

Inflation in scale-invariant theories of gravity

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Abstract

Thanks to the data from Planck Collaboration, the scalar spectral index of primordial fluctuations is known with a very high accuracy. The recent findings of the BICEP2 collaboration, although still under scrutiny, fix also the ratio between the tensor and the scalar power spectra to a value that implies a non-negligible production of gravitational waves in the inflationary Universe. In this letter we show that purely quadratic, renormalizable, and scale-invariant gravity, implemented by loop-corrections, yields very precise predictions when compared to these data. In addition, this model naturally exits inflation towards a standard reheating phase. In contrast to other scale-invariant models, our scenario does not need matter fields coupled to gravity to explain inflation.

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INTRODUCTION

The claim of the BICEP2 collaboration [1] that the tensor-to-scalar ratio has the surprisingly large value $r = 0.2$, which points to a flurry production of gravitational waves during inflation, has polarized the attention of the physics community. The situation has become less clear when some serious criticisms to the BICEP2 analysis appeared in the literature (see for example [2]). Furthermore, very recently, the Planck collaboration has released the data concerning the polarized dust emission [3], while the first attempts to make a joint analysis of Planck and BICEP2 data have been presented (see, for example, [4, 5] and references therein). The general feeling is that the value of r is lower than initially claimed in [1] but still significant.

In a previous paper, we considered cosmic inflation in the context of $f(R)$ gravity and we showed that if the spectral index n_s and r are known, the functional form of $f(R)$ can be uniquely determined [6]. In particular, we have shown that the slow-roll condition during inflation implies that $f(R) \sim R^{2-\delta}$, where δ is a small and weakly time-dependent parameter. These results seem therefore to indicate that, at least during inflation, gravity is best described by a scale-invariant, quadratic Lagrangian, with small deviations that we interpreted as quantum gravitational loop corrections. This is in contrast with the Starobinsky model [7], where the quadratic term is implemented by the usual linear term of general relativity, so that the classical action is no longer scale invariant.

In this Letter, we take inspiration from this work and we study a specific effective Lagrangian of the form $R^2 +$ loop corrections, we calculate r , n_s , and its running and compare the results with the available data. We find that our model reproduces very accurately the observed spectral indices and predicts a value of r that might be in line with the combined BICEP2 and Planck findings. We also show that the Universe evolves smoothly out of inflation into a radiation-dominated phase and, therefore, that there is no need of non-gravitational fields to describe the early Universe, from inflation to reheating.

THE SCALE INVARIANT R^2 MODEL

To begin with, let us recall the most general scale-invariant Jordan frame action containing the square of the Ricci scalar, the Weyl invariant, and the Higgs doublet \mathcal{H} , and a non-

minimal coupling between the two:

$$S_J = \int d^4x \sqrt{g} \left[bR^2 + \xi R\mathcal{H}^2 - (\partial\mathcal{H})^2 - \frac{\lambda}{4!} \mathcal{H}^4 + \dots \right]. \quad (1)$$

The dots stand for the other quadratic invariants of the metric and scale invariant operators of the standard model. Here, the parameters b , ξ , and λ are all dimensionless. This action has been thoroughly investigated in the past, see e.g. [8]. Recently, it was reconsidered in [9], where it was shown that this model leads to an inflationary model consistent with observations, provided one adds a new scalar field degree of freedom and takes in account the running of b , ξ , and λ . In addition, this model is particularly attractive as it is believed to be renormalizable and asymptotically free [10–13], although ghosts are in general present (for a review, see [14]).

In this paper, we take a different look at this action and we wish to show that inflation and reheating do not need the matter sector of the theory. In fact, they can be realized exclusively by the gravitational sector, identified by just the first term of eq. (1), implemented by suitable loop corrections.

In Ref. [6], we showed that the effective, one-loop corrected effective Lagrangian takes the form

$$f_{\text{eff}}(R) = \frac{R^2}{\mu} \left[1 - \gamma \ln \left(\frac{R^2}{\mu^2} \right) \right], \quad (2)$$

which was derived upon the results presented in [16]. In this expression, γ is a small positive parameter and μ is a constant that fixes the scale at which the logarithmic correction comes into play (see also [17], and, for the asymptotic safety approach, [18]). With some approximations, it was shown that this model predicts values of the spectral index n_s and of the tensor-to-scalar ratio r that are in line with the combined data of Planck and BICEP2.

Here, we would like to take a step further by assuming that all loop corrections in the gravitational sector of the effective Lagrangian can be resummed so that f_{eff} becomes, mimicking higher loops,

$$f_{\text{eff}}(R) = \frac{R^2}{\mu \left[1 + \gamma \ln \left(\frac{R^2}{\mu^2} \right) \right]}. \quad (3)$$

The scale μ^{-1} , which has the dimension of a squared mass, makes the functional derivative of f_{eff} with respect to R conveniently dimensionless. Note that when the logarithmic term is

small, the expressions (2) and (3) are basically the same, thus we are not changing the one-loop corrected Lagrangian dramatically and the results below are in fact valid for both cases. As we will shortly see, the surprising feature of (3) is that it yields an inflationary phase such that the spectral index, its running, and the tensor-to-scalar ratio depend exclusively on the number of e-foldings. The only constraints on γ and μ comes from the measured amplitude of the power spectrum.

INFLATION IN $f(R)$ THEORIES

In order to obtain the inflationary observables, we introduce a simple and transparent formalism that is valid for all $f(R)$ theories. Let us consider the generic action in Jordan frame (for reviews on $f(R)$ gravity see e.g. [19–21])

$$S_J = \int d^4x \sqrt{|g|} f(R). \quad (4)$$

Our goal is to express the usual inflationary observables in both Einstein and Jordan frame in a simple and universal form. The equations of motion for a homogeneous and isotropic Universe with metric $ds^2 = -dt^2 + a^2 d\vec{x}^2$ are

$$3XH^2 = \frac{1}{2}(XR - f) - 3H\dot{X}, \quad (5)$$

$$\ddot{X} = -2X\dot{H} + H\dot{X}, \quad (6)$$

where the dot represents a derivative with respect to the (Jordan frame) cosmic time t , $H = a^{-1}\dot{a}$ is the Hubble function, $R \equiv 6(2H^2 + \dot{H})$, and $X \equiv df(R)/dR$. The conformal transformation $\tilde{g}_{\mu\nu} = Xg_{\mu\nu}$ brings the action (4) into the canonical form in Einstein frame

$$S_E = \int d^4x \sqrt{|\tilde{g}|} \left[\frac{M^2}{2} \tilde{R} - \frac{1}{2} (\tilde{\partial}\tilde{\phi})^2 - V(\tilde{\phi}) \right], \quad (7)$$

where

$$V(\tilde{\phi}) = \frac{M^2}{2} \left(\frac{XR - f(R)}{2X^2} \right), \quad (8)$$

and X and $\tilde{\phi}$ are related by

$$\tilde{\phi} = \sqrt{\frac{3}{2}} M \ln(X). \quad (9)$$

Usually, the parameter M is identified with the Planck mass m_p under the hypothesis that the action (7) describes also the low-energy limit of the theory. However, since we will deal with the scale-invariant Lagrangian (3), this identification is not strictly speaking justified. Nevertheless, for now we keep a conservative point of view by setting $M = m_p$ and we will comment below on alternative choices. If $X(R)$ is positive definite and invertible, we can always write a derivative with respect to $\tilde{\phi}$ in terms of a derivative with respect to R . In particular, we can express the slow-roll parameters as (the prime indicates a functional derivative with respect to R)

$$\begin{aligned}\epsilon &= \frac{M^2}{2} \left(\frac{d \ln(V)}{d\tilde{\phi}} \right)^2 = \frac{(XR - 2f)^2}{(XR - f)^2}, \\ \eta &= \frac{M^2}{V} \frac{d^2 V}{d\tilde{\phi}^2} = \frac{2(XR - 4f)}{3(XR - f)} + \frac{2X^2}{3(XR - f)X'}, \\ \xi^2 &= \frac{M^2}{V^2} \frac{dV}{d\tilde{\phi}} \frac{d^3 V}{d\tilde{\phi}^3} = \frac{4(XR - 2f)(X^3 X'' + X'^3 X R - 8X'^3 f + 3X^2 X'^2)}{9X'^3 (XR - f)^2},\end{aligned}\tag{10}$$

from which we construct the spectral index, its running, and the tensor-to-scalar ratio defined as

$$n_s = 1 - 6\epsilon + 2\eta, \quad \frac{dn_s}{d \ln k} = 16\epsilon\eta - 24\eta^2 - 2\xi^2, \quad r = 16\epsilon.\tag{11}$$

With the help of the definition (9), we can also define the number of e-folding as a function of $\tilde{\phi}$ or R according to

$$N(\tilde{\phi}) = \frac{1}{M^2} \int V \left(\frac{dV}{d\tilde{\phi}} \right)^{-1} d\tilde{\phi} = \frac{3}{2} \int \frac{V}{V'} \frac{X'^2}{X^2} dR.\tag{12}$$

We stress that these formulae are valid for any $f(R)$ theory and hold whenever $X(R)$ is positive definite and invertible.

GLOBAL DYNAMICS

Before turning to inflation, It is useful to recall well-known properties of the classical, scale-invariant Lagrangian $L = b\sqrt{|g|}R^2$ in the physical Jordan frame. For this theory, the equations of motion (5)-(6) reduce to the single equation

$$2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} = 0,\tag{13}$$

which has only two exact solutions, namely $H = \text{const}$, which corresponds to a de Sitter space with arbitrary cosmological constant, and $H = (2t)^{-1}$, which describes a radiation-dominated Universe with $R = 0$. Upon quantization, it is also known that this theory has no ghosts [15].

If we consider loop corrections as in (3), it is easy to see from the equations of motion that (6) identically vanishes when R (and so H) is a constant. Then, (5) determines the curvature, yielding $R_{\text{div}} = \mu \exp\left(-\frac{1}{2\gamma}\right)$, where the subscript indicates that, at that value of R , $f_{\text{eff}}(R)$ diverges, see (3). Thus, the system formally approaches a de Sitter vacuum when $R \rightarrow R_{\text{div}}$. On the other hand, also $H = 1/(2t)$ is an asymptotic solution for large t . To see this it is sufficient to note that, since $R \rightarrow 0$ in this case, all terms of the type $R \ln R$ vanish and the equations of motion reduce again to the expression (13). These qualitative features can be easily checked by solving numerically the full system of equations, which confirms that the solution to the equations of motion for the model (3) smoothly connect a (quasi) de Sitter space to a radiation-dominated Universe.

INFLATIONARY OBSERVABLES

We now turn to the inflationary phase. With the help of eqs. (10) and (11), we find that

$$\begin{aligned} \epsilon &= \frac{4\gamma^2}{3(1 + \gamma z - 2\gamma)^2}, & \eta &= -\frac{16\gamma^3}{3(1 + \gamma z - 2\gamma) [(1 + \gamma z)^2 - 3\gamma(1 + \gamma z) + 4\gamma^2]}, \\ \xi^2 &= \frac{128\gamma^5 [2 + (2z^3 - 3z^2 - 3z + 8)\gamma^3 + (6z^2 - 6z - 3)\gamma^2 + (6z - 3)\gamma]}{9 [1 + (z^2 - 3z + 4)\gamma^2 + (2z - 3)\gamma]^3 [1 + (z - 2)\gamma]^2}. \end{aligned} \quad (14)$$

where we set $z = \ln(R^2/\mu^2)$.

Conventionally, inflation ends at the largest value of $\tilde{\phi}$ (or the corresponding R) such that $(\epsilon = 1, |\eta| = 1)$. By inspection of the first two equations above, we find that the relation between the largest roots is $R(\epsilon = 1) \simeq 0.991 \times R(|\eta| = 1)$ *independently* of the value of γ . Therefore, we fix the end of inflation at $|\eta| = 1$, which corresponds to

$$z_{\text{end}} = \frac{5}{3} - \frac{1}{\gamma} - \frac{(15\sqrt{29} - 80)^{2/3} - 5}{3(15\sqrt{29} - 80)^{1/3}} \simeq 3.1727 - \frac{1}{\gamma}. \quad (15)$$

With $f(R)$ of the form (3), we can integrate eq. (12) and the result, in terms of z , reads

$$N(z) = \frac{3z^2}{16} - \frac{3z}{2} + \frac{3z}{8\gamma} + \frac{3}{4} \ln \left[\frac{(1 + \gamma z)^4}{(1 + \gamma z - \gamma)} \right]. \quad (16)$$

At a given number N^* of e-folding before the end of inflation, when the relevant scales exit the horizon, the corresponding value of z_{ex} is implicitly determined by the equation

$$N(z_{\text{ex}}) - N(z_{\text{end}}) = N^*. \quad (17)$$

The spectral index, its running, and the tensor-to-scalar ratio are finally obtained numerically by inserting z_{ex} in the expressions (14) and (11). One surprising characteristic is that the results do not depend on γ but only on N^* . In table I, we report the numerical values of n_s , r , and $dn_s/d \ln k$ for a range of N^*

N^*	n_s	r	$dn_s/d \ln k$
40	0.9661	0.084	-0.0008
45	0.9697	0.075	-0.0007
50	0.9727	0.068	-0.0005

TABLE I: Values of n_s , its running, and r corresponding to three values of the number of e-foldings before the end of inflation.

We note that, in order to fit the experimental value $n_s = 0.9603 \pm 0.0073$ [22] we need to take a number of e-fold which is lower than the standard interval $50 < N^* < 60$. However, it is known that for non-polynomial (in Einstein frame) models of inflation such a range can be safely extended [23]. We also note that the tensor-to-scalar ratio is about ten times larger than the one predicted by the Starobinsky model [27] and it might fit the combined Planck-BICEP2 experimental value. The running of the spectral index is negative but quite low compared to other models and to the Planck result $dn_s/d \ln k = -0.015$ which is, however, uncertain up to 1.5σ [22]. Although n_s , $dn_s/d \ln k$, and r are independent of γ and μ , the amplitude A_s of the power spectrum of the curvature perturbations is not. In our model (with the assumption that M is the same as the Planck mass) we find the expression

$$A_s = \frac{V}{24\pi^2 M^4 \epsilon} = \frac{\mu(1 + \gamma z)^2 (1 + \gamma z - 2\gamma)^3}{512 M^2 \pi^2 \gamma^2 (1 + \gamma z - \gamma)^2}, \quad (18)$$

which must be evaluated at the horizon exit $z = z_{\text{ex}}$. By assuming the typical value $A_s \simeq 10^{-12}$, we find that $\sqrt{\mu}/M \simeq 10^{-6}/\sqrt{\gamma}$. The parameter γ is assumed to be a small number, and only when $\gamma \sim 10^{-12}$ the mass scale $\sqrt{\mu}$ approaches the value of the Planck mass M . With the help of eqs. (9) and (17), we can write $\tilde{\phi}$ at a generic N^* as

$$\frac{\tilde{\phi}^*}{M} = f(N^*) - \frac{\sqrt{6}}{2} \left(\ln \gamma + \frac{1}{2\gamma} \right), \quad (19)$$

where the first term is a complicate function of N^* only. If, for example, we require that the value of $\tilde{\phi}$ at horizon exit is of the order of $5M$, as in the Starobinsky model [7], we find that, for $N^* = 40$, $\gamma \simeq 0.087$ in line with the requirement that $\gamma \ll 1$. In Jordan frame, this value corresponds to $R_{\text{ex}} \simeq 3 \times 10^{-8} M^2$ and to a Hubble parameter that can be estimated to be of the order of $H_{\text{ex}} = \sqrt{R_{\text{ex}}/12} \simeq \sqrt{2\mu} = 5 \times 10^{-6} M$, similarly to the Starobinsky model. This shows that our model can be compared to the Starobinsky one in terms of energy scales and spectral index.

REHEATING

After the end of inflation, marked by the condition $|\eta| = 1$, our model of the early Universe enter a radiation-dominated phase, as explained above. As it is well-known, the transition from inflation to radiation domination can trigger a conspicuous particle production [24] and reheat the Universe. The details of the reheating phase are strongly model-dependent and go beyond the scope of this latter. However, we notice that the “graceful” exit in our case differs from most $f(R)$ models. In general, these are built in such a way that at the end of inflation a term linear in R takes over so that the effective potential has a minimum, where the scalaron can oscillate and ignite the reheating phase, as in the Starobinsky model [25]. In our case such a linear term is not present and reheating is entirely due to the smooth transition from a de Sitter to a conformally invariant metric.

There is however another possibility to mention, namely that the reheating is induced by the matter part of the Lagrangian (1), which includes the linear (in R) term $\xi R\mathcal{H}^2$. If we assumes that, at the end of inflation, the Higgs field stays “frozen” in some vacuum state, and that the term $\xi R\mathcal{H}^2$ is of the order of bR^2 , then reheating can take place along the lines of the Starobinsky model. Whether this mechanism can compete with the transition described above is an interesting and open question.

CONCLUSIONS

In this letter we entertained the idea that the early Universe can be entirely described by a purely quadratic gravitational effective theory, provided loop corrections are taken in account. In fact, these are crucial as they allow to smoothly patch two classical solutions of

quadratic gravity, namely the de Sitter space and the radiation-dominated Universe. During the (quasi) de Sitter phase, the predicted tilt of the spectrum of scalar perturbations matches the Planck data. Our model also predict a non-negligible gravitational wave production with a spectral index in between Planck and BICEP2 measurements. The outcome of the ongoing joint analysis of the two data set will be a crucial test for our proposal.

It is worth noticing that our model is not, in principle, affected by a transplanckian problem. It is know, from the Lyth bound [26], that in single-field inflation, in order to have a non-negligible value of r , one needs a planckian excursion of the inflaton field, according to $\Delta\phi/m_p \sim \int_0^{N^*} dN \sqrt{r}$. If future data analysis will confirm that r is of the order of 10^{-1} it will be difficult to claim that quantum gravitational effects should not be taken in account in these models. In our case, the situation is different as the scale M that appears in (9) should not in general be identified with the Planck mass. Thus, the expression $\Delta\phi/m_p$ implicitly contains the ratio M/m_p and the Lyth bound can be relaxed. This is consistent with the fact that the classical part of our model is scale-invariant and physical scales appear only during reheating, when standard model particles emerge.



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