Minimal decomposition of binary forms with respect to tangential projections

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Abstract

Let $C \subset \mathbb{P}^{n+1}$ be a rational normal curve and let $X \subset \mathbb{P}^n$ be one of its tangential projection. We describe the X-rank of a point $P \in \mathbb{P}^n$ in terms of the schemes evincing the C-rank or the border C-rank of the preimage of P.

Introduction

In many applications, like Biology and Statistics, it turns out to be useful to develop techniques for reducing the dimension of high-dimensional data (like Principal Component Analysis [PCA]) that can be encoded in a tensor. In many cases these tensors turn out have many entries equal to zero (see eg. [9] for an example of a chemical, biological system). One of the main problems is to find a minimal decomposition of such a tensor in terms of other tensors with the same structure but containing as few non-zero terms as possible. We want to address these questions from an Algebraic Geometry point of view (we suggest [8] for a good description about the relation between Biology, Statistics and Algebraic Geometry on these kind of questions). We will direct

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our attention to the very special case of complex symmetric tensors of order $n+1$ with at least one entry equal to zero and defined over a 2-dimensional vector space. In other words, we study the case of homogeneous polynomials $p \in \mathbb{C}[u,t]_{n+1}$ of degree $n+1$ having at least one coefficient equal to zero. Assume, for a moment, we have fixed an ordered basis $\{x_0, \ldots, x_{n+1}\}$ for $\mathbb{C}[u,t]_{n+1}$. A binary form with the coefficient in the i-th position equal to zero can be obtained by projecting a binary form to the hyperplane $H_i \subset \mathbb{C}[u,t]_{n+1}$ identified by the equation $x_i = 0$. We will focus on projections ℓ_O from a point $O \in \mathbb{P}(\mathbb{C}[u,t]_{n+1}) \simeq \mathbb{P}^{n+1}$ to $\mathbb{P}(H_i) \simeq \mathbb{P}^n$ that corresponds to tangential projections to the rational normal curve that is canonically embedded in $\mathbb{P}^{n+1}$. This will allow us to relate the minimal decomposition of a binary form p of degree $n+1$ as a sum of $(n+1)$-th powers of linear forms $L_1^{n+1}, \ldots, L_r^{n+1} \in \mathbb{C}[u,t]_{n+1}$, with the minimal decomposition of the projected $\ell_O(p) \in \mathbb{P}(H_i)$ (that is a binary form of the same degree $n+1$ but with the i-th coefficient equal to zero) in terms of $\ell_O(L_1^{n+1}), \ldots, \ell_O(L_r^{n+1})$. Explicitly, if r is the minimum number of addenda that are required to write $p \in \mathbb{C}[u,t]_{n+1}$ as

$$p = L_1^{n+1} + \cdots + L_r^{n+1}$$

then we will prove in Theorem 1 and in Theorem 2 that there is a dense subset of $\mathbb{P}(H_i) \simeq \mathbb{P}^n$ where r is also the minimum number of addenda that are required to write $\ell_O(p)$ as follows:

$$\ell_O(p) = \ell_O(L_1^{n+1}) + \cdots + \ell_O(L_r^{n+1}).$$

We will also describe what is the relation between the minimal decomposition of p and the minimal decomposition of an $\ell_O(p)$ from this dense subset.

The minimal decomposition of a generic binary form of degree $n+1$ in terms of $(n+1)$-th powers of binary linear forms was first studied by J. J. Sylvester then formalized with an algorithm in [6] (see also [4] for a more recent proof).

What we want to study in this paper is obviously a very special case for applications (in applications one often needs linear projections from a large dimensional linear subspace) but we hope to give in this way some ideas for further works on wider classes of analogous problems. In any case these kinds of questions lead to a nice geometrical problem that is the computation of X-ranks with respect to a degree $n+1$ cuspidal linearly normal curve $X \subset \mathbb{P}^n$.

If $Y \subset \mathbb{P}^N$ is any non-degenerate projective variety, the minimum integer ρ for which there exists a reduced 0-dimensional subscheme $S \subset Y$ of degree ρ whose linear span $\langle S \rangle$ contains a point $P \in \mathbb{P}^N$ is often called the *Y-rank* $r_Y(P)$ of P with respect to Y and we say that S *evinces* P.

The classical *s-th secant variety* $\sigma_s(Y) \subset \mathbb{P}^N$ is defined to be the Zariski closure of the set of points of Y-rank less or equal than s.

This allows one to introduce the concept of *Y-border rank $br_Y(P)$* of a point $P \in \mathbb{P}^N$ as a minimum integer s such that $P \in \sigma_s(Y)$.

Let us fix the following notation that we will use throughout the paper.

Notation 1. Let $C \subset \mathbb{P}^{n+1}$ be a smooth rational normal curve of degree $n+1$. Fix $A \in C$. Let $2A$ denotes the degree 2 effective divisor of C with A as its reduction. The tangent line T_AC is the line $\langle 2A \rangle$. Fix also a point $O \in T_AC \setminus \{A\}$ to be the center of the projection $\ell_O : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ that sends C into a curve $X := \ell_O(C) \subset \mathbb{P}^n$. The curve X is a linearly normal curve of $\mathbb{P}^n$ with degree $n+1$, arithmetic genus 1 and the ordinary cusp $\ell_O(A) \in X \subset \mathbb{P}^n$ as its unique singular point.

The main results of this paper are Theorem 1 and Theorem 2 where we give a description of both the X-rank and the X-border rank of a point $P \in \mathbb{P}^n$ and we relate them with the C-rank and the C-border rank of its preimage via ℓ_O.

Theorem 1. Fix integers n, ρ such that $n \geq 3$ and $2 \leq \rho \leq \lfloor (n+3)/2 \rfloor$. Let $C, \{A, O\} \subset \mathbb{P}^{n+1}$ and $X \subset \mathbb{P}^n$ be as in Notation 1. Fix $M \in \mathbb{P}^{n+1} \setminus \{O\}$ such that $r_C(M) = \rho$. Let $E \subset C$ be a finite set that evinces the C-rank of M. Set $P := \ell_O(M)$. Then the following hold:

(i) If $2\rho \leq n$, then $r_X(P) = \rho$ and $\ell_O(E)$ is the unique subset of X computing $r_X(P)$.

(ii) If $n+1 \leq 2\rho \leq n+2$, then $\rho - 1 \leq r_X(P) \leq \rho$.

(iii) If n is odd and $2\rho = n+3$, then there is a non-empty open subset $\mathcal{U}$ of $\mathbb{P}^{n+1}$ such that $r_C(M) = \rho$ and $r_X(\ell_O(M)) = \rho - 1$ for all $M \in \mathcal{U}$.

We remark that part (iii) of Theorem 1 is true for the image, X, of a linear projection of any integral and non-degenerate curve $Y \subset \mathbb{P}^{n+1}$ from an arbitrary $O \in \mathbb{P}^{n+1} \setminus Y$.

In Theorem 2 we take as P a point $\ell_O(B)$ such that the border C-rank of B is not computed by a reduced scheme, i.e. such that the C-rank of B is strictly bigger than the border C-rank of B.

1 Preliminary Lemmas

We borrow from [5] the following result (we only need the case in which Y is a rational normal curve of $\mathbb{P}^{n+1}$ with $2t \leq n+2$; thus the case we use is a particular case of [5, Lemma 2.1.5]). See [5, Theorem 1.5.1] for part (iv) at least for reduced schemes for arbitrary Veronese varieties.

Lemma 1. Let $Y \subset \mathbb{P}^N$ be a smooth and non-degenerate subvariety of dimension at most 2. Let $\beta(Y)$ be the maximal integer t such that $\dim\langle Z\rangle = \deg(Z) - 1$ for every 0-dimensional subscheme $Z \subset Y$ such that $\deg(Z) \le t$. Fix $P \in \mathbb{P}^N$ and assume $t \le \beta(Y)$.

(i) $P \in \sigma_t(Y)$ if and only if there is a 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) \le t$ and $P \in \langle Z\rangle$.

(ii) $P \in \sigma_t(Y) \setminus \sigma_{t-1}(Y)$ if and only if t is the first integer such that there is a 0-dimensional subscheme $Z \subset Y$ with $\deg(Z) = t$ and $P \in \langle Z\rangle$.

(iii) If $2t \le \beta(Y)$ and $P \in \sigma_k(Y) \setminus \sigma_t(Y)$, then there is a unique 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) \le t$ and $P \in \langle Z\rangle$. Moreover, $\deg(Z) = t$.

(iv) If $Y \subset \mathbb{P}^N$ is a rational normal curve, then $\beta(Y) = N + 1$. In this case for each $P \in \mathbb{P}^N$ there is a unique 0-dimensional scheme Z evincing $br_Y(P)$, i.e. a unique 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) \le br_Y(P)$ and $P \in \langle Z\rangle$. Moreover, $\deg(Z) = br_Y(P)$.

Proof. Since Y is smooth and $\dim(Y) \le 2$, every 0-dimensional subscheme A of Y is smoothable, i.e. it is a flat limit of a family of unions of $\deg(A)$ distinct points ([7]). As remarked in the proof of [5, Lemma 2.1.5], the assumption "$\dim\langle Z\rangle = \deg(Z) - 1$ for every 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) \le t$" is sufficient to use [4, Proposition 11], and get part (i).

Part (ii) follows from part (i) applied to the integers t and $t-1$. Take P and t as in part (iii) and assume the existence of schemes $Z, W \subset Y$ such that $\deg(Z) \le t$, $\deg(W) \le t$, $P \in \langle Z\rangle \cap \langle W\rangle$ and $Z \ne W$. Part (ii) gives $\deg(Z) = \deg(W) = t$, $P \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$ and $P \notin \langle W'\rangle$ for any $W' \subsetneq W$. Hence $h^1(\mathcal{I}_{Z\cup W}(1)) > 0$ ([3, Lemma 1]), contradicting the inequalities $\deg(Z \cup W) \le 2t \le \beta(Y)$.

Part (iv) follows from (iii) and a theorem of Sylvester ([6], [4, §3], [10, §4]). $\square$

Take Z as in parts (ii) and (iii) of Lemma 1. We say that Z *evinces the Y-border rank of P*.

Lemma 2. Fix an integral and non-degenerate subvariety $Y \subset \mathbb{P}^{n+x}$, $n > 0$, $x > 0$, and a linear $(x-1)$-dimensional subspace $V \subset \mathbb{P}^{n+x}$ such that $V \cap Y = \emptyset$. Set $X := \ell_V(Y)$. Then

$$r_X(\ell_V(Q)) = \min_{P \in (\langle V \cup \{Q\}\rangle \setminus V)} r_Y(P) \quad \text{for all } Q \in \mathbb{P}^{n+x} \setminus V. \tag{1}$$

There are $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ and a finite set $S \subset X$ such that S evinces $r_Y(P)$, $\sharp(S) = \sharp(\ell_V(S))$ and $\ell_V(S)$ evinces $r_X(\ell_V(Q))$.

Proof. First of all let us prove the inequality "$\geq$" in (1). Since $V \cap Y = \emptyset$, then obviously $\ell_V|Y$ is a finite morphism. Since $\ell_V|Y : Y \to X$ is surjective, for each finite set of points $S \subset X$ we may fix another finite subset $S_V \subset Y$ such that $\ell_V(S_V) = S$ and $\sharp(S_V) = \sharp(S)$. Since $S_V \subseteq Y$, then $S_V \cap V = \emptyset$. Thus the set $S \subset X$ turns out to be linearly independent if and only if S_V is linearly independent and $\langle S_V\rangle \cap V = \emptyset$. Now fix $Q \in \mathbb{P}^{n+x} \setminus V$ and take $S \subset X$ computing $r_X(\ell_V(Q))$. Thus $\sharp(S) = r_X(\ell_V(Q))$ and S is linearly independent by definition of a set that evinces the X-rank of a point. Since S is linearly independent, the set S_V is linearly independent and $\langle S_V\rangle \cap V = \emptyset$. Now, if $\ell_V(Q)$ is an element of $\langle S\rangle$, then $\langle S_V\rangle \cap \langle V \cup \{Q\}\rangle \neq \emptyset$. Since $\langle S_V\rangle \cap V = \emptyset$, there is a unique $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ such that $\{P\} = \langle S_V\rangle \cap \langle V \cup \{Q\}\rangle$. Since $S_V \subset Y$, we have $r_Y(P) \leq \sharp(S_V) = \sharp(S) = r_X(\ell_V(Q))$.

To get the reverse inequality we may just quote [2, Lemma 14] but since it is quite easy to be proved, we show here a shorter proof. Fix any $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ and any $A \subset Y$ computing $r_Y(P)$. Since $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ we have $\ell_V(P) = \ell_V(Q)$. Since $\ell_V(P) \in \langle \ell_V(A)\rangle$, we have $r_X(\ell_V(Q)) \leq r_Y(P)$. $\square$

2 Theorems

We can now focus on tangential projections $X \subset \mathbb{P}^n$ of rational normal curves $C \subset \mathbb{P}^{n+1}$ for $n \geq 3$. We give both a description of the schemes that realize the X-border rank (Theorem 2) and the X-rank (Theorem 1) of a point $P \in \mathbb{P}^n$ with respect to a curve X just described and the precise value of the X-rank of such a point P (except in the critical range $2w \geq n$ or $2\rho \geq n$, respectively). In Theorem 2 we give the X-rank of a point $P \in \mathbb{P}^n$ that is the image via ℓ_O of a point $B \in \mathbb{P}^{n+1}$ whose C-border rank is smaller that its C-rank. In Theorem 1, the point $P \in \mathbb{P}^n$ is the image of a point $M \in \mathbb{P}^{n+1}$ whose C-border rank is equal to its C-rank. Moreover we will explain the relation between the schemes that evince $br_X(P)$ and $r_X(P)$ and the schemes that evince $br_C(B)$ and $r_C(B)$ where $B \in \mathbb{P}^{n+1}$ is a point that is sent into $P \in \mathbb{P}^n$ by the tangential projection.

Theorem 2. Let $C \subset \mathbb{P}^{n+1}$, $n \geq 3$, be a rational normal curve and let also $X := \ell_O(C) \subset \mathbb{P}^n$ and $O \in T_AC \setminus \{A\}$ for a fixed $A \in C$ be as in Notation 1. Fix $B \in \sigma_w(C) \setminus \sigma_w^0(C) \subset \mathbb{P}^{n+1}$, $w \geq 2$, and set $P := \ell_O(B)$. Let $W \subset C$ be the degree w subscheme which evinces $br_C(B)$ (part (iv) of Lemma 1).

1. (1) Assume $O \in \langle W\rangle$ (this case occurs if and only if A appears in W with multiplicity ≥ 2). If $w = 2$, then $r_X(P) = 1$ and $P = \ell_O(A)$. Assume $w \geq 3$. If $W \neq 2A \cup S_1$ with $S_1 \subset C \setminus \{A\}$ and S_1 reduced, then $r_X(Q) = n+3-w$. If $W \neq 2A \cup S_1$ with $S_1 \subset C \setminus \{A\}$ and S_1 reduced, then either $r_X(P) = w-1$ and $\ell_O(W_{red})$ evinces $r_X(P)$ or $r_X(P) = w-2$ and $\ell_O(W \setminus 2A)$ evinces $r_X(P)$. All cases may occur for some W, B.

2. Assume $O \notin \langle W\rangle$ and $A \in W_{red}$. If $2w \leq n$, then $r_X(P) = n+2-w$.

   Assume $2w = n+1$. Then:

   (a) $n+2-w \leq r_X(P) \leq n+3-w$.

   (b) Let Δ be the set of all 0-dimensional schemes $U \subset C$ such that U is not reduced, $\deg(U) = w$ and A appears with multiplicity 1 in U. The set Δ is an irreducible variety of dimension $w-2$. If W is general in Δ and B is general in $\langle W\rangle$, then W evinces $br_C(B)$ and $r_X(\ell_O(B)) = n+2-w$.

3. Assume $O \notin \langle W\rangle$ and $A \notin W_{red}$. We have $n+1-w \leq r_X(P) \leq n+3-w$. If $2w \leq n-1$, then $r_X(P) = n+3-w$.

   If $2w = n+1$, then $n+1-w \leq r_X(P) \leq n+3-w$.

Proof. Since $b_C(B) \neq r_C(P)$, we have $2w \leq n+1$ ([6, eq. (4)]).

Part (iv) of Lemma 1 gives the uniqueness of the scheme $W \subset C$.

Since we took $B \in \sigma_w(C) \setminus \sigma_w^0(C)$, then $r_C(B) = n+3-w$ (see [4, Theorem 23], [6]). Thus $r_C(P) \leq n+3-w$. Since $\deg(2A) + \deg(W) = w+2 \leq n+2$, we have $\langle 2A\rangle \cap \langle W\rangle = \langle 2A \cap W\rangle$. Since $O \in \langle 2A\rangle \cap \langle W\rangle$ and $O \neq A$, we get that $2A \subseteq W$ if and only if $O \in \langle W\rangle$.

(a) Assume $O \in \langle W\rangle$, i.e. $2A \subseteq W$. If $w = 2$, then $P = \ell_O(A)$ and in this case $r_X(P) = 1 = w-1$ and $r_X(P)$ is evinced by $\{\ell_O(A)\}$.

Now assume $w \geq 3$. We have $r_C(B) = n+3-w$ ([6]). Fix any $Q \in \langle\{O, B\}\rangle \setminus \{O, B\}$. Since $\{O, B\} \subset \langle W\rangle$, we get $Q \in \langle W\rangle$. Hence $br_C(Q) \leq w$. If $br_C(Q) \neq r_C(Q)$, then $r_C(Q) = n+3-br_C(Q) \geq n+3-w$. If this is the case for all Q, then Lemma 2 gives $r_X(P) = n+3-w$. Fix any finite set $S' \subset C \setminus \{A\}$ with $\sharp(S') = w-2$ and set $W' := 2A \cup S'$; take any $B' \in \langle W'\rangle$ with $B' \notin \langle W''\rangle$ for any $W'' \subsetneq W$; obviously $r_X(\ell_O(B')) = w-1$ and $\ell_O(S' \cup \{A\})$ evinces $r_X(\ell_O(B'))$.

Now assume $br_C(Q) = r_C(Q)$. Take $S_3 \subset C$ evincing $r_C(Q)$. Since $\deg(W) + \sharp(S_3) \leq n+2$, we have $\langle W\rangle \cap \langle S_3\rangle = \langle W \cap S_3\rangle$. Since $Q \in \langle W\rangle$ and S_3 evinces $r_C(Q)$, we get $S_3 \subseteq W$. We claim that $2A \cup S_3 = W$, i.e. $W = 2A \cup S_1$ with $S_1 \subset C \setminus \{A\}$, S_1 reduced and either $S_3 = S_1$ or $S_3 = S_1 \cup \{A\}$. Indeed, since $Q \in \langle S_3 \rangle$, $O \in \langle 2A \rangle$ and $B \in \langle \{O, Q\} \rangle$, we have $B \in \langle S_3 \cup 2A \rangle$. Since $S_3 \cup 2A \subseteq W$ and $B \notin \langle W' \rangle$ for any $W' \subsetneq W$, we get $2A \cup S_3$, as claimed. Hence either $r_C(Q) = w - 1$ (case $A \in S_3$) or $r_C(Q) = w$. Taking $W \neq 2A \cup S_1$ with $S_1 \subset C \setminus \{A\}$ and S_1 reduced, we get the existence of (B, W) with $O \in \langle W \rangle$ and $r_X(\ell_O(B)) = n + 3 - w$.

Now we check that if $w \geq 3$ then both cases with $r_X(P) \neq n + 3 - w$ may occur for certain B and W. Take any $W = 2A \cup S_2$ with $S_2 \subset C \setminus \{A\}$, S_2 reduced and $\sharp(S_2) = w - 2$. Let B_1 be a general point of $\langle W \rangle$. Since $\langle 2A \rangle \cap \langle S_2 \rangle = \emptyset$, $\langle S_2 \rangle$ has codimension 2 in $\langle W \rangle$ and B_1 is general in $\langle W \rangle$, we get $\langle \{O, B_1\} \rangle \cap \langle S_2 \rangle = \emptyset$. Lemma 2 gives $r_X(P) \leq w - 1$. Since we proved that $r_X(\ell_O(B_1)) \geq w - 1$, we get $r_X(\ell_O(B_1)) = w - 1$ and that $\ell_O(\{A\} \cup S_2)$ evinces $r_X(P)$. Fix a general $O' \in \langle S_2 \rangle$. Notice that W has only finitely many subschemes. Hence the set $\mathcal{T}$ of all proper subspaces of $\langle W \rangle$ spanned by a subscheme of W is finite. Since O' is general in $\langle S_2 \rangle$ and $\langle 2A \rangle \cap \langle S_2 \rangle = \emptyset$, a general $B_2 \in \langle \{O', O\} \rangle$ is not contained in any $\Lambda \in \mathcal{T}$. Hence W evinces $br_C(B_2)$. Since $\ell_O(O') = \ell_O(B_2)$, we have $r_X(\ell_O(B_2)) \leq w - 2$. We proved that in this case we have $r_X(\ell_O(B_2)) = w-2$ and $\ell_O(S_2)$ evinces $r_X(\ell_O(B_2))$.

(b) Here we assume $O \notin \langle W \rangle$ and $A \in W_{red}$. Part (a) gives that A appears with multiplicity 1 in W.

First assume $2w \leq n$. Set $W_1 := W \setminus \{A\}$ and $W_2 := W_1 \cup 2A$. Thus $\deg(W_2) = \deg(W_1) + 2 = w + 1$. Fix any $Q \in \langle \{O, B\} \rangle \setminus \{O, B\}$. Since $B \in \langle W \rangle$ and $O \notin \langle W \rangle$, then $Q \notin \langle W \rangle$. Since $O \in \langle 2A \rangle$, then $Q \in \langle W_2 \rangle$. Thus $br_C(Q) \leq w + 1$. Let Γ be the only scheme evincing $br_C(Q)$. Since $\deg(W_2) + \deg(\Gamma) \leq 2w + 2 \leq n + 2$, we have $\langle \Gamma \rangle \cap \langle W_2 \rangle = \langle \Gamma \cap W_2 \rangle$. Since $Q \in \langle \Gamma \rangle \cap \langle W_2 \rangle$ and Q is not in the linear span of a proper subscheme of Γ, we get $\Gamma \subseteq W_2$. Since $B \in \langle 2A \cup \Gamma \rangle$, we also have $W \subseteq \Gamma \cup 2A$. Hence either $\Gamma = W_2$ or $\Gamma = W$ or $\Gamma = W_1$. Since $Q \notin \langle W \rangle$, we have $\Gamma = W_2$. Thus $br_C(Q) = w + 1$ and $br_C(Q)$ is evinced by a non-reduced scheme. Hence $r_C(Q) = n + 3 - br_C(Q) = n + 2 - w$ ([4, Theorem 23]). Hence Lemma 2 gives $r_X(P) = n + 2 - w$.

Now assume $2w = n + 1$. Assume $r_X(P) \leq n + 1 - w$ and take $Q_1 \in \langle \{O, B\} \rangle$ and $E \subset C$ such that $\sharp(E) = r_X(P) = r_C(Q_1)$, E evinces $r_C(Q_1)$ and $\ell_O(E)$ evinces $r_X(P)$. We have $\deg(W_2) + \sharp(E) \leq (w+1) + (n+1-w)$. Hence $\langle W_2 \rangle \cap \langle E \rangle = \langle W_2 \cap E \rangle$. As above we get $E \subset W_2$. Since E is reduced, we get $E \subseteq (W_2)_{red} = W_{red}$. Since W is not reduced, we get $\sharp(E) \leq w - 1$. Since $B \in \langle 2A \cup E \rangle$ and $\deg(2A \cup E) + \deg(W) \leq n + 2$, we also get $B \in \langle W \cap (2A \cup E) \rangle$, i.e. $W \subseteq 2A \cup E$. Since A appears with multiplicity 1 in W and E is reduced, we get that W is reduced, a contradiction.

We have $U \in \Delta$ if and only if $U = \{A\} \cup U'$ with U' a degree $w - 1$

subscheme of $C \setminus \{A\}$ and U' not reduced. Hence Δ is an irreducible variety of dimension $w-2$. Set $\Lambda := \sigma_w(C) \setminus \sigma_w^0(C)$. Let Λ' be the set of all $B \in \Lambda$ whose border rank is evinced by a 0-dimensional scheme containing A with multiplicity 1. The uniqueness of the scheme computing the C-border rank implies that Λ and Λ' are non-empty irreducible constructible subsets with dimension $2w-1 = n-1$ and $n-2$, respectively. Set $\Theta := (\cup_{W \in \Delta} \langle W \rangle) \cap \Lambda'$. Let $\mathcal{J} \subset \mathbb{P}^{n+1}$ denote the join of $\langle 2A \rangle$ and Θ, i.e. the closure in $\mathbb{P}^{n+1}$ of the planes $\langle 2A \cup \{K\} \rangle$ with $K \in \Theta \setminus \langle 2A \rangle$. Fix a general $W \in \Delta$ and a general $B \in \langle W \rangle$. Since B is general, W is linearly independent and W has only finitely many subschemes, we have $B \notin \langle W' \rangle$ for every $W' \subsetneq W$. Since $2w \le n+1$, we get $b_C(B) = w$ and that W evinces the border rank of B (Lemma 1). Since W is not reduced, we have $B \in \sigma_w(C) \setminus \sigma_w^0(C)$. We just proved that $n+2-w \le r_X(P) \le n+3-w$. By Lemma 2, to prove that $r_X(P) = n+2-w = w+1$, it is sufficient to prove the existence of $Q \in \langle \{O, B\} \rangle \setminus \{O\}$ with $r_C(Q) \le w+1$. Since $r_C(Q) = w+1$ if and only if $br_C(Q) = w+1$ ([6, eq. (4)]), it is sufficient to find $Q \in \langle \{O, B\} \rangle$ with $Q \notin \sigma_w(C)$. Assume that this is not the case (for general W, B). Varying W in Δ and B in Λ' we get that $\mathcal{J} \subseteq \sigma_w(C)$. Hence the hypersurface $\sigma_w(C)$ is a cone with vertex containing $\langle 2A \rangle$. Since $\mathrm{Aut}(\mathbb{P}^1)$ acts transitively on $\sigma_1(C)$, we get that $\sigma_w(C)$ is a cone whose vertex contains $\sigma_1(C)$. Since $\langle \sigma_1(C) \rangle = \mathbb{P}^n$, we get $\sigma_w(C) = \mathbb{P}^{n+1}$.

(c) Here we assume $A \notin W_{red}$. The proof of part (b) works verbatim, taking $W_2 := W \cup 2A$, i.e. now W_2 has degree $w+2$. $\square$

Proof of Theorem 1. Let us first check that $O \notin \langle E \rangle$.
Assume $O \in \langle E \rangle$. Thus $O \in \langle 2A \rangle \cap \langle E \rangle$. Since $\deg(2A) + \deg(E) = 2 + \rho \le n+2$, we get $O \in \langle 2A \rangle \cap \langle E \rangle = \langle 2A \cap E \rangle$, where $2A \cap E$ denote the scheme-theoretic intersection. Since E is reduced, we get $2A \cap E \subseteq \{A\}$. Since $O \ne A$, we get a contradiction. Hence $O \notin \langle E \rangle$.

Since $P \in \ell_O(E)$ and $\sharp(E) = r_C(M)$, we have $r_X(P) \le \rho$.

(a) Here we assume $2\rho \le n$. Take $S \subset X$ computing $r_X(\ell_O(M))$. In this case it is sufficient to prove that $S = \ell_O(E)$. Since $\ell_O|C$ is injective, there is a unique $S' \subset C$ such that $\ell_O(S') = S$. Since $P = \ell_O(M) \in \langle S \rangle$, we have $M \in \langle \{O\} \cup S' \rangle \subset \langle 2A \cup S' \rangle$. Thus $M \in \langle 2A \cup S' \rangle \cap \langle E \rangle$. Since $\deg(2A \cup S') + \deg(E) \le 2 + 2\rho \le n+2$, the scheme $2A \cup S' \cup E$ is linearly independent. Thus $\langle 2A \cup S' \rangle \cap \langle E \rangle$ is the linear span of the scheme-theoretic intersection $(2A \cup S') \cap E$. Since E is reduced and $M \notin \langle E' \rangle$ for any $E' \subsetneq E$, we get that either $S' = E$ or $S' \cup \{A\} = E$. If $A \notin E$, then we get $S' = E$, as wanted.

Assume $A \in E$, if $S' = E$, then we are done. Assume that $S' \neq E$, i.e. $S' = E \setminus \{A\}$. Since $\ell_O(M) \in \langle \ell_O(S') \rangle$, we have $\{O, M\} \cap \langle S' \rangle \neq \emptyset$. In the ρ-dimensional linear space $\langle 2A \cup S' \rangle$, the linear subspaces $\langle E \rangle$ and $\langle \{O\} \cup S' \rangle$ are different hyperplanes, because $O \notin \langle E \rangle$. Hence the line $\langle \{O, M\} \rangle \subset \langle 2A \cup S' \rangle$ intersects E in a unique point. Call P' this point. Since $M \in \langle E \rangle$, we have $P' = M$. Since $\ell_O(M) \in \langle \ell_O(S') \rangle$, we have $\{O, M\} \cap \langle S' \rangle \neq \emptyset$. Hence $P' \in \langle S' \rangle$. Hence $r_C(M) \leq \rho - 1$, a contradiction.

(b) Here we assume $n + 1 \leq 2\rho \leq n + 2$. Assume $r_X(P) \leq \rho - 2$ and take $S' \subset C$ such that $\sharp(S') = r_X(P)$ and $\ell_O(S')$ evinces $r_X(P)$. Since $\deg(2A \cup S' \cup E) \leq n + 2$, as in step (a) we get that $\langle 2A \cup S' \rangle \cap \langle E \rangle$ is the linear span of the scheme-theoretic intersection $(2A \cup S') \cap E$. Since $O \in \langle 2A \rangle$ and $M \in \langle \{O\} \cup S' \rangle \cap \langle E \rangle$, while $M \notin \langle E' \rangle$ for any $E' \subsetneq E$, we get a contradiction.

(c) Assume n odd and $2\rho = n + 3$. A general $P_1 \in \mathbb{P}^{n+1}$ satisfies $r_C(P_1) = br_C(P_1) = (n+3)/2$. A general $P' \in \mathbb{P}^n$ satisfies $P' \in \sigma_{(n+1)/2}(X) \setminus \sigma_{(n-1)}(X)$ ([1, Remark 1.6]). A general $P' \in \mathbb{P}^n$ is of the form $\ell_O(P_1)$ with P_1 general in $\mathbb{P}^{n+1}$. $\square$

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