A new quaternion based kinematic model for the operation and the identification of an articulated arm coordinate measuring machine inspired by the geodetic methodology

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Abstract

In this paper we describe the mathematical model of a kinematic chain and its use in the design and implementation of the algorithms that are necessary for the operation and the identification of an AACMM. The mathematical model is based on the use of quaternions for the representation of rotations. The direct kinematic problem is solved by a quite straightforward application of the model. The identification problem is solved with an iterative procedure based on linearized equations and the application of the least squares principle. The analytical linearization of the equations and the definition of some constraint equations are a significant part of the paper. The model can be also used to describe the kinematic chain of a robotic arm. The description of some experiments performed with a functioning AACMM demonstrate the effectiveness of the model and of the algorithms.

Keywords: AACMM, identification, quaternions, least-squares

1. Introduction

Coordinate measuring machines are widely used in many technical and scientific applications. A particular kind of these instruments are the articulated arm coordinate measuring machine (AACMM) that have a mechanical structure similar to a robotic arm. A mechanical articulated arm is composed of a set of rigid bodies (links) connected by joints and realizing a kinematic chain. Each joint realizes a rotation. The similarity between an AACMM and a robotic arm has been described by various authors (see e.g. Santolaria and Aguilar in [1], Aguilar et al. in [2], Gao et al. in [3, 4] and Santolaria et al. in [5]). Despite the similarity between the geometrical structure of an AACMM and a robotic arm, important differences between the two types of device exist. The various degrees of freedom of a robotic arm are associated with actuators, the degrees of freedom of an AACMM are associated with measuring devices. The purpose of a robotic arm is to move a manipulator or a tool to a specific position, while the purpose of an AACMM is to determine the coordinates of a point reached by a probe that is positioned by an operator. Another important difference is the following: in an AACMM only the position of the extremal point of the last element of the chain (the probe) is important, in a robotic arm also the orientation of the last element of the chain (the tool) is important.

The operation of both a robotic arm and an AACMM is governed by a system of computer programs. The algorithms implemented are quite different in the two cases, but they can be based on the same analytical representation of the kinematic chain.

In the following we describe a mathematical model for the representation of the chain, the algorithms for the operation and the identification of an AACMM and their implementation and testing.

The problem at hand clearly belongs to the realm of mechanical metrology, but it is quite similar to a survey problem and it is solved using methodological expertise that are typical of the geodetic disciplines.

It is worth noticing that the algorithms and the computer programs are remarkably simple due to the proper choice of the most convenient mathematical tools for the representation of the system: we used quaternions for the representation of the orientation of the various elements (links and joints) of the kinematic chain. The advantages of the use of quaternions are explained in the paper.

2. The main problems in the operation and the identification of an articulated arm

The articulated arm and its pose in space can be described by three sets of parameters:

- the geometrical parameters,
- the readings of the encoders that measure the rotations of the joints,
- the coordinates of the probe.

All these parameters must be related within a suitable analytical model that will be described in a following Section.

The parameters are named here in a manner that refers to an AACMM.

There are three main problems that must be solved using the model: they are the so called *direct kinematic problem*, the *inverse kinematic problem* and *identification*. In each problem one of the three sets of parameters is unknown and it must be computed or estimated from the other two.

In the direct kinematic problem the geometric parameters of the system are known (by design and/or estimation), the rotations of the joints are known and the coordinates of the last element of the chain must be computed. This is the typical main problem for the operation of an AACMM where the rotation of the joints are measured by the embedded encoders.

In the inverse kinematic problem the geometric parameters of the system are known, the desired position of the last element of the chain is known and the rotations of the joints must be computed. This is the typical problem for the operation of a robotic arm where the rotation of the joints are realized by actuators. This is an inverse problem that in general can not be solved directly and does not have a unique solution. Furthermore, the complete trajectory from an initial to a final pose must be determined taking into account various constraints and some optimality criteria.

The identification is another important and not so easy inverse problem. It is necessary to collect the readings of the encoders that measure the rotations when the probe is visiting known or partly known points. The geometric parameters of the system must be estimated. The paper by Goswami et al. [6] clearly explains the need of evaluating the parameters of a kinematic chain by using an estimation procedure to obtain high positional accuracy. The estimation procedure that is examined in the present article is essentially the level-2-calibration according to the classification given by Roth et al. in [7]. (The use of the words *calibration*, *identification* and *estimation* is not completely uniform in the technical literature. We mainly use *identification* and *estimation* in the present article.)

A further problem to be considered is the use of different reference frames. In general there is a reference frame attached to the mechanism and another reference frame attached to the object that must be measured. The two reference frames are related by the well known *seven parameters transformation* also known as *Helmert transformation* in geodesy or *absolute orientation* in photogrammetry.

3. The geometrical scheme of the mechanical system and its describing parameters

The articulated arm is a chain of rigid links that can be geometrically modeled as an open polygon in space. Each side of the polygon can be described by means of a vector and the vector can be parametrized by means of its three Cartesian components in a proper reference frame. Each joint can be characterized by the unit vector that gives the direction of the rotational axis and by the reading of its encoder. In order to describe the intrinsic geometry of the arm its degrees of freedom are frozen in a particular pose, the *basic pose*. In the basic pose the following parameters completely describe the geometry of the system:

- the components of the vectors associated to the sides of the polygon,
- the components of the unit vectors of the direction of the axes of the joints,
- the readings of the encoders of the joints.

Any generic pose of the arm different from the basic pose is essentially described by the actual readings of the encoders.

Now the various variables can be defined:

- n is the number of joints and of links in the arm,
- $\underline{r}_k$ $(k = 1..n)$ is the vector associated to link k in the basic pose,
- $\underline{u}_k$ is the unit vector associated to the axis of joint k in the basic pose,
- α_{0k} is the reading of the encoder k in the basic pose,
- α_k is the reading of the encoder k in a generic pose,
- $\underline{x}$ is the vector of coordinates of the probe (that is the end-point of link n).

The meaning of the defined variables is illustrated in Fig. 1.

It is worth noting that the proposed model can deal with any number of joints and links in any configuration. Joint k connects link k with link $k-1$ except for joint 1 that connects the first link to a support.

The geometric description of the arm is given, as already stated, by the parameters $\underline{r}_k$, $\underline{u}_k$ and α_{0k}. This is a convenient description because all the involved entities have a clear and intuitive meaning. On the other hand some comments are necessary to completely understand their nature:

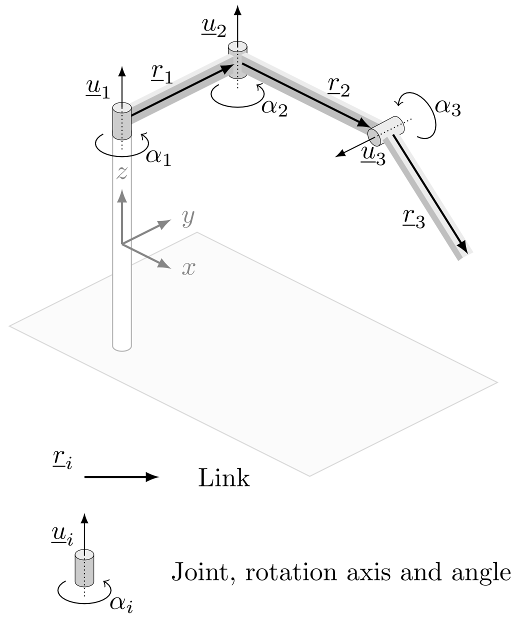

Figure 1: Schematic view of an hypothetical kinematic arm with $n = 3$.

- the geometrical description of the arm is only loosely related to its physical structure,
- the choice of the reference frame attached to the mechanism is arbitrary,
- the definition of the basic pose is quite convenient to build a simple and flexible model,
- the choice of the basic pose is arbitrary,
- the coordinates of the probe are invariant to some transformation of the parameters that describe the arm

All these points will be developed in Section (7).

4. Rotations and quaternions

A rotation in three dimensional space can be characterized by its rotational axis and its angle, both highly significant geometric entities. The axis in turn can be represented by a unit vector. In the problem at hand the rotational axis is mechanically realized by the joint and the angle (α) is measured by the encoder. The axis and the angle give the most natural and most significant description of the rotation, but other sets of parameters are more appropriate to perform computations. In fact there are many different sets of parameters and different algebraic techniques to handle rotations in space. A general treatment can be found in the papers by Nitschke and Knickmeyer [8], by Stuelpnagel [9] and by Dai [10]. The same topic is treated by Waldron and Schmiedeler [11] in the framework of kinematics of robots. Any non singular parametric representation of rotations needs 4 parameters as a minimum, even though the group of rotations is a three-dimensional manifold, as proved by Stuelpnagel [9].

A convenient representation of rotations is based on the algebra of quaternions. Quaternions are hyper-complex numbers with four components. In this Section some "facts" about quaternions and rotations are presented in an informal manner as an introduction to the plain formalism used in the next Sections for the solution of the problems at hand. A complete treatment about rotations and quaternions can be found in the books by Altmann [12] and by Kuipers [13], but the essential elements of quaternion algebra can be exposed in a few phrases.

The following notations will be used:

- $\mathbf{q}$: bold letters are quaternionic variables,

- (q_0, q_1, q_2, q_3) : the four components of a quaternion are real numbers designated with a subscript,

- $\underline{q}$: underlined letters are vectors in $\mathbb{R}^3$,

- $\mathbf{q} = q_0 + i\underline{q}$: a quaternion is written as the "sum" of a real scalar component and an imaginary vector component storing the last three components of the quaternion.

The last expression resembles the notation generally used for complex numbers: a quaternion can be considered as a "complex number" with scalar real part and vector imaginary part ($\in \mathbb{R}^3$).

The elementary operations of ordinary real algebra can be redefined on quaternions. The identity of two quaternions, the sum and the difference between two quaternions are defined component-wise. The product of quaternions is defined by:

$$\mathbf{qp} = (q_0 p_0 - \underline{q} \cdot \underline{p}) + i(q_0 \underline{p} + p_0 \underline{q} + \underline{q} \wedge \underline{p}) \tag{1}$$

where $\underline{q} \cdot \underline{p}$ is the dot product and $\underline{q} \wedge \underline{p}$ is the cross product. The product of quaternions is associative and not commutative. The product of a real number and a quaternion can be performed by first "upgrading" the real number to a real quaternion (i.e. a quaternion with null imaginary component).

Three more definitions are needed. The *conjugate* of the quaternion $\mathbf{q}$ has the same real part of $\mathbf{q}$ and opposite imaginary part and it is denoted by $\overline{\mathbf{q}}$. The *norm* of $\mathbf{q}$ is $|\mathbf{q}| = (\mathbf{q}\overline{\mathbf{q}})^{1/2} = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}$. The reciprocal of a not-null quaternion is defined by:

$$\mathbf{q}^{-1} = |\mathbf{q}|^{-2} \overline{\mathbf{q}}. \tag{2}$$

Quaternions are specially useful for the representation of rotations in $\mathbb{R}^3$ as already stated at the beginning of this Section. Each vector in space can be associated to a purely imaginary quaternion:

$$\underline{x} \leftrightarrow \mathbf{x} = 0 + i\underline{x}. \tag{3}$$

Given a unimodular quaternion $\mathbf{q}$ the expression:

$$\mathbf{y} = \mathbf{q}\mathbf{x}\overline{\mathbf{q}} \tag{4}$$

is a linear function of $\mathbf{x}$ and it is itself an imaginary quaternion. The transformation (4) preserves both the dot product and the box product in $\mathbb{R}^3$ (via (3)). Therefore the transformation (4) is a rotation. If we look at quaternions as a 4-dimensional linear space we can distinguish two three-dimensional manifolds to represent vectors and rotations in $\mathbb{R}^3$ respectively. Vectors in ordinary space are represented by imaginary quaternions, which form a three-dimensional linear subspace; rotations in ordinary space are represented by unimodular quaternions, which form a hypersphere. Indeed this is not a one-to-one correspondence because $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation.

The quaternion associated with a rotation is related to the geometric entities of the same rotation by the following relations:

- the real part is $q_0 = \cos(\alpha/2)$,
- the imaginary part is a vector with the same orientation of the rotational axis and the modulus equal to $\sin(\alpha/2)$.

The geometrical meaning of the quaternion that represents a rotation in expression (4) is now clear.

Now we consider a rotation represented by a quaternion $\mathbf{q}_a$ and a second rotation represented by a quaternion $\mathbf{q}_b$ applied to the result of the first rotation. The repeated application of (4) and the associativity of the quaternionic product easily shows that the resulting rotation is represented by $\mathbf{q}_b\mathbf{q}_a$.

The use of quaternions to represent rotations by means of (4) presents several advantages over the choice of other parametrizations and other expressions. Two peculiar properties of the chosen representation play an important role in the realization of the analytical model of an articulated arm, namely:

- the quaternion that represents a rotation has a clear geometrical meaning,
- the composition of two (or more) rotations is represented by the product of the corresponding quaternions.

Another important advantage of the use of quaternions is the possibility to solve the absolute orientation problem in a direct way for any configuration (Sansò [14]).

5. The analytical model of the articulated arm and the solution of the direct kinematic problem

The joint k imposes a rotation of angle $\alpha_k - \alpha_{0k}$ around an axis oriented as $\underline{u}_k$ to all the subsequent links i.e. to links from k to n. This rotation is represented by the quaternion:

$$\mathbf{q}_k = \cos((\alpha_k - \alpha_{0k})/2) + i\sin((\alpha_k - \alpha_{0k})/2)\underline{u}_k. \tag{5}$$

The link k is rotated by the combined action of all the joints that precede it i.e. the joints from 1 to k. The quaternion that represents the total rotation acting on link k is therefore:

$$\mathbf{p}_k = \prod_{l=1}^{k} \mathbf{q}_l. \tag{6}$$

Each link k is described in the basic pose by the vector $\underline{r}_k$ that is considered the imaginary part of the quaternion: $\mathbf{r}_k = 0 + i\underline{r}_k$. Now the quaternion associated to the same link in a generic pose is:

$$\mathbf{p}_k\mathbf{r}_k\overline{\mathbf{p}}_k \tag{7}$$

where the due rotation $\mathbf{p}_k$ has been applied. The imaginary quaternion associated to the position of the probe is simply obtained by:

$$\mathbf{x} = \mathbf{x}_0 + \sum_{k=1}^{n} \mathbf{p}_k\mathbf{r}_k\overline{\mathbf{p}}_k \tag{8}$$

where the constant term $\mathbf{x}_0$ accounts for the arbitrary position of the origin. The set of equations (5, 6, 7, 8) form the kinematic model of the chain. Equations (6), (7) and (8) can be combined to express the model in a single equation:

$$\mathbf{x} = \mathbf{x}_0 + \sum_{k=1}^{n} \left(\left(\prod_{l=1}^{k} \mathbf{q}_l \right) \mathbf{r}_k \overline{\left(\prod_{l=1}^{k} \mathbf{q}_l \right)} \right). \tag{9}$$

The Eq. (9) immediately solves the direct kinematic problem, i.e. the computation of the coordinates of the probe. The same equation can be properly elaborated to solve the identification problem as will be shown in Section (8).

The model represented by (9) describes the kinematic of the arm that is ideally composed of rigid links, a rigid ground support and perfect joints. This assumption coincides with the conditions required by Roth et al. in [7] for a purely kinematic model.

6. Kinematic redundancy of the structure of the arm

In this work, *Kinematic redundancy of the structure of the arm* refers to two different aspects of the structure of a kinematic arm.

At first, an arm with just the minimum number of degrees of freedom (DoF) is considered. In this situation a certain position of the probe can be generally attained with more than one pose of the arm. The different poses that correspond to a certain position of the probe are a discrete limited set. This is a first kind of redundancy that is always present.

If the arm has more than the minimum number of degrees of freedom, position of the probe can be generally attained with an infinite number of different poses, that form an infinite continuous set. This is a second kind of redundancy.

In a robotic arm, where the orientation of the last element of the chain is also of interest, the situation is almost identical. Only the count of minimal DoF must be changed.

The kinematic redundancy of the structure of the arm has no relevant consequences in the solution of the direct kinematic problem. The redundancy makes an AACMM easier to handle in the measuring process. The computation of $\mathbf{x}$ by means of Eq. (9) obviously includes the proper value n of the DoF. The kinematic redundancy of the structure of the arm is extremely relevant, on the contrary, when the inverse kinematic problem must be solved in the operation of a robotic arm.

The kinematic redundancy is also important in the identification because it allows the collection of a richer set of data.

7. Arbitrariness and redundancy of the parametric model of the arm

The model of the articulated arm described in Sections (3) and (5) contains some arbitrariness that has already been mentioned in the last part of Section (3). Furthermore the model is redundant, as it is evident from the invariance of the coordinates of the probe with respect to some change of the parameters. A proper knowledge of the various forms of arbitrariness and of redundancy is necessary to properly understand all the aspects of the model and to properly select the parameters that can be estimated in the identification procedure.

The use of coordinates, that numerically represent position of points, and the use of Cartesian components, that numerically represent vectors, require the choice of a reference frame, i.e. to fix in a more or less arbitrary manner the degrees of freedom of a rigid body. The reference frame used in the description of the model of the articulated arm is in fact all arbitrary because the coordinates generated by the AACMM must eventually be transformed into a more significant environment-related or object-related reference frame. Furthermore there are no special elements in the AACMM that can be used to identify a privileged frame.

The use of a basic pose is quite convenient to describe the geometric features of the arm in a simple, intuitive and computationally convenient manner, but it is quite evident that from a physical and geometrical point of view no privileged pose exists. The choice of the basic pose is therefore arbitrary.

The set of vectors associated to the elements of the arm (i.e the $\underline{r}_k$) are generic three-dimensional vectors. The choice to represent each link of the arm with a three-dimensional vector is quite natural in many respects and it is computationally convenient, but it implies some redundancy and some arbitrariness. The redundancy of some parametric models and the relation between a mathematical model of the chain and its physical structure are well known problems. The problems related to model redundancy and model minimality are the main concern of several authors, see e.g. the papers by Schröer et al. [15] and by He et al. [16]. The same problems are mentioned by many others including Goswami and Bosnik [17], Santolaria and Aguilar [1] and Hollerbach et al. [18].

In our model the coordinates of the probe ($\underline{x}$) are invariant to the following transformation of the vectors that describe the links:

$$\begin{aligned} \underline{r}_{(k-1),2} &= \underline{r}_{(k-1),1} - a\underline{u}_k \\ \underline{r}_{k,2} &= \underline{r}_{k,1} + a\underline{u}_k \end{aligned} \tag{10}$$

where the values of the second index $1,2$ mean *before transformation, after transformation* and a is any real number. In fact $\underline{x}$ is independent of $\underline{u}_k \cdot (\underline{r}_k - \underline{r}_{(k-1)})$. The described invariance does not affect the choice of the components of the $\underline{r}_k$, but special care is required in the selection of the components of the $\underline{r}_k$ that have to be estimated in the identification process. For the sake of completeness we note that the case of $\underline{r}_1$ is quite peculiar. The quantity $(\underline{u}_1 \cdot \underline{r}_1)\underline{u}_1$ enters in the computation of $\underline{x}$ as an additive quantity independent of the pose of the arm. Its variation is therefore equivalent to a change in the position of the origin of the reference frame which is in turn arbitrary.

In practice we have a model, expressed by Eq. (9), that contains more than the minimal number of parameters, but only some of them will be estimated in the identification process. More precisely: the estimation process will take place in a properly chosen linear subspace of the space of parameters. The use of a redundant model is convenient for many reasons:

- it allows the use of parameters that are connected to a physical description of the arm,
- it is easy to treat numerically,
- the selection of the proper subspace where the estimation takes place is defined analytically in a very general manner (see Section (8)) and it allows a quite flexible and efficient computational procedure.

8. The linearization of the analytical model and the solution of the identification problem

The identification is the computation of the geometrical parameters of the mechanical system from a set of data. The data used for these operation are the readings of the encoders and complete or partial information on the position of the probe. The data must be collected for a suitable set of points. The identification is a kind of inverse problem. Our solution is based on the least squares estimation principle and on the already defined mathematical model. The least squares principle gives optimal estimations of the parameters when redundant measurements are available. The mathematical model must be manipulated to isolate the various unknowns and it must be linearized to enter in the iterative procedure for the solution of the estimation problem.

The Eq. (8) can be easily transformed into:

$$\mathbf{x} = \mathbf{x}_0 + \sum_{j=1}^{k-1} \mathbf{p}_j \mathbf{r}_j \overline{\mathbf{p}}_j + \mathbf{p}_k \mathbf{r}_k \overline{\mathbf{p}}_k + \sum_{j=k+1}^{n} \mathbf{p}_j \mathbf{r}_j \overline{\mathbf{p}}_j \tag{11}$$

where the term $\mathbf{r}_k$ has been isolated. (We stipulate that in Eq. (11) the summations with starting index larger than final index are null. With this convention a separated treatment of first and last arm links is unnecessary). From Eq. (6) it follows: $\overline{\mathbf{p}}_j \mathbf{p}_i = \mathbf{q}_{j+1}\mathbf{q}_{j+2}\dots\mathbf{q}_i$ and $\mathbf{p}_j = \mathbf{p}_{j-1}\mathbf{q}_j$, therefore Eq. (11) becomes:

$$\mathbf{x} = \mathbf{x}_0 + \sum_{j=1}^{k-1} \mathbf{p}_j \mathbf{r}_j \overline{\mathbf{p}}_j + \mathbf{p}_{k-1}\mathbf{q}_k \left(\mathbf{r}_k + \sum_{j=k+1}^{n} \left(\prod_{l=k+1}^{j} \mathbf{q}_l \right) \mathbf{r}_j \overline{\left(\prod_{l=k+1}^{j} \mathbf{q}_l \right)} \right) \overline{\mathbf{q}}_k \overline{\mathbf{p}}_{k-1} \tag{12}$$

where the term $\mathbf{q}_k$ has also been isolated. In order to obtain a more compact notation in the following developments the quaternions $\mathbf{s}_k$ are defined by:

$$\mathbf{s}_k = \mathbf{r}_k + \sum_{j=k+1}^{n} \left(\prod_{l=k+1}^{j} \mathbf{q}_l \right) \mathbf{r}_j \overline{\left(\prod_{l=k+1}^{j} \mathbf{q}_l \right)} \tag{13}$$

and the Eq. (12) is rewritten as:

$$\mathbf{x} = \mathbf{x}_0 + \sum_{j=1}^{k-1} \mathbf{p}_j \mathbf{r}_j \overline{\mathbf{p}}_j + \mathbf{p}_{k-1}\mathbf{q}_k \mathbf{s}_k \overline{\mathbf{q}}_k \overline{\mathbf{p}}_{k-1}. \tag{14}$$

The first steps toward the construction of a linearized model suitable for identification are the choice of the parameters that have to be estimated and the linearization of Eq. (14) with respect to the same parameters. The parameters that describe the geometry of the articulated arm are $\underline{r}_k$, $\underline{u}_k$ and α_{0k}. The simplest way to define the basic pose is to fix the α_{0k}. The values of the α_{0k}, or of some other parameter related to the zero-offset of the encoders, are generally considered as the most important parameters that need to be estimated in the identification process. As a consequence their arbitrariness may seem a bit strange, but they can be arbitrarily fixed just because this is the very definition of the so called basic pose and because the parameters that remain free ($\underline{u}_k$ and $\underline{r}_k$) are able to fully represent the geometrical identification. The choice of the values of the α_{0k} is therefore a matter of convenience. It is advisable to chose a set of values that allows to obtain the approximate values of the other parameters in an easy way.

Any unknown variable can be represented as the sum of a known value denoted by $\tilde{\cdot}$ plus an unknown small variation denoted by $\delta\cdot$ as in $\underline{u}_k = \underline{\tilde{u}}_k + \delta\underline{u}_k$. Small variation means that the variations of the functions that will appear in any equation can be approximated by linear functions of the variations of the unknowns.

The vectors $\underline{u}_k$ must be estimated taking into account that they are unit vectors. The constraint $\underline{u}_k \cdot \underline{u}_k = 1$ in first order approximation becomes $\underline{\tilde{u}}_k \cdot \delta\underline{u}_k = 0$. This linearized constraint bounds $\delta\underline{u}_k$ into a two-dimensional subspace. The variation of $\underline{r}_k$ is just confined in the same subspace to avoid the singularity that would derive by the invariance of $\underline{x}$ with respect to transformation (10). To handle the variations of $\underline{u}_k$ and $\underline{r}_k$ it is convenient to use a basis $(\underline{v}_k, \underline{w}_k)$ of the two-dimensional subspace orthogonal to $\underline{u}_k$. The two vectors $\underline{v}_k$ and $\underline{w}_k$ are chosen orthogonal to $\underline{\tilde{u}}_k$, orthogonal to each other and with unitary norm. These last two conditions are not mandatory, but they are quite convenient. The variations of $\underline{u}_k$ and $\underline{r}_k$ are expressed as $\delta\underline{u}_k = a_k\underline{v}_k + b_k\underline{w}_k$ and $\delta\underline{r}_k = c_k\underline{v}_k + d_k\underline{w}_k$ respectively. Therefore we have:

$$\underline{u}_k = \underline{\tilde{u}}_k + a_k\underline{v}_k + b_k\underline{w}_k \tag{15}$$

and

$$\underline{r}_k = \underline{\tilde{r}}_k + c_k\underline{v}_k + d_k\underline{w}_k. \tag{16}$$

Equation (8) can now be linearized with respect to all the unknown variables and the result is:

$$\begin{aligned}\tilde{\mathbf{x}} + \delta\mathbf{x} = \tilde{\mathbf{x}}_0 + \delta\mathbf{x}_0 + \sum_{k=1}^{n}\tilde{\mathbf{p}}_k\tilde{\mathbf{r}}_k\tilde{\overline{\mathbf{p}}}_k + \\ + \sum_{k=1}^{n}(a_k\sin((\alpha_k - \alpha_{0k})/2)\tilde{\mathbf{p}}_{k-1}\left(\mathbf{v}_k\tilde{\mathbf{s}}_k\tilde{\overline{\mathbf{q}}}_k + \tilde{\mathbf{q}}_k\tilde{\mathbf{s}}_k\overline{\mathbf{v}}_k\right)\tilde{\overline{\mathbf{p}}}_{k-1} + \\ + b_k\sin((\alpha_k - \alpha_{0k})/2)\tilde{\mathbf{p}}_{k-1}\left(\mathbf{w}_k\tilde{\mathbf{s}}_k\tilde{\overline{\mathbf{q}}}_k + \tilde{\mathbf{q}}_k\tilde{\mathbf{s}}_k\overline{\mathbf{w}}_k\right)\tilde{\overline{\mathbf{p}}}_{k-1} + \\ + c_k\tilde{\mathbf{p}}_k\mathbf{v}_k\tilde{\overline{\mathbf{p}}}_k + d_k\tilde{\mathbf{p}}_k\mathbf{w}_k\tilde{\overline{\mathbf{p}}}_k).\end{aligned} \tag{17}$$

The decomposition of $\mathbf{x}$ as $\tilde{\mathbf{x}} + \delta\mathbf{x}$ is necessary because $\underline{x}$ appears in non linear expression in other equations that will be stated in the sequel. The similar decomposition applied to $\mathbf{x}_0$ is just to handle all the variables in the same manner, but it is not mandatory.

The practical use of Eq. (17) in the estimation process requires an analysis of the measurements that can be performed to that purpose.

It is necessary to consider some different situations that can arise when collecting the measurements to be used for the identification:

(*i*) measuring of points with already known coordinates referred to a unique reference frame,

(*ii*) measuring of several points with already known coordinates referred to a set of reference frames,

(*iii*) repeated measuring of points with unknown coordinates with different poses of the arm,

(iv) measuring of points that belong to a partially known surface.

In case (i) the most complete possible information about points are available. This is also the simplest case. Case (ii) can arise by the use of a complex calibrated reference object that is moved in different positions in space. This situation requires the use of additional parameters to account for the different positions and orientations of the reference object. This aspect is not treated analitically. In case (iii) the useful information comes from the measurement of the *same* point with *different* poses of the arm. Case (iv) is quite common. Points can be measured on the surface of a sphere. The sphere generally has unknown center and can have known or unknown radius. A quite common reference object is a dumbbell composed of two spheres with the centers at a fixed distance. Another possibility is the use of a reference object composed of many spheres connected in a more complex structure. The use of other surfaces, like cylinders, is less common.

In cases (i), (ii) and (iii) there are three significant scalar equations arising from Eq. (17). In fact the two sides of the Eq. (17) are imaginary quaternions and therefore they are equivalent to usual vectors in $\mathbb{R}^3$. Furthermore the measurements involve points that are individually identified. The linearized observation equation for a point on a sphere (case iv) can be obtained by the combination of Eq. (17) with the implicit equation of the sphere. It is necessary to introduce new parameters for the center of the sphere ($\underline{x}_s$) and for its radius (ρ). The equation of the sphere is:

$$\rho = \sqrt{(\underline{x} - \underline{x}_s) \cdot (\underline{x} - \underline{x}_s)} \tag{18}$$

which in linearized form becomes:

$$\sqrt{(\underline{\tilde{x}} - \underline{\tilde{x}}_s) \cdot (\underline{\tilde{x}} - \underline{\tilde{x}}_s)} - \tilde{\rho} = -\frac{1}{\tilde{\rho}}(\underline{\tilde{x}} - \underline{\tilde{x}}_s) \cdot (\delta\underline{x} - \delta\underline{x}_s) + \delta\rho. \tag{19}$$

The unknown coordinates of the point can be treated as nuisance parameters. Their elimination from Eq. (17) and Eq. (19) leaves just one scalar equation.

The presence of spheres with known distance between the centers require the use of pseudo-observation equations. They have the form:

$$D = \sqrt{(\underline{x}_{si} - \underline{x}_{sl}) \cdot (\underline{x}_{si} - \underline{x}_{sl})} \tag{20}$$

where D is the known distance while $\underline{x}_{si}$ and $\underline{x}_{sl}$ are the coordinates of the centers of the two spheres. Equation (20) can be linearized in the form:

$$D - \sqrt{(\underline{\tilde{x}}_{si} - \underline{\tilde{x}}_{sl}) \cdot (\underline{\tilde{x}}_{si} - \underline{\tilde{x}}_{sl})} = \frac{1}{\sqrt{(\underline{\tilde{x}}_{si} - \underline{\tilde{x}}_{sl}) \cdot (\underline{\tilde{x}}_{si} - \underline{\tilde{x}}_{sl})}}(\underline{\tilde{x}}_{si} - \underline{\tilde{x}}_{sl}) \cdot (\delta\underline{x}_{si} - \delta\underline{x}_{sl}) \tag{21}$$

and then it is used in the estimation process with a suitable overweighting.

All the considered observation equations involve coordinates and therefore a reference frame must be fixed. This can be done in many different manners. In case (i) the simplest choice is to use the reference frame that must be already defined to describe the set of known points. With this choice the reference frame is directly related to the environment and all the $3 + 4n$ parameters ($\underline{x}_0$ and $\{a_k, b_k, c_k, d_k\}$) must be estimated. Case (ii) is similar, but one of the positions and orientations of the reference object must be chosen to fix the reference frame, and a set of parameters must be considered to describe the other positions and orientations. An alternative, which is mandatory in cases (*iii*) and (*iv*), is to fix the reference frame by means of the arbitrary choice of the values of some parameters. The origin is fixed by an arbitrary value imposed to $\underline{x}_0$. The orientation can be fixed by the choice of the direction of one of the axes of the joints and the direction of one of the vectors of the links of the arm. In practice this can be done by setting $a_1 = 0$, $b_1 = 0$, $c_1 = 0$ (or $d_1 = 0$, depending on the component that is more significant for the orientation). If there are no dimensional information (as in case *iii*) one more parameter must be fixed. In practice this can be done fixing both $c_1 = 0$ and $d_1 = 0$. This is obviously just one possible choice: fixing other parameters can be more appropriate in some situations.

The set of observation equations and pseudo-observation equations form a redundant linear system, i.e. a system with more equations than unknowns. The system of equations is treated in a weighted least-squares (WLS) estimation procedure.

The linearized kinematic model can be used both for the identification of an AACMM and of a robotic arm. On the other hand all the information about the different kind of data that can be collected and about their use in the identification process can not be directly applied to a robotic arm.

9. Comments about the least squares criterion and the solution strategy

There are several different optimization criteria and solution methods that are mentioned in the literature for the identification of an AACMM or of a robotic arm. They include least squares based techniques as mentioned by Hollerbach et al. in [18], by Goswami et al. in [6] and by other authors; Simulated Annealing as mentioned by Gao et al. in [3]; Particle Swarm Optimization as mentioned by Gao et al. in [4] and Levenberg-Marquardt method as mentioned by Aguilar et al. in [2] and by many other authors. Good practical results are reported for all the mentioned techniques.

We believe that the use of weighted least-squares presents several theoretical advantages and that it is practically quite effective. The WLS can be described as an optimization criterion for the approximation of the measurement data with a geometrically based parametric model. The WLS can be rigorously treated in the framework of the probabilistic estimation theory. The WLS produce the best (minimum variance) unbiased linear estimation (BLUE). A complete treatment can be found e.g. in the book by Karl-Rudolf Koch [19].

From a practical point of view the computation of the least squares solution presents no difficulty as far as an appropriate non singular model has been defined and linearized. In the present application the iteration process itself is quite fast because the design of the instrument and a proper choice of the basic pose give a quite good set of starting values of the parameters.

The critical aspect of the application of WLS is the choice of the variance-covariance matrix of the quantities that appear as known term in the observation equations. We have not developed a general stochastic model to this purpose, but the requested variance-covariance matrix can be reasonably built by some rules that worked in all the practical situations we have met so far. We assume that there are no correlations and that the equations that arise from the same operational procedure must have the same weight. In many situations all the measurements are of the same kind and they are collected in similar conditions. In this case the computation of WLS reduces to the computation of ordinary LS, apart from the overweighting of Eq. (21).

Our careful selection of the parameter sub-space in which the optimization is carried out guarantees that the normal matrix is non-singular and no regularization is needed.

Our solution procedure is based on the analytical expression of the derivatives of the observation equation with respect to all the unknowns and on a Newton-like iteration that takes care of the non linearity. This choice is quite more effective than the use of numerical derivatives and of the use of different minimum searching algorithms.

Lastly it is important to recognize the role of the linearized model. The unknowns of the linearized equations are variations to be applied to the preliminary values of the unknowns of the original non linear equations. The procedure must therefore be iterated until it reaches the convergence. The use of the linearized equations is necessary for the numerical iterative computation of the solution, but the final result is the solution of the original non linear problem. The use of the linearized equations in the WLS estimation process does not imply any approximation of the final result.

10. More about the kinematic model of the arm

We present some more comments about the most important aspects of our model and we outline some differences with respect to other models that are presented in scientific literature and are probably well known to the readers. The comments are not meant to evaluate different merits, but just to better clarify the peculiarities of our proposal.

We have found just one model with some similarity to ours. It is described in an unpublished paper by Horn [20] in a purely theoretical manner.

The two models more often mentioned are the model proposed by Denavit and Hartenberg [21] and the model proposed by Sheth and Uicker [22]. The model proposed by Denavit and Hartenberg [21] is by far the most frequently used. Many papers and textbooks present the DH model as the de-facto standard. This is the choice of Spong, Hutchinson and Vidyasagar [23], of Hollerbach et al. [18], of Santolaria and Aguilar [1] and of many others. Some authors also notice the main drawback of the DH model, i.e. the singularity (or instability) of the model for parallel (or almost parallel) consecutive rotary axes, and mention some improved versions of the DH model; see e.g. Roth et al. [7], Hollerbach et al. [18] and Ozgoren [24]. Many other authors use the Steth-Uicker model; see e.g. the papers by Goswami et al. [6] and by Bongardt [25].

All the mentioned models describe the kinematic chain using a sequence of reference frames attached to the different parts of the mechanism. We prefer to use just one reference frame attached to (some selected part of) the mechanism and one reference frame attached to the object. The paper by Chen-Gang el al. ([26]) in fact distinguishes between models based on *local-link coordinate systems* and models based on *global coordinate systems*.

Most of the models, and specially the Denavit-Hartenberg model and its variations, make extensive use of so called homogeneous coordinates and homogeneous matrices that are the algebraic tools of projective geometry. The use of homogeneous transformations allows a unified notation to represent both translations and rotations. We prefer to stay with the more classical and elementary set-up of analytical geometry, linear algebra and vector calculus mainly because we think that our set-up is simpler. Most of the models, including the Denavit-Hartenberg model, can also treat prismatic joints. Our model can be also generalized in order to include prismatic joints.

Most of the models described in literature present some redundancy in the set of parameters. This problem is treated in various manners that include:

- the application of the Levenberg-Marquardt method that is essentially a numerical regularization of the normal system,
- the selection of some parameters to be eliminated from the identification (see e.g. Zhuang et al. [27]),
- the regularization from the spectral decomposition of the normal matrix (see e.g. Goswami and Bosnik [17]).

We have described a different procedure that is based on an invariance analysis and direct construction of a base of the subspace of the estimable parameters (see Sections 7 and 8). The resulting algorithm is both flexible and effective. All the elements of our analytical model have a clear geometrical interpretation, and this property is of great value when the redundancy must be analyzed and the singularities must be avoided.

11. Ideas for a more complete model

The kinematic model described in this article has been conceived for a specific application, i.e. the operation and the identification of an AACMM with rotary joints.

Here after we give some ideas for possible generalizations and extensions that can be useful for the application of the model to a robotic arm, for the presence of prismatic joints and to account for some deviations from the geometric and mechanical assumptions.

In an AACMM only the three coordinates of the extremal point are relevant as final output. The pose of the various links determine the end-point coordinates, but the pose itself is of no interest in the operations. On the contrary a robotic arm ends with a tool that in some cases must be considered as a rigid body to be properly placed with all its 6 DoF. This can be easily done with the described model. The tool can be considered as the last element of the chain and the expression (6) with $k = n$ gives the quaternion that describes its orientation in space.

A prismatic joins can be described using a unit vector to represent the direction of the displacement and a scalar variable to describe the amplitude of the displacement. When a prismatic joint is present the parameters to be estimated must be selected with a further analysis, which is indeed very simple.

The model can be improved with developments in three other directions: by taking into account elastic deformation under gravity, thermal deformation and by including a black box compensation of residual errors.

12. Computer programs

We have developed several computer programs based on the described mathematical model. The two main operations realized by the software system are the computation of the coordinates and the estimation of the geometric parameters. Therefore the programs solve two of the three problems described in Section (2), namely the *direct kinematic problem*, and the *identification*.

The programs are strictly based on the theory exposed in the previous Sections. Rotations are represented in three different ways:

- axis-and-angle representation is the most significant one: the angles come from the readings of the encoders and the unit vectors of the axes are part of the parameters of the AACMM;
- quaternions are used for various computations, notably to combine the sequence of rotations;
- matrix-vector multiplication is used to practically apply rotations to vectors (rotation matrices are generated on the fly from quaternions when needed).

We also implemented a purely static version of the *inverse kinematic problem*, but it is not a general solution and we used it only for simulation purposes.

Other programs take care of the absolute orientation and of some data analysis for the assessment of the results.

The program that computes the estimates is obviously the most complex. It can accept data collected in various situations that correspond to cases (*i*), (*iii*) and (*iv*) described in Section (8). In case (*iv*) we can have points on single spheres or on the spheres of a dumbbell.

We tested the programs with a large variety of simulated data before the application to real data that is reported in the next Section. Simulations include cases (*i*), (*iii*) and (*iv*), applications to real measurements only include cases (*iii*) and (*iv*).

13. Practical results

We applied the software system summarily described in the previous Section for the operation and the identification of an AACMM intended for use in the automotive industry.

The arm contains five rotary joints. The probe can terminate with different kind of ends, but in the main experiments we only used a spherical end. The probe is passive (hard probe). The manufacturer supplied two samples of the AACMM for the tests.

We performed the identification of the two measuring machines and the assessment of the attainable precision and accuracy. It must be stressed, however, that the true purpose of the tests is a demonstration of the effectiveness of the procedure and of the operational capabilities of the software.

The data necessary to perform the identification and to assess the obtained performance have been collected by measuring some points on a conic seat that allows repeated measurements of the same point and points on two reference objects: a sphere with a diameter of two inches and a dumbbell consisting of two spheres with a diameter of one inch and distance between the centers of 800 millimeters.

The values of the parameters available before the identification are obtained from the instrument design and from a reading of the encoders. The reading is performed after placing the mechanism in a particular pose with all the links aligned. The alignment is roughly evaluated by visual inspection. The chosen pose allows an easy evaluation of the direction of all the vectors. The obtained readings of the encoders are then assumed as the definition of the already mentioned basic pose.

The extremely low accuracy of the alligment obtained by visual inspection does not affect the final result for two reasons:

- the basic pose is in fact defined by frozing the values of the readings of the rotarional encoders,
- any other outcome of the alignment generates the approximate values of some parameters that are estimater later on.

The identification is computed as a least squares estimation, therefore its overall quality is represented by the ratio $\hat{\sigma}_0^2/\sigma_0^2$ where the parameter σ_0^2 is the *a-priori* value of the so called variance of unit weight and $\hat{\sigma}_0^2$ is the estimated value of the same quantity. The ratio is important because it is an indicator of anomalous situations (when it is large) and to perform quick comparisons between different identification tests. In the present case we have a reasonable yet not fully rigorous stochastic model of the data, therefore we do not perform any acceptance test based on the mentioned ratio.

A naturally appealing assessment of the quality of the identification is obtained by a test performed with data different from the data used in the identification.

In the description of the measurements and of the computations we will simply denote the two measuring machines as "instrument A" and "instrument B". The instrument A was used to collect data in 8 different sessions. The data was divided into two sets simply named dataset 1 and dataset 2. The instrument B was used to collect data in 13 different sessions and the data was again partitioned into two sets. The spatial distribution of the measured objects within the measuring volume is shown in Figs. (2), (3), (4) and (5). The measuring volume is ideally enclosed in an upright cylinder. The measured objects are projected both on the base of the cylinder and on the side surface of the cylinder. In each figure the circular scheme is a view of the base of the cylinder, while the rectangular scheme represents the unwrapped side surface of the cylinder where the up direction in the measuring space is along the short side of the rectangle and the azimuth is along the long side.

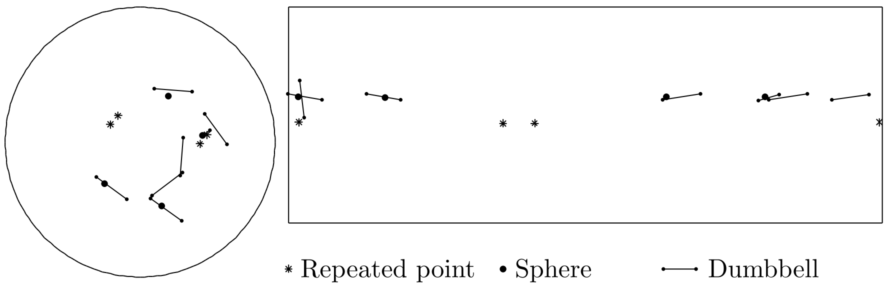

Figure 2: Dataset 1 of instrument A

The usual quality index of an estimation process is $\hat{\sigma}_0$ as we have already stated. The values of $\hat{\sigma}_0$ are reported in Table (1) for all the treated datasets along with the corresponding number of Degree of Freedom. Degree of Freedom (DoF) is intended here in the statistical meaning, i.e. the number of measures

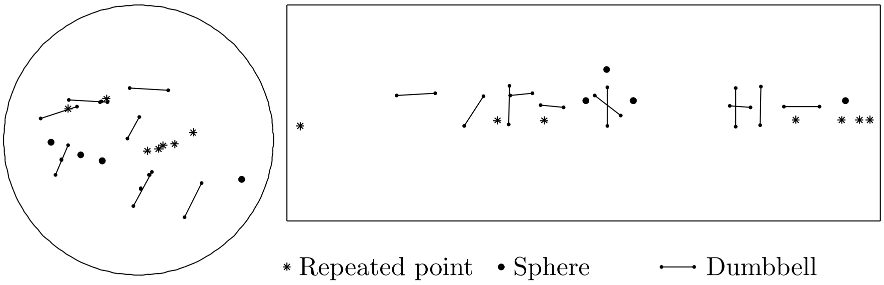

Figure 3: Dataset 2 of instrument A

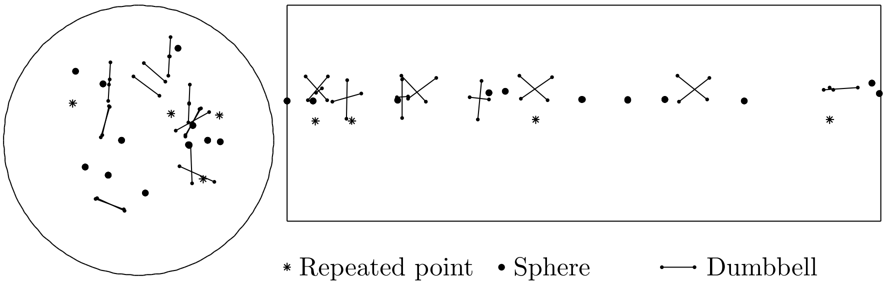

Figure 4: Dataset 1 of instrument B

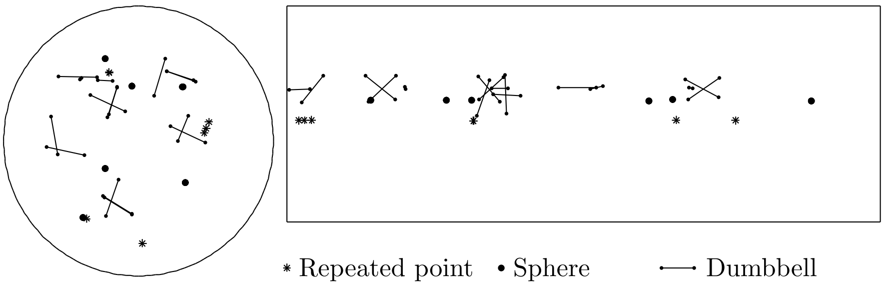

Figure 5: Dataset 2 of instrument B

in a dataset that exceed the theoretical minimum. The result seems satisfactory. The last column of Table (1) contains the dispersion index of the coordinates that can be obtained using the *a-priory* parameters of the measuring machines. These numbers are representative of the modest performance that can be achieved with the parameters obtained from the design and from some empirical identification.

Data Set	Value of $\hat{\sigma}_0$ (DoF) with **estimated** parameters	Value of $\hat{\sigma}_0$ (DoF) with **preliminary** parameters
Instrument A, dataset 1	0.48 mm (771)	38.93 mm (855)
Instrument A, dataset 2	0.55 mm (1030)	43.32 mm (1162)
Instrument A, all the data	0.54 mm (1819)	41.52 mm (2017)
Instrument B, dataset 1	0.33 mm (525)	185.49 mm (756)
Instrument B, dataset 2	0.75 mm (560)	209.37 mm (782)
Instrument B, all the data	0.63 mm (1103)	197.99 mm (1538)

Table 1: Values of $\hat{\sigma}_0$ for various data set and coddesponding number of Degree of Freedom.

A further assessment of the attained accuracy can be obtained with some tests loosely inspired by the ASME standard [28]. We determined the diameter of a sphere from the coordinates of some points measured on the surface of the sphere itself. The obtained values are compared with the nominal value of the diameter (2 inch). In a different test we determined the distance between the center of two spheres of a dumbbell. Again the obtained values are compared with the nominal value of the distance (800 mm). The datasets 1 and 2 of each instrument are combined to obtain a kind of crosscheck. The mean and the extremal values of the differences between the determined and the nominal values are reported in Tables (2) and (3). The most significant and reliable results are obtained when different data sets are used for the parameter identification and for accuracy accesment by comparison aginst nominal values. (See cases A1-A2, A2-A1, B1-B2 and B2-B1 in Tables (2) and (3).)

Dataset used for identification	Dataset used in the test	Number of measured objects	Mean difference (mm)	Max abs difference (mm)
A 1	A 1	4	0.087	0.310
A 1	A 2	9	0.170	0.760
A 2	A 1	4	0.100	0.300
A 2	A 2	9	0.130	0.620
B 1	B 1	33	0.007	0.613
B 1	B 2	29	0.120	0.825
B 2	B 1	33	0.120	1.167
B 2	B 2	29	0.294	1.060

Table 2: Mean value and maximun absolute value of the differences between the determined and the nominal diameter of a 2 inch sphere.

Dataset used for identification	Dataset used in the test	Number of measured objects	Mean difference (mm)	Max abs difference (mm)
A 1	A 1	7	0.080	0.630
A 1	A 2	11	-0.084	1.340
A 2	A 1	7	0.330	0.980
A 2	A 2	11	-0.120	1.244
B 1	B 1	17	-0.019	0.807
B 1	B 2	16	-0.191	1.110
B 2	B 1	17	0.698	2.740
B 2	B 2	16	-0.068	1.860

Table 3: Mean value and maximun absolute value of the differences between the determined and the nominal distance of the centers of the spheres of a dumbbell.

14. Final remarks, conclusions and perspectives

We have presented an analytical model of a kinematic chain that is, to our knowledge, quite original. In the identification and estimation process we used some knowledge and some mathematical tools that are typical (although not exclusive) of geodesy: application of the least-squares criterion, direct linearization, Newton-like iterations to account for the non-linearity and proper selection of the estimable parameters to avoid any singularity.

The practical results obtained in the identification of two measuring machines are completely satisfactory, because they show the validity of the mathematical model and of the designed procedures and the operational capabilities of the software.

The work described in this paper can be completed and developed in many directions. On the theoretical side our model of the kinematic chain deserves a more in-depth comparison with other available models. On the practical side we plan to test the procedure with different reference objects and with different measuring machines. The overall procedure can be improved by the joint use of our model and a black-box model of the residual errors.

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