $\vec{v}_j := \vec{v}(q_j)$, $j = 0, \ldots, J$. Furthermore we associate with $\vec{v}$ the following finite difference operators:

$$\delta^- \vec{v}_j := \frac{\vec{v}_j - \vec{v}_{j-1}}{h}, \qquad j = 1, \dots, J; \tag{2.6a}$$

$$\delta^+ \vec{v}_j := \delta^- \vec{v}_{j+1} = \frac{\vec{v}_{j+1} - \vec{v}_j}{h}, \qquad j = 0, \dots, J-1; \tag{2.6b}$$

$$\delta^1 \vec{v}_j := \tfrac{1}{2}(\delta^+ \vec{v}_j + \delta^- \vec{v}_j) = \frac{\vec{v}_{j+1} - \vec{v}_{j-1}}{2h}, \qquad j = 1, \dots, J-1; \tag{2.6c}$$

$$\delta^2 \vec{v}_j := \frac{\delta^+ \vec{v}_j - \delta^- \vec{v}_j}{h} = \frac{\vec{v}_{j+1} - 2\vec{v}_j + \vec{v}_{j-1}}{h^2}, \quad j = 1, \dots, J-1. \tag{2.6d}$$

Two grid functions $\vec{v}$ and $\vec{w}$ satisfy the following summation by parts formula:

$$h \sum_{j=1}^{J} \delta^- \vec{v}_j \cdot \delta^- \vec{w}_j = -h \sum_{j=1}^{J-1} \vec{v}_j \cdot \delta^2 \vec{w}_j + \vec{v}_J \cdot \delta^- \vec{w}_J - \vec{v}_0 \cdot \delta^+ \vec{w}_0. \tag{2.7}$$

In addition, we introduce the following discrete norms and seminorms:

$$|\vec{v}|_{0,h}^2 := \tfrac{1}{2} h |\vec{v}_0|^2 + h \sum_{j=1}^{J-1} |\vec{v}_j|^2 + \tfrac{1}{2} h |\vec{v}_J|^2; \quad |\vec{v}|_{1,h}^2 := h \sum_{j=1}^{J} |\delta^- \vec{v}_j|^2; \tag{2.8}$$

$$\|\vec{v}\|_{1,h}^2 := |\vec{v}|_{0,h}^2 + |\vec{v}|_{1,h}^2; \quad |\vec{v}|_{2,h}^2 := h \sum_{j=1}^{J-1} |\delta^2 \vec{v}_j|^2.$$

We also recall the following inverse inequality, as well as a discrete version of a well–known Sobolev type inequality.

LEMMA 2.2. *Let $\vec{v} : \mathcal{G}_h \to \mathbb{R}^2$ be an arbitrary grid function. Then*

$$\max_{1 \le k \le J} |\delta^- \vec{v}_k| \le h^{-\frac{1}{2}} |\vec{v}|_{1,h}, \tag{2.9}$$

$$\max_{0 \le k \le J} |\vec{v}_k|^2 \le |\vec{v}|_{0,h}^2 + 2 |\vec{v}|_{0,h} |\vec{v}|_{1,h}, \tag{2.10}$$

$$\max_{1 \le k \le J} |\delta^- \vec{v}_k|^2 \le |\vec{v}|_{1,h}^2 + 2 |\vec{v}|_{1,h} |\vec{v}|_{2,h}. \tag{2.11}$$

In addition, if $\vec{v}_0 \cdot \vec{e}_1 = \vec{v}_J \cdot \vec{e}_1 = 0$, then

$$|\vec{v}_j \cdot \vec{e}_1| \le 2 q_j (1 - q_j) \max_{1 \le k \le J} |\delta^- \vec{v}_k|, \quad 0 \le j \le J. \tag{2.12}$$

Proof. The inverse inequality (2.9) follows immediately from the definition (2.8). Let $0 \le k \le J$. For $0 \le j \le k$ it follows from (2.6b), the elementary inequality

$$(a+b)^2 \le 2(a^2 + b^2), \quad a, b \in \mathbb{R},$$

and (2.8) that

$$\begin{aligned} |\vec{v}_k|^2 &= |\vec{v}_j|^2 + \sum_{\ell=j}^{k-1} (|\vec{v}_{\ell+1}|^2 - |\vec{v}_\ell|^2) = |\vec{v}_j|^2 + h \sum_{\ell=j}^{k-1} (\vec{v}_{\ell+1} + \vec{v}_\ell) \cdot \delta^+ \vec{v}_\ell \\ &\le |\vec{v}_j|^2 + \sqrt{2} \left(h \sum_{\ell=j}^{k-1} (|\vec{v}_{\ell+1}|^2 + |\vec{v}_\ell|^2) \right)^{\frac{1}{2}} |\vec{v}|_{1,h} \le |\vec{v}_j|^2 + 2 |\vec{v}|_{0,h} |\vec{v}|_{1,h}. \end{aligned} \tag{2.13}$$

Similarly, for $k+1 \le j \le J$, we have

$$|\vec{v}_k|^2 = |\vec{v}_j|^2 - \sum_{\ell=k}^{j-1} (|\vec{v}_{\ell+1}|^2 - |\vec{v}_\ell|^2) \le |\vec{v}_j|^2 + 2|\vec{v}|_{0,h}|\vec{v}|_{1,h}. \tag{2.14}$$

Combining (2.13) and (2.14) yields that $\max_{0\le k\le J} |\vec{v}_k|^2 \le |\vec{v}_j|^2 + 2|\vec{v}|_{0,h}|\vec{v}|_{1,h}$ for $0 \le j \le J$. Multiplication by $\frac{h}{2}$ for $j = 0, J$ and by h for $1 \le j \le J-1$ followed by summation over $j = 0, \dots, J$ yields (2.10). The inequality (2.11) is obtained in an analogous manner, taking into account that $\delta^+\delta^-\vec{v}_j = \delta^2\vec{v}_j$.

In order to prove (2.12), we observe that $\vec{v}_0 \cdot \vec{e}_1 = \vec{v}_J \cdot \vec{e}_1 = 0$ implies

$$|\vec{v}_j \cdot \vec{e}_1| \le h \sum_{k=1}^{j} |\delta^-\vec{v}_k| \le jh \max_{1\le k\le J} |\delta^-\vec{v}_k| \quad \text{and} \quad |\vec{v}_j \cdot \vec{e}_1| \le (J-j)h \max_{1\le k\le J} |\delta^-\vec{v}_k|,$$

so that

$$|\vec{v}_j \cdot \vec{e}_1| \le \min\{q_j, 1-q_j\} \max_{1\le k\le J} |\delta^-\vec{v}_k| \le 2q_j(1-q_j) \max_{1\le k\le J} |\delta^-\vec{v}_k|, \quad 0 \le j \le J. \qquad \square$$

We consider the following fully discrete approximation, where in order to discretize in time, we let $t_m = m\,\Delta t$, $m = 0, \dots, M$ with the uniform time step $\Delta t = \frac{T}{M} > 0$. Let $\vec{X}_j^0 = \vec{x}_0(q_j)$, $j = 0, \dots, J$. Then, for $m = 0, \dots, M-1$ find $\vec{X}^m : \mathcal{G}_h \to \mathbb{R}^2$ such that for $j = 1, \dots, J-1$

$$\frac{\vec{X}_j^{m+1} - \vec{X}_j^m}{\Delta t} = \frac{\delta^2 \vec{X}_j^{m+1}}{|\delta^1 \vec{X}_j^m|^2} - \frac{1}{|\delta^1 \vec{X}_j^m|^2} \frac{\delta^1 \vec{X}_j^{m+1} \cdot \vec{e}_2}{\vec{X}_j^m \cdot \vec{e}_1} (\delta^1 \vec{X}_j^m)^\perp \tag{2.15a}$$

together with the boundary conditions,

$$\vec{X}_0^{m+1} \cdot \vec{e}_1 = 0, \quad \delta^+ \vec{X}_0^{m+1} \cdot \vec{e}_2 = \tfrac{1}{4} h |\delta^+ \vec{X}_0^m|^2 \frac{\vec{X}_0^{m+1} - \vec{X}_0^m}{\Delta t} \cdot \vec{e}_2, \tag{2.15b}$$

$$\vec{X}_J^{m+1} \cdot \vec{e}_1 = 0, \quad \delta^- \vec{X}_J^{m+1} \cdot \vec{e}_2 = -\tfrac{1}{4} h |\delta^- \vec{X}_J^m|^2 \frac{\vec{X}_J^{m+1} - \vec{X}_J^m}{\Delta t} \cdot \vec{e}_2. \tag{2.15c}$$

The above scheme requires the solution of a linear system in each time step. We will address the existence and uniqueness of this system in section 3 within the error analysis. Furthermore, we remark that (2.15b) and (2.15c) are obtained from inserting (1.6a) and (1.6b) into a Taylor expansion at $\rho \in \{0, 1\}$, yielding a consistency error that is small enough to derive optimal error bounds. At the same time, the form of these conditions turns out to be crucial in order to handle the degeneracy of the equation close to the axis of rotation.

LEMMA 2.3 (consistency). *Suppose that $\vec{x} : [0,1] \times [0,T] \to \mathbb{R}^2$ satisfies Assumption 2.1. Let $\vec{x}_j^m := \vec{x}(q_j, t_m)$ for $j = 0, \dots, J$ and $m = 0, \dots, M$. Define the consistency errors of the finite difference scheme (2.15) by*

$$\vec{R}_j^{m+1} := \frac{\vec{x}_j^{m+1} - \vec{x}_j^m}{\Delta t} - \frac{\delta^2 \vec{x}_j^{m+1}}{|\delta^1 \vec{x}_j^m|^2} + \frac{1}{|\delta^1 \vec{x}_j^m|^2} \frac{\delta^1 \vec{x}_j^{m+1} \cdot \vec{e}_2}{\vec{x}_j^m \cdot \vec{e}_1} (\delta^1 \vec{x}_j^m)^\perp, \quad 1 \le j \le J-1, \tag{2.16a}$$

as well as

$$R_0^{m+1} := \delta^+ \vec{x}_0^{m+1} \cdot \vec{e}_2 - \tfrac{1}{4} h |\delta^+ \vec{x}_0^m|^2 \frac{\vec{x}_0^{m+1} - \vec{x}_0^m}{\Delta t} \cdot \vec{e}_2, \tag{2.16b}$$

$$R_J^{m+1} := \delta^- \vec{x}_J^{m+1} \cdot \vec{e}_2 + \tfrac{1}{4} h |\delta^- \vec{x}_J^m|^2 \frac{\vec{x}_J^{m+1} - \vec{x}_J^m}{\Delta t} \cdot \vec{e}_2. \tag{2.16c}$$

Then there exists a constant $C > 0$ such that, for $m = 0, \dots, M-1$,

$$|\vec{R}_j^{m+1}| \le C\big(h^2 + \Delta t\big), \quad j = 1, \dots, J-1, \quad \text{and} \quad |R_0^{m+1}| + |R_J^{m+1}| \le Ch\big(h^2 + \Delta t\big). \tag{2.17}$$

Proof. Simple Taylor expansions yield the well-known results:

$$\left| \frac{\vec{x}_j^{m+1} - \vec{x}_j^m}{\Delta t} - \vec{x}_t(q_j, t_m) \right| \le C\Delta t, \qquad 0 \le j \le J,\ 0 \le m \le M-1, \tag{2.18a}$$

$$|\delta^- \vec{x}_j^m - \vec{x}_\rho(q_j, t_m)| \le Ch, \qquad 1 \le j \le J,\ 0 \le m \le M, \tag{2.18b}$$

$$|\delta^1 \vec{x}_j^m - \vec{x}_\rho(q_j, t_m)| + |\delta^2 \vec{x}_j^m - \vec{x}_{\rho\rho}(q_j, t_m)| \le Ch^2, \qquad 1 \le j \le J-1,\ 0 \le m \le M, \tag{2.18c}$$

$$\delta^1 \vec{x}_j^m - \vec{x}_\rho(q_j, t_m) - \tfrac{1}{6} h^2 \vec{x}_{\rho\rho\rho}(q_j, t_m) = \mathcal{O}(h^3), \quad 1 \le j \le J-1,\ 0 \le m \le M, \tag{2.18d}$$

where we have observed that $\vec{x}(q_j, \cdot) \in C^2([0,T])$ and $\vec{x}(\cdot, t_m) \in C^4([0,1])$. Evaluating (1.6a) at $(\rho, t) = (q_j, t_m)$, $j = 1, \dots, J-1$, $m = 0, \dots, M-1$, we find that

$$\vec{x}_t(q_j, t_m) = \frac{\vec{x}_{\rho\rho}(q_j, t_m)}{|\vec{x}_\rho(q_j, t_m)|^2} - \frac{1}{|\vec{x}_\rho(q_j, t_m)|^2} \frac{\vec{x}_\rho(q_j, t_m) \cdot \vec{e}_2}{\vec{x}_j^m \cdot \vec{e}_1} \vec{x}_\rho^{\perp}(q_j, t_m), \tag{2.19}$$

where the assumed regularity of $\vec{x}$ allows us to use (1.6a) also at time $t = 0$. If we combine (2.16a) with (2.19) and note (2.18a) as well as (2.18c), we obtain

$$\begin{aligned} |\vec{R}_j^{m+1}| &\le \left| \frac{\vec{x}_j^{m+1} - \vec{x}_j^m}{\Delta t} - \vec{x}_t(q_j, t_m) \right| + \left| \frac{\delta^2 \vec{x}_j^{m+1}}{|\delta^1 \vec{x}_j^m|^2} - \frac{\vec{x}_{\rho\rho}(q_j, t_m)}{|\vec{x}_\rho(q_j, t_m)|^2} \right| \\ &\quad + \frac{|\vec{x}_\rho(q_j, t_m) \cdot \vec{e}_2|}{\vec{x}_j^m \cdot \vec{e}_1} \left| \frac{(\delta^1 \vec{x}_j^m)^\perp}{|\delta^1 \vec{x}_j^m|^2} - \frac{\vec{x}_\rho^{\perp}(q_j, t_m)}{|\vec{x}_\rho(q_j, t_m)|^2} \right| \\ &\quad + \frac{1}{|\delta^1 \vec{x}_j^m|} \frac{|\big(\delta^1 \vec{x}_j^{m+1} - \vec{x}_\rho(q_j, t_m)\big) \cdot \vec{e}_2|}{\vec{x}_j^m \cdot \vec{e}_1} \\ &\le C\big(h^2 + \Delta t\big) + C \frac{|\big(\delta^1 \vec{x}_j^{m+1} - \vec{x}_\rho(q_j, t_m)\big) \cdot \vec{e}_2|}{\vec{x}_j^m \cdot \vec{e}_1}. \end{aligned} \tag{2.20}$$

In addition, it follows from (2.18d), (A.1b), (A.9), and (A.8) that

$$\begin{aligned} &|\big(\delta^1 \vec{x}_j^{m+1} - \vec{x}_\rho(q_j, t_m)\big) \cdot \vec{e}_2| \\ &\le |\big(\delta^1 \vec{x}_j^{m+1} - \delta^1 \vec{x}_j^m\big) \cdot \vec{e}_2| + |\big(\delta^1 \vec{x}^m - \vec{x}_\rho(q_j, t_m)\big) \cdot \vec{e}_2| \\ &\le \Delta t \sup_{t_m \le t \le t_{m+1}} |\delta^1 \vec{x}_t(q_j, t) \cdot \vec{e}_2| \\ &\quad + Ch^2 \min_{q \in \{0,1\}} |\vec{x}_{\rho\rho\rho}(q_j, t_m) \cdot \vec{e}_2 - \vec{x}_{\rho\rho\rho}(q, t_m) \cdot \vec{e}_2| + Ch^3 \\ &\le K\Delta t \sup_{t_m \le t \le t_{m+1}} \vec{x}(q_j, t) \cdot \vec{e}_1 + Ch^2 \min_{q \in \{0,1\}} |\vec{x}_{\rho\rho\rho}(q_j, t_m) - \vec{x}_{\rho\rho\rho}(q, t_m)| + Ch^3 \\ &\le C\Delta t q_j(1 - q_j) + Ch^2 \min_{q \in \{0,1\}} |q_j - q| + Ch^3 \le Cq_j(1 - q_j)\big(\Delta t + h^2\big). \end{aligned}$$

If we insert this bound into (2.20) and note that $\vec{x}_j^m \cdot \vec{e}_1 \geq c_2 q_j(1-q_j)$, $0 \leq j \leq J$, in view of (A.8), we obtain (2.17) for $\vec{R}_j^{m+1}$, $j = 1, \ldots, J-1$. Let us next examine R_0^{m+1}. A Taylor expansion yields

$$\delta^+ \vec{x}_0^{m+1} = \frac{\vec{x}_1^{m+1} - \vec{x}_0^{m+1}}{h} = \vec{x}_\rho(0, t_{m+1}) + \tfrac{1}{2} h \vec{x}_{\rho\rho}(0, t_{m+1}) + \tfrac{1}{6} h^2 \vec{x}_{\rho\rho\rho}(0, t_{m+1}) + \mathcal{O}(h^3),$$

which together with (1.6b), (A.1b), $\vec{x}_{\rho\rho}(0, \cdot) \in C^1([0,T])$, and (A.1c) implies that

$$\begin{aligned}
\delta^+ \vec{x}_0^{m+1} \cdot \vec{e}_2 &= \tfrac{1}{2} h \vec{x}_{\rho\rho}(0, t_{m+1}) \cdot \vec{e}_2 + \mathcal{O}(h^3) = \tfrac{1}{2} h \vec{x}_{\rho\rho}(0, t_m) \cdot \vec{e}_2 + \mathcal{O}\big(h(h^2 + \Delta t)\big) \\
&= \tfrac{1}{4} h (\vec{x}_t(0, t_m) \cdot \vec{e}_2) |\vec{x}_\rho(0, t_m)|^2 + \mathcal{O}\big(h(h^2 + \Delta t)\big) \\
&= \tfrac{1}{4} h \, \frac{\vec{x}_0^{m+1} - \vec{x}_0^m}{\Delta t} \cdot \vec{e}_2 |\delta^+ \vec{x}_0^m|^2 + \mathcal{O}\big(h(h^2 + \Delta t)\big),
\end{aligned}$$

where we have used (2.18a) as well as

$$\begin{aligned}
|\delta^+ \vec{x}_0^m|^2 - |\vec{x}_\rho(0, t_m)|^2 &= \big(\delta^+ \vec{x}_0^m - \vec{x}_\rho(0, t_m)\big) \cdot \big(\delta^+ \vec{x}_0^m + \vec{x}_\rho(0, t_m)\big) \\
&= \big(\tfrac{1}{2} h \vec{x}_{\rho\rho}(0, t_m) + \mathcal{O}(h^2)\big) \cdot \big(2 \vec{x}_\rho(0, t_m) + \mathcal{O}(h)\big) \\
&= h \vec{x}_{\rho\rho}(0, t_m) \cdot \vec{x}_\rho(0, t_m) + \mathcal{O}(h^2) = \mathcal{O}(h^2),
\end{aligned}$$

recall (A.1a). The bound for R_J^{m+1} is obtained in a similar way. ∎

THEOREM 2.4. *Suppose that $\vec{x} : [0,1] \times [0,T] \to \mathbb{R}^2$ satisfies Assumption 2.1. Then there exist $h_0 > 0, \gamma > 0$ such that the discrete solution $(\vec{X}^m)_{m=1,\ldots,M}$ to (2.15) exists, and the error*

$$\vec{E}_j^m := \vec{x}_j^m - \vec{X}_j^m, \quad j = 0, \ldots, J; \ m = 0, \ldots, M \tag{2.21}$$

satisfies

$$\max_{1 \leq m \leq M} \left[\|\vec{E}^m\|_{1,h}^2 + \max_{0 \leq j \leq J} |\vec{E}_j^m|^2 \right] \leq C\big(h^4 + (\Delta t)^2\big), \tag{2.22}$$

$$\Delta t \sum_{m=1}^{M} \left[|\vec{E}^m|_{2,h}^2 + \left| \frac{\vec{E}^m - \vec{E}^{m-1}}{\Delta t} \right|_{0,h}^2 \right] \leq C\big(h^4 + (\Delta t)^2\big), \tag{2.23}$$

provided that $0 < h \leq h_0$ and $\Delta t \leq \gamma h$.

3. Proof of Theorem 2.4. Assumption 2.1 assures the existence of positive constants c_0, C_0, c_1, δ such that (2.2) and (2.3) hold. Let $h \leq \delta$. We set $J_1 := \lfloor \frac{\delta}{h} \rfloor \in \mathbb{Z}_{\geq 1}$ so that $q_{J_1} = J_1 h \in [\frac{1}{2}\delta, \delta]$. We shall prove Theorem 2.4 with the help of an induction argument. In particular, we will prove that there exist $h_0 > 0$, $0 < \gamma \leq 1$, and $\mu > 0$ such that if $0 < h \leq h_0$ and $\Delta t \leq \gamma h$, then for $m \in \{0, \ldots, M\}$ the discrete solution $\vec{X}^m$ exists and satisfies

$$\|\vec{E}^m\|_{1,h}^2 \leq \big(h^4 + (\Delta t)^2\big) e^{\mu t_m}. \tag{3.1}$$

The assertion (3.1) clearly holds for $m = 0$, for arbitrary $h_0 \leq \delta$, $0 < \gamma \leq 1$, and $\mu > 0$. On assuming that (3.1) holds for a fixed $m \in \{0, \ldots, M-1\}$, we will now show that it also holds for $m+1$.

To begin, let us choose $0 < h_0 \le \delta$ and $0 < \gamma \le 1$ so small that

$$(h_0^2 + \gamma^2)e^{\mu T} \le 1.$$

Then, since $\Delta t \le \gamma h$, (3.1) implies that

$$\|\vec{E}^m\|_{1,h}^2 \le h^2(h^2+\gamma^2)e^{\mu t_m} \le h^2, \qquad 0 < h \le h_0.$$

In particular, we infer from Lemma 2.2 that

$$\max_{0\le j\le J} |\vec{E}_j^m| + \max_{1\le j\le J} |\delta^- \vec{E}_j^m| + \max_{1\le j\le J-1} |\delta^1 \vec{E}_j^m| \le Ch^{\frac12}. \tag{3.2}$$

This implies for $1 \le j \le J$, on recalling (2.18b) and (2.2), that

$$|\delta^- \vec{X}_j^m| \le |\delta^- \vec{x}_j^m| + |\delta^- \vec{E}_j^m| \le |\vec{x}_\rho(q_j, t_m)| + Ch^{\frac12} \le C_0 + Ch^{\frac12},$$

and similarly $|\delta^- \vec{X}_j^m| \ge c_0 - Ch^{\frac12}$. Arguing in the same way for $\delta^1 \vec{X}_j^m$, we infer that

$$\tfrac12 c_0 \le |\delta^- \vec{X}_j^m| \le 2C_0,\ 1 \le j \le J, \quad \tfrac12 c_0 \le |\delta^1 \vec{X}_j^m| \le 2C_0,\ 1 \le j \le J-1, \tag{3.3}$$

provided that $0 < h \le h_0$, with $h_0 > 0$ being chosen smaller if necessary. A similar argument together with (2.3a) and (2.3b) shows that

$$\delta^- \vec{X}_j^m \cdot \vec{e}_1 \ge \tfrac14 c_0, \quad 1 \le j \le J_1, \qquad \delta^- \vec{X}_j^m \cdot \vec{e}_1 \le -\tfrac14 c_0, \quad J - J_1 \le j \le J. \tag{3.4}$$

Next, since $\vec{x}_\rho(0, t_m) \cdot \vec{e}_2 = 0$, recall (1.6b), we have from (3.2) and (2.15b) that

$$\begin{aligned} h\,|\delta^- \vec{X}_1^m| &\le h\,|\delta^- \vec{x}_1^m| + h\,|\delta^- \vec{E}_1^m| \le h\,\delta^- \vec{x}_1^m \cdot \vec{e}_1 + h\,|\delta^- \vec{x}_1^m \cdot \vec{e}_2| + Ch^{\frac32} \\ &\le h\,\delta^- \vec{X}_1^m \cdot \vec{e}_1 + Ch^{\frac32} = \vec{X}_1^m \cdot \vec{e}_1 + Ch^{\frac32} \le (1 + Ch^{\frac12})\vec{x}_1^m \cdot \vec{e}_1 + Ch^{\frac32}, \end{aligned}$$

where in the last step we have observed that

$$\vec{X}_1^m \cdot \vec{e}_1 - \vec{x}_1^m \cdot \vec{e}_1 = -h\delta^- \vec{E}_1^m \cdot \vec{e}_1 \le Ch^{\frac32} \le Ch^{\frac12} \vec{x}_1^m \cdot \vec{e}_1$$

on noting $\vec{x}_1^m \cdot \vec{e}_1 \ge \frac12 c_0 h$. Arguing in the same way at the right boundary, we obtain

$$\tfrac34 h\,|\delta^- \vec{X}_1^m| \le \vec{X}_1^m \cdot \vec{e}_1 \le \tfrac43 \vec{x}_1^m \cdot \vec{e}_1, \qquad \tfrac34 h |\delta^- \vec{X}_J^m| \le \vec{X}_{J-1}^m \cdot \vec{e}_1 \le \tfrac43 \vec{x}_{J-1}^m \cdot \vec{e}_1 \tag{3.5}$$

for a possibly smaller $h_0 > 0$. Next, (3.2), (2.3c), and the fact that $J_1 h \ge \frac12 \delta$ imply that

$$\vec{X}_j^m \cdot \vec{e}_1 \ge c_1 - Ch^{\frac12} \ge \tfrac12 c_1, \quad J_1 \le j \le J - J_1 \tag{3.6}$$

after choosing h_0 smaller again if required. In addition, there exists $c_3 > 0$ such that

$$\vec{X}_j^m \cdot \vec{e}_1 \ge c_3 q_j (1 - q_j), \quad 0 \le j \le J. \tag{3.7}$$

To see this, note that $\vec{X}_0^m \cdot \vec{e}_1 = 0$ and (3.4) imply that

$$\vec{X}_j^m \cdot \vec{e}_1 \ge \tfrac14 c_0 jh \ge \tfrac14 c_0 q_j (1 - q_j), \quad 0 \le j \le J_1, \tag{3.8a}$$

and similarly

$$\vec{X}_j^m \cdot \vec{e}_1 \geq \tfrac{1}{4} c_0 q_j (1 - q_j), \quad J - J_1 \leq j \leq J. \tag{3.8b}$$

Combining these estimates with (3.6) proves the bound (3.7). If we combine (3.7) with (2.12) and (3.3), we obtain

$$\frac{\vec{X}_{j\pm1}^m \cdot \vec{e}_1}{\vec{X}_j^m \cdot e_1} \leq \frac{4C_0}{c_3} \frac{q_{j\pm1}(1 - q_{j\pm1})}{q_j(1 - q_j)} \leq \frac{8C_0}{c_3}. \quad 1 \leq j \leq J - 1. \tag{3.9}$$

Finally, (2.2) and (3.3) imply that

$$\left| \frac{1}{|\delta^- \vec{x}_j^m|^2} - \frac{1}{|\delta^- \vec{X}_j^m|^2} \right| \leq C |\delta^- \vec{E}_j^m|, \quad 1 \leq j \leq J, \tag{3.10a}$$

$$\left| \frac{1}{|\delta^1 \vec{x}_j^m|^2} - \frac{1}{|\delta^1 \vec{X}_j^m|^2} \right| \leq C |\delta^1 \vec{E}_j^m|, \quad 1 \leq j \leq J - 1. \tag{3.10b}$$

LEMMA 3.1 (existence and uniqueness). *Let $\vec{X}^m : \mathcal{G}_h \to \mathbb{R}^2$ be as above. Then (2.15) has a unique solution $\vec{X}^{m+1} : \mathcal{G}_h \to \mathbb{R}^2$, provided that h_0 is small enough.*

Proof. The relations (2.15) form a linear system with $2(J+1)$ unknowns for the $2(J+1)$ values of $\vec{X}^{m+1}$ at the nodes q_j, $j = 0, \ldots, J$. It is therefore sufficient to show that the corresponding homogeneous system,

$$\frac{1}{\Delta t} \vec{X}_j - \frac{1}{|\delta^1 \vec{X}_j^m|^2} \delta^2 \vec{X}_j = - \frac{1}{|\delta^1 \vec{X}_j^m|^2} \frac{\delta^1 \vec{X}_j \cdot \vec{e}_2}{\vec{X}_j^m \cdot \vec{e}_1} (\delta^1 \vec{X}_j^m)^\perp, \quad j = 1, \ldots, J - 1, \tag{3.11a}$$

$$\vec{X}_0 \cdot \vec{e}_1 = 0, \quad \delta^+ \vec{X}_0 \cdot \vec{e}_2 = \tfrac{1}{4} \frac{h}{\Delta t} (\vec{X}_0 \cdot \vec{e}_2) \, |\delta^+ \vec{X}_0^m|^2, \tag{3.11b}$$

$$\vec{X}_J \cdot \vec{e}_1 = 0, \quad \delta^- \vec{X}_J \cdot \vec{e}_2 = - \tfrac{1}{4} \frac{h}{\Delta t} (\vec{X}_J \cdot \vec{e}_2) \, |\delta^- \vec{X}_J^m|^2, \tag{3.11c}$$

only has the trivial solution $\vec{X} = \vec{0}$. If we multiply (3.11a) with $-h\delta^2 \vec{X}_j$ and sum from $j = 1, \ldots, J-1$ we obtain with the help of (2.7) that

$$\begin{aligned} &\frac{1}{\Delta t} |\vec{X}|_{1,h}^2 + \frac{1}{\Delta t} \big(\vec{X}_0 \cdot \delta^+ \vec{X}_0 - \vec{X}_J \cdot \delta^- \vec{X}_J \big) + h \sum_{j=1}^{J-1} \frac{1}{|\delta^1 \vec{X}_j^m|^2} |\delta^2 \vec{X}_j|^2 \\ &\quad = h \sum_{j=1}^{J-1} \frac{1}{|\delta^1 \vec{X}_j^m|^2} \frac{\delta^1 \vec{X}_j \cdot \vec{e}_2}{\vec{X}_j^m \cdot \vec{e}_1} (\delta^1 \vec{X}_j^m)^\perp \cdot \delta^2 \vec{X}_j. \end{aligned}$$

In view of (3.11b) and (3.5) we have

$$\frac{1}{\Delta t} \vec{X}_0 \cdot \delta^+ \vec{X}_0 = \frac{1}{\Delta t} (\vec{X}_0 \cdot \vec{e}_2)(\delta^+ \vec{X}_0 \cdot \vec{e}_2) = \frac{4}{h} \frac{(\delta^+ \vec{X}_0 \cdot \vec{e}_2)^2}{|\delta^+ \vec{X}_0^m|^2} \geq \tfrac{9}{4} h \frac{(\delta^- \vec{X}_1 \cdot \vec{e}_2)^2}{(\vec{X}_1^m \cdot \vec{e}_1)^2}.$$

Using a similar argument at the right endpoint, as well as (3.3), we deduce

$$\begin{aligned}(3.12)\qquad &\frac{1}{\Delta t}|\vec{X}|^2_{1,h}+\frac{1}{4C_0^2}|\vec{X}|^2_{2,h}+\tfrac{9}{4}h\left(\frac{(\delta^-\vec{X}_1\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}+\frac{(\delta^+\vec{X}_{J-1}\cdot\vec{e}_2)^2}{(\vec{X}_{J-1}^m\cdot\vec{e}_1)^2}\right)\\ &\le h\sum_{j=1}^{J-1}\frac{\delta^1\vec{X}_j\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}\left(\frac{(\delta^1\vec{X}_j^m)^\perp}{|\delta^1\vec{X}_j^m|^2}-\frac{(\delta^1\vec{x}_j^m)^\perp}{|\delta^1\vec{x}_j^m|^2}\right)\cdot\delta^2\vec{X}_j\\ &\quad+h\sum_{j=1}^{J-1}\frac{1}{|\delta^1\vec{x}_j^m|^2}\frac{\delta^1\vec{X}_j\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}(\delta^1\vec{x}_j^m)^\perp\cdot\delta^2\vec{X}_j\\ &=:h\sum_{j=1}^{J-1}\vec{S}_j^1\cdot\delta^2\vec{X}_j+h\sum_{j=1}^{J-1}\vec{S}_j^2\cdot\delta^2\vec{X}_j.\end{aligned}$$

Using (3.10b) and (3.2) we infer that

$$|\vec{S}_j^1|\le C\frac{|\delta^1\vec{X}_j\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1}\,|\delta^1\vec{E}_j^m|\,|\delta^2\vec{X}_j|\le Ch^{\frac12}\frac{|\delta^1\vec{X}_j\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1}\,|\delta^2\vec{X}_j|,$$

and hence

$$h\sum_{j=1}^{J-1}\vec{S}_j^1\cdot\delta^2\vec{X}_j\le\frac{1}{8C_0^2}|\vec{X}|^2_{2,h}+Ch^2\sum_{j=1}^{J-1}\frac{(\delta^1\vec{X}_j\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}.$$

The term $\vec{S}_j^2$ corresponds exactly to $-\vec{T}_j^{m,3}$ in (3.13) below if we replace $\vec{E}^{m+1}$ by $\vec{X}$. We may therefore deduce from Lemma 3.4 that

$$\begin{aligned}h\sum_{j=1}^{J-1}\vec{S}_j^2\cdot\delta^2\vec{X}_j&\le-c_4h\sum_{j=1}^{J-1}\frac{(\delta^1\vec{X}_j\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}+\tfrac{5}{3}h\left(\frac{(\delta^-\vec{X}_1\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}+\frac{(\delta^+\vec{X}_{J-1}\cdot\vec{e}_2)^2}{(\vec{X}_{J-1}^m\cdot\vec{e}_1)^2}\right)\\&\quad+\frac{1}{8C_0^2}|\vec{X}|^2_{2,h}+C\,|\vec{X}|^2_{1,h}.\end{aligned}$$

If we insert the above bounds into (3.12) and recall that $\Delta t\le\gamma h\le h$ we infer that

$$\left(\frac1h-C\right)|\vec{X}|^2_{1,h}+(c_4-Ch)h\sum_{j=1}^{J-1}\frac{(\delta^1\vec{X}_j\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}\le0,$$

which implies that $\vec{X}\equiv\vec{X}_0$ provided that $0<h\le h_0$, where h_0 is chosen smaller if necessary. The boundary conditions (3.11b), on noting (3.3), then yield $\vec{X}\equiv0$. ∎

We begin our error analysis by combining (2.21), (2.15a), and (2.16a) in order to derive the following error relation:

(3.13)

$$\begin{aligned}&\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}-\frac{\delta^2\vec{E}_j^{m+1}}{|\delta^1\vec{X}_j^m|^2}=\left(\frac{1}{|\delta^1\vec{x}_j^m|^2}-\frac{1}{|\delta^1\vec{X}_j^m|^2}\right)\delta^2\vec{x}_j^{m+1}\\&\quad+\frac{\delta^1\vec{X}_j^{m+1}\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}\left[\left(\frac{1}{|\delta^1\vec{X}_j^m|^2}-\frac{1}{|\delta^1\vec{x}_j^m|^2}\right)(\delta^1\vec{X}_j^m)^\perp-\frac{1}{|\delta^1\vec{x}_j^m|^2}(\delta^1\vec{E}_j^m)^\perp\right]\\&\quad-\frac{1}{|\delta^1\vec{x}_j^m|^2}\frac{\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}(\delta^1\vec{x}_j^m)^\perp\end{aligned}$$

$$+\frac{1}{|\delta^1\vec{x}_j^m|^2}\left(\frac{1}{\vec{X}_j^m\cdot\vec{e}_1}-\frac{1}{\vec{x}_j^m\cdot\vec{e}_1}\right)(\delta^1\vec{x}_j^{m+1}\cdot\vec{e}_2)(\delta^1\vec{x}_j^m)^\perp+\vec{R}_j^{m+1}$$
$$=:\sum_{i=1}^5\vec{T}_j^{m,i},\qquad 1\le j\le J-1.$$

Furthermore, for the boundary points we have in view of (1.6b), (2.15b), (2.15c), (2.16b), and (2.16c) that

$$(\vec{E}_0^{m+1}-\vec{E}_0^m)\cdot\vec{e}_1=(\vec{E}_J^{m+1}-\vec{E}_J^m)\cdot\vec{e}_1=0, \tag{3.14a}$$

$$\frac{\vec{E}_0^{m+1}-\vec{E}_0^m}{\Delta t}=\frac{4}{h}\frac{\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2-R_0^{m+1}}{|\delta^+\vec{X}_0^m|^2}\vec{e}_2+\Big(1-\frac{|\delta^+\vec{x}_0^m|^2}{|\delta^+\vec{X}_0^m|^2}\Big)\frac{\vec{x}_0^{m+1}-\vec{x}_0^m}{\Delta t}, \tag{3.14b}$$

$$\frac{\vec{E}_J^{m+1}-\vec{E}_J^m}{\Delta t}=-\frac{4}{h}\frac{\delta^-\vec{E}_J^{m+1}\cdot\vec{e}_2-R_J^{m+1}}{|\delta^-\vec{X}_J^m|^2}\vec{e}_2+\Big(1-\frac{|\delta^-\vec{x}_J^m|^2}{|\delta^-\vec{X}_J^m|^2}\Big)\frac{\vec{x}_J^{m+1}-\vec{x}_J^m}{\Delta t}. \tag{3.14c}$$

Our strategy for the proof of (3.1) with m replaced by $m+1$ is now as follows. In a discrete analog to the formal procedure in (2.4), we are going to multiply (3.13) with a second order difference of the error $\vec{E}^{m+1}$. The ensuing analysis is technical, and so we split it into three steps. In a first step, we control the terms generated on the left-hand side of (3.13) in order to obtain Lemma 3.2. Next we estimate four of the five terms generated by the right-hand side of (3.13); see Lemma 3.3. The remaining term, which is generated by $\vec{T}^{m,3}$ and loosely corresponds to the last integral in (2.4), requires a particularly careful analysis. We present the derived estimate in Lemma 3.4, where in the proof we will mimic the formal calulations from (2.5).

The induction step is then completed by combining the three lemmas.

LEMMA 3.2. *There exists $C_1>0$ such that for all $0<\lambda\le 1$*

(3.15)

$$\begin{aligned}&\frac{1+\lambda}{2\Delta t}\big(|\vec{E}^{m+1}|_{1,h}^2-|\vec{E}^m|_{1,h}^2\big)+\frac{1}{2\Delta t}|\vec{E}^{m+1}-\vec{E}^m|_{1,h}^2+\frac{1}{4C_0^2}|\vec{E}^{m+1}|_{2,h}^2\\&\quad+\tfrac14 c_0^2\lambda\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2+(2-C_1\lambda)h\left[\frac{\big(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2\big)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}+\frac{\big(\delta^+\vec{E}_{J-1}^{m+1}\cdot\vec{e}_2\big)^2}{(\vec{X}_{J-1}^m\cdot\vec{e}_1)^2}\right]\\&\le Ch\big(h^4+(\Delta t)^2\big)+C|\vec{E}^m|_{1,h}^2\\&\quad+h\sum_{i=1}^5\sum_{j=1}^{J-1}\vec{T}_j^{m,i}\cdot\left(\lambda|\delta^1\vec{X}_j^m|^2\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}-\delta^2\vec{E}_j^{m+1}\right).\end{aligned}$$

Proof. Fix $0<\lambda\le 1$. If we multiply (3.13) by $h(\lambda|\delta^1\vec{X}_j^m|^2\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}-\delta^2\vec{E}_j^{m+1})$ and sum over $j=1,\dots,J-1$, we obtain

$$\begin{aligned}&-(1+\lambda)\frac{h}{\Delta t}\sum_{j=1}^{J-1}\big(\vec{E}_j^{m+1}-\vec{E}_j^m\big)\cdot\delta^2\vec{E}_j^{m+1}+h\sum_{j=1}^{J-1}\frac{|\delta^2\vec{E}_j^{m+1}|^2}{|\delta^1\vec{X}_j^m|^2}\\&\qquad+h\lambda\sum_{j=1}^{J-1}|\delta^1\vec{X}_j^m|^2\left|\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}\right|^2\\&\quad=h\sum_{i=1}^5\sum_{j=1}^{J-1}\vec{T}_j^{m,i}\cdot\left(\lambda|\delta^1\vec{X}_j^m|^2\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}-\delta^2\vec{E}_j^{m+1}\right).\end{aligned} \tag{3.16}$$

Applying summation by parts, (2.7), to the first term in (3.16), and noting (2.8) and $2(a-b)a = a^2 - b^2 + (a-b)^2$, yields

$$
\begin{aligned}
(3.17)\qquad & -\frac{h}{\Delta t}\sum_{j=1}^{J-1}\big(\vec{E}_j^{m+1}-\vec{E}_j^{m}\big)\cdot\delta^2\vec{E}_j^{m+1}\\
&=\frac{h}{\Delta t}\sum_{j=1}^{J}\delta^-\big(\vec{E}_j^{m+1}-\vec{E}_j^{m}\big)\cdot\delta^-\vec{E}_j^{m+1}\\
&\qquad-\frac{\vec{E}_J^{m+1}-\vec{E}_J^{m}}{\Delta t}\cdot\delta^-\vec{E}_J^{m+1}+\frac{\vec{E}_0^{m+1}-\vec{E}_0^{m}}{\Delta t}\cdot\delta^+\vec{E}_0^{m+1}\\
&=\frac{1}{2\Delta t}\big(|\vec{E}^{m+1}|_{1,h}^2-|\vec{E}^{m}|_{1,h}^2\big)+\frac{1}{2\Delta t}|\vec{E}^{m+1}-\vec{E}^{m}|_{1,h}^2\\
&\qquad-\frac{\vec{E}_J^{m+1}-\vec{E}_J^{m}}{\Delta t}\cdot\delta^-\vec{E}_J^{m+1}+\frac{\vec{E}_0^{m+1}-\vec{E}_0^{m}}{\Delta t}\cdot\delta^+\vec{E}_0^{m+1}.
\end{aligned}
$$

On noting (3.14b), (2.6b), Young's inequality, (2.17), (3.3), (3.10a), and (3.5), we can estimate the last term on the right-hand side of (3.17) as

$$
\begin{aligned}
\frac{\vec{E}_0^{m+1}-\vec{E}_0^{m}}{\Delta t}\cdot\delta^+\vec{E}_0^{m+1}&=\frac{4}{h}\frac{\big(\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2\big)^2}{|\delta^+\vec{X}_0^{m}|^2}-\frac{4}{h}\frac{\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2}{|\delta^+\vec{X}_0^{m}|^2}R_0^{m+1}\\
&\qquad+\left(1-\frac{|\delta^+\vec{x}_0^{m}|^2}{|\delta^+\vec{X}_0^{m}|^2}\right)\frac{\vec{x}_0^{m+1}-\vec{x}_0^{m}}{\Delta t}\cdot\vec{e}_2(\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2)\\
&\ge\frac{4-\varepsilon}{h}\frac{\big(\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2\big)^2}{|\delta^-\vec{X}_1^{m}|^2}-C_\varepsilon h\big(h^4+(\Delta t)^2\big)-C_\varepsilon h\,|\delta^+\vec{E}_0^{m}|^2\\
&\ge\tfrac{9}{16}(4-\varepsilon)h\,\frac{\big(\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2\big)^2}{(\vec{X}_1^{m}\cdot\vec{e}_1)^2}-C_\varepsilon h\big(h^4+(\Delta t)^2\big)-C_\varepsilon h\,|\delta^+\vec{E}_0^{m}|^2.
\end{aligned}
$$

On choosing ε sufficiently small, and arguing similarly for $\frac{\vec{E}_J^{m+1}-\vec{E}_J^{m}}{\Delta t}\cdot\delta^-\vec{E}_J^{m+1}$, we find that (3.17) implies

$$
\begin{aligned}
(3.18)\qquad -(1+\lambda)\frac{h}{\Delta t}\sum_{j=1}^{J-1}\big(\vec{E}_j^{m+1}-\vec{E}_j^{m}\big)\cdot\delta^2\vec{E}_j^{m+1}\ge\ & \frac{1+\lambda}{2\Delta t}\big(|\vec{E}^{m+1}|_{1,h}^2-|\vec{E}^{m}|_{1,h}^2\big)\\
&+\frac{1}{2\Delta t}|\vec{E}^{m+1}-\vec{E}^{m}|_{1,h}^2+2h\left[\frac{\big(\delta^+\vec{E}_0^{m+1}\cdot\vec{e}_2\big)^2}{(\vec{X}_1^{m}\cdot\vec{e}_1)^2}+\frac{\big(\delta^-\vec{E}_J^{m+1}\cdot\vec{e}_2\big)^2}{(\vec{X}_{J-1}^{m}\cdot\vec{e}_1)^2}\right]\\
&-Ch\big(|\delta^-\vec{E}_1^{m}|^2+|\delta^-\vec{E}_J^{m}|^2\big)-Ch\big(h^4+(\Delta t)^2\big).
\end{aligned}
$$

In addition, we deduce from (3.14b), (3.10a), (3.3), and the fact that, thanks to (3.7), we have $h|\delta^+\vec{X}_0^{m}|^2\ge h^{-1}\,(\vec{X}_1^{m}\cdot\vec{e}_1)^2\ge C\,\vec{X}_1^{m}\cdot\vec{e}_1$ that

$$
\left|\frac{\vec{E}_0^{m+1}-\vec{E}_0^{m}}{\Delta t}\right|\le C\frac{|\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2|}{\vec{X}_1^{m}\cdot\vec{e}_1}+C\frac{1}{h}|R_0^{m+1}|+C|\delta^-\vec{E}_1^{m}|,
$$

so that (2.17) yields

$$
(3.19)\qquad \frac{h}{2}\left|\frac{\vec{E}_0^{m+1}-\vec{E}_0^{m}}{\Delta t}\right|^2\le Ch\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_1^{m}\cdot\vec{e}_1)^2}+Ch(h^2+\Delta t)^2+Ch|\delta^-\vec{E}_1^{m}|^2.
$$

Inserting (3.18) into (3.16) and using (2.8), (3.3), and (2.6b), as well as (3.19) and a corresponding estimate at the right boundary, we obtain the desired result (3.15). ∎

LEMMA 3.3. *Let* $0 < \lambda \le 1$ *and* $\Delta t \le \gamma h$. *Then*

$$
\begin{aligned}
(3.20)\quad & h\sum_{i=1}^{5}\sum_{j=1}^{J-1}\vec{T}_j^{m,i}\cdot\left(\lambda|\delta^1\vec{X}_j^m|^2\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}-\delta^2\vec{E}_j^{m+1}\right)+h\sum_{j=1}^{J-1}\vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1}\\
&\le\frac{1}{8C_0^2}|\vec{E}^{m+1}|_{2,h}^2+C\big(|\vec{E}^m|_{1,h}^2+|\vec{E}^{m+1}|_{1,h}^2\big)+C\,\frac{\gamma}{\Delta t}|\vec{E}^{m+1}-\vec{E}^m|_{1,h}^2\\
&\quad+\tfrac{1}{8}c_0^2\lambda\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2+C\big(h^4+(\Delta t)^2\big)+C(\lambda+h)h\sum_{j=1}^{J-1}\frac{(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}.
\end{aligned}
$$

Proof. We recall the definitions of the terms $\vec{T}^{m,i}$ in (3.13). Then we note from (3.10b), (3.3), (3.7), (A.8), and (3.2) that

$$
\begin{aligned}
|\vec{T}_j^{m,1}|+|\vec{T}_j^{m,2}|&\le C\left(|\delta^2\vec{x}_j^{m+1}|+\frac{|\delta^1\vec{X}_j^{m+1}\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1}\right)|\delta^1\vec{E}_j^m|\\
&\le C\left(1+\frac{|\delta^1\vec{x}_j^{m+1}\cdot\vec{e}_2|}{\vec{x}_j^{m+1}\cdot\vec{e}_1}\frac{\vec{x}_j^{m+1}\cdot\vec{e}_1}{\vec{X}_j^m\cdot\vec{e}_1}+\frac{|\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1}\right)|\delta^1\vec{E}_j^m|\\
&\le C|\delta^1\vec{E}_j^m|+Ch^{\frac12}\frac{|\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1},\quad 1\le j\le J-1,
\end{aligned}
$$

so that (2.6c), (2.6b), (3.3), and (2.8) imply that

$$
\begin{aligned}
(3.21)\quad & h\sum_{i=1}^{2}\sum_{j=1}^{J-1}\vec{T}_j^{m,i}\cdot\left(\lambda\,|\delta^1\vec{X}_j^m|^2\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}-\delta^2\vec{E}_j^{m+1}\right)\\
&\le Ch\sum_{j=1}^{J-1}\left(|\delta^-\vec{E}_j^m|+|\delta^-\vec{E}_{j+1}^m|+h^{\frac12}\frac{|\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1}\right)\left(\lambda\left|\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}\right|+|\delta^2\vec{E}_j^{m+1}|\right)\\
&\le\varepsilon\lambda\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2+\varepsilon\,|\vec{E}^{m+1}|_{2,h}^2+C_\varepsilon\,|\vec{E}^m|_{1,h}^2+C_\varepsilon h^2\sum_{j=1}^{J-1}\frac{(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}.
\end{aligned}
$$

The term involving the product of $\vec{T}^{m,3}$ with $\delta^2\vec{E}^{m+1}$ is not estimated, while

$$
\begin{aligned}
(3.22)\quad & h\sum_{j=1}^{J-1}\vec{T}_j^{m,3}\cdot\lambda\,|\delta^1\vec{X}_j^m|^2\frac{\vec{E}_j^{m+1}-\vec{E}_j^m}{\Delta t}\\
&\le C\lambda\left(h\sum_{j=1}^{J-1}\frac{(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}\right)^{\frac12}\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}\\
&\le\varepsilon\lambda\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2+C_\varepsilon\lambda h\sum_{j=1}^{J-1}\frac{(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}.
\end{aligned}
$$

Next, we have

$$
(3.23)\qquad |\vec{T}_j^{m,4}|\le C\frac{|\vec{E}_j^m\cdot\vec{e}_1|}{(\vec{x}_j^m\cdot\vec{e}_1)(\vec{X}_j^m\cdot\vec{e}_1)}\,|\delta^1\vec{x}_j^{m+1}\cdot\vec{e}_2|,\quad 1\le j\le J-1.
$$

In addition, (A.8) and (A.9) yield

$$|\delta^1 \vec{x}_j^{m+1} \cdot \vec{e}_2| = \frac{|\delta^1 \vec{x}_j^{m+1} \cdot \vec{e}_2|}{\vec{x}_j^{m+1} \cdot \vec{e}_1} \vec{x}_j^{m+1} \cdot \vec{e}_1 \le C q_j (1-q_j), \quad 1 \le j \le J-1,$$

so that (3.23), (2.12), (A.8), (3.7), and (2.9) imply

$$\begin{aligned} |\vec{T}_j^{m,4}| &\le C \, \frac{q_j^2 (1-q_j)^2}{c_2 c_3 q_j^2 (1-q_j)^2} \max_{1 \le k \le J} |\delta^- \vec{E}_k^m| \le C \max_{1 \le k \le J} |\delta^- \vec{E}_k^m| \\ &\le C \big(\max_{1 \le k \le J} |\delta^- \vec{E}_k^{m+1}| + \max_{1 \le k \le J} |\delta^- (\vec{E}_k^{m+1} - \vec{E}_k^m)| \big) \\ &\le C \max_{1 \le k \le J} |\delta^- \vec{E}_k^{m+1}| + C h^{-\frac{1}{2}} |\vec{E}^{m+1} - \vec{E}^m|_{1,h}, \quad 1 \le j \le J-1. \end{aligned}$$

Hence we obtain with the help of (2.11) and the fact that $\Delta t \le \gamma h$

(3.24)

$$\begin{aligned} & h \sum_{j=1}^{J-1} \vec{T}_j^{m,4} \cdot \left(\lambda |\delta^1 \vec{X}_j^m|^2 \frac{\vec{E}_j^{m+1} - \vec{E}_j^m}{\Delta t} - \delta^2 \vec{E}_j^{m+1} \right) \\ & \quad \le C \left(\max_{1 \le j \le J} |\delta^- \vec{E}_j^{m+1}| + h^{-\frac{1}{2}} |\vec{E}^{m+1} - \vec{E}^m|_{1,h} \right) \left(\lambda \left| \frac{\vec{E}^{m+1} - \vec{E}^m}{\Delta t} \right|_{0,h} + |\vec{E}^{m+1}|_{2,h} \right) \\ & \quad \le \varepsilon |\vec{E}^{m+1}|_{2,h}^2 + \varepsilon \lambda \left| \frac{\vec{E}^{m+1} - \vec{E}^m}{\Delta t} \right|_{0,h}^2 + C_\varepsilon \, |\vec{E}^{m+1}|_{1,h}^2 + C_\varepsilon \, \frac{\gamma}{\Delta t} |\vec{E}^{m+1} - \vec{E}^m|_{1,h}^2. \end{aligned}$$

Finally, we infer from (2.17) that

$$\begin{aligned} & h \sum_{j=1}^{J-1} \vec{T}_j^{m,5} \cdot \left(\lambda |\delta^1 \vec{X}_j^m|^2 \frac{\vec{E}_j^{m+1} - \vec{E}_j^m}{\Delta t} - \delta^2 \vec{E}_j^{m+1} \right) \\ & \quad \le C \, \big(h^2 + \Delta t \big) \left(\lambda \left| \frac{\vec{E}^{m+1} - \vec{E}^m}{\Delta t} \right|_{0,h} + |\vec{E}^{m+1}|_{2,h} \right) \\ & \quad \le \varepsilon \, |\vec{E}^{m+1}|_{2,h}^2 + \varepsilon \lambda \left| \frac{\vec{E}^{m+1} - \vec{E}^m}{\Delta t} \right|_{0,h}^2 + C_\varepsilon \, \big(h^4 + (\Delta t)^2 \big). \end{aligned} \tag{3.25}$$

If we add (3.21), (3.22), (3.24), and (3.25) and then choose ε sufficiently small, the bound (3.20) follows. ∎

In the next lemma we mimic the formal calulations in (2.5), thereby closing our estimates.

LEMMA 3.4. *There exists a constant $c_4 > 0$ such that*

$$\begin{aligned} & h \sum_{j=1}^{J-1} \vec{T}_j^{m,3} \cdot \delta^2 \vec{E}_j^{m+1} \\ & \quad \ge c_4 h \sum_{j=1}^{J-1} \frac{(\delta^1 \vec{E}_j^{m+1} \cdot \vec{e}_2)^2}{(\vec{X}_j^m \cdot \vec{e}_1)^2} - \tfrac{5}{3} h \left(\frac{(\delta^- \vec{E}_1^{m+1} \cdot \vec{e}_2)^2}{(\vec{X}_1^m \cdot \vec{e}_1)^2} + \frac{(\delta^+ \vec{E}_{J-1}^{m+1} \cdot \vec{e}_2)^2}{(\vec{X}_{J-1}^m \cdot \vec{e}_1)^2} \right) \\ & \qquad - \varepsilon |\vec{E}^{m+1}|_{2,h}^2 - C_\varepsilon \, |\vec{E}^{m+1}|_{1,h}^2. \end{aligned} \tag{3.26}$$

Proof. Let us start by writing

(3.27)
$$\sum_{j=1}^{J-1} \vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1} = \sum_{j=1}^{J_1} \vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1} + \sum_{j=J_1+1}^{J-J_1-1} \vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1} + \sum_{j=J-J_1}^{J-1} \vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1},$$

and we begin by estimating the first sum on the right-hand side of (3.27). On recalling (3.13), we can write

(3.28)
$$\vec{T}_j^{m,3} = \frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}\,\vec{e}_2 - \frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}\left(\frac{(\delta^1\vec{x}_j^m)^\perp}{|\delta^1\vec{x}_j^m|} + \vec{e}_2\right) =: \vec{S}_j^1 + \vec{S}_j^2.$$

Observing from (2.6) that

$$\begin{aligned}\big(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2\big)\delta^2\vec{E}_j^{m+1}\cdot\vec{e}_2 &= \frac{1}{h}(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2)(\delta^+\vec{E}_j^{m+1} - \delta^-\vec{E}_j^{m+1})\cdot\vec{e}_2\\ &= \frac{1}{2h}\big((\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2)^2 - (\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2\big),\end{aligned}$$

we find that

(3.29)
$$\begin{aligned} h\sum_{j=1}^{J_1}\vec{S}_j^1\cdot\delta^2\vec{E}_j^{m+1} &= \tfrac{1}{2}\sum_{j=1}^{J_1}\frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{1}{\vec{X}_j^m\cdot\vec{e}_1}\big((\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2)^2 - (\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2\big)\\ &= \tfrac{1}{2}\sum_{j=1}^{J_1}\frac{(\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2)^2}{|\delta^1\vec{x}_j^m|\,\vec{X}_j^m\cdot\vec{e}_1} - \tfrac{1}{2}\sum_{j=0}^{J_1-1}\frac{(\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2)^2}{|\delta^1\vec{x}_{j+1}^m|\,\vec{X}_{j+1}^m\cdot\vec{e}_1}\\ &= \tfrac{1}{2}\sum_{j=1}^{J_1-1}\left(\frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{1}{\vec{X}_j^m\cdot\vec{e}_1} - \frac{1}{|\delta^1\vec{x}_{j+1}^m|}\,\frac{1}{\vec{X}_{j+1}^m\cdot\vec{e}_1}\right)(\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2)^2\\ &\quad - \tfrac{1}{2}\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{|\delta^1\vec{x}_1^m|\,\vec{X}_1^m\cdot\vec{e}_1} + \tfrac{1}{2}\frac{(\delta^-\vec{E}_{J_1+1}^{m+1}\cdot\vec{e}_2)^2}{|\delta^1\vec{x}_{J_1}^m|\,\vec{X}_{J_1}^m\cdot\vec{e}_1}.\end{aligned}$$

In order to estimate the sum on the right-hand side, we observe that $|\delta^1\vec{x}_{j+1}^m| \le \frac{1}{2h}\int_{q_j}^{q_{j+2}}|\vec{x}_\rho^m|\,\mathrm{d}\rho \le C_0$, recall (2.2), and observe that $\vec{X}_j^m\cdot\vec{e}_1 \le \vec{X}_{j+1}^m\cdot\vec{e}_1 \le C\vec{X}_j^m\cdot\vec{e}_1$ for $1\le j\le J_1-1$; recall (3.4) and (3.9). Hence we obtain with the help of (3.4) that

$$\begin{aligned}&\frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{1}{\vec{X}_j^m\cdot\vec{e}_1} - \frac{1}{|\delta^1\vec{x}_{j+1}^m|}\,\frac{1}{\vec{X}_{j+1}^m\cdot\vec{e}_1}\\ &\quad= h\,\frac{1}{|\delta^1\vec{x}_{j+1}^m|}\,\frac{\delta^-\vec{X}_{j+1}^m\cdot\vec{e}_1}{(\vec{X}_{j+1}^m\cdot\vec{e}_1)\vec{X}_j^m\cdot\vec{e}_1} - \frac{1}{\vec{X}_j^m\cdot\vec{e}_1}\left(\frac{1}{|\delta^1\vec{x}_{j+1}^m|} - \frac{1}{|\delta^1\vec{x}_j^m|}\right)\\ &\quad\ge h\frac{1}{C_0}\,\frac{\frac{c_0}{4}}{(\vec{X}_{j+1}^m\cdot\vec{e}_1)^2} - Ch\frac{1}{\vec{X}_{j+1}^m\cdot\vec{e}_1} \ge h\frac{c_0}{8C_0}\,\frac{1}{(\vec{X}_{j+1}^m\cdot\vec{e}_1)^2} - Ch, \quad 1\le j\le J_1-1.\end{aligned}$$

Furthermore, on noting $|\delta^1\vec{x}_1^m| \ge \delta^1\vec{x}_1^m\cdot\vec{e}_1 = \frac{1}{2h}\vec{x}_2^m\cdot\vec{e}_1 \ge \frac{1}{2h}\vec{x}_1^m\cdot\vec{e}_1 \ge \frac{3}{8}\frac{1}{h}\vec{X}_1^m\cdot\vec{e}_1$, recall (3.5); we have that

$$\tfrac{1}{2}\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{|\delta^1\vec{x}_1^m|\,\vec{X}_1^m\cdot\vec{e}_1} \le \tfrac{4}{3}h\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}.$$

Inserting the above two estimates into (3.29) yields, on recalling (2.8), that

$$h\sum_{j=1}^{J_1}\vec{S}_j^1\cdot\delta^2\vec{E}_j^{m+1}\geq h\frac{c_0}{16C_0}\sum_{j=1}^{J_1-1}\frac{(\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_{j+1}^m\cdot\vec{e}_1)^2}-\tfrac{4}{3}h\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}-C|\vec{E}^{m+1}|_{1,h}^2. \tag{3.30}$$

Note that in view of (1.6b), (2.2), and (2.3a) we have $\frac{\vec{x}_\rho(0,t)}{|\vec{x}_\rho(0,t)|}=\vec{e}_1$, so that $\frac{\vec{x}_\rho^{\perp}(0,t)}{|\vec{x}_\rho(0,t)|}=\vec{e}_1^{\perp}=-\vec{e}_2$. Hence (2.18c) and the smoothness of $\vec{x}$ imply

$$\begin{aligned}\left|\frac{(\delta^1\vec{x}_j^m)^\perp}{|\delta^1\vec{x}_j^m|}+\vec{e}_2\right|&\leq\left|\frac{(\delta^1\vec{x}_j^m)^\perp}{|\delta^1\vec{x}_j^m|}-\frac{\vec{x}_\rho^{\perp}(q_j,t_m)}{|\vec{x}_\rho(q_j,t_m)|}\right|+\left|\frac{\vec{x}_\rho^{\perp}(q_j,t_m)}{|\vec{x}_\rho(q_j,t_m)|}-\frac{\vec{x}_\rho^{\perp}(0,t_m)}{|\vec{x}_\rho(0,t_m)|}\right|\\&\leq C(h+q_j)\leq Cq_j\,,\end{aligned}$$

for $1\leq j\leq J_1$, which means that with the help of (3.8a) we obtain

$$\begin{aligned}h\sum_{j=1}^{J_1}\vec{S}_j^2\cdot\delta^2\vec{E}_j^{m+1}&\geq -Ch\sum_{j=1}^{J_1}\big(|\delta^-\vec{E}_j^{m+1}|+|\delta^+\vec{E}_j^{m+1}|\big)\,|\delta^2\vec{E}_j^{m+1}|\\&\geq-\varepsilon|\vec{E}^{m+1}|_{2,h}^2-C_\varepsilon|\vec{E}^{m+1}|_{1,h}^2.\end{aligned} \tag{3.31}$$

Combining (3.30) and (3.31) with (3.28) we obtain

$$h\sum_{j=1}^{J_1}\vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1}\geq h\frac{c_0}{16C_0}\sum_{j=2}^{J_1}\frac{(\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}-\tfrac{4}{3}h\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}-\varepsilon|\vec{E}^{m+1}|_{2,h}^2-C_\varepsilon\,|\vec{E}^{m+1}|_{1,h}^2. \tag{3.32}$$

In order to estimate the third sum on the right-hand side of (3.27), we start from

$$\vec{T}_j^{m,3}=-\frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}\vec{e}_2-\frac{1}{|\delta^1\vec{x}_j^m|}\,\frac{\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2}{\vec{X}_j^m\cdot\vec{e}_1}\left(\frac{(\delta^1\vec{x}_j^m)^\perp}{|\delta^1\vec{x}_j^m|}-\vec{e}_2\right)$$

and use similar arguments as above to obtain

$$h\sum_{j=J-J_1}^{J-1}\vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1}\geq h\frac{c_0}{16C_0}\sum_{j=J-J_1+1}^{J-1}\frac{(\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}-\tfrac{4}{3}h\frac{(\delta^-\vec{E}_J^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_{J-1}^m\cdot\vec{e}_1)^2}-\varepsilon|\vec{E}^{m+1}|_{2,h}^2-C_\varepsilon\,|\vec{E}^{m+1}|_{1,h}^2. \tag{3.33}$$

Moreover, it follows from (2.2), (3.3), and (3.6) that

$$\begin{aligned}&h\sum_{j=J_1+1}^{J-J_1-1}\vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1}-h\frac{c_0}{16C_0}\sum_{j=J_1+1}^{J-J_1}\frac{(\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}\\&\quad\geq-Ch\sum_{j=1}^{J-1}\big(|\delta^-\vec{E}_j^{m+1}|+|\delta^-\vec{E}_{j+1}^{m+1}|\big)|\delta^2\vec{E}_j^{m+1}|-Ch\sum_{j=1}^{J-1}|\delta^-\vec{E}_j^{m+1}|^2\\&\quad\geq-\varepsilon\,|\vec{E}^{m+1}|_{2,h}^2-C_\varepsilon|\vec{E}^{m+1}|_{1,h}^2.\end{aligned} \tag{3.34}$$

If we combine (3.32), (3.33), and (3.34) we obtain

$$h\sum_{j=1}^{J-1}\vec{T}_j^{m,3}\cdot\delta^2\vec{E}_j^{m+1}\geq h\frac{c_0}{16C_0}\sum_{j=2}^{J-1}\frac{(\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}\\ -\tfrac{4}{3}h\left(\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}+\frac{(\delta^+\vec{E}_{J-1}^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_{J-1}^m\cdot\vec{e}_1)^2}\right)-\varepsilon|\vec{E}^{m+1}|_{2,h}^2-C_\varepsilon\,|\vec{E}^{m+1}|_{1,h}^2. \tag{3.35}$$

Observing that in view of (2.6b) and (3.9)

$$\frac{|\delta^+\vec{E}_j^{m+1}\cdot\vec{e}_2|}{\vec{X}_j^m\cdot\vec{e}_1}\leq\frac{8C_0}{c_3}\frac{|\delta^-\vec{E}_{j+1}^{m+1}\cdot\vec{e}_2|}{\vec{X}_{j+1}^m\cdot\vec{e}_1},\quad 1\leq j\leq J-2,$$

we have

$$h\sum_{j=1}^{J-1}\frac{(\delta^1\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}\leq\frac{h}{2}\sum_{j=1}^{J-1}\frac{(\delta^+\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}+\frac{h}{2}\sum_{j=1}^{J-1}\frac{(\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}\\ \leq\frac{h}{2}\left(\frac{(\delta^-\vec{E}_1^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_1^m\cdot\vec{e}_1)^2}+\frac{(\delta^+\vec{E}_{J-1}^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_{J-1}^m\cdot\vec{e}_1)^2}\right)+Ch\sum_{j=2}^{J-1}\frac{(\delta^-\vec{E}_j^{m+1}\cdot\vec{e}_2)^2}{(\vec{X}_j^m\cdot\vec{e}_1)^2}.$$

If we insert this bound into (3.35), we deduce (3.26) provided that c_4 is small enough. ∎

Combining Lemmas 3.2, 3.3, and 3.4 we obtain after choosing ε, γ, and λ sufficiently small that

$$\frac{1+\lambda}{2\Delta t}\big(|\vec{E}^{m+1}|_{1,h}^2-|\vec{E}^m|_{1,h}^2\big)+\frac{1}{16C_0^2}|\vec{E}^{m+1}|_{2,h}^2+\tfrac{1}{8}c_0^2\lambda\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2\\ \leq C\big(|\vec{E}^m|_{1,h}^2+|\vec{E}^{m+1}|_{1,h}^2\big)+C\big(h^4+(\Delta t)^2\big). \tag{3.36}$$

Furthermore, we have

$$\frac{1}{2\Delta t}\big(|\vec{E}^{m+1}|_{0,h}^2-|\vec{E}^m|_{0,h}^2\big)\leq\tfrac{1}{2}\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}\big(|\vec{E}^{m+1}|_{0,h}+|\vec{E}^m|_{0,h}\big)\\ \leq\tfrac{1}{16}c_0^2\lambda\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2+C\big(|\vec{E}^{m+1}|_{0,h}^2+|\vec{E}^m|_{0,h}^2\big). \tag{3.37}$$

On inserting (3.37) into (3.36), divided by $(1+\lambda)$, we obtain that there exist constants $c_6>0$ and $C_2>0$ such that

$$\frac{1}{\Delta t}\big(\|\vec{E}^{m+1}\|_{1,h}^2-\|\vec{E}^m\|_{1,h}^2\big)+c_6\left(|\vec{E}^{m+1}|_{2,h}^2+\left|\frac{\vec{E}^{m+1}-\vec{E}^m}{\Delta t}\right|_{0,h}^2\right)\\ \leq C_2\big(\|\vec{E}^{m+1}\|_{1,h}^2+\|\vec{E}^m\|_{1,h}^2\big)+C_2\big(h^4+(\Delta t)^2\big). \tag{3.38}$$

Combining (3.38) with the induction hypothesis (3.1) completes the proof of Theorem 2.4. In fact, if we choose h_0 so small that $C_2\Delta t\leq\frac{1}{2}$ for $\Delta t\leq\gamma h_0$, then $0<(1-C_2\,\Delta t)^{-1}\leq 1+2C_2\Delta t$, and so it follows from (3.38) and (3.1) that

$$\begin{aligned}\|\vec{E}^{m+1}\|_{1,h}^2 &\le (1-C_2\Delta t)^{-1}\Big[\big(1+C_2\Delta t\big)\|\vec{E}^m\|_{1,h}^2 + C_2\Delta t\big(h^4+(\Delta t)^2\big)\Big]\\ &\le \big(1+2C_2\Delta t\big)^2\|\vec{E}^m\|_{1,h}^2 + C_2\big(1+2C_2\Delta t\big)\Delta t\big(h^4+(\Delta t)^2\big)\\ &\le \big(1+2C_2\Delta t\big)^2\big(h^4+(\Delta t)^2\big)e^{\mu t_m} + 2C_2\Delta t\big(h^4+(\Delta t)^2\big)\\ &\le \big(1+3C_2\Delta t\big)^2\big(h^4+(\Delta t)^2\big)e^{\mu t_m}\\ &\le \big(h^4+(\Delta t)^2\big)e^{6C_2\Delta t}e^{\mu t_m} = \big(h^4+(\Delta t)^2\big)e^{\mu t_{m+1}}\end{aligned}$$

if we choose $\mu = 6C_2$. Since μ, as well as γ, were chosen independently of h and Δt, we have shown (3.1) by induction. Together with (2.10) this proves the inequality (2.22). Finally, multiplying (3.38) by Δt and summing for $m = 0, \dots, M-1$ yields the bound (2.23).

4. Numerical results. It is easy to show that a shrinking sphere with the radius $[1-4\,t]^{\frac{1}{2}}$ is a solution to (1.1). In fact, the parameterization

$$\vec{x}(\rho,t) = [1-4\,t]^{\frac{1}{2}}\begin{pmatrix}\sin(\pi\,\rho)\\ \cos(\pi\,\rho)\end{pmatrix} \tag{4.1}$$

solves (1.6). On letting $\vec{x}_j^m = \vec{x}(q_j, t_m)$, $j = 0, \dots, J$, we compare (4.1) to the discrete solutions $(\vec{X}^m)_{m=0,\dots,M}$ of (2.15) and perform two convergence experiments. In particular, we choose either $\Delta t = h$ or $\Delta t = h^2$ for $h = J^{-1} = 2^{-k}$, $k = 5, \dots, 9$.

The results in Tables 1 and 2 confirm the theoretical results proved in Theorem 2.4. We stress that the quadratic convergence rate for the H^1–seminorm in Table 2 is better than the linear rate observed in [2, Table 4] for the finite element scheme considered there. This suggests that the delicate treatment of the boundary nodes in our finite difference scheme (2.15) is crucial to obtain the optimal convergence rate in Theorem 2.4.

In Figure 1 we show a simulation for the mean curvature flow of a sphere with an inscribed torus. In particular, the initial surface self-intersects on the equator of the sphere and has genus zero. For the scheme (2.15) we choose $J = 1024$ and $\Delta t = 10^{-4}$.

TABLE 1
Errors for the convergence test for (4.1) over the time interval $[0, 0.125]$ with $\Delta t = h$, together with their experimental orders of convergence (EOC).

J	$\max_{m=0,\dots,M} \lvert\vec{x}^m - \vec{X}^m\rvert_{0,h}$	EOC	$\max_{m=0,\dots,M} \lvert\vec{x}^m - \vec{X}^m\rvert_{1,h}$	EOC
32	3.5744e-02	—	1.1225e-01	—
64	2.0034e-02	0.84	6.2934e-02	0.83
128	1.0690e-02	0.91	3.3582e-02	0.91
256	5.5352e-03	0.95	1.7389e-02	0.95
512	2.8185e-03	0.97	8.8546e-03	0.97

TABLE 2
Errors for the convergence test for (4.1) over the time interval $[0, 0.125]$ with $\Delta t = h^2$.

J	$\max_{m=0,\dots,M} \lvert\vec{x}^m - \vec{X}^m\rvert_{0,h}$	EOC	$\max_{m=0,\dots,M} \lvert\vec{x}^m - \vec{X}^m\rvert_{1,h}$	EOC
32	1.0024e-03	—	3.1480e-03	—
64	2.5201e-04	1.99	7.9165e-04	1.99
128	6.3093e-05	2.00	1.9821e-04	2.00
256	1.5779e-05	2.00	4.9571e-05	2.00
512	3.9451e-06	2.00	1.2394e-05	2.00

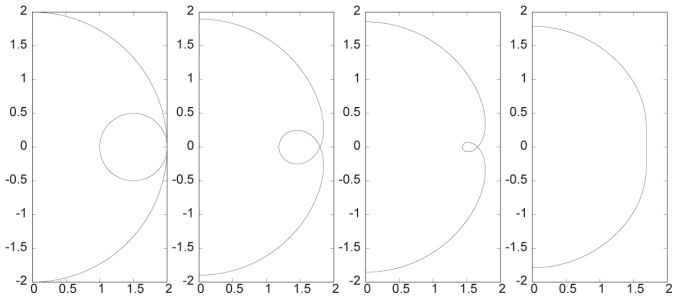

FIG. 1. *Evolution for a torus inscribed within a sphere. Plots are at times* $t = 0, 0.1, 0.14, 0.2$.

Under the mean curvature flow, the torus attempts to shrink to a circle. For the generating curve, this means that the cusp or swallow tail tries to disappear. Of course, for the approximated partial differential equation this represents a singularity, where the curvatures of the curve, and of the corresponding axisymmetric surface, blow up. However, the discrete scheme (2.15) is blind to the self-intersection and the associated singularity. Hence the finite difference approximation simply integrates across the singularity. The same behavior can be seen, for example, in [8, Figure 4.2] and [3, Figure 6]. Continuing the evolution in Figure 1 would show the curve approaching a shrinking semicircle that eventually vanishes at the origin.

In the recent article [2], Barrett, Deckelnick, and Nürnberg numerically studied the Angenent torus (see [1, 17]) as an example of a self-shrinker for the mean curvature flow. Here we recall that the surface $\mathcal{S}(0)$ is called a self-shrinker if the self-similar family of surfaces

$$\mathcal{S}(t) = [1-t]^{\frac{1}{2}}\mathcal{S}(0)$$

is a solution to (1.1). In what follows, we would like to use our approximation (2.15) in order to investigate self-shrinkers of genus zero. It was shown in [15] that the only bounded embedded genus-0 self-shrinker in $\mathbb{R}^3$ is the sphere of radius 2. Note that the unit sphere has an extinction time of $T_0 = \frac{1}{4}$; recall (4.1). On the other hand, in [9] the existence of infinitely many immersed self-shrinkers with rotational symmetry was proved. Hence, inspired by [9, Figure 3], we would like to compute such a self-similar evolution for the mean curvature flow. To this end, we use the open curve analog of [2, (5.7), (5.8)] in order to calculate a profile curve of a self-shrinker that has three self-intersections. Using the obtained curve as initial data for the scheme (2.15) yields the self-similar evolution displayed in Figure 2. Here we used the discretization parameters $J = 512$ and $\Delta t = 10^{-4}$. Note that the numerical method appears to confirm the unit extinction time. In fact, continuing the evolution until the method breaks down yields the behavior of the approximate surface area

$$A^m = 2\pi h \sum_{j=1}^{J} \vec{X}_j^m \cdot \vec{e}_1 |\delta^- \vec{X}_j^m|,$$

as shown in Figure 3, with the expected linear decay and an approximate extinction time of 1.

Finally, we include a numerical experiment to demonstrate that our scheme can also deal with initial data that violate the 90° contact angle condition in (1.6b). To

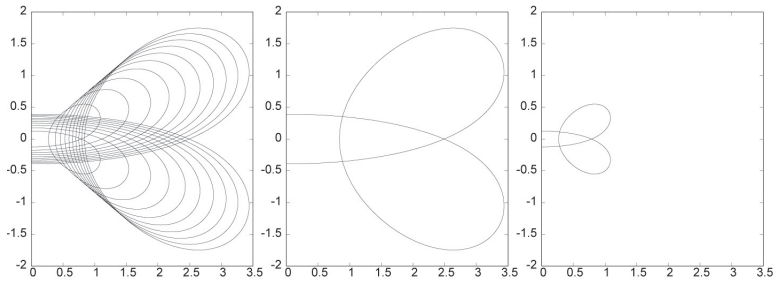

FIG. 2. *Self-similar evolution for a surface with three self-intersections. Plots are at times* $t = 0, 0.1, \dots, 0.9$ *and again at times* $t = 0$ *and* $t = 0.9$.

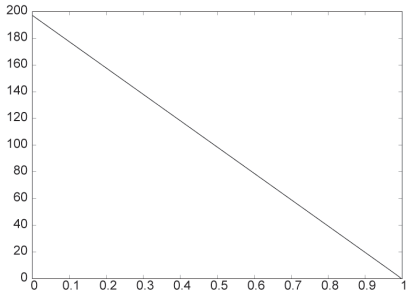

FIG. 3. *A plot of the approximate surface area* A^m *for the simulation in Figure* 2 *over time.*

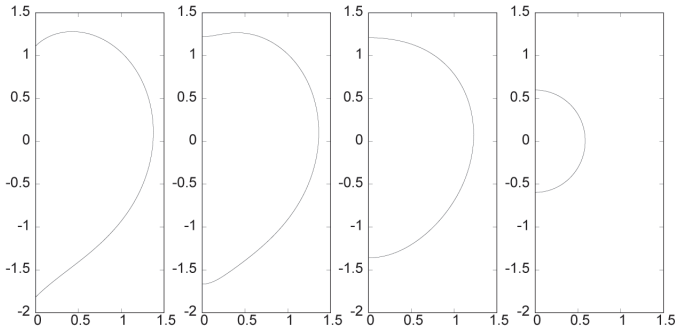

FIG. 4. *Evolution for a surface with two cone singularities. Plots are at times* $t = 0, 0.01, 0.1, 0.4$.

this end, in Figure 4 we start a simulation for a surface that has two cone singularities: an inward cone and an outward cone. The generating curve has a 45° contact angle at the axis of rotation, which induces a discontinuous jump in time for the solution of the partial differential equation. For the simulation we choose $J = 512$ and $\Delta t = 10^{-4}$ for the scheme (2.15). It can be observed that the outward cone very quickly smoothes to a rounded tip, while the inward cone also smoothes and rises at the same time. Eventually the curve approaches a shrinking semicircle that will shrink to a point.

Appendix A. Properties of the solution.

LEMMA A.1 (behavior at the boundary). *Let* $\vec{x} : [0,1] \times [0,T] \to \mathbb{R}^2$ *satisfy Assumption* 2.1. *Then we have*

$$\vec{x}_{\rho\rho}\cdot\vec{e}_1=\vec{x}_{\rho\rho}\cdot\vec{x}_\rho=0 \qquad \text{on } \{0,1\}\times[0,T], \tag{A.1a}$$

$$\vec{x}_{\rho\rho\rho}\cdot\vec{e}_2=0 \qquad \text{on } \{0,1\}\times[0,T], \tag{A.1b}$$

$$\vec{x}_t=2\frac{\vec{x}_{\rho\rho}\cdot\vec{e}_2}{|\vec{x}_\rho|^2}\vec{e}_2 \quad \text{on } \{0,1\}\times[0,T]. \tag{A.1c}$$

Proof. We have from (1.6b) that

$$\vec{x}_\rho^{\,\perp}(0,t)=(|\vec{x}_\rho(0,t)|\,\vec{e}_1)^\perp=-|\vec{x}_\rho(0,t)|\,\vec{e}_2, \tag{A.2}$$

and so we obtain with the help of L'Hospital's rule that

$$\begin{aligned}\lim_{\rho\searrow0}\frac{1}{|\vec{x}_\rho(\rho,t)|^2}\frac{\vec{x}_\rho(\rho,t)\cdot\vec{e}_2}{\vec{x}(\rho,t)\cdot\vec{e}_1}\vec{x}_\rho^{\,\perp}(\rho,t)&=\frac{1}{|\vec{x}_\rho(0,t)|^2}\frac{\vec{x}_{\rho\rho}(0,t)\cdot\vec{e}_2}{\vec{x}_\rho(0,t)\cdot\vec{e}_1}\vec{x}_\rho^{\,\perp}(0,t)\\&=-\frac{\vec{x}_{\rho\rho}(0,t)\cdot\vec{e}_2}{|\vec{x}_\rho(0,t)|^2}\vec{e}_2.\end{aligned} \tag{A.3}$$

Thus (1.6a) implies that

$$\vec{x}_t(0,t)=\frac{\vec{x}_{\rho\rho}(0,t)}{|\vec{x}_\rho(0,t)|^2}+\frac{\vec{x}_{\rho\rho}(0,t)\cdot\vec{e}_2}{|\vec{x}_\rho(0,t)|^2}\vec{e}_2. \tag{A.4}$$

Observing from (1.6b) that $\vec{x}_t(0,t)\cdot\vec{e}_1=0$, we infer from (A.4) that $\vec{x}_{\rho\rho}(0,t)\cdot\vec{e}_1=0$, which together with (1.6b) proves (A.1a) at $\rho=0$. In particular, $\vec{x}_{\rho\rho}(0,t)=(\vec{x}_{\rho\rho}(0,t)\cdot\vec{e}_2)\vec{e}_2$. Combining this with (A.4) yields (A.1c) at $\rho=0$. In order to prove (A.1b), we differentiate (1.6a) with respect to ρ and obtain

$$\vec{x}_{t\rho}=\frac{\vec{x}_{\rho\rho\rho}}{|\vec{x}_\rho|^2}-2\frac{\vec{x}_{\rho\rho}\cdot\vec{x}_\rho}{|\vec{x}_\rho|^4}\vec{x}_{\rho\rho}+\frac{\vec{x}_\rho\cdot\vec{e}_2}{\vec{x}\cdot\vec{e}_1}\big(2\frac{\vec{x}_{\rho\rho}\cdot\vec{x}_\rho}{|\vec{x}_\rho|^4}\vec{x}_\rho^{\,\perp}-\frac{1}{|\vec{x}_\rho|^2}\vec{x}_{\rho\rho}^{\,\perp}\big)-\big(\frac{\vec{x}_\rho\cdot\vec{e}_2}{\vec{x}\cdot\vec{e}_1}\big)_\rho\frac{1}{|\vec{x}_\rho|^2}\vec{x}_\rho^{\,\perp} \tag{A.5}$$

in $(0,1)\times(0,T]$. A further application of L'Hospital's rule implies that

$$\begin{aligned}\lim_{\rho\searrow0}\left(\frac{\vec{x}_\rho\cdot\vec{e}_2}{\vec{x}\cdot\vec{e}_1}\right)_\rho(\rho,t)&=\lim_{\rho\searrow0}\frac{(\vec{x}_{\rho\rho}(\rho,t)\cdot\vec{e}_2)(\vec{x}(\rho,t)\cdot\vec{e}_1)-(\vec{x}_\rho(\rho,t)\cdot\vec{e}_2)(\vec{x}_\rho(\rho,t)\cdot\vec{e}_1)}{(\vec{x}(\rho,t)\cdot\vec{e}_1)^2}\\&=\lim_{\rho\searrow0}\frac{(\vec{x}_{\rho\rho\rho}(\rho,t)\cdot\vec{e}_2)(\vec{x}(\rho,t)\cdot\vec{e}_1)-(\vec{x}_\rho(\rho,t)\cdot\vec{e}_2)(\vec{x}_{\rho\rho}(\rho,t)\cdot\vec{e}_1)}{2(\vec{x}(\rho,t)\cdot\vec{e}_1)(\vec{x}_\rho(\rho,t)\cdot\vec{e}_1)}\\&=\tfrac{1}{2}\frac{\vec{x}_{\rho\rho\rho}(0,t)\cdot\vec{e}_2}{\vec{x}_\rho(0,t)\cdot\vec{e}_1},\end{aligned} \tag{A.6}$$

since $\vec{x}_{\rho\rho}(0,t)\cdot\vec{e}_1=0$. Combining (A.5) and (A.6), on noting (A.1a), (1.6b), and (A.2), yields that

$$\vec{x}_{t\rho}(0,t)=\frac{\vec{x}_{\rho\rho\rho}(0,t)}{|\vec{x}_\rho(0,t)|^2}+\tfrac{1}{2}\frac{\vec{x}_{\rho\rho\rho}(0,t)\cdot\vec{e}_2}{|\vec{x}_\rho(0,t)|^2}\vec{e}_2. \tag{A.7}$$

Since $\vec{x}_{t\rho}(0,t)\cdot\vec{e}_2=0$ in view of (1.6b), we deduce from (A.7) that also (A.1b) holds at the left boundary point. The proof of (A.1) for the other boundary point is analogous. ∎

LEMMA A.2. *Let $\vec{x}:[0,1]\times[0,T]\to\mathbb{R}^2$ satisfy Assumption 2.1. Then there exists $0<c_2<\tilde{c}_2$ such that*

$$c_2\rho(1-\rho)\le\vec{x}(\rho,t)\cdot\vec{e}_1\le\tilde{c}_2\rho(1-\rho) \quad \text{for all } (\rho,t)\in[0,1]\times[0,T]. \tag{A.8}$$

Moreover, there exists $K > 0$ such that for all $0 < h \le \frac{1}{2}\delta$, with δ as in (2.3),

$$\text{(A.9)}\quad \frac{1}{\vec{x}\cdot\vec{e}_1}\left|\frac{\partial_t^\ell \vec{x}(\cdot+h,t)-\partial_t^\ell \vec{x}(\cdot-h,t)}{2h}\cdot\vec{e}_2\right| \le K \qquad \text{in } [h,1-h]\times[0,T],\ \ell=0,1.$$

Proof. The result (A.8) is an immediate consequence of (2.3).

Let $t \in [0,T]$ and $h \le \rho \le \frac{1}{2}\delta$. We infer from (1.6b) and (2.3a) that $\vec{x}(\rho,t)\cdot\vec{e}_1 \ge \frac{1}{2}c_0\rho$, and hence

$$\begin{aligned}\frac{1}{\vec{x}(\rho,t)\cdot\vec{e}_1}\left|\frac{\vec{x}(\rho+h,t)-\vec{x}(\rho-h,t)}{2h}\cdot\vec{e}_2\right| &\le \frac{2}{c_0\rho}\left|\frac{1}{2h}\int_{\rho-h}^{\rho+h}\big(\vec{x}_\rho(\zeta,t)-\vec{x}_\rho(0,t)\big)\cdot\vec{e}_2\,\mathrm{d}\zeta\right|\\ &\le \frac{2}{c_0}\max_{[0,\delta]}|\vec{x}_{\rho\rho}(\cdot,t)\cdot\vec{e}_2|\\ &\le \frac{2}{c_0}\max_{[0,\delta]\times[0,T]}|\vec{x}_{\rho\rho}\cdot\vec{e}_2|.\end{aligned}$$

We can argue in the same way for $1-\frac{1}{2}\delta \le \rho \le 1-h$, while for $\frac{1}{2}\delta \le \rho \le 1-\frac{1}{2}\delta$ we have that

$$\begin{aligned}\frac{1}{\vec{x}(\rho,t)\cdot\vec{e}_1}\left|\frac{\vec{x}(\rho+h,t)-\vec{x}(\rho-h,t)}{2h}\cdot\vec{e}_2\right| &\le \frac{1}{c_1}\left|\frac{1}{2h}\int_{\rho-h}^{\rho+h}\vec{x}_\rho(\zeta,t)\cdot\vec{e}_2\,\mathrm{d}\zeta\right|\\ &\le \frac{1}{c_1}\max_{[\frac{\delta}{2},1-\frac{\delta}{2}]\times[0,T]}|\vec{x}_\rho\cdot\vec{e}_2|,\end{aligned}$$

so that (A.9) holds with $K = \max\{\frac{2}{c_0}\max_{[0,\delta]\times[0,T]}|\vec{x}_{\rho\rho}\cdot\vec{e}_2|, \frac{1}{c_1}\max_{[\frac{\delta}{2},1-\frac{\delta}{2}]\times[0,T]}|\vec{x}_\rho\cdot\vec{e}_2|\}$ in the case $\ell=0$. The case $\ell=1$ can be treated in the same way on noting that $\vec{x}_{t\rho}(q,t)\cdot\vec{e}_2 = 0$ for $q \in \{0,1\}$. ∎

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