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REPRESENTATION TYPE OF SURFACES IN $\mathbb{P}^3$

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ABSTRACT. The goal of this article is to prove that every surface with a regular point in the three-dimensional projective space of degree at least four, is of wild representation type under the condition that either X is integral or $\mathrm{Pic}(X) \cong \langle \mathcal{O}_X(1) \rangle$; we construct families of arbitrarily large dimension of indecomposable pairwise non-isomorphic aCM vector bundles. On the other hand, we prove that every non-integral aCM scheme of arbitrary dimension at least two, is also very wild in a sense that there exist arbitrarily large dimensional families of pairwise non-isomorphic aCM non-locally free sheaves of rank one.

1. INTRODUCTION

An arithmetically Cohen-Macaulay (for short, aCM) sheaf on a projective scheme X is a coherent sheaf supporting X, which has trivial intermediate cohomology and the stalk at each point whose depth equals the dimension of X. ACM vector bundles correspond to maximal Cohen-Macaulay modules over the associated graded ring and they reflect the properties of the graded ring. It is believed that the category generated by aCM sheaves on X measures the complexity of X. Indeed, a classification of aCM varieties was proposed as *finite, tame or wild* representation type according to the complexity of this category in [7] and there are several contributions to this trichotomy such as [8, 3, 6, 10]. It is only recent when such a representation type is determined for each aCM variety that is not a cone; see [11].

In this article, we pay our attention to the representation type of surfaces in three-dimensional projective space. Since the aCM vector bundles on smooth surfaces of degree at most two are completely classified due to the work by Horrocks and [14, 15], we may focus on surfaces of degree at least three. The case of cubic surfaces is dealt in [4, 9] and the case of quartic surfaces is from [16]. Our main result is the following, which implies that the surfaces in Theorem 1.1 are of wild representation type.

Theorem 1.1. *Let $X \subset \mathbb{P}^3$ be a surface of degree at least four with $X_{\mathrm{reg}} \neq \emptyset$ and assume either* $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$ *or that X is integral. For every even and positive integer r, there exists a family $\{\mathcal{E}_\lambda\}_{\lambda \in \Lambda}$ of indecomposable aCM vector bundles of rank r such that Λ is an integral quasi-projective variety with* $\dim \Lambda = r$ *and $\mathcal{E}_\lambda \not\cong \mathcal{E}_{\lambda'}$ for all $\lambda \neq \lambda'$ in Λ.*

It has to be noticed that although the result in [11] is more general than the implication of Theorem 1.1 regarding the wildness of the representation type, Theorem 1.1 provides a concrete way of constructing families of indecomposable aCM 'vector bundles' with prescribed rank, even on singular surfaces.

On the other hand, every non-integral aCM projective schemes of arbitrary dimension at least two is of 'very wild' representation type, in a sense that there exist arbitrarily large dimensional families of pairwise non-isomorphic aCM non-locally free sheaves of rank one; see Proposition 5.3.

Here we summarize the structure of this article. In Section 2 we collect several definitions and basic results that are used throughout the article. In Section 3 we state the main result in Theorem 3.9, which would automatically imply Theorem 1.1. We also give a proof of Theorem 3.9 in special case and suggest a number of its variation to construct aCM vector bundles. Then we spend the whole Section 4 for the proof of Theorem 3.9; basically we use induction on rank and the main ingredient for the proof is Lemma

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4.5 and the use of monodromy argument. Then we show in Section 5 the wildness of any aCM projective scheme of dimension at least two by investigating non-locally free ideal sheaves.

2. Preliminary

Throughout the article our base field **k** is algebraically closed of characteristic 0. We always assume that our projective schemes $X \subset \mathbb{P}^N$ are arithmetically Cohen-Macaulay, namely, $h^1(\mathscr{I}_{X,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$ and $h^i(\mathscr{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and all $i = 1, \dots, \dim X - 1$, of pure dimension at least two. Then by [17, Théorème 1 in page 268] all local rings $\mathscr{O}_{X,x}$ are Cohen-Macaulay of dimension $\dim X$. From $h^1(\mathscr{I}_{X,\mathbb{P}^N}) = 0$ we see that X_{red} is connected. Since in all our results we have $N = \dim X + 1 = 3$, the reader may just assume that X is a surface in $\mathbb{P}^3$. For a vector bundle $\mathscr{E}$ of rank $r \in \mathbb{Z}$ on X, we say that $\mathscr{E}$ *splits* if all its indecomposable factors are $\mathscr{O}_X(t)$ for some $t \in \mathbb{Z}$; $\mathscr{E} \cong \oplus_{i=1}^r \mathscr{O}_X(t_i)$ for some $t_i \in \mathbb{Z}$ with $i = 1, \dots, r$.

We always fix the embedding $X \subset \mathbb{P}^N$ and the associated polarization $\mathscr{O}_X(1)$. For a coherent sheaf $\mathscr{E}$ on a closed subscheme X of a fixed projective space, we denote $\mathscr{E} \otimes \mathscr{O}_X(t)$ by $\mathscr{E}(t)$ for $t \in \mathbb{Z}$. For another coherent sheaf $\mathscr{G}$, we denote by $\hom_X(\mathscr{F}, \mathscr{G})$ the dimension of $\operatorname{Hom}_X(\mathscr{F}, \mathscr{G})$, and by $\operatorname{ext}^i_X(\mathscr{F}, \mathscr{G})$ the dimension of $\operatorname{Ext}^i_X(\mathscr{F}, \mathscr{G})$. Finally we denote the canonical sheaf of X by ω_X.

Definition 2.1. A coherent sheaf $\mathscr{E}$ on X is called *arithmetically Cohen-Macaulay* (for short, aCM) if the following hold:

(i) $\mathscr{E}$ is locally Cohen-Macaulay, that is, the stalk $\mathscr{E}_x$ has depth equal to $\dim \mathscr{O}_{X,x}$ for any point x on X, and
(ii) $H^i(\mathscr{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \dots, \dim(X) - 1$.

Remark 2.2. In the condition (i) of Definition 2.1, we may only require that the stalk $\mathscr{E}_x$ has positive depth for any point $x \in X$; see [2, Remark 2.2] and [17, Théorème 1 in page 268].

If $\mathscr{E}$ is a coherent sheaf on a closed subscheme X of a fixed projective space, then we may consider its Hilbert polynomial $\mathrm{P}_{\mathscr{E}}(t) \in \mathbb{Q}[t]$ with the leading coefficient $\mu(\mathscr{E})/d!$, where d is the dimension of $\operatorname{Supp}(\mathscr{E})$ and $\mu = \mu(\mathscr{E})$ is called the *multiplicity* of $\mathscr{E}$. The *normalized* Hilbert polynomial $p_{\mathscr{E}}(t)$ of $\mathscr{E}$ is defined to be the Hilbert polynomial of $\mathscr{E}$ divided by $\mu(\mathscr{E})$.

Definition 2.3. If $\dim \operatorname{Supp}(\mathscr{E}) = \dim(X)$, then the *rank* of $\mathscr{E}$ is defined to be

$$\operatorname{rank}(\mathscr{E}) = \frac{\mu(\mathscr{E})}{\mu(\mathscr{O}_X)}.$$

Otherwise it is defined to be zero.

For an integral scheme X, the rank of $\mathscr{E}$ is the dimension of the stalk $\mathscr{E}_x$ at the generic point $x \in X$. But in general $\operatorname{rank}(\mathscr{E})$ needs not be integer.

Lemma 2.4. *Let $(X, \mathscr{O}_X(1))$ be an aCM projective scheme of dimension $n \ge 2$. For a fixed coherent sheaf $\mathscr{G}$ with pure depth n on X, assume the existence of $t_0 \in \mathbb{Z}$ such that $s := h^1(\mathscr{G}(t_0)) > 0$. Then the vector space $W := H^1(\mathscr{G}(t_0))$ induces the following unique extension up to isomorphisms*

$$0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{O}_X(-t_0) \otimes W^\vee \to 0 \tag{1}$$

and the sheaf $\mathscr{E}$ in the middle satisfies the following:

(i) $h^1(\mathscr{E}(t)) = h^1(\mathscr{G}(t))$ *for all $t \neq t_0$, and* $h^1(\mathscr{E}(t_0)) = 0$;
(ii) $h^i(\mathscr{E}(t)) = h^i(\mathscr{G}(t))$ *for all $t \in \mathbb{Z}$ and all i with $2 \le i \le n-1$.*

If $\mathscr{G}$ is locally free, then $\mathscr{E}$ is locally free.

Proof. All statements, except the one concerning $h^1(\mathscr{E}(t_0))$, are true for any sheaf $\mathscr{E}$ fitting into (1). The vanishing of $H^1(\mathscr{E}(t_0))$ is equivalent to the bijectivity of the coboundary map $\delta : H^0(\mathscr{O}_X) \otimes W^\vee \to H^1(\mathscr{G}(t_0))$ associated to the twist by $\mathscr{O}_X(t_0)$ of (1). The bijectivity of δ is a standard result on the extension functor. □

Theorem 2.5. *Let $X \subset \mathbb{P}^N$ be a projective Gorenstein scheme with pure dimension two and pure depth two, satisfying that*

- $h^1(\mathscr{O}_X(t)) = 0$ *for all* $t \in \mathbb{Z}$ *and* $h^1(\mathscr{I}_{X,\mathbb{P}^N}) = 0$*;*
- $X_{\text{reg}} \neq \emptyset$ *and* $\deg(\omega_X) + \deg(X) \geq 0$.

Then there exists a two-dimensional family of pairwise non-isomorphic aCM vector bundles of rank two on X whose very general member is indecomposable; here "very general" means outside countably many proper subvarieties.

Proposition 2.6. *Let $X \subset \mathbb{P}^N$ be as in Theorem 2.5. Assume $X_{\text{reg}} \neq \emptyset$ and fix $p \in X_{\text{reg}}$. Then there exists an aCM vector bundle $\mathscr{E}_p$ of rank two on X fitting into the exact sequence*

$$0 \to \omega_X(1) \to \mathscr{E}_p \to \mathscr{I}_{p,X} \to 0. \tag{2}$$

Moreover, if $\deg(\omega_X) + \deg(X) \geq 0$ *and $p, q \in X_{\text{reg}}$ with $p \neq q$, then we have $\mathscr{E}_p \not\cong \mathscr{E}_q$.*

Proof. Since X is Gorenstein, $\omega_X(1)$ is a line bundle and we get

$$\text{Ext}^1_X(\mathscr{I}_{p,X}, \omega_X(1)) \cong H^1(\mathscr{I}_{p,X}(-1))^\vee \cong \mathbf{k}.$$

So up to isomorphism there exists a unique sheaf $\mathscr{E}_p$ fitting into an extension (2) with a nonzero extension class. Since $h^0(\mathscr{O}_X(-1)) = 0$ and $p \in X_{\text{reg}}$, the Cayley-Bacharach condition is satisfied for (2) and so $\mathscr{E}_p$ is locally free; see [5]. Note that the restriction map

$$H^0(\mathscr{O}_X(t)) \to H^0(\mathscr{O}_X(t)_{|\{p\}})$$

is surjective for any $t \geq 0$. This implies that $h^1(\mathscr{I}_{p,X}(t)) = 0$ for any $t \geq 0$, because we have $h^1(\mathscr{O}_X(t)) = 0$. Then we see from (2) that $h^1(\mathscr{E}_p(t)) = 0$ for any $t \geq 0$. On the other hand, from $\det(\mathscr{E}_p) \cong \omega_X(1)$, we get that $h^1(\mathscr{E}_p(t)) = h^1(\mathscr{E}_p^\vee \otimes \omega_X(-t)) = h^1(\mathscr{E}_p(-t-1)) = 0$ for $t < 0$ by Serre's duality. Thus $\mathscr{E}_p$ is aCM.

For the second assertion, assume $\mathscr{E}_p \cong \mathscr{E}_q$. From the assumption $\deg(\omega_X(1)) \geq 0$, we get $h^0(\omega_X^\vee(-1)) \leq 1$ with equality if and only if $\omega_X \cong \mathscr{O}_X(-1)$. In particular, we have $h^0(\mathscr{I}_{p,X} \otimes \omega_X^\vee(-1)) = 0$. Then from the assumption $h^1(\mathscr{O}_X) = 0$ and (2), we get $h^0(\mathscr{E}_p \otimes \omega_X^\vee(-1)) = 1$ and that p is the only zero of a nonzero section of $H^0(\mathscr{E}_p \otimes \omega_X^\vee(-1))$. Thus we get $p = q$. □

Proof of Theorem 2.5: By assumption X_{reg} is a two-dimensional quasi-projective smooth variety. By Proposition 2.6 there is a flat family of aCM vector bundles $\{\mathscr{E}_p\}_{p \in X_{\text{reg}}}$ of rank two such that if $p, q \in X_{\text{reg}}$ and $p \neq q$, then $\mathscr{E}_p \not\cong \mathscr{E}_q$. Now assume that $\mathscr{E}_p$ is decomposable for some $p \in X_{\text{reg}}$, say $\mathscr{E}_p \cong \mathscr{A}_1 \oplus \mathscr{A}_2$ with each $\mathscr{A}_i$ a line bundle on X. Since $\det(\mathscr{E}_p) \cong \omega_X(1)$, we have $\mathscr{A}_2 \cong \mathscr{A}_1^\vee \otimes \omega_X(1)$. Now from the assumption that $h^1(\mathscr{O}_X) = 0$, we see that $\text{Pic}(X)$ is discrete and countable. This implies that there can exist only countably many decomposable vector bundles in the family. Since the base field $\mathbf{k}$ is algebraically closed and so uncountable, there exists some indecomposable vector bundle in the family $\{\mathscr{E}_p\}_{p \in X_{\text{reg}}}$ and for a very general point o on any connected component of X_{reg} the vector bundle $\mathscr{E}_o$ is indecomposable. □

Throughout the article, as in Proposition 2.6, our construction of aCM sheaf of rank two on X is in terms of the following extension

$$0 \to \omega_X \to \mathscr{E} \to \mathscr{I}_{Z,X}(a) \to 0 \tag{3}$$

with Z a locally complete intersection of codimension two in X and $a \in \mathbb{Z}$. Such extensions are parametrized by $\text{Ext}^1_X(\mathscr{I}_{Z,X}(a), \omega_X)$. In case when X is a surface, the coboundary map associated to (3) is

$$\delta_1 : H^1(\mathscr{I}_{Z,X}(a)) \to H^2(\omega_X) \cong \mathbf{k}$$

and by Serre's duality in [13, Theorem 3.12] its dual is

$$\mathbf{k} \cong \text{Hom}_X(\omega_X, \omega_X) \to \text{Ext}^1_X(\mathscr{I}_{Z,X}(a), \omega_X),$$

which is obtained by applying the functor $\text{Hom}_X(-, \omega_X)$ to (3). Thus the coboundary map δ_1 is surjective if and only if (3) is a non-trivial extension. Since we assume $h^1(\mathscr{O}_X) = h^1(\omega_X) = 0$, this implies that $h^1(\mathscr{E}) = h^1(\mathscr{I}_{Z,X}(a)) - 1$.

3. aCM VECTOR BUNDLE ON SURFACES IN $\mathbb{P}^3$

We always assume that $X \subset \mathbb{P}^3$ is a surface of degree m, not necessarily smooth. In particular, its dualizing sheaf is $\omega_X \cong \mathscr{O}_X(m-4)$ and we get $h^2(\mathscr{O}_X) = \binom{m-1}{3}$. We also have $h^0(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = 0$.

Lemma 3.1. *Each line bundle $\mathscr{O}_X(t)$ with $t \in \mathbb{Z}$, is stable as an $\mathscr{O}_{\mathbb{P}^3}$-sheaf with pure depth 2.*

Proof. It is enough to deal with the case $t = 0$. Assume the contrary and take a subsheaf $\mathscr{A} \subsetneq \mathscr{O}_X$ such that $\mathscr{B} := \mathscr{O}_X/\mathscr{A}$ has depth 2 and normalized Hilbert polynomial at least the one of $\mathscr{O}_X$. Since $\mathscr{B}$ is a quotient of $\mathscr{O}_X$ with depth 2 and X has no embedded component, we get $\mathscr{B} \cong \mathscr{O}_T$ for T a union of some of the irreducible components of X_{red} with at most the multiplicities appearing in X. This implies that $T \in |\mathscr{O}_{\mathbb{P}^3}(d)|$ for some integer d with $1 \le d < m$. Now the Hilbert polynomial of $\mathscr{O}_X$ is

$$\begin{aligned}\mathrm{P}_{\mathscr{O}_X}(t) &= \binom{t+3}{3} - \binom{t-m+3}{3}\\ &= \left(\frac{m}{2}\right)t^2 + \left(2m - \frac{m^2}{2}\right)t + \left(\frac{m^3}{6} - m^2 + \frac{11m}{6}\right).\end{aligned}$$

Similarly, we get the Hilbert polynomial $\mathrm{P}_{\mathscr{O}_T}(t)$ of $\mathscr{O}_T$ by replacing m in $\mathrm{P}_{\mathscr{O}_X}(t)$ by d. Then we see that $p_{\mathscr{O}_X}(t) < p_{\mathscr{O}_T}(t)$ for $t \gg 0$, a contradiction. □

Remark 3.2. If either $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$ or X is integral, then every line bundle is stable. Note also that the proof of Lemma 3.1 shows that the ideal sheaf $\mathscr{I}_{Z,X}$ for any zero-dimensional subscheme $Z \subset X$, is also stable. If X is integral, then any sheaf of rank 1 with positive depth is stable. Thus these sheaves are indecomposable.

Proposition 3.3. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \ge 2$ with $X_{\text{reg}} \neq \emptyset$. Fix $p \in X_{\text{reg}}$, and let $\mathscr{E}_p$ be the unique non-trivial extension*

$$0 \to \mathscr{O}_X(m-3) \to \mathscr{E}_p \to \mathscr{I}_{p,X} \to 0. \tag{4}$$

Then $\mathscr{E}_p$ is an aCM vector bundle of rank two on X and $\mathscr{E} \not\cong \mathscr{O}_X(a) \oplus \mathscr{O}_X(b)$ for any $a, b \in \mathbb{Z}$. If one of the following holds, then $\mathscr{E}$ is indecomposable.

(i) $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$,
(ii) *$\mathscr{O}_X(t)$ for $t \in \mathbb{Z}$ are the only aCM line bundles on X, or*
(iii) *$m \ge 4$ and X is integral.*

Proof. By Proposition 2.6 it remains to deal with indecomposability of $\mathscr{E}_p$. First show that there are no integers a, b such that $\mathscr{E}_p \cong \mathscr{O}_X(a) \oplus \mathscr{O}_X(b)$. Assume that such a, b exist, say $a \ge b$. Since $h^0(\mathscr{E}_p(3-m)) = 1$ and $h^0(\mathscr{E}_p(2-m)) = 0$, we get $(a,b) = (m-3, 0)$ and $m \ge 3$. Then we get $h^0(\mathscr{E}_p) = \binom{m}{3} + 1$, while (4) gives $h^0(\mathscr{E}_p) = \binom{m}{3}$.

Now assume that $\mathscr{E}_p$ is decomposable. Since $\mathscr{E}_p$ is locally free and it has rank 2, we have $\mathscr{E}_p \cong \mathscr{A}_1 \oplus \mathscr{A}_2$ with each $\mathscr{A}_i \in \mathrm{Pic}(X)$. Since $\mathscr{E}_p$ is aCM, each $\mathscr{A}_i$ is aCM. In cases (i) and (ii) the assertion holds by above. Thus we assume the case (iii). By Lemma 3.1 and Remark 3.2, (4) is the HN filtration of $\mathscr{E}_p$. Applying the functor $\mathrm{Hom}_X(\mathscr{E}_p, -)$ to (4), we get

$$0 \to \mathrm{Hom}_X(\mathscr{E}_p, \mathscr{O}_X(m-3)) \to \mathrm{Hom}_X(\mathscr{E}_p, \mathscr{E}_p) \to \mathrm{Hom}_X(\mathscr{E}_p, \mathscr{I}_{p,X}) \to \mathrm{Ext}^1_X(\mathscr{E}_p, \mathscr{O}_X(m-3)).$$

Note that $\hom_X(\mathscr{E}_p, \mathscr{O}_X(m-3)) = h^2(\mathscr{E}_p(-1)) = h^0(\mathscr{E}_p) = \binom{m}{3}$ by Serre's duality. By applying the functor $\mathrm{Hom}_X(-, \mathscr{I}_{p,X})$ to (4), we get

$$\hom_X(\mathscr{E}_p, \mathscr{I}_{p,X}) = \hom_X(\mathscr{I}_{p,X}, \mathscr{I}_{p,X}) = 1.$$

Thus we have

$$\binom{m}{3} \le \hom_X(\mathscr{E}_p, \mathscr{E}_p) \le 1 + \binom{m}{3}.$$

Since $h^0(\mathcal{O}_X) = 1$, we have $\hom_X(\mathcal{A}_i, \mathcal{A}_i) = 1$ for each i. So we get

$$\hom_X(\mathcal{E}_p, \mathcal{E}_p) = 2 + \hom_X(\mathcal{A}_1, \mathcal{A}_2) + \hom_X(\mathcal{A}_2, \mathcal{A}_1).$$

Since X is integral, each $\mathcal{A}_i$ is stable and we get either $\mathcal{A}_1 \cong \mathcal{A}_2$ or $\hom_X(\mathcal{A}_i, \mathcal{A}_{3-i}) = 0$ for each i. In the latter case we have $\hom_X(\mathcal{E}_p, \mathcal{E}_p) = 2 < \binom{m}{3}$, a contradiction. In the former case, we have $\hom_X(\mathcal{E}_p, \mathcal{E}_p) = 4$ and the only possibility is $m = 4$. But this is also impossible, since we would get $\mathcal{A}_1^{\otimes 2} \cong \det(\mathcal{E}_p) \cong \mathcal{O}_X(1)$. □

Proposition 3.4. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 2$ and let $Z \subset X$ be a zero-dimensional subscheme of degree 3, which is not collinear. Assume that Z is a locally complete intersection inside X, i.e. for each $p \in Z_{\mathrm{red}}$ the ideal sheaf of Z at $\mathcal{O}_{X,p}$ is generated by two elements of $\mathcal{O}_{X,p}$. Then there is a vector bundle $\mathcal{G}$ of rank two fitting into an exact sequence*

$$0 \to \mathcal{O}_X(m-4) \to \mathcal{G} \to \mathcal{I}_{Z,X} \to 0 \tag{5}$$

with $h^1(\mathcal{G}(t)) = 0$ for all $t \neq 0$ and $h^1(\mathcal{G}) = 1$. There is also an exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \xrightarrow{u} \mathcal{O}_X \to 0, \tag{6}$$

where $\mathcal{E}$ is an aCM vector bundle of rank three such that $\mathcal{E} \not\cong \mathcal{O}_X(a_1) \oplus \mathcal{O}_X(a_2) \oplus \mathcal{O}_X(a_3)$ for any $(a_1, a_2, a_3) \in \mathbb{Z}^{\oplus 3}$. Moreover, if $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$, then $\mathcal{E}$ is indecomposable.

Proof. Since $\omega_X \cong \mathcal{O}_X(m-4)$, we have $h^0(\mathcal{O}_X(4-m) \otimes \omega_X) = 1$ and $\mathcal{O}_X(4-m) \otimes \omega_X$ is globally generated. Since $\mathcal{O}_X(4-m) \otimes \omega_X$ is globally generated, we have $h^0(\mathcal{I}_{p,X} \otimes \mathcal{O}_X(4-m) \otimes \omega_X) = 0$ for all $p \in Z_{\mathrm{red}}$. Since Z is a locally complete intersection, the Cayley-Bacharach condition is satisfied and so there is a locally free $\mathcal{G}$ fitting into (5); see [5]. From (5) we immediately get $h^1(\mathcal{G}(t)) = 0$ for all $t > 0$, because Z is not collinear. Note that $\det(\mathcal{G}) \cong \mathcal{O}_X(m-4)$ and $\mathcal{G}$ is a vector bundle of rank two. This implies $\mathcal{G}^\vee \cong \mathcal{G}(4-m)$. For $t < 0$, we have $h^1(\mathcal{G}(t)) = h^1(\mathcal{G}^\vee(-t) \otimes \omega_X) = h^1(\mathcal{G}(-t)) = 0$ by Serre's duality. Now consider the coboudnary map $\delta_1 : H^1(\mathcal{I}_{Z,X}) \to H^2(\mathcal{O}_X(m-4)) \cong \mathbf{k}$ with $\ker(\delta_1) = H^1(\mathcal{G})$. The dual of δ_1 is the map

$$\mathrm{Hom}_X(\mathcal{O}_X(m-4), \mathcal{O}_X(m-4)) \to \mathrm{Ext}^1_X(\mathcal{I}_{Z,X}, \mathcal{O}_X(m-4))$$

sending the identity map to the element corresponding to $\mathcal{G}$. This implies that δ_1 is surjective and $h^1(\mathcal{G}) = 1$.

Now we apply Lemma 2.4 to $\mathcal{G}$ to obtain an aCM vector bundle $\mathcal{E}$ of rank three fitting into (6). Since $h^1(\mathcal{G}) = 1$ and $h^1(\mathcal{E}) = 0$, (5) and (6) give $h^0(\mathcal{E}) = h^0(\mathcal{G}) = \binom{m-1}{3}$. Assume the existence of integers $a_1 \geq a_2 \geq a_3$ such that $\mathcal{E} \cong \oplus_{i=1}^3 \mathcal{O}_X(a_i)$. Since $\det(\mathcal{E}) \cong \mathcal{O}_X(m-4)$, we have $a_1 + a_2 + a_3 = m-4$. If $2 \leq m \leq 3$, then we have $a_1 \geq 0$ from $a_1 + a_2 + a_3 = m-4$. This implies that $h^0(\mathcal{O}_X(a_1)) > 0 = \binom{m-1}{3} = h^0(\mathcal{E})$, a contradiction. If $m = 4$, then we have $h^0(\mathcal{E}) = 1$. Since $a_1 + a_2 + a_3 = 0$, we have $\sum_{i=1}^3 h^0(\mathcal{O}_X(a_i)) > 1$, a contradiction. Finally assume $m > 4$. From (5) and (6) we see that $\mathcal{O}_X(m-2)$ is the first non-trivial sheaf in the HN filtration of $\mathcal{E}$. Thus $a_1 = m-4$ and $h^0(\mathcal{O}_X(a_1)) = \binom{m-1}{3}$. Since $a_2 + a_3 = 0$, we have $h^0(\mathcal{O}_X(a_2)) > 0$ and so $h^0(\mathcal{E}) > \binom{m-1}{3}$, a contradiction. Hence we get $\mathcal{E} \not\cong \oplus_{i=1}^3 \mathcal{O}_X(a_i)$ for any triple of integers (a_1, a_2, a_3).

It remains to show the last assertion. Assume $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$ and that $\mathcal{E}$ is decomposable; by the previous assertion we have $\mathcal{E} \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with $\mathrm{rank}(\mathcal{A}_i) = i$ for each i and $\mathcal{A}_2$ indecomposable. Set $\mathcal{A}_1 \cong \mathcal{O}_X(a)$ for $a \in \mathbb{Z}$. Since $h^0(\mathcal{E}) = \binom{m-1}{3}$, we have $a \leq m-4$. From (5) and (6) we get the existence of a subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $\mathcal{F} \cong \mathcal{O}_X(m-4)$ and $\mathcal{E}/\mathcal{F}$ is an extension $\mathcal{H}$ of $\mathcal{O}_X$ by $\mathcal{I}_{Z,X}$. Note that $\mathcal{H}$ is not locally free, because $\mathcal{I}_{Z,X}$ has not depth 2. In particular, $\mathcal{H}$ is not isomorphic to $\mathcal{A}_2$ and we get $\mathcal{A}_1 \not\cong \mathcal{F}$. So we have $a < m-4$. Now consider a restriction map

$$u_{|\{0\} \oplus \mathcal{A}_2} : \{0\} \oplus \mathcal{A}_2 \to \mathcal{O}_X.$$

If this restriction map is surjective, then its kernel is a line bundle, say $\mathcal{O}_X(b)$. Since X is aCM, we get $\mathcal{A}_2 \cong \mathcal{O}_X \oplus \mathcal{O}_X(b)$, a contradiction. Thus the restriction map is not surjecitve and so the other restriction map $u_{|\mathcal{A}_1 \oplus \{0\}}$ is not zero. In particular, we get $a \leq 0$. If $a = 0$, then we have $\mathcal{A}_1 \cong \mathcal{O}_X$ and the map $u_{|\mathcal{A}_1 \oplus \{0\}}$ is an isomorphism. Thus (6) splits and we get $h^1(\mathcal{E}) \geq h^1(\mathcal{G}) > 0$, a contradiction. Hence we get $a < 0$.

Since there is no nonzero map $\mathscr{F} \to \mathscr{A}_1$ from $a < m-4$, $\mathscr{F}$ is isomorphic to a subsheaf $\mathscr{F}_1$ of $\mathscr{A}_2$ and we get $\mathscr{H} \cong \mathscr{O}_X(a) \oplus \mathscr{A}_2/\mathscr{F}_1$. From $a < 0$ we see that there is no nonzero map $\mathscr{I}_{Z,X} \to \mathscr{O}_X(a)$. Since $\mathscr{H}$ is an extension of $\mathscr{O}_X$ by $\mathscr{I}_{Z,X}$, we get that $\mathscr{I}_{Z,X} \cong \mathscr{A}_2/\mathscr{F}_1$ and so $\mathscr{O}_X(a) \cong \mathscr{O}_X$, a contradiction. □

Remark 3.5. In case $m = 1$, i.e. $X = \mathbb{P}^2$, we fail in obtaining an indecomposable aCM vector bundle of rank three, using the method in Proposition 3.4. Indeed, we get $\mathscr{G} \cong \Omega^1_{\mathbb{P}^2}$ and the corresponding vector bundle of rank three is $\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$.

Corollary 3.6. *Let $X \subset \mathbb{P}^3$ be union of multiple planes in which at least one plane occurs with multiplicity* 1*. Then there is an indecomposable aCM vector bundle of rank three on X. If $m > 4$, we have a family of such aCM vector bundles of dimension* 6.

Proof. Assume that X has one component H with multiplicity 1. In this case we take as Z a set of 3 general points in H. Then the first assertion follows from Proposition 3.4. Note that the set of all such Z has dimension 6. Now assume that X has a component H with multiplicity 3. Fix a general point $p \in H$ and take a general line $L \subset \mathbb{P}^3$ with $p \in L$. Then set Z to be the connected component of the scheme $X \cap L$ with p as its reduction. Then we may get the assertion from Proposition 3.4 and that $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$ by [2, Lemma 2.5]. □

Proposition 3.7. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 4$ with an irreducible component Y appearing with multiplicity* 2 *in X. Fix $p \in Y_{\mathrm{reg}}$ so that T is the only irreducible component of X containing p. For a general line $L \subset \mathbb{P}^3$ containing p, let $Z \subset X$ be the connected component of $L \cap X$ with p as its reduction. We have* $\deg(Z) = 2$ *and there is an aCM vector bundle $\mathscr{E}_Z$ of rank two fitting into an exact sequence*

$$0 \to \mathscr{O}_X(m-4) \to \mathscr{E}_Z \to \mathscr{I}_{Z,X} \to 0. \tag{7}$$

The set of all isomorphism classes of $\mathscr{E}_Z$ is uniquely parametrized by a 4*-dimensional irreducible quasi-projective variety* Δ *satisfying the following.*

- (i) *For any $\mathscr{E}_Z \in \Delta$, there are no integers a, b with $\mathscr{E}_Z \cong \mathscr{O}_X(a) \oplus \mathscr{O}_X(b)$.*
- (ii) *A very general $\mathscr{E}_Z \in \Delta$ is indecomposable.*
- (iii) *If* $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$*, then each $\mathscr{E}_Z \in \Delta$ is indecomposable.*
- (iv) *If $\mathbb{Z}\langle \mathscr{O}_X(1)\rangle$ are the only aCM line bundles on X, then each $\mathscr{E}_Z \in \Delta$ is indecomposable.*

Proof. Since no other component of X than Y contains p and p is a smooth point of X, we have $\deg(Z) = 2$; it is sufficient to take as L any line through p not contained in the tangent plane T_pY of Y at p.

Since $\omega_X \cong \mathscr{O}_X(m-4)$, we have $h^0(\mathscr{O}_X(4-m) \otimes \omega_X) = 1$ and $\mathscr{O}_X(4-m) \otimes \omega_X$ is globally generated. Thus we have $h^0(\mathscr{I}_{p,X} \otimes \mathscr{O}_X(4-m) \otimes \omega_X) = 0$. Since Z is a locally complete intersection, the Cayley-Bacharach condition is satisfied for (7) and so there is a locally free $\mathscr{E}_Z$ fitting into (7); see [5].

Since $\mathscr{O}_X(1)$ is very ample and $\deg(Z) = 2$, we get $h^1(\mathscr{E}_Z(t)) = 0$ for all $t > 0$ by (5). Note that $\det(\mathscr{E}_Z) \cong \mathscr{O}_X(m-4)$ and $\mathscr{E}_Z$ is a vector bundle of rank two. This implies $\mathscr{E}_Z^\vee \cong \mathscr{E}_Z(4-m)$. For $t < 0$, we have $h^1(\mathscr{E}_Z(t)) = h^1(\mathscr{E}_Z^\vee(m-t-4)) = h^1(\mathscr{E}_Z(-t)) = 0$ by Serre's duality. Now consider the coboudary map $\delta_1 : H^1(\mathscr{I}_{Z,X}) \to H^2(\mathscr{O}_X(m-4)) \cong \mathbf{k}$ with $\ker(\delta_1) = H^1(\mathscr{E}_Z)$. The dual of δ_1 is the map

$$\mathrm{Hom}_X(\mathscr{O}_X(m-4), \mathscr{O}_X(m-4)) \to \mathrm{Ext}^1_X(\mathscr{I}_{Z,X}, \mathscr{O}_X(m-4))$$

sending the identity map to the element corresponding to $\mathscr{E}_Z$. This implies that δ_1 is non-zero and hence and $h^1(\mathscr{E}_Z) = 0$. Thus $\mathscr{E}_Z$ is aCM.

The set of all $p \in Y_{\mathrm{reg}}$ such that Y is the only irreducible component of X containing p is an irreducible 2-dimensional variety Δ'. For each $p \in \mathbb{P}^3$ the set of all lines through p is a $\mathbb{P}^2$. Define a variety Δ as follows:

$$\Delta := \{(p, L) \mid p \in \Delta' \text{ and } L \text{ a line in } \mathbb{P}^3 \text{ with } p \in L \text{ and } L \nsubseteq T_pY\}.$$

Since $m \geq 4$, we have $h^0(\mathscr{I}_{Z,X}(4-m)) = 0$. Thus (7) gives $h^0(\mathscr{E}_Z(4-m)) = 1$. Thus the isomorphism classes of $\mathscr{E}_Z$ uniquely determines Z, i.e. if $\mathscr{E}_Z \not\cong \mathscr{E}_{Z'}$, then we get $Z \neq Z'$. For two elements $(p_1, L_1), (p_2, L_2) \in \Delta$, let Z_i be the subscheme of degree 2 determined by (p_i, L_i) for each $i = 1, 2$. Since each p_i is the reduction of Z_i and L_i is the line spanned by Z_i, the variety Δ uniquely parametrizes the isomorphism classes of the aCM vector bundles $\mathscr{E}_Z$.

Assume $\mathscr{E}_Z \cong \mathscr{O}_X(a) \oplus \mathscr{O}_X(b)$ for some integers a, b with $a \geq b$. Since $\det(\mathscr{E}_Z) \cong \mathscr{O}_X(m-4)$, we have $b = m-4-a$. But since $h^0(\mathscr{E}_Z(4-m)) = 1$, the only possibility is that $a = 4-m$ and $b < 0$, a contradiction. Thus we get (i). We may get (ii) as in the proof of Theorem 2.5. Now assume that $\mathscr{E}_Z$ is decomposable, say $\mathscr{E}_Z \cong \mathscr{A}_1 \oplus \mathscr{A}_2$ with each $\mathscr{A}_i$ a line bundle. Since $\mathscr{E}_Z$ is aCM, each $\mathscr{A}_i$ is also aCM. Thus (iii) and (iv) follow from (i). □

Remark 3.8. In case $m = 2$, i.e. $X = 2H$ the double plane with a hyperplane $H \subset \mathbb{P}^3$, the vector bundle $\mathscr{E}_Z$ described in Proposition 3.7 is the vector bundle $\mathscr{O}_X(-1)^{\oplus 2}$.

Theorem 3.9. *Let $X \subset \mathbb{P}^3$ be a surface of degree $m \geq 4$ with $X_{\text{reg}} \neq \emptyset$, i.e. X has an irreducible component Y appearing with multiplicity 1. We further assume that either $\text{Pic}(X) = \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$ or X is integral. For a fixed integer $s > 0$ and a set $S \subset X_{\text{reg}} \cap Y$ with $\sharp(S) = s$, a general sheaf $\mathscr{E}_S$ fitting into an exact sequence*

$$0 \to \mathscr{O}_X(m-3)^{\oplus s} \xrightarrow{v} \mathscr{E}_S \to \oplus_{p \in S} \mathscr{I}_{p,X} \to 0, \tag{8}$$

is a locally free, indecomposable and aCM sheaf of rank $2s$. Moreover, if $S' \subset X_{\text{reg}} \cap Y$ is another set with $\sharp(S') = s$ and $S' \neq S$, then we have $\mathscr{E}_{S'} \not\cong \mathscr{E}_S$.

We have $\text{ext}^1_X(\mathscr{I}_{p,X}, \mathscr{O}_X(m-3)) = h^1(\mathscr{I}_{p,X}(-1)) = 1$ for each $p \in X_{\text{reg}}$ by Serre's duality. So the extension $\mathscr{E}_S$ corresponds to an element in a finite dimensional vector space

$$\mathbb{E}(S) := \text{Ext}^1_X(\oplus_{p \in S} \mathscr{I}_{p,X}, \mathscr{O}_X(m-3)^{\oplus s}) \cong \mathbf{k}^{s^2}.$$

If $s = 1$, say $S = \{p\}$, the dimension of $\mathbb{E}$ is one. Thus there exists a unique non-trivial extension. Denote this non-trivial extension simply by $\mathscr{E}_p$.

In Theorem 3.9, a "general" choice of $\mathscr{E}_S$ means that there exists a non-empty Zariski open subset $\mathbb{U} \subset \mathbb{E}(S)$ such that the middle term of any extension in $\mathbb{U}$ is aCM, locally free and indecomposable.

Proof of Theorem 1.1: The family Σ of all $S \subset X_{\text{reg}}$ with $\sharp(S) = s$ clearly has dimension $2s$. By Theorem 3.9, if S and S' are two distinct sets in Σ, then we get $\mathscr{E}_S \not\cong \mathscr{E}_{S'}$. Now there is a universal family on any Ext^1-group of families of sheaves with $\Sigma \times X$ as its base. Thus, we get a family of aCM locally free and indecomposable vector bundles with as a parameter space a rank s^2 vector bundle over Σ; the fibre of this vector bundle over $S \in \Sigma$ is $\mathbb{E}(S)$, corresponding to S. Taking a non-empty open subset V of Σ on which this vector bundle is trivial we get a family of pairwise non-isomorphic sheaves, at least if we restrict V, so that all sheaves in the family are locally free, aCM and indecomposable. □

Remark 3.10. For a surface X as in Theorem 3.9 and Theorem 1.1, the algebraic group $\text{Aut}(X)$ has finite dimension; it is often zero-dimensional. Hence there exists an integer t_0 such that for every even integer r, X has a family of dimension at least $r - t_0$, consisting of indecomposable aCM vector bundles of rank r on X, such that for any two distinct elements $\mathscr{E}, \mathscr{E}'$ in the family there is no $f \in \text{Aut}(X)$ with $f^*(\mathscr{E}) \cong \mathscr{E}'$.

4. Proof of Theorem 3.9

Set $\mathbb{E}'(S)$ to be the set of all elements in $\mathbb{E}(S)$ whose corresponding middle term is locally free and aCM.

Lemma 4.1. *$\mathbb{E}'(S)$ is a non-empty open subset of $\mathbb{E}(S)$.*

Proof. Since being locally free and aCM are both open properties in a flat family, $\mathbb{E}'(S)$ is an open subset of $\mathbb{E}(S)$. Thus it is sufficient to prove that $\mathbb{E}'(S) \neq \emptyset$. Proposition 3.3 gives the case $s = 1$. For $s > 1$, we may find a direct sum of aCM vector bundles of rank two fitting into (8), i.e. take $\oplus_{p \in S} \mathscr{E}_p$. This implies $\mathbb{E}'(S) \neq \emptyset$. □

Remark 4.2. In the set-up of (8) set $\mathscr{A} := v(\mathscr{O}_X(m-3)^{\oplus s})$. By Lemma 3.1 and Remark 3.2 together with the assumption $m \geq 3$, we see that $\mathscr{A}$ is the first term of the HN filtration of $\mathscr{E}_S$. Thus we get $f(\mathscr{A}) \subseteq \mathscr{A}$ for any $f \in \text{End}(\mathscr{E}_S)$.

Lemma 4.3. *If $\mathscr{E}$ is the middle term of an extension $\varepsilon \in \mathbb{E}'(S)$, then $\mathscr{E}$ has no line bundle as a factor.*

Proof. Assume that $\mathscr{L}$ is a line bundle that is a factor of $\mathscr{E}$, i.e. $\mathscr{E} = \mathscr{L} \oplus \mathscr{G}$ for some aCM vector bundle $\mathscr{G}$ of rank $2s-1$. Since $m \ge 3$, we have

$$h^0(\mathscr{L}(3-m)) + h^0(\mathscr{G}(3-m)) = h^0(\mathscr{E}(3-m)) = s.$$

First assume $h^0(\mathscr{L}(3-m)) = 0$ and $h^0(\mathscr{G}(3-m)) = s$. Then we have $v(\mathscr{O}_X(m-3)^{\oplus s}) \subset \{0\} \oplus \mathscr{G}$ in (8) and so $\mathscr{L} \cong \mathscr{I}_{p,X}$ for some $p \in S$, a contradiction. Thus we have $h^0(\mathscr{L}(3-m)) > 0$ and so $h^0(\mathscr{G}(3-m)) < s$. In particular, there is a nonzero map $u : \mathscr{O}_X(m-3) \to \mathscr{L}$. Assume for the moment that $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$ and write $\mathscr{L} \cong \mathscr{O}_X(a)$ for some $a \in \mathbb{Z}$. The map u gives $a \ge m-3$. Since $m \ge 3$, (8) is the HN-filtration of $\mathscr{E}$ and we get $a = m-3$. Thus $\mathscr{G}$ fits into an exact sequence

$$0 \to \mathscr{O}_X(m-3)^{\oplus(s-1)} \to \mathscr{G} \to \oplus_{p\in S}\mathscr{I}_{p,X} \to 0.$$

Then we get $h^1(\mathscr{G}(-1)) \ge 1$ from $h^1(\mathscr{I}_{p,X}(-1)) = 1$ and $h^2(\mathscr{O}_X(m-4)) = 1$. Thus $\mathscr{G}$ is not aCM, a contradiction. If X is integral, then every line bundle is stable and so (8) is the HN-filtration of $\mathscr{E}$, we get either $\mathscr{L} \cong \mathscr{O}_X(m-3)$; we get a contradiction as above, or $\mathscr{L}$ is a factor of $\oplus_{p\in S}\mathscr{I}_{p,X}$, which is not locally free, a contradiction. □

Let $\mathbb{F}(S)$ (resp. $\mathbb{F}'(S)$) be the set of isomorphism classes of middle terms of extensions in $\mathbb{E}(S)$ (resp. $\mathbb{E}'(S)$). Let us denote by $\mathscr{E} = \mathscr{E}(\varepsilon)$ the middle term of the extension corresponding to $\varepsilon \in \mathbb{E}'(S)$.

Lemma 4.4. *For two non-empty finite sets $S_1, S_2 \subset X_{\mathrm{reg}}$ with $\sharp(S_i) = s_i$, take $\mathscr{E}_i \in \mathbb{F}'(S_i)$ and call $\mathscr{A}_i$ the subsheaf of $\mathscr{E}_i$ isomorphic to $\mathscr{O}_X(m-3)^{\oplus s_i}$ for each $i = 1,2$. If there exists a map $f : \mathscr{E}_1 \to \mathscr{E}_2$ with $f(\mathscr{E}_1) \subset \mathscr{A}_2$, then we have $S_1 \cap S_2 \ne \emptyset$.*

Proof. Since $\mathrm{Hom}_X(\mathscr{O}_X(m-3), \mathscr{I}_{p,X}) = 0$ for all $p \in X$, we have $f(\mathscr{A}_1) \subseteq \mathscr{A}_2$. In particular, f induces a nonzero map $\tilde{f} : \oplus_{p\in S_1}\mathscr{I}_{p,X} \to \oplus_{q\in S_2}\mathscr{I}_{q,X}$. This implies that $S_1 \cap S_2 \ne \emptyset$. □

Lemma 4.5. *Assume that $\mathscr{E} \in \mathbb{F}'(S)$ is decomposable; $\mathscr{E} \cong \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_h$ with each $\mathscr{E}_i$ indecomposable. Then there is a partition $S = \sqcup_{i=1}^h S_i$ with $\mathscr{E}_i \in \mathbb{F}'(S_i)$ for each i. If there is another decomposition $\mathscr{E} \cong \mathscr{E}_1' \oplus \cdots \oplus \mathscr{E}_k'$ with each $\mathscr{E}_j'$ indecomposable, then we get $k = h$ and there is a permutation $\sigma : \{1,\dots,h\} \to \{1,\dots,h\}$ such that $\mathscr{E}_{\sigma(i)}' \cong \mathscr{E}_i$ for all i and $\mathscr{E}_{\sigma(i)}' \in \mathbb{F}(S_{\sigma(i)})$.*

Proof. We use induction on s. The case $s = 1$ is true, because each $\mathscr{E}_p$ for $p \in X_{\mathrm{reg}}$ is indecomposable by Proposition 3.3. Since $\mathscr{E}$ is aCM by the definition of $\mathbb{F}(S)$, each $\mathscr{E}_i$ is also aCM. We consider the subsheaf $\mathscr{A} \cong \mathscr{O}_X(m-3)^{\oplus s} \subset \mathscr{E}$ as in Remark 4.2 and set $\mathscr{G}_i := \mathscr{A} \cap \mathscr{E}_i$. Since the HN filtration of $\mathscr{E}$ is obtained from the ones of each factors, we have

$$\mathscr{A} \cong \oplus_{i=1}^h \mathscr{G}_i \quad \text{and} \quad \oplus_{p\in S}\mathscr{I}_{p,X} \cong \oplus_{i=1}^h \mathscr{E}_i/\mathscr{G}_i.$$

By Lemma 4.3 we have $\mathscr{G}_i \subsetneq \mathscr{E}_i$ for all i. By Remark 3.2 we may write $S = \sqcup_{i=1}^h S_i$ with $\mathscr{E}_i/\mathscr{G}_i \cong \oplus_{p\in S_i}\mathscr{I}_{p,X}$. Since $\mathscr{E}_i/\mathscr{G}_i \ne 0$, we have $S_i \ne \emptyset$ for all i. Thus the set $\{S_1,\dots,S_h\}$ gives a partition of S. To prove the first part of the lemma it is sufficient to prove that $\sharp(S_i) = \mathrm{rank}(\mathscr{G}_i)/2$ for all i. If this is not true, then there is $i \in \{1,\dots,h\}$ with $\sharp(S_i) > \mathrm{rank}(\mathscr{G}_i)/2$, i.e. $\mathrm{rank}(\mathscr{G}_i \cap \mathscr{A}) > \sharp(S_i)$. The exact sequence

$$0 \to \mathscr{A} \cap \mathscr{G}_i \to \mathscr{G}_i \to \oplus_{p\in S_j}\mathscr{I}_{p,X} \to 0$$

gives $h^1(\mathscr{G}_i) \ge \sharp(S_i) - \mathrm{rank}(\mathscr{G}_i \cap \mathscr{A}) > 0$. In particular, $\mathscr{G}_i$ is not aCM, a contradiction.

Now we check the last assertion of the lemma. Take two partitions

$$S = S_1 \sqcup \cdots \sqcup S_h = S_1' \sqcup \cdots \sqcup S_k'$$

such that there is a decomposition

$$\mathscr{E} \cong \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_h \cong \mathscr{E}_1' \oplus \cdots \oplus \mathscr{E}_k'$$

with $\mathscr{E}_i \in \mathbb{F}'(S_i)$ and $\mathscr{E}'_j \in \mathbb{F}'(S'_j)$ indecomposable. By the Krull-Schmidt theorem in [1], we get $h = k$ and there is a permutation $\sigma : \{1,\dots,h\} \to \{1,\dots,h\}$ such that $\mathscr{B}_{\sigma(i)} \cong \mathscr{E}_i$ for all i. By renaming $\{\mathscr{E}'_1,\dots,\mathscr{E}'_h\}$, we may assume that $\mathscr{E}'_i \cong \mathscr{E}_i$ for all i. This implies

$$\sharp(S_i) = \mathrm{rank}(\mathscr{E}_i)/2 = \mathrm{rank}(\mathscr{E}'_i)/2 = \sharp(S'_i).$$

Now fix an isomorphism $f_i : \mathscr{E}_i \to \mathscr{E}'_i$ for each i. Since (8) gives the HN filtrations of $\mathscr{E}_i$ and $\mathscr{E}'_i$, the map f induces an isomorphism $\tilde{f}_i : \oplus_{p\in S_i}\mathscr{I}_{p,X} \to \oplus_{p\in S'_i}\mathscr{I}_{p,X}$. Since p is the unique point of X at which $\mathscr{I}_{p,X}$ is not locally free, we get $S_i = S'_i$. For each i, let $\mathscr{A}_i$ be the unique subsheaf of $\mathscr{E}_i$ isomorphic to $\mathscr{O}_X(m-3)^{\sharp(S_i)}$. Then for any embedding $u : \mathscr{E}_i \to \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_h$, the composition $v_j \circ \pi_j \circ u$

$$\mathscr{E}_i \xrightarrow{u} \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_h \xrightarrow{\pi_j} \mathscr{E}_j \xrightarrow{v_j} \oplus_{p\in S_j}\mathscr{I}_{p,X}$$

is zero for any $j \neq i$ by Lemma 4.4, where $\pi_j : \mathscr{E} \to \mathscr{E}_j$ is the projection and $v_j : \mathscr{E}_j \to \oplus_{p\in S_j}\mathscr{I}_{p,X}$ is the surjection in (8) for S_j. Since u is an embedding, we see that $v_i \circ \pi_i \circ u$ is surjective. Thus $\mathscr{G} := \pi_i(u(\mathscr{E}_i))$ is a subsheaf with $v_i(\mathscr{G}) = \oplus_{p\in S_i}\mathscr{I}_{p,X}$. □

Lemma 4.6. *With the setting as in Theorem 3.9, we have* $\mathrm{ext}^1_X(\mathscr{E}_p,\mathscr{E}_q) \geq 2$ *for two points* $p,q \in X_{\mathrm{reg}}$*, possibly* $p = q$.

Proof. Set $\mathscr{F}_o := \mathscr{E}_o(3-m)$ for $o \in \{p,q\}$. Since $\mathrm{Ext}^i_X(\mathscr{E}_p,\mathscr{E}_q) \cong \mathrm{Ext}^i_X(\mathscr{F}_p,\mathscr{F}_q)$, we have $\chi(\mathscr{E}_p \otimes \mathscr{E}_q^\vee) = \chi(\mathscr{F}_p \otimes \mathscr{F}_q^\vee)$. Since Euler's characteristic is constant in a flat family of vector bundles and $p,q \in X_{\mathrm{reg}}$, it is sufficient to compute $\chi(\mathscr{F}_p \otimes \mathscr{F}_q^\vee)$ when X is smooth. Since a smooth surface in $\mathbb{P}^3$ is connected, the same observation applied to a family of vector bundles on X shows $\chi(\mathscr{F}_p \otimes \mathscr{F}_q^\vee) = \chi(\mathscr{F}_p \otimes \mathscr{F}_p^\vee)$.

We have an exact sequence

$$0 \to \mathscr{O}_X \xrightarrow{v} \mathscr{F}_p \xrightarrow{w} \mathscr{I}_{p,X}(3-m) \to 0 \tag{9}$$

with $\det(\mathscr{F}_p) \cong \mathscr{O}_X(3-m)$ and $c_2(\mathscr{F}_p) = 1$. Since $X \subset \mathbb{P}^3$ is a surface of degree m, we have $c_1(\mathscr{F}_p)^2 = m(m-3)^2$. By Riemann-Roch for $\mathscr{E}nd(\mathscr{F}_p)$, we have

$$\begin{aligned}\chi(\mathscr{E}nd(\mathscr{F}_p)) &= c_1(\mathscr{F}_p)^2 - 4c_2(\mathscr{F}_p) + 4\chi(\mathscr{O}_X) = m(m-3)^2 - 4 + 4\binom{m-1}{3} + 4 \\ &= \frac{1}{6}\left(10m^3 - 60m^2 + 98m - 24\right).\end{aligned}$$

In particular, we have $\chi \sim \frac{5}{3}m^3$ for $m \gg 0$. Note that by Serre's duality we have $h^2(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) = h^0(\mathscr{F}_p \otimes \mathscr{F}_p^\vee(m-4))$.

Claim 1: We have $\hom_X(\mathscr{F}_p,\mathscr{F}_p) = 1 + \binom{m}{3}$.

Proof of Claim 1: We have $\hom_X(\mathscr{I}_{p,X}(3-m),\mathscr{O}_X) = h^0(\mathscr{O}_X(m-3)) = \binom{m}{3}$ and any nonzero map $\mathscr{I}_{p,X}(3-m) \to \mathscr{O}_X$ induces an element in $\mathrm{Hom}_X(\mathscr{F}_p,\mathscr{F}_p)$ with rank one as the following composition:

$$\mathscr{F}_p \xrightarrow{w} \mathscr{I}_{p,X}(3-m) \to \mathscr{O}_X \xrightarrow{v} \mathscr{F}_p.$$

The vector space $\mathrm{Hom}_X(\mathscr{F}_p,\mathscr{F}_p)$ also contains the nonzero multiples of the identity map $\mathscr{F}_p \to \mathscr{F}_p$ and these maps have rank two. Thus we get $h^0(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) \geq 1 + \binom{m}{3}$. On the other hand, for any $f \in \mathrm{Hom}_X(\mathscr{F}_p,\mathscr{F}_p)$ we get $w \circ f \circ (v(\mathscr{O}_X)) \subseteq v(\mathscr{O}_X)$ from $h^0(\mathscr{I}_{p,X}(3-m)) = 0$. Thus $w \circ f \circ v$ induces a map $f_1 : \mathscr{O}_X \to \mathscr{O}_X$, which is induced by the multiplication by $c \in \mathbf{k}$. Hence $f - c\cdot\mathrm{Id}_{\mathscr{F}_p}$ is induced by a unique $g \in \mathrm{Hom}_X(\mathscr{I}_{p,X}(3-m),\mathscr{F}_p)$. Since $\mathscr{F}_p$ is locally free and X is smooth at p, we have $\mathrm{Hom}_X(\mathscr{I}_{p,X}(3-m),\mathscr{F}_p) = H^0(\mathscr{F}_p(m-3))$. By (9) we have $h^0(\mathscr{F}_p(m-3)) = \binom{m}{3}$ and so $\hom_X(\mathscr{F}_p,\mathscr{F}_p) \leq 1 + \binom{m}{3}$. □

Claim 2: We have $\hom_X(\mathscr{F}_p,\mathscr{F}_p(m-4)) \geq \binom{2m-4}{3} + 2\binom{m-1}{3} - \binom{m-4}{3} - 1$.

Proof of Claim 2: For any $f \in \mathrm{Hom}_X(\mathscr{F}_p,\mathscr{F}_p(4-m))$, set $f_1 := f_{|v(\mathscr{O}_X)}$. Since $h^0(\mathscr{O}_X(-1)) = 0$, we have $w \circ f_1 = 0$ and so $f_1(v(\mathscr{O}_X)) \subset v(\mathscr{O}_X(m-4)))$. Take f with $f_1 \equiv 0$. Such a map f is uniquely determined by an element in $\mathrm{Hom}_X(\mathscr{I}_{p,X}(3-m),\mathscr{F}_p(m-4))$ and the converse also holds. Since $\mathscr{F}_p(m-4)$ is locally free and X is smooth at p, we have $\mathrm{Hom}_X(\mathscr{I}_{p,X}(3-m),\mathscr{F}_p(m-4)) = \mathrm{Hom}_X(\mathscr{O}_X(3-m),\mathscr{F}_p(m-4)) = H^0(\mathscr{F}_p(2m-7))$. Since $h^1(\mathscr{O}_X(t)) = 0$ for any $t \in \mathbb{Z}$, (9) gives

$$h^0(\mathscr{F}_p(2m-7)) = h^0(\mathscr{O}_X(2m-7)) + h^0(\mathscr{O}_X(m-4)) - 1 = \binom{2m-4}{3} - \binom{m-4}{3} + \binom{m-1}{3} - 1.$$

Note that a map f obtained by a composition

$$\mathscr{F}_p \xrightarrow{w} \mathscr{I}_{p,X}(3-m) \to \mathscr{O}_X(m-4) \xrightarrow{v} \mathscr{F}_p(m-4)$$

has $f_1 \equiv 0$. Now for any linear subspace $W \subset \mathrm{Hom}_X(\mathscr{F}_p,\mathscr{F}_p(m-4))$ such that $f_1 \not\equiv 0$ for any $f \in W \setminus \{0\}$, we would get

$$\hom_X(\mathscr{F}_p,\mathscr{F}_p(m-4)) \geq \binom{2m-4}{3} - \binom{m-4}{3} + \binom{m-1}{3} - 1 + \dim W.$$

We may choose W to consist of the compositions of the identity map $\mathscr{F}_p \to \mathscr{F}_p$ with the multiplication by an element of $H^0(\mathscr{O}_X(m-4))$. Then we have $\dim W = \binom{m-1}{3}$. □

Combining Claims 1 and 2, we get

$$\begin{aligned} h^0(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) + h^2(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) &\geq \binom{2m-4}{3} + \binom{m}{3} + 2\binom{m-1}{3} - \binom{m-4}{3} \\ &= \frac{1}{6}\left(10m^3 - 60m^2 + 98m - 12\right). \end{aligned}$$

Thus we have

$$h^1(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) = h^0(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) + h^2(\mathscr{F}_p \otimes \mathscr{F}_p^\vee) - \chi(\mathscr{E}nd(\mathscr{F}_p)) \geq 2$$

and so we get the assertion. □

Proof of Theorem 3.9: By Remark 4.2 (8) is the HN filtration of $\mathscr{E}_S$. Proposition 3.3 gives the case $s = 1$. For $s > 1$, we may find a direct sum of s vector bundles of rank 2 from the case $s = 1$, fitting into (8): just take $\oplus_{p \in S}\mathscr{E}_p$. So a general extension in $\mathbb{E}(S)$ has a locally free and aCM middle term, because being local free and aCM are both open conditions.

Note that $h^0(\mathscr{E}_S(3-m)) = s$ from (8). In particular there is a unique subsheaf $\mathscr{A} \subset \mathscr{E}_S$ isomorphic to $\mathscr{O}_X(m-3)^{\oplus s}$ and for each $f \in \mathrm{Hom}(\mathscr{O}_X(m-3),\mathscr{E}_S)$ we have $f(\mathscr{O}_X(m-3)) \subseteq \mathscr{A}$. Now by Lemma 3.1 and Remark 3.2, the extension (8) is the HN filtration of $\mathscr{E}_S$. By uniqueness of the HN filtration, we get $\mathscr{E}_S \not\cong \mathscr{E}_{S'}$ for $S \neq S'$.

Now it remains to show the indecomposability of $\mathscr{E}_S$. By Lemma 4.3, there is no rank one factor of $\mathscr{E}_S$.

Claim 1: For two distinct points p, q in X_{reg}, we have

$$\mathrm{Hom}_X(\mathscr{I}_{p,X},\mathscr{I}_{q,X}) = 0, \mathrm{Hom}_X(\mathscr{E}_p,\mathscr{I}_{q,X}) = 0 \text{ and } \mathrm{Ext}^1_X(\mathscr{I}_{p,X},\mathscr{I}_{q,X}) = 0.$$

Proof of Claim 1: By an extension theorem for locally free sheaves in [12, Exercise I.3.20], we have $\mathrm{Hom}_X(\mathscr{I}_{p,X},\mathscr{I}_{q,X}) = \mathrm{Hom}_X(\mathscr{O}_X,\mathscr{I}_{q,X}) = 0$. The second vanishing is obtained from the first vanishing and $\mathrm{Hom}_X(\mathscr{O}_X(m-3),\mathscr{I}_{q,X}) = 0$. For the last vanishing, we apply the functor $\mathrm{Hom}_X(\mathscr{I}_{p,X},-)$ to the standard exact sequence for $\mathscr{I}_{q,X} \subset \mathscr{O}_X$ and obtain an exact sequence

$$0 \to \mathrm{Hom}_X(\mathscr{I}_{p,X},\mathscr{O}_X) \to \mathrm{Hom}_X(\mathscr{I}_{p,X},\mathscr{O}_q) \to \mathrm{Ext}^1_X(\mathscr{I}_{p,X},\mathscr{I}_{q,X}) \to \mathrm{Ext}^1_X(\mathscr{I}_{p,X},\mathscr{O}_X)$$

by the first vanishing in the Claim. Here we have

$$\mathrm{Hom}_X(\mathscr{I}_{p,X},\mathscr{O}_X) \cong \mathrm{Hom}_X(\mathscr{I}_{p,X},\mathscr{O}_q) \cong \mathbf{k}$$

and $\mathrm{Ext}^1_X(\mathscr{I}_{p,X},\mathscr{O}_X) \cong H^1(\mathscr{I}_{p,X}(m-4))^\vee$ by Serre's duality. Then we get the assertion from the assumption that $m \geq 4$. □

(a) First assume $s = 2$ and take two distinct points p, q in X_{reg}.

Claim 2: If there exists a sheaf $\mathscr{G} \not\cong \mathscr{E}_p \oplus \mathscr{E}_q$ fitting into the exact sequence

$$0 \to \mathscr{E}_p \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{E}_q \to 0, \tag{10}$$

then the case $s = 2$ is true.

Proof of Claim 2: Such a sheaf $\mathscr{G}$ would be locally free and aCM with rank 4. Since $h^1(\mathscr{O}_X) = 0$ and (8) gives the HN filtrations of $\mathscr{E}_p$ and $\mathscr{E}_q$ by Lemmas 3.1 and Remark 3.2, $\mathscr{G}$ has a subsheaf $\mathscr{F} \cong \mathscr{O}_X(m-3)^{\oplus 2}$ such that $\mathscr{G}/\mathscr{F}$ is an extension of $\mathscr{I}_{q,X}(1)$ by $\mathscr{I}_{p,X}(1)$. Claim 1 gives $\mathscr{G}/\mathscr{F} \cong \mathscr{I}_{p,X} \oplus \mathscr{I}_{q,X}$ and so we get $\mathscr{G} \cong \mathscr{E}_S$ with $S = \{p, q\}$. □

Claim 3: If $\mathscr{G} \cong \mathscr{E}_p \oplus \mathscr{E}_q$ for all $\mathscr{G}$ in (10), then we have $\mathrm{Ext}^1_X(\mathscr{E}_q, \mathscr{E}_p) = 0$.

Proof of Claim 3: Let $\mathscr{G} \cong \mathscr{E}_p \oplus \mathscr{E}_q$ fitting into (10) correspond to $\varepsilon \in \mathrm{Ext}^1_X(\mathscr{E}_q, \mathscr{E}_p)$. Then it is sufficient to prove that $\varepsilon = 0$, or $\ker(v) \cong \mathscr{E}_p \oplus \{0\}$. But since $\ker(v) \cong \mathscr{E}_p$, it is sufficient to prove that either $\mathscr{E}_p \oplus \{0\} \supseteq \ker(v)$ or $\mathscr{E}_p \oplus \{0\} \subseteq \ker(v)$. Assume $v(\mathscr{E}_p \oplus \{0\}) \neq 0$. Since $\mathrm{Hom}_X(\mathscr{E}_p, \mathscr{I}_{q,X}) = 0$ by Claim 1, we have $v(\mathscr{E}_p \oplus \{0\}) \subseteq \mathscr{O}_X(m-3)$. This implies that the restriction of the surjection $\mathscr{E}_q \to \mathscr{I}_{q,X}$ to $v(\{0\} \oplus \mathscr{E}_q)$ is surjective. Since $h^0(\mathscr{O}_X) = 1$ and $\mathrm{Hom}_X(\mathscr{O}_X(m-3), \mathscr{I}_{q,X}) = 0$, we get either $v(\{0\} \oplus \mathscr{O}_X(m-3)) = 0$ or v induces an isomorphism $\{0\} \oplus \mathscr{O}_X(m-3) \to \mathscr{O}_X(m-3)$. Assume for the moment $v(\{0\} \oplus \mathscr{O}_X(m-3)) = 0$. Since $v(\mathscr{E}_p \oplus \{0\})$ maps to 0 in $\mathscr{I}_{q,X}$, we get that $v(\{0\} \oplus \mathscr{E}_q)$ is a subsheaf of $\mathscr{E}_q$ which maps isomorphically onto $\mathscr{I}_{q,X}$. So we get $\mathscr{E}_q \cong \mathscr{O}_X(m-3) \oplus \mathscr{I}_{q,X}$, a contradiction. Now assume $v(\{0\} \oplus \mathscr{O}_X(m-3)) = \mathscr{O}_X(m-3)$. Since $v(\{0\} \oplus \mathscr{E}_q)$ maps surjectively onto $\mathscr{I}_{q,X}$, the surjection v induces an isomorphism $\{0\} \oplus \mathscr{E}_q \to \mathscr{E}_q$. Hence we get $\mathscr{E}_p \oplus \{0\} \subseteq \ker(v)$. □

Since $\mathrm{Ext}^1_X(\mathscr{E}_q, \mathscr{E}_p) \neq 0$ by Lemma 4.6, Claim 3 concludes the proof of the case $s = 2$.

(b) Assume $s > 2$ and that Theorem 3.9 holds for smaller numbers. On $\mathbb{E}(S)$ there is a universal family of extensions, i.e. a coherent sheaf $\mathscr{V}$ over $\mathbb{E}(S) \times X$ such that for each $\varepsilon \in \mathbb{E}(S)$ the sheaf $\mathscr{V}_{|\{\varepsilon\} \times X}$ is the middle term $\mathscr{E}(\varepsilon)$ of the extension corresponding to ε; in general, if we take $\mathbb{P}(\mathbb{E}(S))$ as a parameter space, then no such a universal sheaf exists. We call $\mathscr{V}'$ the restriction of of $\mathscr{V}$ to $\mathbb{E}'(S) \times X$; we thus consider the family of aCM vector bundles induced from the extensions in $\mathbb{E}'(S)$.

Define a set $\Gamma(S)$ as follows:

$$\Gamma(S) := \left\{(\varepsilon, \varphi) \mid \varepsilon \in \mathbb{E}'(S) \text{ and } \varphi \in \mathrm{End}(\mathscr{E}(\varepsilon)) \text{ with } \varphi^2 = \varphi\right\}.$$

Note that φ is a projection of $\mathscr{E}(\varepsilon)$ onto a factor of $\mathscr{E}(\varepsilon)$, with the exception when $\varphi = \mathrm{Id}_{\mathscr{E}(\varepsilon)}$ or $\varphi \equiv 0$; if $\mathscr{E}(\varepsilon)$ is indecomposable, only $(\varepsilon, \mathrm{Id}_{\mathscr{E}(\varepsilon)})$ and $(\varepsilon, 0)$ are contained in $\Gamma(S)$. Indeed, for any vector bundle $\mathscr{G}$, there exists a one-to-one correspondence:

$$\{\varphi \in \mathrm{End}(\mathscr{G}) \mid \varphi^2 = \varphi\} \leftrightarrow \{\text{factors of } \mathscr{G}\}$$

via $\varphi \mapsto \mathrm{Im}(\varphi) = \ker(\mathrm{Id}_{\mathscr{G}} - \varphi)$, with $\mathscr{G}$ being associated to $\mathrm{Id}_{\mathscr{G}}$ and 0 associated to the zero map. Thus $\mathscr{G}$ is decomposable if and only if $\mathrm{End}(\mathscr{G})$ has a non-trivial idempotent. Note that $\Gamma(S)$ is a closed in the total space of the vector bundle $\mathscr{H}om(\mathscr{V}', \mathscr{V}')$ over $\mathbb{E}'(S) \times X$. By Lemma 4.5, for each $\mathscr{E}(\varepsilon)$ there is a unique partition of S associated to any decomposition of $\mathscr{E}(\varepsilon)$ with only finitely many indecomposable factors by the Krull-Schmidt theorem in [1]. By Lemma 4.5 for each $\mathscr{E} \in \mathbb{F}'(S)$ each isomorphism class of factors of $\mathscr{E}$ corresponds to a unique subset of S; $\mathscr{E}$ and 0 correspond to S and $\emptyset$, respectively. For each $(\varepsilon, \varphi) \in \Gamma(S)$, let $S(\varphi)$ be the subset of S associated to $\mathrm{Im}(\varphi)$ by Lemma 4.5. Set

$$\Gamma_0(S) := \left\{(\varepsilon, \varphi) \in \Gamma(S) \mid \varphi \neq 0 \text{ and } \varphi \neq \mathrm{Id}_{|\mathscr{E}(\varepsilon)}\right\}.$$

The goal is to show that $\Gamma_0(S)$ is not dominant over $\mathbb{F}(S)$ for a general S.

Note that up to now we did not use that S is contained in the same connected component $Y \cap X_{\mathrm{reg}}$ of X_{reg}. In particular the case $s = 2$ holds even if X has more than one irreducible components with multiplicity one and the two points of S belong to different connected components of X_{reg}.

Now we use a monodromy argument, which requires that S is contained in a connected component of $T := X_{\mathrm{reg}} \cap Y$ and that S is general in Y. Set $S = \{p_1, \dots, p_s\}$ and fix an ordering of the points in S, along which we get an ordering of the indecomposable factors of the sheaf $\oplus_{p \in S} \mathscr{I}_{p,X}$. Together with the usual ordering on the factors of $\mathscr{O}_X(m-3)^{\oplus s}$, we may see any $\varepsilon \in \mathbb{E}(S)$ as an $(s \times s)$-square matrix, say $\varepsilon = (\varepsilon_{ij})$

with $1 \le i,j \le s$, where ε_{ij} is an element of the 1-dimensional vector space $\mathrm{Ext}^1_X(\mathscr{I}_{p_j,X}, \mathscr{O}_X(m-3))$. Note that for a fixed integer j, each ε_{ij} with $i = 1,\dots,s$, is an element of the same 1-dimensional vector space. We write $\mathscr{O}_X(m-3)^{\oplus s} = \mathbb{C}^s \otimes \mathscr{O}_X(m-3)$.

Claim 4: $\mathscr{E} = \mathscr{E}(\varepsilon)$ has two indecomposable factors, one of them being $\mathrm{Im}(\varphi)$ and the other one being $\ker(\varphi)$.

Proof of Claim 4: Since $\varphi^2 = \varphi$, we have $\mathscr{E} \cong \mathscr{F}_1 \oplus \mathscr{F}_2$ with $\mathscr{F}_1 := \mathrm{Im}(\varphi)$ and $\mathscr{F}_2 = \ker(\varphi)$. By the definition of A, we get an exact sequence

$$0 \to \mathscr{O}_X(m-3)^{\oplus k} \to \mathscr{F}_1 \to \oplus_{p\in A}\mathscr{I}_{p,X} \to 0, \tag{11}$$

with $k := \sharp(A)$. Since neither $\varphi \equiv 0$ nor $\varphi = \mathrm{Id}_{\mathscr{E}}$, we have $0 < k < s$. Then by Lemma 4.5 we get an exact sequence

$$0 \to \mathscr{O}_X(m-3)^{\oplus(s-k)} \to \mathscr{F}_2 \to \oplus_{p\in S\setminus A}\mathscr{I}_{p,X} \to 0. \tag{12}$$

Now we need to prove that each $\mathscr{F}_i$ is indecomposable. By the inductive assumption it is sufficient to prove that $\mathscr{F}_1$ and $\mathscr{F}_2$ are the middle terms of general extensions (11) and (12), respectively. Since (8) gives the HN filtration of each $\mathscr{F}_i$, there are linear subspaces $V_1, V_2 \subset \mathbb{C}^s$ such that $\dim V_1 = k$, $\dim V_2 = s-k$ and

$$v(\mathbb{C}^s \otimes \mathscr{O}_X(m-3)) \cap \mathscr{F}_i = V_i \otimes \mathscr{O}_X(m-3)$$

for each i. From $\mathscr{E} \cong \mathscr{F}_1 \oplus \mathscr{F}_2$ we see that $\mathbb{C}^s = V_1 \oplus V_2$. Now we reorder the points in S so that all points of A are smaller than any points of $S\setminus A$. Then ε can be understood as an $(s\times s)$-square matrix in a block form:

$$\varepsilon = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22}\end{array}\right]$$

Here the $(k\times k)$-matrix B_{11} in the upper left corner, is associated to the extension (11) and similarly the $((s-k)\times(s-k))$-matrix B_{22} in the lower right corner, is associated to the extension (12). The matrix of ε also has a $(k\times(s-k))$-submatrix B_{12} and an $((s-k)\times k)$-submatrix B_{21}. Since ε is general, all the entries in each B_{ij} are also general. In particular, B_{11} and B_{22} are general and this implies that each $\mathscr{F}_i$ is general. The inductive assumption gives that each $\mathscr{F}_i$ is indecomposable. □

Assume that a general $\mathscr{E} = \mathscr{E}(\varepsilon)$ has two indecomposable factor, i.e. the set $\Gamma_0(S)$ is dominant over $\mathbb{F}(S)$. Let $\Gamma'(S)$ be an irreducible component of $\Gamma_0(S)$ dominant over $\mathbb{F}(S)$ and set $A := S(\varphi)$, where (ε,φ) is any element of $\Gamma'(S)$. Now assume that (ε,φ) is general in $\Gamma'(S)$ and set $\mathscr{E} := \mathscr{E}(\varepsilon)$. Note that the subset $A \subset S$ is invariant as (ε,φ) varies in $\Gamma_0(S)$, due to the irreducibility of $\Gamma_0(S)$. Below we find a contradiction under the assumptions that $\mathscr{E}$ is decomposable and that S is general in $\mathrm{Sym}^s(T)$.

Let $\widetilde{\Gamma}$ be the set of all triples $(S,\mathscr{E},\varphi)$ with $S\in\mathrm{Sym}^s(T)$ and $(\mathscr{E},\varphi)\in\Gamma_0(S)$. Then $\widetilde{\Gamma}$ is an algebraic subset whose fibre over $S\in\mathrm{Sym}^s(T)$ is $\Gamma_0(S)$, with a projection map $u:\widetilde{\Gamma}\to\mathrm{Sym}^s(T)$. If u is not dominant, then it would imply that there exists a $2s$-dimensional family of pairwise not isomorphic indecomposable aCM vector bundles of rank $2s$ on X. Thus we may assume that u is dominant. We fix a general $S\in\mathrm{Sym}^s(T)$ and fix an irreducible component $\Gamma'(S)$ of $\Gamma(S)$ to which we apply the previous construction with the partition $A\sqcup(S\setminus A)$ of S attached to $\Gamma'(S)$. Let $\widetilde{\Gamma}'$ be any irreducible component of $\widetilde{\Gamma}$ containing $\Gamma'(S)$ such that $u_{|\widetilde{\Gamma}'}$ is dominant.

Let $\mathscr{V}$ denote a non-empty Zariski open subset of $\mathrm{Sym}^s(T)$ containing S such that for every $T\in\mathscr{V}$ a general $\mathscr{E}_T\in\mathbb{E}(T)$ has exactly two indecomposable factors, one associated to a subset F of T with $|F| = |A| = k$ and the other one associated to $T\setminus E$. Now we fix $p\in A$ and $q\in S\setminus A$. Since Y_{reg} is a connected manifold and $p,q\in Y_{\mathrm{reg}}$, there exists a connected smooth affine curve $U\subset\mathbb{A}^1(\mathbf{k})$ with a map $\varphi: U\to Y_{\mathrm{reg}}$ such that $\varphi(t_0) = p$ and $\varphi(t_1) = q$ for some $t_0,t_1\in U$, and $\varphi(U)$ passes no other points of S. Similarly we may consider a map $\varphi': U\to Y_{\mathrm{reg}}$ with $\varphi'(t_1) = p$ and $\varphi'(t_0) = q$ such that $\varphi(t)\neq\varphi'(t)$ for any $t\in U$. For each $t\in U$, set

$$A_t := (A\setminus\{p\})\cup\{\varphi(t)\} \quad,\quad S_t := (S\setminus\{p,q\})\cup\{\varphi(t),\varphi'(t)\},$$

e.g. $(A_{t_0}, S_{t_0}) = (A_{t_1}, S_{t_1}) = (A, S)$. Restricting U to an open neighborhood of $\{t_0, t_1\}$, we may assume that $S_t \in \mathscr{V}$ for all $t \in U$. Then for each $t \in U$ we have a partition $S_t = A_t \sqcup (S_t \setminus A_t)$ such that a general $\mathscr{E}_{S_t} \in \Gamma'(S_t)$ has exactly two indecomposable factors, one associated to A_t and the other associated to $S_t \setminus A_t$, due to the choice of $\widetilde{\Gamma}'$.

We start from $t = t_0$ and vary t in U to arrive at $t = t_1$, where we have $S_{t_1} = S = A_q \sqcup (S \setminus A_q)$ with $A_q = (A \setminus \{p\}) \cup \{q\}$. Since $s > 2$, we have $\{A, S \setminus A\} \neq \{A_q, S \setminus A_q\}$, contradicting the assumption that $\mathscr{E}_S$ has exactly two indecomposable factors. □

5. Non-locally free aCM sheaf

In this section, we let $X \subset \mathbb{P}^N$ be a closed subscheme with pure dimension n at least two. Assume that each local ring $\mathscr{O}_{X,x}$ with $x \in X$, has depth n and that X is aCM with respect to $\mathscr{O}_X(1)$, i.e. $h^i(\mathscr{I}_{X,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$ and all $1 \le i \le n-1$. The exact sequence

$$0 \to \mathscr{I}_{X,\mathbb{P}^N}(t) \to \mathscr{O}_{\mathbb{P}^N}(t) \to \mathscr{O}_X(t) \to 0$$

shows that $h^i(\mathscr{I}_{X,\mathbb{P}^N}(t)) = h^{i-1}(\mathscr{O}_X(t))$ for all $i \ge 2$. Hence we may restate our assumption as $h^1(\mathscr{I}_{X,\mathbb{P}^N}(t)) = 0$ and $h^i(\mathscr{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \dots, n-2$. By a theorem of Serre, the condition that $h^i(\mathscr{O}_X(-x)) = 0$ for $x \gg 0$ and $i = 1, \dots, n-2$, plus having positive depth at each $x \in X$, is equivalent to all $\mathscr{O}_{X,x}$ having depth n. Since $h^1(\mathscr{I}_{X,\mathbb{P}^N}) = 0$, we have $h^0(\mathscr{O}_X) = 1$ and in particular X is connected. Since $h^1(\mathscr{I}_{X,\mathbb{P}^N}(1)) = 0$, X is linearly normal in the linear subspace of $\mathbb{P}^N$ spanned by X. Since $n \ge 2$ we have $h^1(\mathscr{O}_X) = 0$ an so $\operatorname{Pic}(X)$ is a finitely generated abelian group.

Fix an irreducible component Y of X_{red}. If X is a hypersurface in $\mathbb{P}^N$, then the multiplicity $\mu \ge 1$ is well-defined. In the general case we do not need the notion of the multiplicity μ of Y in X at a general point of Y. In this section we need knowledge only on whether $\mu = 1$ or $\mu > 1$. We say that Y has multiplicity $\mu = 1$ if X is reduced at a general $x \in Y$, i.e. there is a non-empty open subset $U \subseteq Y$ such that $\mathscr{O}_{X,x} = \mathscr{O}_{Y,x}$ for all $x \in U$. Otherwise we say that Y has multiplicity $\mu > 1$. We are interested only in the case X not integral; if Y has multiplicity 1, then we have other irreducible components of X_{red}.

Lemma 5.1. *Let $C \subset X$ be a reduced aCM subvariety of pure dimension $n-1$. Then its ideal sheaf $\mathscr{I}_{C,X}$ is an aCM $\mathscr{O}_X$-sheaf such that*

- *it is locally free outside C and*
- *for any closed subscheme $Y \subsetneq X$, it is not an $\mathscr{O}_Y$-sheaf.*

Proof. Since C is aCM as a closed subscheme of $\mathbb{P}^N$ and C has pure dimension $n-1$, we have $h^1(\mathscr{I}_{C,\mathbb{P}^N}(t)) = 0$ for all $t \in \mathbb{Z}$. Thus the restriction map $\rho_t : H^0(\mathscr{O}_{\mathbb{P}^N}(t)) \to H^0(\mathscr{O}_C(t))$ is surjective for any $t \in \mathbb{Z}$. Since ρ_t factors through the restriction map $\eta_t : H^0(\mathscr{O}_X(t)) \to H^0(\mathscr{O}_C(t))$, η_t is surjective. Since η_t is surjective and $h^1(\mathscr{O}_X(t)) = 0$, we have $h^1(\mathscr{I}_{C,X}(t)) = 0$. This implies that $\mathscr{I}_{C,X}$ is aCM. From $\mathscr{I}_{C,X \setminus C} \cong \mathscr{O}_{X \setminus C}$, we see that $\mathscr{I}_{C,X}$ is locally free and of rank 1 outside C. Since C is not an irreducible component of X_{red} and $\mathscr{I}_{C,X}$ is locally free of positive rank outside C, there is no closed subscheme $Y \subsetneq X$ with $\mathscr{I}_{C,X}$ an $\mathscr{O}_Y$-sheaf. □

Proposition 5.2. *Fix an irreducible component Y of X_{red}. For a fixed integer $e > 0$ and any integral divisor $C \in |\mathscr{O}_Y(e)|$, define*

$$\Sigma_C := \left\{ p \in Y \mid \mathscr{I}_{C,X} \text{ is not locally free at } p \right\}.$$

(i) *If Y has multiplicity $\mu > 1$ in X, then we have $\Sigma_C = C$, i.e. for all $p \in C$ the sheaf $\mathscr{I}_{C,X}$ is not locally free at p. For any two integral curves $C_1, C_2 \in |\mathscr{O}_Y(e)|$, we have $\mathscr{I}_{C_1,X} \cong \mathscr{I}_{C_2,X}$ if and only if $C_1 = C_2$.*

(ii) *Assume that Y has multiplicity $\mu = 1$ and that X is not integral. Let $F \in |\mathscr{O}_Y(m-1)|$ be the complete intersection of Y with the other components of X, counting multiplicities. If $F \neq \emptyset$, then F has pure dimension $n-1$ and $F \cap C \neq \emptyset$ with $\Sigma_C = (F \cap C)_{\mathrm{red}}$.*

(iii) *For any two integral divisors $C_1, C_2 \in |\mathscr{O}_Y(e)|$ such that $\mathscr{I}_{C_1,X} \cong \mathscr{I}_{C_2,X}$, we have $\Sigma_{C_1} = \Sigma_{C_2}$; in case (i) we have the converse.*

Proof. By Lemma 5.1 the sheaf $\mathscr{I}_{C,X}$ is aCM and locally free with rank 1 at all $p \in X \setminus C$. Fix $p \in C$ and assume that $\mathscr{I}_{C,X}$ is locally free at p. Then there is $w \in (\mathscr{I}_{C,X})_p$ such that w is not a zero-divisor of $\mathscr{O}_{X,p}$ and $(\mathscr{I}_{C,X})_p \cong w\mathscr{O}_{X,p}$ as a module over the local ring $\mathscr{O}_{X,p}$. We get that in a neighborhood of p the divisor C is a Cartier divisor of X. Let $I \subset \mathscr{O}_{X,p}$ be the ideal of Y and $J \subset \mathscr{O}_{X,p}$ the ideal of C. We have $I \subset J$. First assume that X is not reduced at a general point of X. Since the support of the nilradical $\eta \subset \mathscr{O}_X$ of the structural sheaf $\mathscr{O}_Y$ is a closed subset of X_{red}, X is not reduced at any point of Y and in particular it is not reduced at p. Thus there is a nonzero $h \in I$ such that $h^m = 0$ for some $m > 0$. Since $I \subset J$, we have $h \in J$ and so h is divided by w. Thus we get $w^m = 0$ and so w is a zero-divisor, a contradiction.

Now assume that X is reduced at a general point of Y. Since X is not integral and it has pure depth n, X_{red} has at least one another irreducible component. Since $h^0(\mathscr{O}_X) = 1$, X is connected and so $F \neq \emptyset$. Fix any $x \in F$. Since $\mathscr{O}_{X,x}$ has depth $n \geq 2$, it is connected in dimension $\leq n-1$, i.e. for any open neighborhood W of x in X and any closed subscheme V of W, there is a neighborhhod U of x in W such that $U \setminus (U \cap V)$ is connected. Thus F has pure dimension $n-1$. Since $C \in |\mathscr{O}_Y(e)|$, C is a Cartier divisor of Y. Thus C is a Cartier divisor of X at all points of $C \setminus (C \cap F)$. Since $e > 0$, C is an ample divisor of Y. In particular, we get $F \cap C \neq \emptyset$. Fix $p \in F \cap C$. Any local equation w of C at p vanishes on each irreducible component of X_{red} containing p, because w is assumed to be a non-zero divisor of $\mathscr{O}_{X,p}$. There is at least one another irreducible component of X_{red} containing p, because $p \in F$.

Part (iii) is obvious. □

As a corollary of Proposition 5.2 we get the following result, which shows that X is of wild representation type in a very strong form.

Proposition 5.3. *Take X as above. For a fixed integer $w > 0$, there is an integral quasi-projective variety Δ and a flat family $\{\mathscr{F}_a\}_{a\in\Delta}$ of aCM sheaf on X with each $\mathscr{F}_a$ locally free outside a one-codimensional subscheme C_a and for each $a \in \Delta$ the set of all $b \in \Delta$ such that $\mathscr{F}_b \cong \mathscr{F}_a$ is contained in an algebraic subscheme $\Delta_a \subset \Delta$ with $\dim \Delta - \dim \Delta_a \geq w$.*

Proof. First assume that X has at least one irreducible component Y with multiplicity at least 2. Fix a positive integer e such that $\dim |\mathscr{O}_Y(e)| \geq w$ and take as Δ the family of all integral $C \in |\mathscr{O}_Y(e)|$. Then we may apply (i) of Proposition 5.2. In this case we may find Δ with the additional condition that for all $a, b \in \Delta$ we have $\mathscr{F}_a \cong \mathscr{F}_b$ if and only if $a = b$.

Now assume that each irreducible component of X has multiplicity 1 and fix one of them, say Y. Write $F \subset Y$ as in (ii) of Proposition 5.2. Fix an integer $e > 0$ such that $h^0(\mathscr{O}_X(e)) - h^0(\mathscr{O}_X(e)(-F)) > w$ and let Δ be the set of all integral divisors $C \in |\mathscr{O}_X(e)|$ not contained in F and such that the scheme $F \cap C$ is reduced. Since F has pure dimension $n-1$ and C is an ample divisor, the set $(F \cap C)_{\mathrm{red}}$ has pure dimension 2. Note that if $C, D \in \Delta$ and $(C \cap F)_{\mathrm{red}} = (D \cap F)_{\mathrm{red}}$, then any equation of C in $H^0(\mathscr{O}_X(e))$ differs from an equation of D by an element of $H^0(\mathscr{O}_X(e)(-F))$. Then we may apply (ii) of Proposition 5.2. □

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