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ABSTRACT

A geometric environment for the study of non-holonomic Lagrangian systems is developed. A definition of admissible displacement valid in the presence of arbitrary non-linear kinetic constraints is proposed. The meaning of *ideality* for non-strictly mechanical systems is analyzed. The concepts of geometric and/or dynamical symmetry of a constrained system are discussed and embodied in a subsequent non-holonomic formulation of Noether theorem. A revisitation of the results in an "extrinsic" variational language is worked out. A few examples and an appendix illustrating some properties of the manifold of admissible kinetic states are presented.

I. INTRODUCTION

Despite its relatively ancient origin, non-holonomic mechanics is still a challenging field of research. The bibliography on the subject is quite rich: besides the foundational aspects,[1–3] a largely incomplete list of recent contributions includes Refs. 4–15 and references therein. An extension of the symplectic reduction theory[16,17] to non-holonomic mechanics is presented in Ref. 18. An analysis of the connection between constrained systems and control theory may be found in Ref. 19. A variational principle for mechanical systems subject to kinetic constraints and a corresponding formulation of Noether theorem are proposed in Ref. 20 and exemplified in Ref. 21. Alternative formulations of the theorem are developed in Refs. 22–25.

Despite these contributions, work still remains to be done. Among the arguments deserving more in-depth analysis, a significant one is the relationship between non-holonomic symmetries and conservation laws. This and other related topics are the subject of the present paper.

The presentation includes a preliminary discussion of the geometric setup. The concept of ideal constraints and the resulting equations of motion are briefly reviewed.

A more subtle analysis concerns the concept of admissible displacement: the argument, straightforward in the case of linear constraints, is extended to arbitrary non-linear ones through a process of "pointwise linearization," associating to each kinetic state of the system a linear constraint "osculating" the original one. The construction, based on the affine nature of the first jet of the configuration manifold, relies on the introduction of an algorithm, here called the Poincaré–Cartan map, assigning to each function $f(t,q,\dot q)$ a semibasic 1-form $\vartheta(f)$, expressing the first order Taylor expansion of f along the fibers of the jet.

On this basis, a formulation of d'Alembert's principle valid for both positional and kinetic constraints is obtained.

The interplay between admissible displacements and canonical lifts of specific classes of vector fields from the configuration manifold to the velocity space is subsequently considered.[26] The argument leads to a natural characterization of the geometric and/or dynamic symmetries consistent with the constraints and to a subsequent analysis of the relation between symmetry properties and conservation laws for non-holonomic systems.

For completeness, the whole setup is finally placed in an "extrinsic" variational context, centered on the use of an action functional I defined on the totality of sections of the configuration manifold and not only on the admissible ones.

The aim is to verify that an admissible section γ is a solution of the equations of motion if and only if it is an extremal of I with respect to the totality of deformations with fixed end points tangent to fields of admissible virtual displacements along γ. Given the above, the analysis of the behavior of the action functional under arbitrary admissible deformations of the extremals provides a further insight into the relation between dynamic symmetries and conservation laws.

The presentation is completed by a (perhaps too) thorough discussion of a few examples and by an appendix illustrating the relation between specific attributes of the constraints—notably sub-linearity and partial integrability—and geometric properties of the family of admissible displacements.

II. GEOMETRIC SETUP

A. Preliminaries

For convenience of the reader, we review here a few aspects of jet bundle geometry, especially relevant to the subsequent discussion.[26,27]

(i) Let $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ denote an $(n+1)$ dimensional fiber bundle referred to fibered coordinates $t, q^1, \ldots, q^n$, with t representing *absolute time*. The first jet $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ is an affine bundle over $\mathcal{V}_{n+1}$, modeled on the vertical space $V(\mathcal{V}_{n+1})$ and canonically isomorphic to the affine submanifold $\{x \,|\, \dot{t}(x) = 1\}$ in $T(\mathcal{V}_{n+1})$. We interpret $\mathcal{V}_{n+1}$ as the *configuration manifold* of a Lagrangian system and $j_1(\mathcal{V}_{n+1})$ as the associated *velocity space*.

The concepts of the *vertical bundle* $V(j_1(\mathcal{V}_{n+1}))$, *contact bundle* $C(j_1(\mathcal{V}_{n+1}))$, *vertical lift of vectors*, and the definition of the *fundamental tensor* of $j_1(\mathcal{V}_{n+1})$ are regarded as known. For details, the reader is referred to Ref. 27 and references therein.

Unless otherwise stated, we refer to $j_1(\mathcal{V}_{n+1})$ to local jet coordinates $t, q^i, \dot{q}^i$. The notation $\omega^i := dq^i - \dot{q}^i dt$ is used throughout. The symbol $\langle \;\|\; \rangle$ indicates the non-singular pairing between $V(j_1(\mathcal{V}_{n+1}))$ and $C(j_1(\mathcal{V}_{n+1}))$ based on the duality relations $\left\langle \frac{\partial}{\partial \dot{q}^r} \,\middle\|\, \omega^k \right\rangle = \delta^k_r$.

For any function $f \in \mathscr{F}(j_1(\mathcal{V}_{n+1}))$, the contact 1-form $d_v f = \frac{\partial f}{\partial \dot{q}^k}\,\omega^k$, uniquely characterized by the requirement $\langle V \,\|\, d_v f \rangle = V(f)\ \forall\, V \in V(j_1(\mathcal{V}_{n+1}))$, is called the *fiber differential* of f.[27] The fundamental tensor of $j_1(\mathcal{V}_{n+1})$ is denoted by $J = \frac{\partial}{\partial \dot{q}^k} \otimes \omega^k$.

(ii) Every vector field $X \in \mathcal{D}^1(\mathcal{V}_{n+1})$ determines a corresponding *variational vector field* $T(X)$ on the tangent space $T(\mathcal{V}_{n+1})$.[26,28] The latter is tangent to the submanifold $j_1(\mathcal{V}_{n+1}) \xrightarrow{i} T(\mathcal{V}_{n+1})$ if and only if the field X satisfies the condition $\mathscr{L}_X(dt) = 0, \Leftrightarrow \langle X, dt \rangle = \alpha = \text{const.}$ In the stated circumstance, the field X is called *isochronous*.

The totality of isochronous fields over $\mathcal{V}_{n+1}$ form an infinite dimensional Lie algebra, henceforth denoted by $\mathfrak{H}$. Every element $X = \alpha\frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} \in \mathfrak{H}$ can be lifted to a vector field $\hat{X}$ over $j_1(\mathcal{V}_{n+1})$, i-related to $T(X)$. In coordinates, the resulting expression reads

$$\hat{X} := \alpha\frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i}, \tag{1a}$$

with $\dot{X}^i = \frac{\partial X^i}{\partial t} + \frac{\partial X^i}{\partial q^k}\dot{q}^k$ denoting the *symbolic time derivative* of X^i.

A significant feature of the algorithm (1a) is its localizability on curves: given any section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$, the restriction of $\hat{X}$ to the first jet extension $j_1(\gamma)$ depends only on the knowledge of X on γ, thereby inducing a correspondence sending each isochronous vector field $X(t) = \alpha\left(\frac{\partial}{\partial t}\right)_\gamma + X^i(t)\left(\frac{\partial}{\partial q^i}\right)_\gamma$ along γ into the field,

$$\hat{X}(t) = \alpha\left(\frac{\partial}{\partial t}\right)_{j_1(\gamma)} + X^i(t)\left(\frac{\partial}{\partial q^i}\right)_{j_1(\gamma)} + \frac{dX^i}{dt}\left(\frac{\partial}{\partial \dot{q}^i}\right)_{j_1(\gamma)} \tag{1b}$$

along $j_1(\gamma)$.

(iii) An important result, borrowed from classical mechanics, is the existence of a correspondence $\vartheta : \mathscr{F}(j_1(\mathcal{V}_{n+1})) \to \mathcal{D}_1(j_1(\mathcal{V}_{n+1}))$, henceforth called the *Poincaré–Cartan map*, converting each function $f(t, q, \dot{q})$ into the semibasic 1-form,

$$\vartheta(f) = f\,dt + d_v f = f\,dt + \frac{\partial f}{\partial \dot{q}^k}\,\omega^k. \tag{2}$$

Through the algorithm (2), every function $f \in \mathscr{F}(j_1(\mathcal{V}_{n+1}))$ gives rise to a linear map $\mathfrak{g} : V(j_1(\mathcal{V}_{n+1})) \to C(j_1(\mathcal{V}_{n+1}))$ sending each vertical vector $U = U^i \frac{\partial}{\partial \dot{q}^i}$ into the contact 1-form $\mathfrak{g}(U) = U \,\lrcorner\, d[\vartheta(f)]$. In coordinates, we have the expression

$$\mathfrak{g}(U) = U \,\lrcorner\, \left(df \wedge dt + d\left(\frac{\partial f}{\partial \dot q^k}\right) \wedge \omega^k - \frac{\partial f}{\partial \dot q^k}\, d\dot q^k \wedge dt \right) = U^h \frac{\partial^2 f}{\partial \dot q^h\, \partial \dot q^k}\, \omega^k. \tag{3}$$

In view of the latter, the map $\mathfrak{g}$ is non-singular if and only if $\det\left(\frac{\partial^2 f}{\partial \dot q^h\, \partial \dot q^k}\right) \neq 0$ everywhere on $j_1(\mathcal{V}_{n+1})$.

Remark 1. For later use, it is worth spending a few words on the relation between the map (2) and the affine nature of the fibration $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$. To this end, given any $\varsigma \in \mathcal{V}_{n+1}$, we recall that the $(n+1)$-dimensional vector space $P_1(\varsigma) \subset \mathscr{F}(\pi^{-1}(\varsigma))$ formed by the totality of affine polynomials of degree 1 over the fiber $\pi^{-1}(\varsigma)$ is canonically isomorphic to the cotangent space $T^*_\varsigma(\mathcal{V}_{n+1})$ through the correspondence

$$p = a_0 + a_k\, \dot q^k \quad \rightleftarrows \quad \tilde p := a_0\, dt_{|\varsigma} + a_k\, dq^k_{\,|\varsigma}. \tag{4}$$

Furthermore, each point $x \in \pi^{-1}(\varsigma)$ determines a projection $\mathscr{F}(\pi^{-1}(\varsigma)) \to P_1(\varsigma)$ assigning to every function $f(\dot q^1, \ldots, \dot q^n)$ the unique polynomial $p_{(x,f)} \in P_1(\varsigma)$ defined by the conditions $p_{(x,f)}(x) = f(x)$, $\left(dp_{(x,f)}\right)_{|x} = (df)_{|x}$, namely,

$$p_{(x,f)}(\dot q^1, \ldots, \dot q^n) = f\left(\dot q^1(x), \ldots, \dot q^n(x)\right) + \left(\frac{\partial f}{\partial \dot q^k}\right)_x \left(\dot q^k - \dot q^k(x)\right). \tag{5}$$

On account of Eq. (4), polynomial (5) identifies a 1-form

$$\tilde p_{(x,f)} = \left[f(x) - \left(\frac{\partial f}{\partial \dot q^k}\right)_x \dot q^k(x) \right] dt_{|\varsigma} + \left(\frac{\partial f}{\partial \dot q^k}\right)_x dq^k_{\,|\varsigma} \tag{6}$$

in the cotangent space $T^*_\varsigma(\mathcal{V}_{n+1})$. By means of the pull-back $(\pi_x)_*{}^*$, the latter can be lifted to a 1-form

$$(\pi_x)_*{}^*\left(\tilde p_{(x,f)}\right) = \left[f\, dt + \frac{\partial f}{\partial \dot q^k}\left(dq^k - \dot q^k\, dt\right) \right]_x = \left[f\, dt + \frac{\partial f}{\partial \dot q^k}\, \omega^k \right]_x$$

at the point x, synthetically denoted by $\vartheta(f)_{|x}$.

In this way, by varying x in $\pi^{-1}(\varsigma)$ and ς in $\mathcal{V}_{n+1}$, we obtain a correspondence ϑ assigning to each function $f \in \mathscr{F}(j_1(\mathcal{V}_{n+1}))$ a semibasic 1-form $\vartheta(f) \in \mathcal{D}_1(j_1(\mathcal{V}_{n+1}))$, expressed in coordinates by Eq. (2). In particular, if $p = a_0 + a_k\, \dot q^k$ is the linear polynomial over $j_1(\mathcal{V}_{n+1})$ associated with the 1-form $\tilde p = a_0\, dt + a_k\, dq^k \in \mathcal{D}_1(\mathcal{V}_{n+1})$, a straightforward check shows that the 1-form $\vartheta(p)$ coincides with the pull-back $\pi^*(\tilde p)$.

(iv) The presence of non-holonomic constraints is accounted for by restricting the space of admissible kinetic states to a fibered submanifold,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ i\ } & j_1(\mathcal{V}_{n+1}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1} \end{array}.$$

Referring the space $\mathcal{A}$ to fibered coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$, the imbedding $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ is locally expressed as

$$\dot q^k = \psi^k(t, q^1, \ldots, q^n, z^1, \ldots, z^r)\,, \qquad k = 1, \ldots, n, \tag{7a}$$

with $\operatorname{rank}\left\| \frac{\partial(\psi^1 \cdots \psi^n)}{\partial(z^1 \cdots z^r)} \right\| = r$. Alternatively, $\mathcal{A}$ can be represented implicitly as

$$g^\sigma(t, q^1, \ldots, q^n, \dot q^1, \ldots, \dot q^n) = 0\,, \qquad \sigma = 1, \ldots, n-r, \tag{7b}$$

with $\operatorname{rank}\left\| \frac{\partial(g^1 \cdots g^{n-r})}{\partial(\dot q^1 \cdots \dot q^n)} \right\| = n - r$.

A section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ is *admissible* if and only if there exists a section $\hat{\gamma} : \mathbb{R} \to \mathcal{A}$, called the *lift* of γ, satisfying the conditions $\pi \cdot \hat{\gamma} = \gamma,\ i \cdot \hat{\gamma} = j_1(\gamma)$.

Given any bundle $\mathcal{W}$ over $j_1(\mathcal{V}_{n+1})$, its restriction to the submanifold $\mathcal{A}$ is denoted by $\mathcal{W}_{\mathcal{A}}$. For simplicity, no distinction is made between $\mathcal{A}$ and its image $i(\mathcal{A})$, between semibasic 1-forms in $T^*_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$ and semibasic 1-forms in $T^*(\mathcal{A})$, or between the tangent space $T(\mathcal{A})$ and its push-forward $i_*(T(\mathcal{A})) \subset T_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$.

(v) The vertical and the contact bundles are easily adapted to the submanifold $\mathcal{A}$, giving rise to analogous bundles, respectively denoted by $V(\mathcal{A})$, $C(\mathcal{A})$, identified with corresponding sub-bundles of $T_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$ and $T^*_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$.

The annihilator of the vertical space $V(\mathcal{A})$ under the pairing $\langle\ \|\ \rangle$ identifies a distinguished sub-bundle $\chi(\mathcal{A}) \subset C(\mathcal{A})$, called the *Chetaev bundle*.[5] Its elements are contact 1-forms $\nu = \nu_i \omega^i$ along $\mathcal{A}$, satisfying the conditions

$$\left\langle \nu \,\middle\|\, \frac{\partial}{\partial z^A} \right\rangle = \nu_k \frac{\partial \psi^k}{\partial z^A} = 0, \qquad A = 1, \dots, r.$$

If the submanifold $\mathcal{A}$ is represented in the implicit form (7b), a local basis for $\chi(\mathcal{A})$ is provided by the (pull-back of) the fiber differentials $d_{\mathcal{V}} g^\sigma = \frac{\partial g^\sigma}{\partial \dot{q}^k} \omega^k$.

(vi) The vector sub-bundle $(\chi(\mathcal{A}))^0 \subset T_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$ formed by the totality of vectors along $\mathcal{A}$ annihilating the Chetaev 1-forms under *ordinary* pairing will play a central role in all subsequent developments. For the time being, referring $\chi(\mathcal{A})$ to the local basis $d_{\mathcal{V}} g^\sigma$ induced by the representation (7b), we note that every $Y \in (\chi(\mathcal{A}))^0$ satisfies the relation

$$0 = \langle Y, d_{\mathcal{V}} g^\sigma \rangle = \langle Y, \omega^i \rangle \frac{\partial g^\sigma}{\partial \dot{q}^i} = \langle J(Y), dg^\sigma \rangle, \qquad \sigma = 1, \dots, n - r,$$

with J denoting the fundamental tensor of $j_1(\mathcal{V}_{n+1})$.

Therefore, for a vector $Y \in T_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$, being an element of $(\chi(\mathcal{A}))^0$ does not mean being tangent to the submanifold $\mathcal{A}$ but having an image $J(Y)$ tangent to $\mathcal{A}$.

B. Ideal constraints

The equations of motion of a mechanical system subject to ideal kinetic constraints are well known.[1–5,9–12,24] In this subsection, we add a few remarks on the subject.

The situation we will consider is summarized as follows: in the manifold $j_1(\mathcal{V}_{n+1})$, we are given a semispray $\hat{Z}$, called the *free dynamical flow*, determined by a corresponding "extrinsic" Lagrangian $\hat{L} = \hat{L}(t, q, \dot{q})$, describing what the evolution of a given system $\mathfrak{B}$ would be in the absence of kinetic constraints. For generality, the function $\hat{L}$ is not assumed to be a polynomial of degree 2 in the variables $\dot{q}^i$ but only to satisfy the regularity condition $\det\left(\frac{\partial^2 \hat{L}}{\partial \dot{q}^h \partial \dot{q}^k}\right) \neq 0$, i.e., to induce a non-singular map $\mathfrak{g} : V(j_1(\mathcal{V}_{n+1})) \to C(j_1(\mathcal{V}_{n+1}))$.

Composed with the pairing $\langle\ \|\ \rangle$, the map $\mathfrak{g}$ determines a (possibly non-positive) scalar product between vertical vectors, expressed in components as

$$(U, V) := \langle\, U \,\|\, \mathfrak{g}(V) \rangle = \frac{\partial^2 \hat{L}}{\partial \dot{q}^h \partial \dot{q}^k} U^h V^k.$$

In the stated geometric setup, the kinetic constraints are accounted for by the introduction of the submanifold $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ and of the associated class of *admissible evolutions*. The pull-back $L := i^*(\hat{L}) = \hat{L}\big(t, q^k, \psi^k(t, q^k, z^A)\big)$ is called the intrinsic Lagrangian; the pull-back $p_k = i^*\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right)$, related to L by the identities $p_k \frac{\partial \psi^k}{\partial z^A} = \frac{\partial L}{\partial z^A}$, are called the kinetic momenta.

In a mechanical context, the constraints are thought of as devices interacting with the system through corresponding *reactive forces*, modifying the original free flow $\hat{Z}$ and converting it into an *effective dynamical flow* Z, tangent to the submanifold $\mathcal{A}$. Being a physical operation determined by the nature of the reactive forces, the conversion process has no natural counterpart in the pseudo-mechanical context considered here, the term "pseudo" referring to the broader class of allowed Lagrangians.

To address this aspect, a natural approach, similar to the one adopted in classical mechanics, consists in *postulating* a specific conversion algorithm, incorporating it into the definition of a corresponding class of constraints.

In doing this, attention must be paid to the fact that when applied to mechanical systems, the resulting scheme has a precise *physical* meaning. To achieve this goal, we note that being a vertical vector, the difference $Z - \hat{Z}$ is completely determined by the knowledge of the contact 1-form $\mathfrak{g}(Z - \hat{Z}) = (Z - \hat{Z}) \,\lrcorner\, d\vartheta(\hat{L})$, where, by definition, the contribution $\hat{Z} \,\lrcorner\, d\vartheta(\hat{L})$ vanishes identically. Specifying the conversion algorithm $\hat{Z} \to Z$ is therefore equivalent to characterizing the internal product $Z \,\lrcorner\, d\vartheta(\hat{L})$.

On the other hand, since the effective flow Z is (the push forward of) a vector field over $\mathcal{A}$, we have the identifications

$$Z\rfloor\, d\vartheta(\hat{L}) \;=\; Z\rfloor\, i^*\big(d\vartheta(\hat{L})\big) = Z\rfloor\, d\big(Ldt + p_k\,\omega^k\big), \tag{8}$$

with L and p_k, respectively, denoting the intrinsic Lagrangian and the kinetic momenta. With this in mind, we introduce the following definition:[29]

Definition 1. A kinetic constraint is called ideal if and only if the 1*-form described by Eq.* (8) *is a Chetaev* 1*-form.*

Let us verify that when applied to mechanical systems, Definition 1 reproduces the standard notion of ideality based on Gauss principle of *least constraint*.[4,5] For simplicity, we consider a discrete system, formed by N points P_i of masses m_i, subject to arbitrary positional and/or kinetic constraints.

We denote by $\underline{F}_i$ the active forces, by $\underline{\phi}_i$ the reactive ones, and by Z the effective dynamical flow and adopt the standard notation of analytical mechanics for positions, velocities, etc. The constraint function is then

$$C := \frac{1}{2}\sum_{i=1}^{N} m_i\left|\underline{a}_i - \frac{\underline{F}_i}{m_i}\right|^2, \tag{9}$$

with

$$\underline{a}_i = \frac{\partial^2 P_i}{\partial t^2} + 2\,\frac{\partial^2 P_i}{\partial t\,\partial q^k}\,\psi^k + \frac{\partial^2 P_i}{\partial q^k\,\partial q^r}\,\psi^k\psi^r + \frac{\partial \psi^i}{\partial q^k}\left(\frac{\partial \psi^k}{\partial t} + \frac{\partial \psi^k}{\partial q^r}\,\psi^r + \frac{\partial \psi^k}{\partial z^A}\,\dot{z}^A\right).$$

Minimizing function (9) within the class of admissible accelerations at a given kinetic state $x = (t, q^i, z^A)$ yields the equations

$$0 = \frac{\partial C}{\partial \dot{z}^A} = \sum_{i=1}^{N}\big(m_i\underline{a}_i - \underline{F}_i\big)\cdot\frac{\partial \underline{a}_i}{\partial \dot{z}^A} = \sum_{i=1}^{N}\big(m_i\underline{a}_i - \underline{F}_i\big)\cdot\frac{\partial P_i}{\partial q^k}\,\frac{\partial \psi^k}{\partial z^A}. \tag{10}$$

Recalling the standard representation of $\sum_i m_i\underline{a}_i\cdot\frac{\partial P_i}{\partial q^k}$ in terms of the kinetic energy $\hat{T} = \frac{1}{2}\sum_i m_i v_i^2$ and assuming the existence of a generalized potential $\hat{U}(t, q, \dot{q})$ satisfying the relation $\sum_i \underline{F}_i\cdot\frac{\partial P_i}{\partial q^k} = \frac{\partial \hat{U}}{\partial q^k} - \frac{d}{dt}\frac{\partial \hat{U}}{\partial \dot{q}^k}$, Eq. (10) can be written in the form

$$\left[Z\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right) - \frac{\partial \hat{L}}{\partial q^k}\right]\frac{\partial \psi^k}{\partial z^A} \;=\; 0, \tag{11}$$

with $\hat{L} = \hat{T} + \hat{U}$ denoting the extrinsic Lagrangian.

On the other hand, definition (3) of the Poincaré–Cartan map entails the identity

$$Z\rfloor\, d\big(\vartheta(\hat{L})\big) = Z\rfloor\,\left[d\hat{L}\wedge dt + d\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right)\wedge\omega^k + \frac{\partial \hat{L}}{\partial \dot{q}^k}d\omega^k\right] = \left[Z\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right) - \frac{\partial \hat{L}}{\partial q^k}\right]\omega^k.$$

Together with Eq. (11), the latter shows that $Z\rfloor\, d\big(\vartheta(\hat{L})\big)$ is a Chetaev 1-form, thus proving that in a mechanical context, Definition 1 reproduces the content of Gauss principle.

Returning to the general case, Definition 1 itself is a statement about the fact that the evolution of *any* Lagrangian system subject to ideal constraints is described in terms of the free Lagrangian $\hat{L}$ by the equation[30]

$$Z\rfloor\, d\big(\vartheta(\hat{L})\big) = \left[Z\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right) - \frac{\partial \hat{L}}{\partial q^k}\right]\omega^k \in \chi(\mathcal{A}), \tag{12}$$

commonly written in the form (11) or in the equivalent one

$$Z\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right) - \frac{\partial \hat{L}}{\partial q^k} \;=\; \lambda_\sigma\frac{\partial g^\sigma}{\partial \dot{q}^k}.$$

As anticipated in Eq. (8), Eq. (11) can be formulated in terms of geometric objects pertaining to the environment $\mathcal{A}$, namely, the intrinsic Lagrangian L and the kinetic momenta p_k. The argument is entirely straightforward;[4,5] on account of the identity $\frac{\partial L}{\partial q^k} = \frac{\partial \hat{L}}{\partial q^k} + p_r\frac{\partial \psi^r}{\partial q^k}$, the resulting equation reads

$$\left[Z(p_k) - \frac{\partial L}{\partial q^k} + p_r \frac{\partial \psi^r}{\partial q^k}\right]\frac{\partial \psi^k}{\partial z^A} = 0. \tag{13}$$

Remark 2. The approach adopted so far focuses on the genesis of the equations of motion, rather than on their technical aspects. The conclusion, expressed in intrinsic form by Eq. (13), is that for Lagrangian systems subject to ideal constraints, the significant dynamical objects are the intrinsic Lagrangian L and the momenta p_k. We could therefore do without the extrinsic Lagrangian and set up a formalism based solely on intrinsic quantities.

Although formally correct, this procedure would spoil the analysis of the first integrals, reducing it to a mere formalism, devoid of any geometrical insight. We will therefore continue to use the extrinsic Lagrangian $\hat{L}$, bearing in mind that, as a dynamic object, the latter is defined up to an equivalence relation of the form

$$\hat{L} \sim \hat{L}' \quad \Longleftrightarrow \quad \big(d\hat{L}\big)_x - \big(d\hat{L}\big)_x = \big(d\dot{f}\big)_x \quad \forall f \in \mathscr{F}(\mathcal{V}_{n+1}),\ \forall x \in \mathcal{A},$$

accounting both for the arbitrariness in the prolongation of $\hat{L}$ outside the submanifold $\mathcal{A}$ and for the invariance of the Lagrange equations under arbitrary gauge transformations $\hat{L}' = \hat{L} + \dot{f}$, $f \in \mathscr{F}(\mathcal{V}_{n+1})$. We let the reader verify that equivalent Lagrangians $\hat{L}, \hat{L}'$ do indeed lead to the same intrinsic equations (13).

C. Admissible displacements

(i) An important element of rationalization comes from the analysis of the concept of *admissible displacement* in the presence of non-holonomic constraints.

Matters are straightforward when all constraints are *linear*: in this case, $\mathcal{A}$ is an affine sub-bundle of $j_1(\mathcal{V}_{n+1})$, formed by the totality of kinetic states $x = \big(\frac{\partial}{\partial t}\big)_{\pi(x)} + \dot{q}^i(x)\Big(\frac{\partial}{\partial q^i}\Big)_{\pi(x)}$ annihilating the linear module spanned by $n-r$ independent 1-forms $\mu^\sigma = \mu_0^\sigma dt + \mu_i^\sigma dq^i$, $\sigma = 1,\ldots,n-r$, or, what is the same, fulfilling a set of linear equations of the form $g^\sigma(t,q,\dot{q}) = \mu_0^\sigma(t,q) + \mu_i^\sigma(t,q)\dot{q}^i = 0$.

In any configuration $\varsigma \in \mathcal{V}_{n+1}$, the admissible displacements are then vectors $X = X^0\big(\frac{\partial}{\partial t}\big)_\varsigma + X^i\Big(\frac{\partial}{\partial q^i}\Big)_\varsigma \in T_\varsigma(\mathcal{V}_{n+1})$ satisfying the condition $\langle X, \mu^\sigma\rangle = 0$.

In the stated circumstance, the Poincaré–Cartan map (2) entails the identification

$$\vartheta(g^\sigma) = g^\sigma dt + \frac{\partial g^\sigma}{\partial \dot{q}^k}\,\omega^k = \Big(\mu_0^\sigma + \mu_k^\sigma \dot{q}^k\Big)dt + \mu_k^\sigma\Big(dq^k - \dot{q}^k dt\Big) = \pi^*(\mu^\sigma).$$

For any kinetic state $x \in \pi^{-1}(\varsigma)$ and any vector $\hat{X} \in T_x(j_1(\mathcal{V}_{n+1}))$, the latter implies the relation

$$\big\langle \pi_*(\hat{X}), \mu^\sigma\big\rangle = \big\langle \hat{X}, \pi^*(\mu^\sigma)\big\rangle = \big\langle \hat{X}, \vartheta(g^\sigma)\big\rangle = \big\langle \hat{X}, \cancel{g^\sigma(x)dt} + d_v g^\sigma\big\rangle$$

showing that the vanishing of $\big\langle(\pi_x)_*\hat{X}, \mu^\sigma\big\rangle$, $\sigma = 1,\ldots,n-r$ is equivalent to the vanishing of $\big\langle \hat{X}, \nu\big\rangle\ \forall\, \nu \in \chi(\mathcal{A})$.

Summing up, we conclude that in the presence of linear constraints, the admissible displacements are push-forward of vectors belonging to the sub-bundle $(\chi(\mathcal{A}))^0 \subset T(j_1(\mathcal{V}_{n+1}))$.

Dropping the linearity assumption does not change the result but requires a more careful interpretation. In fact,

- the lack of a description of the constraints based solely on the environment $\mathcal{V}_{n+1}$ precludes any possibility of defining the admissible displacements regardless of the knowledge of the kinetic state of the system;
- if $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$ is not an affine sub-bundle of $j_1(\mathcal{V}_{n+1})$, for each $\varsigma \in \mathcal{V}_{n+1}$, the image space $\pi_*(T_x(j_1(\mathcal{V}_{n+1}))$ may generally depend on the point x in the fiber $\pi^{-1}(\varsigma)$.

Both aspects are accounted for by the interpretation of the Poincaré–Cartan map outlined in Remark 1. To clarify this point, we represent $\mathcal{A}$ in the implicit form (7b), and denote by $\pi_{\mathcal{A}}$ the restriction to $\mathcal{A}$ of the projection $\pi : j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$.

We then observe that for any configuration $\varsigma \in \mathcal{V}_{n+1}$, the fiber $\mathcal{A}_\varsigma := \pi_{\mathcal{A}}^{-1}(\varsigma)$ is a submanifold of the affine space $\pi^{-1}(\varsigma)$, locally described by the equations $g^\sigma(t(\varsigma), q^i(\varsigma), \dot{q}^i) = 0$, and that for any kinetic state $x \in \mathcal{A}_\varsigma$, the tangent plane to $\mathcal{A}_\varsigma$ at x is the affine submanifold of $\pi^{-1}(\varsigma)$ described by the equations $\cancel{g^\sigma(x)} + \Big(\frac{\partial g^\sigma}{\partial \dot{q}^k}\Big)_x\big(\dot{q}^k - \dot{q}^k(x)\big) = 0$, the cancellation being due to the vanishing of g^σ on $\mathcal{A}$.

For any $x \in \mathcal{A}_\varsigma$, there exists therefore a *linear* constraint *tangent* to the original one, i.e., approximating it in a neighborhood of x, up to second order terms in the velocities.

The idea is then that as far as the admissible displacements $X \in T_\varsigma(\mathcal{V}_{n+1})$ are concerned, the restrictions imposed by the original constraint when the system is in the kinetic state x coincide with those imposed by the corresponding linearized one, i.e., with notation (6),

$$X \text{ admissible } \Longleftrightarrow \left\langle X, \tilde{p}_{(x,g^\sigma)} \right\rangle = \left\langle X, \left(\frac{\partial g^\sigma}{\partial \dot{q}^k}\right)_x \left((dq^k)_\varsigma - \dot{q}^k(x)(dt)_\varsigma\right) \right\rangle = 0. \tag{14}$$

However, as already pointed out, the requirement [Eq. (14)] is equivalent to the validity of a representation of the form $X = (\pi_x)_*(\hat{X})$, with $\hat{X} \in T_x(j_1(\mathcal{V}_{n+1}))$ satisfying $\langle \hat{X}, (d_v g^\sigma)_x \rangle = 0$, $\sigma = 1, \ldots, n-r$, i.e., with $\hat{X}$ annihilating the totality of Chetaev 1-forms at x.

We can therefore state:

Proposition 1. If the system is in a kinetic state x projecting into a configuration ς, the admissible displacements $X \in T_\varsigma(\mathcal{V}_{n+1})$ are images under the map π_* of vectors belonging to the annihilator $(\chi(\mathcal{A})_x)^0 \subset T_x(j_1(\mathcal{V}_{n+1}))$.

In coordinates, on account of Eq. (14), the displacements allowed by the constraints when the system is in the kinetic state x have components $X^0\left(\frac{\partial}{\partial t}\right)_\varsigma + X^i\left(\frac{\partial}{\partial q^i}\right)_\varsigma$ fulfilling the conditions $\left(X^k - X^0 \dot{q}^k(x)\right)\left(\frac{\partial g^\sigma}{\partial \dot{q}^k}\right)_x = 0$, summarized into the representation

$$X^k = X^0\, \psi^k(x) + U^A \left(\frac{\partial \psi^k}{\partial z^A}\right)_x$$

for arbitrary $X^0, U^A \in \mathbb{R}$. For later use, setting $V^A = U^A - X^0 z^A$, we express it in the more convenient form

$$X^k = X^0 \left(\psi^k - z^A \frac{\partial \psi^k}{\partial z^A}\right)_x + V^A \left(\frac{\partial \psi^k}{\partial z^A}\right)_x. \tag{15}$$

An admissible displacement satisfying $\langle X, dt \rangle = 0$ is called a *virtual* displacement.

(ii) In a mechanical context, the previous arguments help extending d'Alembert's *principle of virtual work* to the class of ideal kinetic constraints.

On account of Eq. (15), given a system of points placed in positions $P_i = P_i(\varsigma)$ with velocities $\underline{v}_i(x) = \left(\frac{\partial P_i}{\partial t}\right)_\varsigma + \left(\frac{\partial P_i}{\partial q^k}\right)_\varsigma \dot{q}^k(x)$, the effect of a virtual displacement X is, in fact, the family of displacements

$$\delta P_i = \left(\frac{\partial P_i}{\partial q^k}\right)_\varsigma X^k = \left(\frac{\partial P_i}{\partial q^k}\right)_\varsigma \left(\frac{\partial \psi^k}{\partial z^A}\right)_x V^A.$$

Accordingly, the virtual work done by the reactive force is

$$\delta L = \sum_{i=1}^N \underline{\phi}_i \cdot \delta P_i = \sum_{i=1}^N \underline{\phi}_i \cdot \left(\frac{\partial P_i}{\partial q^k}\right)_\varsigma \left(\frac{\partial \psi^k}{\partial z^A}\right)_x V^A.$$

In particular, the vanishing of δL for all virtual displacements is expressed by the condition

$$\sum_{i=1}^N \underline{\phi}_i \cdot \left(\frac{\partial P_i}{\partial q^k}\right)_\varsigma \left(\frac{\partial \psi^k}{\partial z^A}\right)_x = 0$$

which, on account of the second law $\underline{F}_i + \underline{\phi}_i = m_i \underline{a}_i$, is a restatement of Eq. (10).

In the present context, d'Alembert's characterization of ideality is therefore perfectly meaningful and equivalent to Gauss one.

(iii) Proposition 1 does not exclude the existence of *unconditionally admissible* displacements, namely, of displacements $X \in T_\varsigma(\mathcal{V}_{n+1})$ whose admissibility holds for *any* kinetic state $x \in \mathcal{A}_\varsigma$: a necessary and sufficient condition for this to happen is the existence of vector fields $V = V^A \frac{\partial}{\partial z^A}$ along $\mathcal{A}_\varsigma$ satisfying a relation of the form

$$X^k = X^0 \left(\psi^k_\varsigma - z^A \frac{\partial \psi^k_\varsigma}{\partial z^A}\right) + V^A \frac{\partial \psi^k_\varsigma}{\partial z^A}, \tag{16}$$

with $X^0, X^k = \text{const}$ and with ψ^k_ς denoting the restriction of the function ψ^k to the fiber $\mathcal{A}_\varsigma$.

A typical and somehow foundational example occurs when all constraints are *linear*. For more general constraints, we consider the following two alternatives:

(a) when $X^0 \neq 0$, by a proper choice of the coordinates q^i, we can ensure the validity of the representation $X = X^0 \frac{\partial}{\partial t}$. Up to a reordering of indices, in the neighborhood of each point $x \in \mathcal{A}_\varsigma$, we also entitled to adopt $\dot{q}^1, \ldots, \dot{q}^r$ as fiber coordinates, thus expressing the first r functions $\psi^k_\varsigma(z)$ as $\psi^A_\varsigma = z^A$, $A = 1, \ldots, r$.

In the resulting framework, Eq. (16) entails the relation $\psi^k_\varsigma - z^A \frac{\partial \psi^k_\varsigma}{\partial z^A} = 0$, characterizing the ψ^k_ς's as homogeneous functions of degree 1 in the variables z^A.

This fact, together with the implicitly assumed differentiability, forces each ψ^k_ς to be a homogeneous *polynomial* of degree 1: the existence of unconditionally admissible displacements X satisfying $\langle X, dt \rangle \neq 0$ implies therefore the *affine character* of the submanifold $\mathcal{A}_\varsigma \subset \pi^{-1}(\varsigma)$.

(b) if $X^0 = 0$, Eq. (16) reads

$$X^k = V^A \frac{\partial \psi^k_\varsigma}{\partial z^A}. \tag{17}$$

The virtual displacements X fulfilling Eq. (17) form a (possibly 0-dimensional) subspace $\mathfrak{U}$ of the vertical space $V_\varsigma(\mathcal{V}_{n+1}) \subset T_\varsigma(\mathcal{V}_{n+1})$.

To grasp their meaning, we recall that $V_\varsigma(\mathcal{V}_{n+1})$ is the modeling space of the fiber $\pi^{-1}(\varsigma)$, that every $X = X^k \left(\frac{\partial}{\partial q^k}\right)_\varsigma \in V_\varsigma(\mathcal{V}_{n+1})$ can be lifted to a vertical vector field $X^{(v)} = X^k \frac{\partial}{\partial \dot{q}^k}$ on $\pi^{-1}(\varsigma)$, and that the 1-parameter group of translations $x \to x + \xi X$ induced by X in $\pi^{-1}(\varsigma)$ coincides with the 1-parameter group of diffeomorphisms $\varphi_\xi : \dot{q}^k \to \dot{q}^k + \xi X^k$ determined by $X^{(v)}$.

On this account, the content of Eq. (17) is that a vector X belongs to the subspace $\mathfrak{U}$ if and only if its lift $X^{(v)}$ is tangent to the submanifold $\mathcal{A}_\varsigma \subset \pi^{-1}(\varsigma)$, i.e., if and only if the group of translations associated with X *acts* on $\mathcal{A}_\varsigma$, sending each point $x \in \mathcal{A}_\varsigma$ into a point $x \to x + \xi X \in \mathcal{A}_\varsigma$.

If $s := \dim \mathfrak{U} > 0$, this action makes $\mathcal{A}_\varsigma$ into an affine bundle over the manifold $\mathcal{A}_\varsigma/\mathfrak{U}$, quotient of $\mathcal{A}_\varsigma$ under the equivalence relation

$$x \sim y \qquad \Leftrightarrow \qquad x - y \in \mathfrak{U}.$$

We can therefore specialize the choice of the coordinates in $\mathcal{A}_\varsigma$, assuming $z^1, \ldots, z^s$ as affine coordinates along the fibers of the bundle $\mathcal{A}_\varsigma \to \mathcal{A}_\varsigma/\mathfrak{U}$ and $z^{s+1}, \ldots, z^r$ as local coordinates on the base manifold $\mathcal{A}_\varsigma/\mathfrak{U}$.

In this way, each vector field $\frac{\partial}{\partial z^\lambda}$, $\lambda = 1, \ldots, s$ is tangent to the orbits of the 1-parameter group of translations generated by a vector $X_{(\lambda)} \in \mathfrak{U}$, i.e., it is the restriction to $\mathcal{A}_\varsigma$ of the vertical lift $X^{(v)}_{(\lambda)}$ of $X_{(\lambda)}$, thereby satisfying an equation of the form

$$i_*\left(\frac{\partial}{\partial z^\lambda}\right) = \frac{\partial \psi^k_\varsigma}{\partial z^\lambda} \frac{\partial}{\partial \dot{q}^k} := \psi_\lambda{}^k \frac{\partial}{\partial \dot{q}^k},$$

with $\psi_\lambda{}^k := \frac{\partial \psi^k_\varsigma}{\partial z^\lambda} = \text{const}$.

From this, we conclude that when $s > 0$, i.e., when the family of unconditionally admissible virtual displacements is nonempty, the imbedding $\mathcal{A}_\varsigma \to \pi^{-1}(\varsigma)$ admits a canonical representation of the form

$$\dot{q}^k = \psi^k_\varsigma = \sum_{\lambda=1}^{s} \psi_\lambda{}^k z^\lambda + \psi_0{}^k(z^{s+1}, \ldots, z^r),$$

with $\psi_\lambda{}^k = \text{const}$.

In the stated circumstance, the vectors $X_{(\lambda)} = \psi_\lambda{}^k \left(\frac{\partial}{\partial q^k}\right)_\varsigma$, $\lambda = 1, \ldots, s$ span $\mathfrak{U}$.

(iv) The previous arguments are especially relevant in the study of the class $\mathfrak{H}$ of isochronous vector fields. A field $X \in \mathfrak{H}$ is said to be consistent with the constraints if and only if, for any kinetic state $x \in \mathcal{A}$, the vector $X_{\pi(x)}$ is an admissible displacement, i.e., if and only if $X_{\pi(x)}$ is *unconditionally admissible*.

A necessary and sufficient condition for this to happen is the vanishing of $\langle \hat{X}, \nu \rangle \;\; \forall\; \nu \in \chi(\mathcal{A})$, $\hat{X}$ denoting the lift of the field X described by Eq. (1a).

The totality of isochronous vector fields consistent with the constraints will be denoted by $\mathfrak{H}_0$. In coordinates, setting $X = \alpha \frac{\partial}{\partial t} + X^i(t, q) \frac{\partial}{\partial q^i}$ (with $\alpha = \text{const.}$), the consistency requirement can be expressed in either forms

$$X^k = \alpha\left(\psi^k - z^A \frac{\partial \psi^k}{\partial z^A}\right) + V^A \frac{\partial \psi^k}{\partial z^A}, \tag{18a}$$

with $V = V^A \frac{\partial}{\partial z^A} \in \mathcal{D}^1(\mathcal{A})$, or

$$X^k \frac{\partial g^\sigma}{\partial \dot q^k} = \alpha\, \psi^k \frac{\partial g^\sigma}{\partial \dot q^k}, \qquad \sigma = 1, \dots, n - r. \tag{18b}$$

The situation is pointwise identical to the one discussed in point (iii). We can therefore draw the following conclusions:

- if all constraints are linear, referring the submanifold $\mathcal{A}$ to affine coordinates z^A and adopting the representation $\psi^k = \psi_A{}^k(t,q) z^A + \psi_0{}^k(t,q)$, Eq. (18a) reads

$$X^k = \alpha\, \psi_0{}^k + V^A\, \psi_A{}^k.$$

  The requirement $\frac{\partial X^k}{\partial z^B} = 0$ entails the vanishing of $\frac{\partial V^A}{\partial z^B}$. Conversely, any choice of the coefficient α and of a set of components $V^A = V^A(t,q)$ yield a field $X \in \mathfrak{H}_0$;
- if $\mathcal{A}$ is not an affine sub-bundle of $j_1(\mathcal{V}_{n+1})$, a necessary and sufficient condition for the family $\mathfrak{H}_0$ to be non-empty is the existence of fibered coordinates t, q^i, z^A on $\mathcal{A}$ yielding a representation of the functions ψ^k of the form

$$\psi^k = \sum_{\alpha=1}^{s} \psi_\lambda{}^k(t,q) z^\lambda + \psi_0{}^k\big(t, q, z^{s+1}, \dots, z^r\big). \tag{19}$$

  The latter entails a factorization of the projection $\mathcal{A} \to \mathcal{V}_{n+1}$ into the product of an affine fibration of $\mathcal{A}$ over an intermediate base space $\mathcal{Q}$ referred to coordinates $t, q^i, z^{s+1}, \dots, z^r$, followed by a fibration $\mathcal{Q} \to \mathcal{V}_{n+1}$.

Constraints admitting a representation of the form (19) are called *sub-linear*. In the stated circumstance, the family $\mathfrak{H}_0$ is locally generated by the fields

$$X_{(\lambda)} = \psi_\lambda{}^k(t,q) \frac{\partial}{\partial q^k}, \quad \lambda = 1, \dots, s.$$

III. GEOMETRIC AND DYNAMICAL SYMMETRIES

A. Non-holonomic Noether theorem: A differential approach

Let us now focus on the class $\mathfrak{H}_0$ of isochronous vector fields consistent with the constraints. A field $X \in \mathfrak{H}_0$ whose lift $\hat X = \alpha \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + \dot X^i \frac{\partial}{\partial \dot q^i}$ satisfies the requirement $i^*(\hat X(\hat L)) = 0$ is called a *geometric symmetry* of the system.

Keeping the notation Z for the dynamical flow and ϑ for the Poincaré–Cartan map, for any $X \in \mathfrak{H}_0$, we have the relation

$$Z \,\lrcorner\, \mathscr{L}_{\hat X}\big(\vartheta(\hat L)\big) = Z \,\lrcorner \left[\hat X(\hat L) dt + \hat X\!\left(\frac{\partial \hat L}{\partial \dot q^k}\right)\omega^k + \frac{\partial \hat L}{\partial \dot q^k}\big(dX^k - \dot X^k dt\big)\right] = i^*(\hat X(\hat L)),$$

whence recalling the representation (12) of the evolution equations, as well as the fact that the condition $X \in \mathfrak{H}_0$ implies $\hat X \in (\chi(\mathcal{A}))^0$,

$$\begin{aligned} Z\big(\langle \hat X, \vartheta(\hat L)\rangle\big) &= Z \,\lrcorner\, d\big(\hat X \,\lrcorner\, \vartheta(\hat L)\big) = Z \,\lrcorner \left[\mathscr{L}_{\hat X}\big(\vartheta(\hat L)\big) - \hat X \,\lrcorner\, d\big(\vartheta(\hat L)\big)\right] \\ &= i^*(\hat X(\hat L)) + \big\langle \hat X, Z \,\lrcorner\, d\big(\vartheta(\hat L)\big)\big\rangle = i^*(\hat X(\hat L)). \end{aligned} \tag{20}$$

Equation (20) implies the following non-holonomic counterpart of Noether theorem:

Proposition 2. Whenever a field $X \in \mathfrak{H}_0$ is a geometric symmetry of the system, the scalar $i^*\big(\langle \hat X, \vartheta(\hat L)\rangle\big)$ is a first integral of the equations of motion.

When the submanifold $\mathcal{A}$ is represented in the implicit form $g^\sigma(t,q,\dot q) = 0$, straightforward consequences of Proposition 2 are as follows:

- if all functions g^σ are homogeneous polynomials of degree 1 in the variables $\dot q^k$, the vanishing of $\frac{\partial \hat L}{\partial t}$ entails the conservation of the Hamiltonian,

$$i^*\left(\frac{\partial \hat L}{\partial \dot q^k}\,\dot q^k - \hat L\right) = p_k \psi^k - L$$

- if all functions g^σ satisfy $\frac{\partial g^\sigma}{\partial \dot q^k} = 0$, the vanishing of $\frac{\partial \hat L}{\partial q^k}$ entails the conservation of the momentum $p_k = i^*\left(\frac{\partial \hat L}{\partial \dot q^k}\right)$.

Both statements are entirely obvious. Their proof is left to the reader.

B. Dynamical symmetries

A more general approach to the study of first integrals is obtained by replacing the concept of geometric symmetry with the weaker one of *dynamical symmetry*. To this end, given an admissible section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$, we denote by $\hat\gamma : \mathbb{R} \to \mathcal{A}$ the lift of γ and by $\mathfrak{H}_{|\gamma}$ the restriction to the curve γ of the family $\mathfrak{H}$ of isochronous vector fields.

The elements of $\mathfrak{H}_{|\gamma}$ are vector fields along γ, expressed in coordinates as

$$X = X^0\left(\frac{\partial}{\partial t}\right)_\gamma + X^k(t)\left(\frac{\partial}{\partial q^k}\right)_\gamma, \tag{21}$$

with $X^0 = \text{const}$. The field (21) is called admissible if and only if, for each t, the vector $X(t)$ is an admissible displacement of the system in the kinetic state $\hat\gamma(t)$. A necessary and sufficient condition for this to happen is the validity of the relation

$$\left(X^k - X^0\,\frac{dq^k}{dt}\right)\left(\frac{\partial g^\sigma}{\partial \dot q^k}\right)_{\hat\gamma} = 0\,, \qquad \sigma = 1,\dots,n-r. \tag{22}$$

Every $X \in \mathfrak{H}_{|\gamma}$ can be lifted to a field $\hat X$ along $\hat\gamma$ [Sec. II, Eq. (1b)]. The lift algorithm, expressed in coordinates as $\hat X = X^0\left(\frac{\partial}{\partial t}\right)_{\hat\gamma} + X^k\left(\frac{\partial}{\partial q^k}\right)_{\hat\gamma} + \frac{dX^k}{dt}\left(\frac{\partial}{\partial \dot q^k}\right)_{\hat\gamma}$, allows us to associate to the dynamical flow Z a distinguished family $\mathfrak{H}(Z)$ of vector fields defined along the submanifold $\mathcal{A}$ [formally, a family of sections of the vector bundle $T_{\mathcal{A}}(j_1(\mathcal{V}_{n+1})) \to \mathcal{A}$].

The construction relies on the fact that the integral lines of Z are jet-extensions of sections $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$. Given any field $Y : \mathcal{A} \to T_{\mathcal{A}}(j_1(\mathcal{V}_{n+1}))$ satisfying $Z(\langle Y, dt\rangle) = 0$ and an integral curve $\hat\gamma$ of Z, we can then evaluate the restriction $Y_{\hat\gamma} := Y \cdot \hat\gamma$ and project it to a vector field $X_\gamma = \pi_*(Y_{\hat\gamma})$ along the section $\gamma = \pi\cdot\hat\gamma$.

In view of the stated assumptions, X_γ is automatically in the class $\mathfrak{H}_{|\gamma}$. We can therefore lift it to a vector field $\hat X_{\hat\gamma}$ along $\hat\gamma$ and compare it with the original $Y_{\hat\gamma}$. The family $\mathfrak{H}(Z)$ is then defined as the collection of fields Y fulfilling the condition $\hat X_{\hat\gamma} = Y_{\hat\gamma}$ for all integral curves of Z.

By abuse of notation, the elements of $\mathfrak{H}(Z)$ are henceforth denoted by $\hat X$. In coordinates, every $\hat X \in \mathfrak{H}(Z)$ admits a representation of the form

$$\hat X = X^0(t,q,z)\,\frac{\partial}{\partial t} + X^k(t,q,z)\,\frac{\partial}{\partial q^k} + Z(X^k)\,\frac{\partial}{\partial \dot q^k}, \tag{23a}$$

more conveniently written as

$$\hat X = X^0 Z + U^k\,\frac{\partial}{\partial q^k} + Z(U^k)\,\frac{\partial}{\partial \dot q^k}, \tag{23b}$$

with $Z(X^0) = 0$ and $U^k = X^k - X^0\psi^k$. Note that, in general, $\hat X$ is not required to be tangent to the submanifold $\mathcal{A}$.

The field $\hat X$ is consistent with the constraints if and only if, for each $x \in \mathcal{A}$, the vector $X_{\pi(x)} = \pi_*(\hat X_x)$ is an admissible displacement of the system in the kinetic state x. On account of Proposition 1, the consistency requirement is expressed by the condition $\hat X \in (\chi(\mathcal{A}))^0$, mathematically equivalent to the validity of a representation of the form

$$J(\hat X) \;=\; U^k\,\frac{\partial}{\partial \dot q^k} \;=\; i_*(V) \;=\; V\left(\psi^k\right)\frac{\partial}{\partial \dot q^k} \tag{24}$$

for arbitrary choice of the vector field $V = V^A\,\frac{\partial}{\partial z^A} \in \mathcal{D}^1(\mathcal{A})$.

The subset $\hat X \in \mathfrak{H}(Z)$ formed by the totality of fields consistent with the constraints is denoted by $\mathfrak{H}_0(Z)$. According to Eq. (24), every $\hat X \in \mathfrak{H}_0(Z)$ is uniquely determined by the assignment of a first integral $X^0 \in \mathscr{F}(\mathscr{A})$ and of vertical vector field $V \in \mathcal{D}^1(\mathcal{A})$, without any restriction on the nature of the imbedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$.

After these preliminaries, let us now come to the construction of possible first integrals: a vector field $\hat X \in \mathfrak{H}_0(Z)$ satisfying the condition $\hat X(\hat L) = 0$ is called a *dynamical symmetry* of the system. With this terminology, every lift of a geometric symmetry is a dynamical symmetry.

Conversely, a dynamical symmetry is the lift of a geometric symmetry if and only if the components X^0, U^k involved in the representation (23b) satisfy the conditions $X^0 = \text{const}$, $\frac{\partial U^k}{\partial z^A} = 0$.

Given a dynamical symmetry $\hat{X}$, in order to evaluate the derivative $Z(\langle \hat{X}, \vartheta(\hat{L})\rangle)$, we prolong $\hat{X}$ to a field defined in a neighborhood of $\mathcal{A}$, noting that since the dynamical flow Z is tangent to $\mathcal{A}$, the value of $Z(\langle \hat{X}, \vartheta(\hat{L})\rangle)$ is independent of the prolongation process.

On account Eq. (23a), we have then the relations

$$Z \rfloor \mathscr{L}_{\hat{X}}\, \omega^k = Z \rfloor \left(dX^k - Z(X^k)dt - \dot{q}^k dX^0 \right) = 0,$$
$$Z \rfloor \mathscr{L}_{\hat{X}}\big(\vartheta(\hat{L})\big) = Z \rfloor \left[\hat{X}(\hat{L})dt + \hat{L}dX^0 + \hat{X}\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right)\omega^k + \frac{\partial \hat{L}}{\partial \dot{q}^k}\, \mathscr{L}_{\hat{X}}\, \omega^k \right] = \hat{X}(\hat{L}),$$

whence also recalling the representation (12) of the evolution equations, as well as the condition $\hat{X} \in (\chi(\mathcal{A}))^0$,

$$Z\big(\langle \hat{X}, \vartheta(\hat{L})\rangle\big) = Z \rfloor\, d\big(\hat{X} \rfloor\, \vartheta(\hat{L})\big) = Z \rfloor \left[\mathscr{L}_{\hat{X}}\left(\vartheta(\hat{L})\right) - \hat{X} \rfloor\, d\big(\vartheta(\hat{L})\big)\right] = \hat{X}(\hat{L}) + \big\langle \hat{X}, Z \rfloor\, d\big(\vartheta(\hat{L})\big)\big\rangle = \hat{X}(\hat{L}). \tag{25}$$

We have therefore the following generalization of Proposition 2:

Proposition 3. Whenever a field $\hat{X} \in \mathfrak{H}_0(Z)$ *is a dynamical symmetry, the scalar* $\langle \hat{X}, \vartheta(\hat{L})\rangle$ *is a first integral of the equations of motion.*

Despite the formal analogies, dynamical symmetries are more difficult to evaluate than geometric ones. For this reason, it is often preferable to invert the content of Proposition 3 and use the first integrals as a tool for the determination of the dynamic symmetries. This is achieved through the following corollary:

Corollary 1. If $F(t, q, z)$ *is a first integral of the equations of motion, every vector field of the form*

$$\hat{X} = X^0 Z + V^A \frac{\partial \psi^k}{\partial z^A} \frac{\partial}{\partial q^k} + Z\left(V^A \frac{\partial \psi^k}{\partial z^A}\right) \frac{\partial}{\partial \dot{q}^k}$$

fulfilling the conditions $Z(X^0) = 0$, $\langle \hat{X}, \vartheta(\hat{L})\rangle = F$ *is a dynamical symmetry of the system.*

The proof follows at once from Eqs. (23b), (24), and (25). A specialization of Corollary 1, especially useful for practical purposes, is expressed by the following corollary, whose proof is again a straightforward consequence of Eqs. (23b)–(25):

Corollary 2. Let $\hat{Y}$ *denote a dynamical symmetry satisfying the condition* $\langle \hat{Y}, \vartheta(\hat{L})\rangle = 1$. *Then, given any first integral F, the vector field* $\hat{X} = F\hat{Y}$ *is a dynamical symmetry satisfying* $\langle \hat{X}, \vartheta(\hat{L})\rangle = F$.

C. Intrinsic formulation

Rephrased in terms of the intrinsic Lagrangian $L := i^*(\hat{L})$ and of the kinetic momenta $p_k := i^*\left(\frac{\partial \hat{L}}{\partial \dot{q}^k}\right)$, Proposition 3 entails the following proposition:

Proposition 4. If $X^0(t, q, z)$ *and* $V = V^A \frac{\partial}{\partial z^A}$ *are, respectively, a first integral and a vertical vector field fulfilling the relation*

$$X^0 Z(L) + V(\psi^k)\left(\frac{\partial L}{\partial q^k} - p_r \frac{\partial \psi^r}{\partial q^k}\right) + p_k Z\big(V(\psi^k)\big) = 0, \tag{26}$$

the scalar $X^0 L + V^A \frac{\partial L}{\partial z^A}$ *is a first integral of the equations of motion.*

Proof. Regardless of the results established so far, Eq. (26), together with the identity

$$V^A \frac{\partial L}{\partial z^A} = V^A \frac{\partial \hat{L}}{\partial \dot{q}^k} \frac{\partial \psi^k}{\partial z^A} = p_k V(\psi^k),$$

yields the expression

$$Z\big(X^0 L + p_k V(\psi^k)\big) = X^0 Z(L) + p_k Z\big(V(\psi^k)\big) + V(\psi^k)\, Z(p_k) = V^A \frac{\partial \psi^k}{\partial z^A}\left(Z(p_k) + p_r \frac{\partial \psi^r}{\partial q^k} - \frac{\partial L}{\partial q^k}\right),$$

whose vanishing is ensured by the evolution equation (13). □

To verify that Proposition 4 is a restatement of Proposition 3, we observe that on account of Eq. (23b), given the pair X^0, V, the vector field $\hat{X} = X^0 Z + V(\psi^k)\frac{\partial}{\partial q^k} + Z(V(\psi^k))\frac{\partial}{\partial \dot{q}^k}$ belongs to the class $\mathfrak{H}_0(Z)$.

Furthermore, the identities $\frac{\partial L}{\partial q^k} = \frac{\partial \hat{L}}{\partial q^k} + p_r \frac{\partial \psi^r}{\partial q^k}$, $Z(\hat{L}) = Z(L)$ entail the relation

$$\hat{X}(\hat{L}) = X^0 Z(L) + V(\psi^k)\left(\frac{\partial L}{\partial q^k} - p_r \frac{\partial \psi^r}{\partial q^k}\right) + p_k Z\Big(V(\psi^k)\Big),$$

which, together with Eq. (26), qualifies $\hat{X}$ as a dynamical symmetry of the system.[31] Given the above equation, the identity

$$\left\langle \hat{X}, \vartheta(\hat{L})\right\rangle = X^0 \hat{L} + p_k V(\psi^k) = X^0 L + V^A \frac{\partial L}{\partial z^A} \tag{27}$$

establishes the required conclusion.

IV. THE VARIATIONAL SETUP

A. The action principle for non-holonomic Lagrangian mechanics

The non-holonomic Lagrange equation (11) may be interpreted as stationarity conditions for an action functional with respect to a specified class of variations. The argument is well known. A brief outline is presented below.

In order to encompass all possible situations, we define an isochronous deformation of an admissible section $\gamma : q^k = q^k(t)$ as a 1-parameter family of curves $\gamma_\xi : t = \varphi^0(\xi, \tau)$, $q^k = \varphi^k(\xi, \tau)$ fulfilling the conditions

$$\varphi^0(\xi, \tau) = \tau + a(\xi), \quad a(0) = 0, \quad \varphi^k(0, \tau) = q^k(\tau),$$

expressing the requirements $\frac{dt}{d\tau} = 1$ and $\gamma_0 = \gamma$.

It goes without saying that in the case of deformations with fixed end points on a closed interval $[t_0, t_1]$, the function $a(\xi)$ is identically zero so that the representation of γ_ξ takes the simpler form $q^k = \varphi^k(\xi, t)$.

The vector field $X = a'(0)\left(\frac{\partial}{\partial t}\right)_\gamma + \frac{\partial \varphi^k}{\partial \xi}\Big|_{\xi=0}\left(\frac{\partial}{\partial q^k}\right)_\gamma := X^0\left(\frac{\partial}{\partial t}\right)_\gamma + X^k\left(\frac{\partial}{\partial q^k}\right)_\gamma$ is called the infinitesimal deformation associated with γ_ξ. By Eq. (21), every such field is automatically in the class $\mathfrak{H}_{|\gamma}$.

An isochronous deformation γ_ξ is called *admissible* if an only if the associated infinitesimal deformation X is admissible in the sense of Eq. (22). In the stated circumstance, the differences $U^k := X^k - X^0 \psi^k_{|\hat{\gamma}}$ satisfy a relation of the form

$$U^k = V(\psi^k) = V^A\left(\frac{\partial \psi^k}{\partial z^A}\right)_{\hat{\gamma}},$$

i.e., they form the components of a vertical vector field $U = i_*(V) = V^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$ tangent to the submanifold $\mathcal{A}$.

It should be noted that unlike what happens in ordinary variational calculus, the present notion of admissibility does not impose any restriction on the curves $\gamma_\xi(t), \xi \neq 0$—which, in fact, are not required to belong the submanifold $\mathcal{A}$—but only on the tangent vector to the orbits of the deformation at $\xi = 0$. This makes the extrinsic approach essential in all subsequent developments.

With these premises, let $I[\gamma]$ denote the action functional assigning to each section $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ the value of the integral $\int_{t_0}^{t_1} \hat{L}\left(t, q^k(t), \frac{dq^k}{dt}\right) dt$. The aim is to show that an admissible section γ is a solution of the dynamical equation (11) if and only if it is an extremal of $I[\gamma]$ with respect to arbitrary admissible deformations with fixed end-points.

To this end, given any deformation $\gamma_\xi : q^k = \varphi^k(\xi, t)$ satisfying the admissibility requirement $X^k := \frac{\partial \varphi^k}{\partial \xi}\Big|_{\xi=0} = V^A\left(\frac{\partial \psi^k}{\partial z^A}\right)_{\hat{\gamma}}$, we evaluate

$$\frac{dI[\gamma_\xi]}{d\xi}\bigg|_{\xi=0} = \left[\frac{d}{d\xi}\int_{t_0}^{t_1} \hat{L}\left(t, \varphi^k, \frac{\partial \varphi^k}{\partial t}\right) dt\right]_{\xi=0} = \int_{t_0}^{t_1}\left(\frac{\partial \hat{L}}{\partial q^k} X^k + \frac{\partial \hat{L}}{\partial \dot{q}^k}\frac{dX^k}{dt}\right) dt,$$

i.e., integrating by parts and making use of the end points conditions $X(t_0) = X(t_1) = 0$,

$$\frac{dI[\gamma_\xi]}{d\xi}\bigg|_{\xi=0} = \int_{t_0}^{t_1}\left(\frac{\partial\hat{L}}{\partial q^k} - \frac{d}{dt}\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)X^k dt = \int_{t_0}^{t_1}\left(\frac{\partial\hat{L}}{\partial q^k} - \frac{d}{dt}\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)\frac{\partial\psi^k}{\partial z^A}V^A dt.$$

The vanishing of $\frac{dI[\gamma_\xi]}{d\xi}\big|_{\xi=0}$ for all admissible deformations with fixed end points is therefore expressed by the condition

$$\left(\frac{\partial\hat{L}}{\partial q^k} - \frac{d}{dt}\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)\frac{\partial\psi^k}{\partial z^A} = 0$$

identical to the Lagrange equation (11). This establishes the required identification between solutions of the problem of motion and extremals of the action functional with respect to the prescribed class of admissible deformations.

B. Dynamical symmetries and conservation laws

The study of the behavior of the action functional under admissible deformations with variable end points provides a further insight into the relationship between dynamic symmetries and conservation laws.

The argument is entirely classical: given a section $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ satisfying the Lagrange equations, we evaluate the action integral $I[\gamma_\xi]$ along an admissible deformation of the general type $\gamma_\xi : t = \tau + a(\xi)$, $q^k = \varphi^k(\xi, \tau)$, namely,

$$I[\gamma_\xi] = \int_{t_0}^{t_1}\hat{L}\left(\tau + a(\xi), \varphi^k(\xi,\tau), \frac{\partial\varphi^k}{\partial\tau}\right)d\tau.$$

Let $X = X^0\left(\frac{\partial}{\partial t}\right)_\gamma + X^k\left(\frac{\partial}{\partial q^k}\right)_\gamma$ denote the corresponding infinitesimal deformation, satisfying the prescriptions illustrated in Subsection III A, namely, $X^0 = a'(0) = \text{const}$, $X^k = \frac{\partial\varphi^k}{\partial\xi}\big|_{\xi=0} = X^0\psi^k\big|_{\hat{\gamma}} + U^k$, with $U^k = V(\psi^k)$.

Since, by hypothesis, the lift $\hat{\gamma}$ of the section γ is an integral line of the dynamical flow Z, the lift $\hat{X} = X^0\left(\frac{\partial}{\partial t}\right)_{\hat{\gamma}} + X^k\left(\frac{\partial}{\partial q^k}\right)_{\hat{\gamma}} + \frac{dX^k}{dt}\left(\frac{\partial}{\partial\dot{q}^k}\right)_{\hat{\gamma}}$ of the field X can be expressed in the form [analogous to Eq. (23b)]

$$\hat{X} = \left(X^0 Z + U^k\frac{\partial}{\partial q^k} + \frac{dU^k}{dt}\frac{\partial}{\partial\dot{q}^k}\right)_{\hat{\gamma}}. \tag{28}$$

On account of Eq. (28), a straightforward calculation yields the evaluation

$$\frac{dI[\gamma_\xi]}{d\xi}\bigg|_{\xi=0} = \int_{t_0}^{t_1}\left(X^0\frac{\partial\hat{L}}{\partial t} + X^k\frac{\partial\hat{L}}{\partial q^k} + \frac{dX^k}{dt}\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)X^k dt = \int_{t_0}^{t_1}\hat{X}(\hat{L})dt = \int_{t_0}^{t_1}\left(X^0 Z(\hat{L}) + U^k\frac{\partial\hat{L}}{\partial q^k} + \frac{dU^k}{dt}\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)dt, \tag{29}$$

whence integrating by parts and recalling the identifications $U^k = V(\psi^k)$, $L = i^*(\hat{L})$, $p_k = i^*\left(\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)$ as well as the fact that curve $\gamma(t)$ is a solution of the equations of motion, >

$$\frac{dI[\gamma_\xi]}{d\xi}\bigg|_{\xi=0} = \left[X^0 L + p_k V(\psi^k)\right]_{t_0}^{t_1} + \int_{t_0}^{t_1}\cancel{\left(\frac{\partial\hat{L}}{\partial q^k} - \frac{d}{dt}\frac{\partial\hat{L}}{\partial\dot{q}^k}\right)\frac{\partial\psi^k}{\partial z^A}}V^A dt. \tag{30}$$

As it stands, Eq. (30) is simply a statement concerning a single section γ. However, it is easily converted into a general result valid along *any* solution of the equations of motion.

To this end, rather than focusing on a specific section γ, we regard as our primary object a *dynamical symmetry* $\hat{X} \in \mathfrak{H}_0(Z)$, i.e., a field of the form (23b) along $\mathcal{A}$ satisfying the conditions $\hat{X} \in (\chi(\mathcal{A}))^0$, $\hat{X}(\hat{L}) = 0$. Then,

- given any curve $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ obeying the equations of motion, the restriction of $\hat{X}$ to the lift $\hat{\gamma}$ is the lift of an admissible infinitesimal deformation of γ, i.e., of a field X tangent to an admissible deformation γ_ξ;
- in view of Eqs. (29) and (30), the vanishing of $\hat{X}(\hat{L})$ ensures the vanishing of $\frac{dI[\gamma_\xi]}{d\xi}\big|_{\xi=0}$ and therefore also of the increment $\left[X^0 L + p_k V(\psi^k)\right]_{t_0}^{t_1}$ along γ.

This being true for any solution γ of the equations of motion, recalling the identification (27), we conclude that whenever the field $\hat{X}$ is a dynamical symmetry, the scalar $\langle\hat{X}, \vartheta(\hat{L})\rangle = X^0 L + p_k V(\psi^k)$ is a first integral of the equations of motion.

V. EXAMPLES

A. The Chaplygin sleigh

The Chaplygin sleigh is a plane rigid body $\mathfrak{B}$ with mass center G, sliding without friction on a horizontal plane. The presence of a sharp blade placed in a point $Q \neq G$ forces the velocity of Q to be parallel to $(G - Q)$.

We denote by a the distance $\overline{QG}$, by m the mass of $\mathfrak{B}$, by I its moment of inertia with respect to the vertical axis through Q, and by $r := \sqrt{I/m}$ the so called *radius of gyration*.

The configuration of the system is determined by the Cartesian coordinates x, y of the point Q and by the angle φ between the vector $(G - Q)$ and the x axis.

Setting $q^1 = x, q^2 = y, q^3 = \varphi$ and denoting by $\underline{v} = \dot{x}\,\underline{e}_1 + \dot{y}\,\underline{e}_2$ the velocity of Q, the kinetic constraint is expressed by the condition

$$g(t, q, \dot{q}) = \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0.$$

Setting $z^1 = v = \sqrt{\dot{x}^2 + \dot{y}^2}, z^2 = \dot{\varphi}$, the representation of the submanifold $\mathcal{A}$ takes the form $\dot{q}^k = \psi^k(t, q, z)$, with

$$\psi^1(t, q, z) = v\,\cos\varphi, \quad \psi^2(t, q, z) = v\,\sin\varphi, \quad \psi^3(t, q, z) = \dot{\varphi}. \tag{31}$$

The extrinsic Lagrangian coincides with the kinetic energy

$$\hat{L} = \hat{T} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\varphi}^2 - ma\dot{\varphi}(\dot{x}\sin\varphi - \dot{y}\cos\varphi), \tag{32}$$

while the intrinsic one reads

$$L = i^*(\hat{L}) = \frac{1}{2}m\left(v^2 + r^2\dot{\varphi}^2\right). \tag{33}$$

In view of Eqs. (18a), (21), (31), and (32), the geometric symmetries of the system are represented by vector fields of the form

$$X = \alpha\frac{\partial}{\partial t} + V^A(t, q)\frac{\partial\psi^k}{\partial z^A}\frac{\partial}{\partial q^k} = \alpha\frac{\partial}{\partial t} + V^1\left(\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}\right) + V^2\frac{\partial}{\partial\varphi},$$

satisfying the condition

$$\begin{aligned} i^*\left(\hat{X}(\hat{L})\right) &= i^*\left[V^2\frac{\partial\hat{L}}{\partial\varphi} + \frac{d}{dt}\left(V^1\cos\varphi\right)\frac{\partial\hat{L}}{\partial\dot{x}} + \frac{d}{dt}\left(V^1\sin\varphi\right)\frac{\partial\hat{L}}{\partial\dot{y}} + \frac{dV^2}{dt}\frac{\partial\hat{L}}{\partial\dot{\varphi}}\right] \\ &= m\left(v\frac{dV^1}{dt} + r^2\dot{\varphi}\frac{dV^2}{dt} + a\dot{\varphi}^2V^1 - av\dot{\varphi}V^2\right) = 0. \end{aligned}$$

Evaluating the symbolic time derivatives, the latter reads

$$\begin{aligned} &v\left(\frac{\partial V^1}{\partial t} + \frac{\partial V^1}{\partial x}v\cos\varphi + \frac{\partial V^1}{\partial y}v\sin\varphi + \frac{\partial V^1}{\partial\varphi}\dot{\varphi}\right) + a\dot{\varphi}^2V^1 \\ &+ r^2\dot{\varphi}\left(\frac{\partial V^2}{\partial t} + \frac{\partial V^2}{\partial x}v\cos\varphi + \frac{\partial V^2}{\partial y}v\sin\varphi + \frac{\partial V^2}{\partial\varphi}\dot{\varphi}\right) - av\dot{\varphi}V^2 = 0. \end{aligned} \tag{34}$$

The validity of Eq. (34) requires the simultaneous vanishing of the coefficients of the monomials $v, \dot{\varphi}, v^2, v\dot{\varphi}, \dot{\varphi}^2$, namely,

$$\frac{\partial V^1}{\partial t} = \frac{\partial V^2}{\partial t} = 0, \tag{35a}$$

$$\frac{\partial V^1}{\partial x}\cos\varphi + \frac{\partial V^1}{\partial y}\sin\varphi = 0, \tag{35b}$$

$$r^2\frac{\partial V^2}{\partial\varphi} + aV^1 = 0, \tag{35c}$$

$$\frac{\partial V^1}{\partial \varphi} + r^2\left(\frac{\partial V^2}{\partial x}\cos\varphi + \frac{\partial V^2}{\partial y}\sin\varphi\right) - aV^2 = 0. \tag{35d}$$

Equations (35b) and (35c) imply the relation

$$r^2\frac{\partial}{\partial \varphi}\left(\frac{\partial V^2}{\partial x}\cos\varphi + \frac{\partial V^2}{\partial y}\sin\varphi\right) = -a\left(\frac{\partial V^1}{\partial x}\cos\varphi + \frac{\partial V^1}{\partial y}\sin\varphi\right) = 0.$$

In view of this, Eqs. (35c) and (35d) entail the second order differential equation

$$\frac{\partial^2 V^1}{\partial \varphi^2} = a\frac{\partial V^2}{\partial \varphi} = -\frac{a^2}{r^2}V^1, \tag{36}$$

Together with Eqs. (35a) and (35b), the latter admits the general solution

$$V^1 = A\cos\left(\frac{a}{r}\varphi + \beta\right), \tag{37}$$

with $A, \beta \in \mathbb{R}$.[32] Comparing this with Eqs. (35c) and (35d) yields the expression

$$V^2 = -\frac{1}{r}A\sin\left(\frac{a}{r}\varphi + \beta\right).$$

The geometric symmetries of the Chaplygin sleigh form therefore a three-dimensional vector space, spanned by the totality of vector fields of the form

$$X = \alpha\frac{\partial}{\partial t} + A\left[\cos\left(\frac{a}{r}\varphi + \beta\right)\left(\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}\right) - \frac{1}{r}\sin\left(\frac{a}{r}\varphi + \beta\right)\frac{\partial}{\partial \varphi}\right].$$

The corresponding first integrals are the three parameter family of functions

$$\langle \hat{X}, \vartheta(\hat{L})\rangle = -\frac{1}{2}\alpha m(v^2 + r^2\dot{\varphi}^2) + A\left[\cos\left(\frac{a}{r}\varphi + \beta\right)mv - \sin\left(\frac{a}{r}\varphi + \beta\right)mr\dot{\varphi}\right].$$

In particular, we highlight the conservation laws

$$v^2 + r^2\varphi^2 = k_1^2, \tag{38a}$$

$$v\cos\left(\frac{a}{r}\varphi + \beta\right) - r\dot{\varphi}\sin\left(\frac{a}{r}\varphi + \beta\right) = k_2 \tag{38b}$$

for arbitrary choices of $\beta, k_1, k_2 \in \mathbb{R}$.

Concerning the determination of possible dynamical symmetries, we proceed as indicated in Corollary 2: by Eqs. (13) and (31)–(33), the intrinsic equations of motion read

$$\frac{dv}{dt} = a\dot{\varphi}^2, \tag{39a}$$

$$\frac{d\dot{\varphi}}{dt} = -\frac{a}{r^2}v\,\dot{\varphi}. \tag{39b}$$

Equations (38a) and (39a) entail the relation

$$\frac{dv}{dt} = \frac{a}{r^2}(k_1^2 - v^2),$$

mathematically equivalent to

$$\frac{d}{dt}\left[-\int\frac{dv}{k_1^2-v^2}+\frac{a}{r^2}\,t\right]=0\quad\Longrightarrow\quad-\frac{1}{2k_1}\log\frac{k_1+v}{k_1-v}+\frac{a}{r^2}\,t=\text{const.}$$

The quantity

$$F:=\frac{\sqrt{k_1^2-v^2}}{k_1+v}\,\exp\!\left(\frac{k_1\,at}{r^2}\right)=\frac{r\dot\varphi}{v+\sqrt{v^2+r^2\dot\varphi^2}}\,\exp\!\left(\frac{\sqrt{v^2+r^2\dot\varphi^2}\,at}{r^2}\right)\tag{40}$$

is therefore a first integral of the equations of motion.

At the same time, with the choice $V=\frac{1}{mr^2\dot\varphi}\frac{\partial}{\partial\varphi}$, the field

$$\hat Y=V(\psi^k)\,\frac{\partial}{\partial q^k}+Z\Big(V(\psi^k)\Big)\,\frac{\partial}{\partial\dot q^k}=\frac{1}{mr^2\dot\varphi}\left(\frac{\partial}{\partial\varphi}+\frac{av}{r^2}\,\frac{\partial}{\partial\dot\varphi}\right)$$

is easily recognized as a dynamical symmetry satisfying $i^*\big(\langle\hat Y,\vartheta(\hat L)\rangle\big)=\frac{1}{mr^2\dot\varphi}\,i^*\Big(\frac{\partial\hat L}{\partial\dot\varphi}\Big)=1$. In view of Corollary 2, the field

$$\hat X=F\,\hat Y=\frac{1}{mr\Big[v+\sqrt{v^2+r^2\dot\varphi^2}\,\Big]}\,\exp\!\left(\frac{\sqrt{v^2+r^2\dot\varphi^2}\,at}{r^2}\right)\left(\frac{\partial}{\partial\varphi}+\frac{av}{r^2}\,\frac{\partial}{\partial\dot\varphi}\right)$$

is therefore a dynamical symmetry associated with the first integral (40).

B. The rolling disk

A classical example of a non-holonomic system is a homogeneous disk of mass m and radius R whose plane remains vertical and which rolls without slipping on a horizontal plane. The configuration of the system is determined by the Cartesian coordinates x, y of the mass center of the disk, by the angle θ between the plane of the disk and the xz plane, and by the rotation angle φ of the disk around its symmetry axis.

Setting $q^1=x, q^2=y, q^3=\theta, q^4=\varphi, z^1=\dot\theta,\ z^2=\dot\varphi$, the kinetic constraint is expressed by the conditions

$$g^1=\dot x+R\,\dot\varphi\cos\,\theta=0,\qquad g^2=\dot y+R\,\dot\varphi\sin\,\theta=0$$

or, in parametric form, by the equations $\dot q^k=\psi^k(t,q,z)$, with

$$\psi^1=-R\,\dot\varphi\cos\,\theta,\qquad\psi^2=-R\,\dot\varphi\sin\,\theta,\qquad\psi^3=\dot\theta,\qquad\psi^4=\dot\varphi.\tag{41}$$

The extrinsic and intrinsic Lagrangians read, respectively,

$$\hat L=\hat T=\frac12\,m\big(\dot x^2+\dot y^2\big)+\frac18\,mR^2\big(2\,\dot\varphi^2+\dot\theta^2\big),\tag{42}$$

$$L=i^*\big(\hat L\big)=\frac18\,mR^2\big(6\,\dot\varphi^2+\dot\theta^2\big).\tag{43}$$

In view of Eqs. (18a), (21), (41), and (42), the geometric symmetries of the system are vector fields of the form

$$X=\alpha\,\frac{\partial}{\partial t}+V^A\,\frac{\partial\psi^k}{\partial z^A}\,\frac{\partial}{\partial q^k}=\alpha\,\frac{\partial}{\partial t}+V^1\,\frac{\partial}{\partial\theta}-V^2\left(R\,\cos\,\theta\,\frac{\partial}{\partial x}+R\,\sin\,\theta\,\frac{\partial}{\partial y}-\frac{\partial}{\partial\varphi}\right)$$

fulfilling the condition

$$i^*\big(\hat X(\hat L)\big)=i^*\left[\frac{dV^1}{dt}\,\frac{\partial\hat L}{\partial\dot\theta}-R\left(\frac{dV^2\cos\,\theta}{dt}\,\frac{\partial\hat L}{\partial\dot x}+\frac{dV^2\sin\,\theta}{dt}\,\frac{\partial\hat L}{\partial\dot y}\right)+\frac{dV^2}{dt}\,\frac{\partial\hat L}{\partial\dot\varphi}\right]=\frac14 mR^2\left(\frac{dV^1}{dt}\,\dot\theta+6\,\frac{dV^2}{dt}\,\dot\varphi\right)=0.\tag{44}$$

Evaluating the symbolic time derivatives, Eq. (44) reads

$$\left[\frac{\partial V^1}{\partial t} - R\left(\frac{\partial V^1}{\partial x}\cos\theta + \frac{\partial V^1}{\partial y}\sin\theta\right)\dot{\varphi} + \frac{\partial V^1}{\partial \theta}\dot{\theta} + \frac{\partial V^1}{\partial \varphi}\dot{\varphi}\right]\dot{\theta}$$
$$+ 6\left[\frac{\partial V^2}{\partial t} - R\left(\frac{\partial V^2}{\partial x}\cos\theta + \frac{\partial V^2}{\partial y}\sin\theta\right)\dot{\varphi} + \frac{\partial V^2}{\partial \theta}\dot{\theta} + \frac{\partial V^2}{\partial \varphi}\dot{\varphi}\right]\dot{\varphi} = 0. \tag{45}$$

The validity of Eq. (45) requires the simultaneous vanishing of the coefficients of the monomials $\dot{\theta}$, $\dot{\varphi}$, $\dot{\theta}^2$, $\dot{\varphi}^2$, $\dot{\theta}\,\dot{\varphi}$, namely,

$$\frac{\partial V^1}{\partial t} = \frac{\partial V^2}{\partial t} = 0, \tag{46a}$$

$$\frac{\partial V^1}{\partial \theta} = 0 \quad\Longrightarrow\quad V^1 = V^1(x, y, \varphi), \tag{46b}$$

$$\frac{\partial V^1}{\partial x} R\cos\theta + \frac{\partial V^1}{\partial y} R\sin\theta - \frac{\partial V^1}{\partial \varphi} - 6\,\frac{\partial V^2}{\partial \theta} = 0, \tag{46c}$$

$$\frac{\partial V^2}{\partial x} R\cos\theta + \frac{\partial V^2}{\partial y} R\sin\theta - \frac{\partial V^2}{\partial \varphi} = 0. \tag{46d}$$

On account of Eqs. (46a) and (46b), Eq. (46c) implies the relation

$$V^2 = \frac{1}{6}\left[\frac{\partial V^1}{\partial x} R\sin\theta - \frac{\partial V^1}{\partial y} R\cos\theta - \frac{\partial V^1}{\partial \varphi}\theta + f(x, y, \varphi)\right].$$

In view of the latter, Eq. (46d) reads

$$\begin{aligned}
&\frac{\partial^2 V^1}{\partial \varphi\,\partial x} R\sin\theta - \frac{\partial^2 V^1}{\partial \varphi\,\partial y} R\cos\theta - \frac{\partial^2 V^1}{\partial \varphi^2}\theta + \frac{\partial f}{\partial \varphi}\\
&\quad -R\cos\theta\left(\frac{\partial^2 V^1}{\partial x^2} R\sin\theta - \frac{\partial^2 V^1}{\partial x\,\partial y} R\cos\theta - \frac{\partial^2 V^1}{\partial x\,\partial \varphi}\theta + \frac{\partial f}{\partial x}\right)\\
&\quad -R\sin\theta\left(\frac{\partial^2 V^1}{\partial x\,\partial y} R\sin\theta - \frac{\partial^2 V^1}{\partial y^2} R\cos\theta - \frac{\partial^2 V^1}{\partial y\,\partial \varphi}\theta + \frac{\partial f}{\partial y}\right) = 0.
\end{aligned}$$

Since both functions V^1, f are independent of the variable θ, the validity of the previous equation for all values of ϑ entails the set of conditions

$$\begin{aligned}
&\frac{\partial^2 V^1}{\partial \varphi^2} = \frac{\partial^2 V^1}{\partial x\,\partial \varphi} = \frac{\partial^2 V^1}{\partial y\,\partial \varphi} = 0 \quad\Longrightarrow\quad \frac{\partial V^1}{\partial \varphi} = \beta = \text{const.},\\
&\frac{\partial^2 V^1}{\partial x\,\partial y} = -\frac{\partial^2 V^1}{\partial x^2} + \frac{\partial^2 V^1}{\partial y^2} = 0,\\
&\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0,
\end{aligned}$$

whence by elementary calculations

$$V^1 = \beta\varphi + a(x^2 + y^2) + bx + cy + d, \qquad f = e,$$

with $\beta, a, b, c, d, e = \text{const.}$

Collecting all results, we conclude that the most general geometric symmetry of the system is described by a vector field of the form

$$\begin{aligned}
X = {}&\alpha\frac{\partial}{\partial t} + \left[a\left(x^2 + y^2\right) + bx + cy + d + \beta\varphi\right]\frac{\partial}{\partial \theta}\\
&-\frac{1}{6}\left[(2ax + b)R\sin\theta - (2ay + c)R\cos\theta + e - \beta\theta\right]\left(R\cos\theta\frac{\partial}{\partial x} + R\sin\theta\frac{\partial}{\partial y} - \frac{\partial}{\partial \varphi}\right).
\end{aligned}$$

The resulting vector space is therefore spanned by the fields

$$X_{(1)}=\frac{\partial}{\partial t},\qquad X_{(2)}=\frac{\partial}{\partial\theta},\qquad X_{(3)}=R\cos\theta\,\frac{\partial}{\partial x}+R\sin\theta\,\frac{\partial}{\partial y}-\frac{\partial}{\partial\varphi},$$
$$X_{(4)}=\left(x^2+y^2\right)\frac{\partial}{\partial\theta}-\frac{R}{3}(x\sin\theta-y\cos\theta)X_{(3)},\qquad X_{(5)}=\varphi\,\frac{\partial}{\partial\theta}+\frac{\theta}{6}X_{(3)},$$
$$X_{(6)}=x\,\frac{\partial}{\partial\theta}-\frac{R}{6}\sin\theta\,X_{(3)},\qquad X_{(7)}=y\,\frac{\partial}{\partial\theta}+\frac{R}{6}\cos\theta\,X_{(3)}.$$

In view of the relation

$$i^*\left(\vartheta(\hat L)\right)=-L\,dt-mR\,\dot\varphi(\cos\theta\,dx+\sin\theta\,dy)+\frac{1}{4}\,mR^2\left(2\dot\varphi\,d\varphi+\dot\theta\,d\theta\right)$$

up to numerical factors, these determine the first integrals

$$F_{(1)}=6\,\dot\varphi^2+\dot\theta^2,\qquad F_{(2)}=\dot\theta,\qquad F_{(3)}=\dot\varphi,\tag{47a}$$

$$F_{(4)}=(x^2+y^2)\,\dot\theta+2R(x\sin\theta-y\cos\theta)\,\dot\varphi,\qquad F_{(5)}=\varphi\,\dot\theta-\theta\dot\varphi,\tag{47b}$$

$$F_{(6)}=x\,\dot\theta+R\,\dot\varphi\sin\theta,\qquad F_{(7)}=y\,\dot\theta-R\,\dot\varphi\cos\theta.\tag{47c}$$

Equation (47a), in turn, entails the conservation law

$$\frac{d}{dt}\,h\left(\theta-\dot\theta t,\varphi-\dot\varphi t\right)=\frac{\partial h}{\partial\xi}\left(\frac{d\theta}{dt}-\dot\theta\right)+\frac{\partial h}{\partial\eta}\left(\frac{d\varphi}{dt}-\dot\varphi\right)=0,$$

with $h(\xi,\eta)$ being any differentiable function on the torus $S^1\times S^1$. We have therefore two further independent first integral that, without loss of generality, can be chosen in the form

$$F_{(8)}=\cos(\theta-\dot\theta t),\qquad F_{(9)}=\cos(\varphi-\dot\varphi t).\tag{48}$$

By Corollary 1, noting that the field $\hat Y=\frac{1}{F_{(1)}}X_{(1)}$ is a dynamical symmetry satisfying $\langle\hat Y,\vartheta(\hat L)\rangle=\text{const}$, it is easily seen that the first integrals (48) are, respectively, associated, up to numerical factors, to the dynamic symmetries,

$$\hat X_{(8)}=F_{(8)}\,\hat Y=\frac{\cos(\theta-\dot\theta t)}{6\dot\varphi^2+\dot\theta^2}\,\frac{\partial}{\partial t},\qquad \hat X_{(9)}=F_{(9)}\,\hat Y=\frac{(\varphi-\dot\varphi t)}{6\dot\varphi^2+\dot\theta^2}\,\frac{\partial}{\partial t}.$$

C. Charged particle subject to a sub-linear kinetic constraint

Consider a point particle with rest mass m_0 and electric charge e, moving in an inertial frame under the action of a constant electromagnetic field $\underline{E}=E\,\underline{e}_2$, $\underline{B}=B\,\underline{e}_3$.

Denoting by $q^k, k=1,2,3$ the Cartesian coordinates and by $\underline{v}=\dot q^k\,\underline{e}_k$ the velocity of P, the relativistic equations of motion

$$\frac{d}{dt}\,\frac{m_0\underline{v}}{\sqrt{1-v^2/c^2}}=e(\underline{E}+\underline{v}\wedge\underline{B})\tag{49}$$

can be derived from the extrinsic Lagrangian

$$\hat L=m_0c^2\left(1-\sqrt{1-v^2/c^2}\right)+eq^2\left(E-B\dot q^1\right).\tag{50}$$

A control device forces the velocity of P to fulfill the constraint equation

$$g(\dot q^1,\dot q^2,\dot q^3)=\left(\dot q^1-\dot q^3\right)^2-\left(\dot q^2\right)^2-c^2=0\tag{51}$$

expressed in the parametric form as $\dot{q}^k = \psi^k(z^1, z^2)$, with

$$\psi^1 = \frac{c}{2}\left(z^1 + \cosh z^2\right), \quad \psi^2 = c\,\sinh z^2, \quad \psi^3 = \frac{c}{2}\left(z^1 - \cosh z^2\right). \tag{52}$$

The representation (52) is of the sub-linear type $\psi^k = \psi^k_1 z^1 + \psi^k_0(z^2)$, with

$$\psi^k_1 = \frac{c}{2}\,(1,0,1), \qquad \psi^k_0 = \frac{c}{2}\left(\cosh z^2,\, 2\,\sinh z^2,\, -\cosh z^2\right). \tag{53}$$

This entails the existence of an isochronous vector field

$$X = \psi^k_1 \frac{\partial}{\partial q^k} = \frac{c}{2}\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^3}\right) \tag{54}$$

consistent with the constraints, i.e., satisfying the condition[33]

$$\langle \hat{X}, d_v g\rangle = \frac{c}{2}\left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^3}\,,\, 2\left(\dot{q}^1 - \dot{q}^3\right)\left(\omega^1 - \omega^3\right) - 2\,\dot{q}^2\omega^2 \right\rangle = 0.$$

On account of Eq. (50), the same field satisfies the relation $\hat{X}(\hat{L}) = 0$, thereby representing a geometric symmetry of the system. By Proposition 2, we conclude that the scalar

$$i^*\left(\langle \hat{X}, \vartheta(\hat{L})\rangle\right) = i^*\left(\frac{\partial \hat{L}}{\partial \dot{q}^1} + \frac{\partial \hat{L}}{\partial \dot{q}^3}\right) = \frac{m_0 c\sqrt{2}\,z^1}{\sqrt{1-(z^1)^2 - 3\sinh^2(z^2)}} - Beq^2$$

is a first integral of the equations of motion. As a check, we may observe that in view of the identification $\frac{\partial \psi^k}{\partial z^1} = \frac{c}{2}\,(1,0,1)$, the first Lagrange–Chetaev equation reads

$$\frac{c}{2}\, i^*\left(\frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{q}^1} + \frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{q}^3} - \cancel{\frac{\partial \hat{L}}{\partial q^1}} - \cancel{\frac{\partial \hat{L}}{\partial q^3}}\right) = 0.$$

APPENDIX: PARTIAL INTEGRABILITY AND SUB-LINEARITY CONDITIONS FOR KINETIC CONSTRAINTS

In this appendix, we discuss two arguments that, although not strictly relevant in a dynamical context, provide a deeper insight into the geometry of the submanifold $\mathcal{A}$.

(i) *Partially integrable constraints.*

When the submanifold $\mathcal{A}$ admits an implicit representation involving one or more functions of the form $g^\alpha(t,q,\dot{q}) = \frac{df^\alpha}{dt}$, the conditions $g^\alpha = 0$ can be integrated and converted into positional restrictions, thus decreasing the number of independent variables q^i along any admissible evolution. In this sense, a truly non-holonomic system is one in which no such a reduction process is available.

To analyze this point, let $\mathcal{C}$ denote a completely integrable k-dimensional distribution $(2r+1 \le k \le n+r+1)$ containing the annihilator $(\chi(\mathcal{A}))^0$ of the Chetaev bundle.[34] Then,

(a) in the neighborhood of each point, $\mathcal{C}$ admits $p = n+r+1-k$ functionally independent "first integrals" $f^\alpha(t,q,z)$, meant as functions satisfying $Y(f^\alpha) = 0$ for all $Y \in \mathcal{C}$;

(b) the differentials df^α span the annihilator $\mathcal{C}^0$, which, by construction, is contained in $\chi(\mathcal{A})$: each df^α is therefore a Chetaev 1-form, hence a semibasic 1-form satisfying $\langle Z, df^\alpha\rangle = 0$ for all semisprays $Z \in \mathcal{D}^1(\mathcal{A})$;

In view of (b), each f^α is independent of the variables z^A: more specifically, it is the pull-back of a function $f^\alpha(t,q)$ on $\mathcal{V}_{n+1}$ satisfying $\frac{df^\alpha}{dt} = 0$ along any admissible evolution of the system: of the $n-r$ kinetic constraints $g^\sigma = 0$, p have therefore a positional character.

Conversely, if the kinetic constraints imply the validity of p independent conservation laws $f^\alpha(t,q) = \text{const}$, the symbolic time derivatives $\dot{f}^\alpha = \frac{\partial f^\alpha}{\partial t} + \frac{\partial f^\alpha}{\partial q^k}\dot{q}^k$ satisfy the relations

$$i^*(\dot{f}^\alpha) = \dot{f}^\alpha(t, q^k, \psi^k(t,q,z)) = 0 \quad \Rightarrow \quad \frac{\partial \dot{f}^\alpha}{\partial \dot{q}^k}\frac{\partial \psi^k}{\partial z^A} = 0 \quad \Rightarrow \quad i^*(d_v \dot{f}^\alpha) \in \chi(\mathcal{A}).$$

On the other hand, by the very definition of $\dot{f}^\alpha$, we have

$$i^*(d_v\dot{f}^\alpha) = i^*\left(\frac{\partial f^\alpha}{\partial q^k}\,\omega^k\right) = \frac{\partial f^\alpha}{\partial q^k}\left(dq^k - \psi^k\,dt\right) = \frac{\partial f^\alpha}{\partial q^k}\,dq^k + \frac{\partial f^\alpha}{\partial t}\,dt = df^\alpha,$$

whence $df^\alpha \in \chi(\mathcal{A})$. The annihilator of the p-dimensional module spanned by the differentials df^α is therefore a completely integrable distribution $\mathcal{C} \supset (\chi(\mathcal{A}))^0$.

Summing up, we conclude that the kinetic constraints induce a foliation of the configuration manifold $\mathcal{V}_{n+1}$ into a p-parameter family of leaves $f^\alpha(t,q) = \text{const}$, p being the co-dimension of the smallest completely integrable distribution $\mathcal{C}$ containing the annihilator $(\chi(\mathcal{A}))^0$, namely, the distribution obtained by "closing" $(\chi(\mathcal{A}))^0$ with respect to the Lie bracket operation.

The extreme cases $p = n - r$ ($\Leftrightarrow\ \mathcal{C} = (\chi(\mathcal{A}))^0$) and $p = 0$ ($\Leftrightarrow\ \mathcal{C} = \mathcal{D}^1(\mathcal{A})$) correspond, respectively, to the case of totally integrable constraints and to the strictly non-holonomic case.

(ii) *Sub-linear constraints.*

By definition, the sub-linear constraints are kinetic constraints admitting a parametric representation of the form

$$\psi^k = \sum_{\alpha=1}^{s} \psi_\alpha{}^k(t,q)z^\alpha + \psi_0{}^k\big(t,q,z^{s+1},\ldots,z^r\big). \tag{A1}$$

An intrinsic characterization of this type of constraints can be based on the following considerations:

- given a semispray $Z = \frac{\partial}{\partial t} + \psi^k\frac{\partial}{\partial q^k} + Z^A\frac{\partial}{\partial z^A}$ and a semibasic 1-form $\mu = \mu_0 dt + \mu_k dq^k$, the correspondence

$$\mathcal{P}(\mu) := \mu - \langle Z,\mu\rangle\,dt = \mu_k\,\omega^k$$

does not depend on the specific choice of Z, maps μ into a contact 1-form, and satisfies the relation $\mathcal{P}^2 = \mathcal{P}$: it is therefore a projection of the module of semibasic 1-forms onto the module of contact 1-forms over $\mathcal{A}$;
- given a contact 1-form $\sigma = \sigma_k\omega^k$ and a vertical vector field $V = V^A\frac{\partial}{\partial z^A}$, the Lie derivative $\mathscr{L}_V\,\sigma = V(\sigma_k)\,\omega^k - \sigma_k V(\psi^k)\,dt$ is a semibasic 1-form. The composite map $D_V := \mathcal{P}\circ\mathscr{L}_V$ is therefore a differential operator on the module of contact 1-forms, whose action is expressed in components as

$$D_V\sigma := \mathcal{P}(\mathscr{L}_V\,\sigma) = V(\sigma_k)\omega^k. \tag{A2}$$

Equation (A2) entails the properties $D_V\omega^k = 0$, $D_V(f\sigma) = V(f)\sigma + fD_V\sigma$, $D_{fV}\sigma = fD_V\sigma\ \forall\, f \in \mathscr{F}(\mathscr{A})$. In particular, when ν is a Chetaev 1-form, the relation $\mathscr{L}_V\,\nu = V(\nu_k)\,\omega^k$ and Eq. (A2) provide the identification $D_V\,\nu = \mathscr{L}_V\nu$.

Given the above, we state the following theorem:

Theorem 1. *A necessary and sufficient condition for the validity of representation* (A1) *is the existence of a completely integrable s-dimensional distribution $\mathcal{D} \subset V(\mathcal{A})$ such that, denoted by $\mathcal{D}^0$ the annihilator of $\mathcal{D}$ under the pairing $\langle\ \|\ \rangle$, the image $D_V\big(\mathcal{D}^0\big)$ is contained in $\mathcal{D}^0$ for any vertical vector field $V \in V(\mathcal{A})$.*

Proof. Necessity: assuming the validity of representation (A1), denote by $\mathcal{D}$ the completely integrable distribution spanned by the fields $\frac{\partial}{\partial z^1},\ldots,\frac{\partial}{\partial z^s}$. Then, for all $V \in V(\mathcal{A})$ and all $\sigma \in \mathcal{D}^0$, the relation

$$\left\langle D_V\sigma \,\middle\|\, \frac{\partial}{\partial z^\alpha}\right\rangle = V(\sigma_k)\,\frac{\partial\psi^k}{\partial z^\alpha} = V\Big(\sigma_k\,\psi_\alpha{}^k\Big) = V\left(\left\langle\sigma \,\middle\|\, \frac{\partial}{\partial z^\alpha}\right\rangle\right) = 0$$

proves $D_V\sigma \in \mathcal{D}^0$.

Sufficiency: given a distribution $\mathcal{D}$ with the required properties, in the neighborhood of each point $x \in \mathcal{A}$, there exist local coordinates $t,\bar{q}^k,\bar{z}^A$ satisfying $\mathcal{D} = Span\big\{\frac{\partial}{\partial\bar{z}^1},\ldots,\frac{\partial}{\partial\bar{z}^s}\big\}$. Up to a reordering of $\bar{q}^1,\ldots,\bar{q}^n$, we can ensure the non-vanishing of $\det\left\|\frac{\partial(\psi^1\cdots\psi^s)}{\partial(\bar{z}^1\cdots\bar{z}^s)}\right\|$.

In this way, performing the transformation

$$q^k = \bar{q}^k,\quad z^\alpha = \psi^\alpha(t,\bar{q},\bar{z}),\quad z^l = \bar{z}^l,\qquad \alpha = 1,\ldots,s,\ \ l = s+1,\ldots,r,$$

we get a new coordinate system t,q^k,z^A satisfying the conditions

$$\frac{\partial}{\partial \bar{z}^\alpha} = \frac{\partial z^\beta}{\partial \bar{z}^\alpha}\frac{\partial}{\partial z^\beta} \qquad \Longrightarrow \qquad \mathcal{D} = Span\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^s}\right\}, \tag{A3}$$

$$\psi^\alpha(t, q, z) = z^\alpha. \tag{A4}$$

By Eq. (A3), the annihilator $\mathcal{D}^0$ contains the totality of 1-forms $\sigma = \sigma_k \omega^k$ fulfilling the condition $\sigma_k \frac{\partial \psi^k}{\partial z^\alpha} = 0$, $\alpha = 1, \dots, s$. For any $V = V^A \frac{\partial}{\partial z^A}$, this implies

$$0 = V\left(\sigma_k \frac{\partial \psi^k}{\partial z^\alpha}\right) = V(\sigma_k)\frac{\partial \psi^k}{\partial z^\alpha} + \sigma_k V\left(\frac{\partial \psi^k}{\partial z^\alpha}\right) = \cancel{\left\langle D_V \sigma \,\Big\|\, \frac{\partial}{\partial z^\alpha}\right\rangle} + \sigma_k V^A \frac{\partial^2 \psi^k}{\partial z^A \partial z^\alpha}$$

the cancelation being due to the requirement $D_V(\mathcal{D}^0) \subset \mathcal{D}^0$.

By the arbitrariness of V we conclude that, under the stated assumptions, there exists a local coordinate system in which the vanishing of $\sigma_k \frac{\partial \psi^k}{\partial z^\alpha}$ implies the vanishing of $\frac{\partial^2 \psi^k}{\partial z^A \partial z^\alpha}$ $\forall\, A = 1, \dots, r$. A moment's thought shows that this entails the validity of a linear relation of the form

$$\frac{\partial^2 \psi^k}{\partial z^A \partial z^\alpha} = \gamma^\beta_{\alpha A} \frac{\partial \psi^k}{\partial z^\beta} \tag{A5}$$

On account of Eq. (A4), for $k = \lambda = 1, \dots, s$, Eq. (A5) yields the relations

$$0 = \frac{\partial^2 \psi^\lambda}{\partial z^A \partial z^\alpha} = \gamma^\beta_{\alpha A}\frac{\partial \psi^\lambda}{\partial z^\beta} = \gamma^\beta_{\beta A}\delta^\lambda_\beta = \gamma^\lambda_{\beta A}.$$

In view of these, Eq. (A5) reduces to

$$\frac{\partial^2 \psi^k}{\partial z^A \partial z^\alpha} = 0 \qquad \forall\; k = 1, \dots, n, \quad A = 1, \dots, r, \quad \alpha = 1, \dots, s,$$

mathematically equivalent to Eq. (A1). □

Remark 3. If a distribution $\mathcal{D}$ with the properties stated in Theorem 1 *exists, its annihilator $\mathcal{D}^0$ contains the totality of Chetaev* 1*-forms $\nu \in \chi(\mathcal{A})$ and therefore also the Lie derivatives $\mathscr{L}_V \nu = D_V \nu$ $\forall\, V \in V(\mathcal{A})$. Hence, dim$(\mathcal{D})$ cannot exceed the difference $n - \dim(\Lambda)$, Λ denoting the module generated by the family of* 1*-forms $\{\nu, \mathscr{L}_V \nu \mid \nu \in \chi(\mathcal{A}), V \in V(\mathcal{A})\}$.*

In particular, when $\mathscr{L}_V(\chi(\mathcal{A})) \subset \chi(\mathcal{A})$ and only in that case, the distribution $\mathcal{D} = V(\mathcal{A})$ generated by the totality of vertical vector fields satisfies all requirements of Theorem 1*. The validity of $\mathscr{L}_V(\chi(\mathcal{A})) \subset \chi(\mathcal{A})$ is therefore equivalent to $\dim \mathcal{D} = r$,* i.e., *to the* linearity *of the constraints.*

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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[29]An equivalent statement, more closely related to the "extrinsic" viewpoint, would be as follows: *A kinetic constraint is ideal if and only if the algorithm sending the free dynamical flow into the effective one is an orthogonal projection, i.e., if and only if the vector* $Z - \hat{Z}$ *is orthogonal to the (push-forward of) the vertical space* $V(\mathcal{A})$*, orthogonality being meant with respect to the scalar product induced by the extrinsic Lagrangian.*
[30]Here and in the following, expressions such as $\sigma \in \chi(\mathcal{A})$ and $\hat{X} \in (\chi(\mathcal{A}))^0$ are meant as shorthands for $\sigma_x \in \chi(\mathcal{A})\ \forall x \in \mathcal{A}$ and $\hat{X}_x \in (\chi(\mathcal{A}))^0\ \forall x \in \mathcal{A}$.
[31]In particular, if the pair X^0, V satisfies the conditions $X^0 = \text{const}$, $\frac{\partial V(\psi^k)}{\partial z^A} = 0$, the field $\hat{X}$ is the lift of a geometric symmetry.
[32]Indeed, Eqs. (35) and (36) admit the solution $V^1 = f(x,y)\cos\left(\frac{a}{r}\varphi\right) + g(x,y)\sin\left(\frac{a}{r}\varphi\right)$, which satisfies Eq. (35b) only for constant f, g. Similarly, Eqs. (35c) and (37) imply the relation $V^2 = -\frac{1}{r} A \sin\left(\frac{a}{r}\varphi + \beta\right) + h(x,y)$, which is consistent with Eq. (35d) only for $h = 0$.
[33]Actually, the fact that the field $J(X) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^3}$ is tangent to the submanifold $\mathcal{A}$ can be deduced directly from Eq. (51).
[34]It goes without saying that for $k = n + r + 1$, the distribution $\mathcal{C}$ includes the totality of vectors $X \in T(\mathcal{A})$.