

INFINITE DIMENSIONAL OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE AND APPLICATIONS TO HIGH ORDER HEAT-TYPE EQUATIONS

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ABSTRACT. The definition of infinite dimensional Fresnel integrals is generalized to the case of polynomial phase functions of any degree and applied to the construction of a functional integral representation of the solution to a general class of high order heat-type equations.

Key words: Infinite dimensional integration, partial differential equations, representations of solutions.

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1. INTRODUCTION

Functional integration is a powerful tool for the study of dynamical systems [34]. The main example is the celebrated Feynman-Kac formula (2), which provides a probabilistic representation of the solution to the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) - V(x)u(t, x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), \end{cases} \quad (1)$$

in terms of the expectation with respect to the distribution of the Wiener process W starting at x (see, e.g. [24]),

$$u(t, x) = \mathbb{E}^x [e^{-\int_0^t V(W(s)) ds} u_0(W(t))]. \quad (2)$$

Formula (2) can be established under rather mild requirements on the potential V and the initial datum u_0 (see, e.g. , [34]) and provides an important instrument in the study of heat equation and its solutions.

More generally, an extensively developed theory relates stochastic processes with the solution to parabolic equations associated to second-order elliptic operators [14]. However, that theory cannot be applied

to more general PDEs such as, for instance, the Schrödinger equation

$$\begin{cases} i\frac{\partial}{\partial t}u(t, x) = -\frac{1}{2}\Delta u(t, x) + V(x)u(t, x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d \\ u(0, x) = u_0(x) \end{cases} \quad (3)$$

describing the time evolution of the state of a nonrelativistic quantum particle, or also heat-type equations associated to high-order differential operators, such as for instance

$$\frac{\partial}{\partial t}u(t) = -\Delta^2 u(t) - V(x)u(t, x). \quad (4)$$

Indeed, a Markov process $\{X(s) : 0 \leq s \leq t\}$ playing the same role for Eq. (3) or Eq. (4) as the Brownian motion for the heat equation doesn't exist. Hence there is no "generalized Feynman Kac formula"

$$\begin{aligned} u(t, x) &= \mathbb{E}^x[e^{-\int_0^t V(X(s))ds} u_0(X(t))] \\ &= \int_{\mathbb{R}^{[0,t]}} e^{-\int_0^t V(\omega(s))ds} u_0(\omega(t)) dP(\omega), \end{aligned} \quad (5)$$

representing the solution of Eq. (3) or Eq. (4) in terms of a (Lebesgue type) integral with respect to a probability measure P on $\mathbb{R}^{[0,t]}$ associated to the process $X(s)$.

Contrarily to the heat equation case, for both Eq. (3) and Eq. (4) the fundamental solution $G_t(x, y)$ is not real and positive, even in the simplest case $V \equiv 0$. In particular the Green function $G_t(x, y)$ of the Schrödinger equation is complex, while for the high-order heat-type equation (4) $G_t(x, y)$ is real and attains both positive and negative values [19]. Therefore it cannot be interpreted as the density of a transition probability measure. As a troublesome consequence, the complex (resp. signed) finitely-additive measure μ on $\Omega = \mathbb{R}^{[0,t]}$ defined on the algebra of "cylinder sets" $I_k \subset \Omega$ (where $\Omega \equiv \mathbb{R}^{[0,+\infty)}$) of the form

$$I_k := \{\omega \in \Omega : \omega(t_j) \in [a_j, b_j], j = 1, \dots, k\}, \quad 0 < t_1 < t_2 < \dots < t_k,$$

by

$$\mu(I_k) = \int_{a_1}^{b_1} \dots \int_{a_k}^{b_k} \prod_{j=0}^{k-1} G_{t_{j+1}-t_j}(x_{j+1}, x_j) dx_1 \dots dx_k, \quad (6)$$

doesn't extend to a corresponding σ -additive measure on the generated σ -algebra. As a matter of fact, if this measure existed, it would have infinite total variation.

This problem was addressed in 1960 by Cameron [13] for the Schrödinger equation and by Krylov [25] for Eq. (4). These results may be viewed as particular cases of a general theorem later established by E. Thomas [35], extending Kolmogorov existence theorem to limits of projective

systems of signed or complex measures, instead of probability ones. In fact, these no-go results forbid a functional integral representation of the solution of Eq. (3) or Eq. (4) in terms of a Lebesgue-type integral with respect to a σ -additive complex or signed measure with finite total variation. Consequently, the integral appearing in the generalized Feynman-Kac formula (5) has to be thought in a weaker sense. One possibility is the definition of the “integral” in terms of a linear continuous functional on a suitable Banach algebra of “integrable functions”, in the spirit of Riesz-Markov theorem, that provides a one-to-one correspondence between complex bounded measures (on suitable topological spaces X) and linear continuous functionals on $C_\infty(X)$ (the continuous functions on X vanishing at ∞).

Referring to Schrödinger equation, this issue has been extensively studied, producing a number of different mathematical definitions of Feynman path integrals (see [30] for an account). We mention in particular for future reference the *Parseval approach*, introduced by Itô [22, 23] in the 60s and developed in the 70s by S. Albeverio and R. Hoegh-Krohn [2, 3], and by D. Elworthy and A. Truman [15].

Dealing with the parabolic equation (4) associated to the bilaplacian, various formulations have been proposed. One of the first was introduced by Krylov [25] and extended by Hochberg [19]. Defining a suitable stochastic pseudo-process whose transition probability function is not positive definite, the authors realized formula (5) in terms of the expectation with respect to a signed measure on $\mathbb{R}^{[0,t]}$ with infinite total variation. That is the reason why the integral in (5) is not defined in Lebesgue sense, but is meant as the limit of finite dimensional cylindrical approximations [7]. It is worthwhile mentioning the work by D. Levin and T. Lyons relying on the “rough paths” theory. Indeed, in [27] the authors conjecture that the signed measure (with infinite total variation) associated to the Krylov-Hochberg pseudo-process could become finite if defined on a certain quotient space on the path space (two path paths are equivalent if they differ for reparametrization).

A different approach was proposed by Funaki [16] and continued by Burdzy [10]. It is based on the construction of a complex-valued stochastic process with dependent increments, obtained by composing two independent Brownian motions. In [16], formula (5) with $V = 0$ is realized as an integral with respect to a well defined positive probability measure on a complex space for a suitable class of analytic initial data u_0 at least. These results have been further developed in [16, 20, 32] and are related to Bochner’s subordination theory [9]. Complex-valued processes, related to PDEs of the form (4), were also proposed by other

authors exploiting various techniques [11, 28, 12, 33]. A new construction for the solution of a general class of high order heat-type equations has been recently proposed, where formula (5) has been realized as limit of expectations with respect to a sequence of suitable random walks in the complex plane [8].

We also mention a completely different approach proposed by R. Léandre [26], which shares some analogies with the mathematical construction of Feynman path integrals with the white-noise-calculus approach [17]. It is worthwhile remarking that most of the results appearing in the literature are restricted to the cases where either $V = 0$ or V is linear. The construction of a generalized Feynman-Kac type formula is still lacking for the solution of high-order heat-type equations similar to (4) with a more general V .

This work aims to construct a Feynman-Kac formula for the solution of a general class of high-order heat-type equations of the form

$$\frac{\partial}{\partial t}u(t, x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p}u(t, x) + V(x)u(t, x), \quad t \in [0, +\infty), x \in \mathbb{R}, \quad (7)$$

where $p \in \mathbb{N}$, $p > 2$, $\alpha \in \mathbb{C}$ is a complex constant and $V : \mathbb{R} \rightarrow \mathbb{C}$ a continuous bounded function Fourier transform of a complex Borel measure on \mathbb{R} .

Adopting the *Fresnel integral* formulation of the mathematical definition of Feynman path integrals [3, 2], we introduce infinite dimensional Fresnel integrals with polynomial phase, generalizing the existing results valid for quadratic phase functions. If the phase function is an homogeneous polynomial of order p , we show in particular how this new kind of functional integral is related to the fundamental solution of Eq. (7) with $V \equiv 0$. This relation will be eventually exploited in the proof of a functional integral representation of the solution of Eq. (7), for a suitable class of potentials V and initial data u_0 , giving rise to a new type of generalized Feynman-Kac formula.

In section 2, a detailed study of the fundamental solution of Eq. (7) takes place in the case $V = 0$. In section 3, we introduce the definition of infinite dimensional Fresnel integral with polynomial phase function showing that a particular example is related to the PDE (7) with $V \equiv 0$. In section 4, we build up a representation of the solution of (7) with $V \neq 0$ in terms of an infinite dimensional Fresnel integral.

2. THE FUNDAMENTAL SOLUTION OF HIGH-ORDER HEAT-TYPE EQUATIONS

Let us consider the p -order heat-type equation:

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t, x) \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, t \in [0, +\infty) \end{cases} \quad (8)$$

where $p \in \mathbb{N}$, $p \geq 2$, and $\alpha \in \mathbb{C}$ is a complex constant. In the following we shall assume that $|e^{\alpha t x^p}| \leq 1$ for all $x \in \mathbb{R}$ and $t \in [0, +\infty)$. In particular, if p is even this condition is fulfilled if $\operatorname{Re}(\alpha) \leq 0$, while if p is odd then α will be taken purely imaginary.

In the case where $p = 2$ and $\alpha \in \mathbb{R}$, $\alpha < 0$, we obtain the heat equation, while for $p = 2$ and $\alpha = i$ Eq. (8) is the Schrödinger equation. Since both cases are extensively studied, in the following we shall mainly focus ourselves on the case where $p \geq 3$.

Let $G_t^p(x, y)$ be the fundamental solution of Eq.(8). Given an initial datum u_0 belonging to the space $S(\mathbb{R})$ of Schwartz test functions, the solution of the Cauchy problem (8) is given by:

$$u(t, x) = \int_{\mathbb{R}} G_t^p(x, y) u_0(y) dy. \quad (9)$$

In particular the following equality holds:

$$G_t^p(x - y) = g_t^p(x - y),$$

where $g_t^p \in S'(\mathbb{R})$ is the Schwartz distribution defined by the Fourier transform

$$g_t^p(x) := \frac{1}{2\pi} \int e^{ikx} e^{\alpha t k^p} dk, \quad x \in \mathbb{R}. \quad (10)$$

The following lemmas state some regularity properties of the distribution g_t^p that will be used in the next section.

Lemma 1. *The tempered distribution (10) is a C^∞ function.*

Proof. A priori g_t^p is an element of $S'(\mathbb{R})$, the Schwartz space of distribution, but we shall prove that g_t^p is a C^∞ function defined by an absolutely convergent Lebesgue integral. This can be easily proved in the case where p is even and $\operatorname{Re}(\alpha) < 0$, since the function $k \mapsto e^{\alpha t k^p}$ is an element of $L^1(\mathbb{R})$.

In the case where $\operatorname{Re}(\alpha) = 0$, i.e. $\alpha = ic$ with $c \in \mathbb{R}$, the function $k \mapsto e^{\alpha t k^p}$ is not summable. Let us denote by $\psi \in S'(\mathbb{R})$ the tempered distribution defined by this map and by $\chi_{[-R, R]}$ the characteristic function of the interval $[-R, R] \subset \mathbb{R}$. By the convergence of $\chi_{[-R, R]}\psi$ to ψ

in $S'(\mathbb{R})$ as $R \rightarrow +\infty$ and the continuity of the Fourier transform as a map from $S'(\mathbb{R})$ to $S'(\mathbb{R})$ we have that

$$g_t^p = \hat{\psi} = \lim_{R \rightarrow +\infty} \widehat{\chi_{[-R,R]}\psi}.$$

On the other hand, by a change in the integration path in the complex k -plane, in the case where p is even and $c > 0$ we have:

$$\begin{aligned} g_t^p(x) &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{ikx} e^{ictk^p} dk = \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_0^R (e^{ikx} + e^{-ikx}) e^{ictk^p} dk \\ &= \lim_{R \rightarrow \infty} \frac{e^{i\pi/2p}}{2\pi} \int_0^R (e^{ie^{i\pi/2p}kx} + e^{-ie^{i\pi/2p}kx}) e^{-ctk^p} dk \\ &= \frac{e^{i\pi/2p}}{2\pi} \int_{\mathbb{R}} e^{ie^{i\pi/2p}kx} e^{-ctk^p} dk, \end{aligned} \quad (11)$$

while in the case where p is even and $c < 0$:

$$g_t^p(x) = \frac{e^{-i\pi/2p}}{2\pi} \int_{\mathbb{R}} e^{ie^{-i\pi/2p}kx} e^{-ctk^p} dk. \quad (12)$$

In the case p is odd, a different integration contour in the complex k -plane yields the following representation:

$$\begin{aligned} g_t^p(x) &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{ikx} e^{ictk^p} dk \\ &= \frac{1}{2\pi} \int_{\mathbb{R}+i\eta} e^{ixz} e^{ictz^p} dz \end{aligned} \quad (13)$$

where $\eta > 0$ if $c > 0$ while $\eta < 0$ if $c < 0$. The integrand in the second line of (13) is absolutely convergent since $|e^{ict(\operatorname{Re}(z)+i\eta)^p}| \sim e^{-c\eta(\operatorname{Re}(z))^{p-1}}$ as $|\operatorname{Re}(z)| \rightarrow \infty$.

Eventually representations (11), (12) and (13) show that g_t^p is a C^∞ function of the variable x . \square

Remark 1. *The proof of lemma (1) shows that $g_t^p : \mathbb{R} \rightarrow \mathbb{C}$ can be extended to an entire analytic function of $z \in \mathbb{C}$. The analyticity of g_t^p follows by the application of Fubini's and Morera's theorems.*

Remark 2. *A formula similar to (11) has also been proved in [4] and applied to the study of some asymptotic properties of finite dimensional Fresnel integral with polynomial phase function.*

The following lemma relies on the study of the detailed asymptotic behaviour of $g_t^p(x)$ for $x \rightarrow \infty$.

Lemma 2. *The function g_t^p is bounded. In particular if p is even and $\operatorname{Re}(\alpha) < 0$ then $g_t^p \in L^1(\mathbb{R})$.*

Proof. By lemma 1 the function g_t^p is continuous, hence the proof of its boundedness can be based only on the study of its asymptotic behavior for $x \rightarrow \infty$. This task is accomplished by means of the stationary phase method [31, 21].

For $x \rightarrow +\infty$, a change of variables in (10) gives:

$$g_t^p(x) = \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{x^{p/p-1}(i\xi + \alpha t \xi^p)} d\xi = \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{x^{p/p-1}\phi(\xi)} d\xi, \quad (14)$$

$\phi : \mathbb{R} \rightarrow \mathbb{C}$ being the complex phase function

$$\phi(\xi) = i\xi + \alpha t \xi^p, \quad \xi \in \mathbb{R}.$$

If either $\operatorname{Re}(\alpha) \neq 0$ or p is odd and $\alpha = ic$, with $c \in \mathbb{R}^+$, then the phase function ϕ has no stationary points on the real line, i.e. there are no real solutions of the equation $\phi'(\xi) = 0$. In this cases an integration by parts argument yields:

$$\int e^{x^{\frac{p}{p-1}\phi(\xi)} d\xi = \int \frac{1}{x^{\frac{p}{p-1}\phi'(\xi)} \frac{d}{d\xi} e^{x^{\frac{p}{p-1}\phi(\xi)} d\xi = \frac{1}{x^{\frac{p}{p-1}}} \int e^{x^{\frac{p}{p-1}\phi(\xi)} \frac{\phi''(\xi)}{(\phi'(\xi))^2} d\xi.$$

By iterating this procedure we obtain that for all $N \in \mathbb{N}$:

$$g_t^p(x) \underset{x \rightarrow +\infty}{\ll} (x^{\frac{p}{p-1}})^{-N}.$$

In the case where $\alpha = ic$ with $c \in \mathbb{R}$, Eq. (14) can be written as

$$g_t^p(x) = \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{ix^{p/p-1}(\xi + ct\xi^p)} d\xi, \quad x > 0.$$

If p is even, an application of the stationary phase method [31, 21] gives:

$$\begin{aligned} g_t^p(x) &= \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{ix^{p/p-1}(\xi + ct\xi^p)} d\xi \\ &\underset{x \rightarrow +\infty}{\sim} e^{\operatorname{sign}(c)i\frac{\pi}{4}} \frac{x^{\frac{2-p}{2(p-1)}}}{\sqrt{2\pi}} e^{-ix^{p/p-1}\frac{p-1}{p}(\frac{1}{pct})^{1/p-1}} \sqrt{\frac{(pct)^{\frac{p-2}{p-1}}}{|c|tp(p-1)}}. \end{aligned}$$

In the case where p is odd and $c < 0$, the same technique yields:

$$\begin{aligned} g_t^p(x) &= \frac{x^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{ix^{p/p-1}(\xi + ct\xi^p)} d\xi \\ &\underset{x \rightarrow +\infty}{\sim} e^{-i\frac{\pi}{4}} (p-1)^{-1/2} (p|c|t)^{-\frac{1}{2(p-1)}} \frac{x^{\frac{2-p}{2(p-1)}}}{\sqrt{2\pi}} e^{ix^{p/p-1}\frac{p-1}{p}(-\frac{1}{pct})^{\frac{1}{p-1}}}. \end{aligned}$$

The case where $x \rightarrow -\infty$ can be studied in the same way. In particular, if p is an even integer the behaviour of g_t^p for $x \rightarrow -\infty$ coincides

with the one for $x \rightarrow +\infty$.

For p odd and $x < 0$, a change of variable argument gives:

$$g_t^p(x) = \frac{(-x)^{\frac{1}{p-1}}}{2\pi} \int_{\mathbb{R}} e^{i(-x)^{p/p-1}(-\xi+ct\xi^p)} d\xi.$$

If $c < 0$ then the phase function $\phi(\xi) = -\xi + ct\xi^p$ has no real stationary points, hence

$$g_t^p(x) \underset{x \rightarrow -\infty}{\ll} x^{-N}, \quad \forall N \in \mathbb{N}.$$

In the case where $c > 0$ and $x \rightarrow -\infty$ the stationary phase method yields

$$g_t^p(x) \underset{x \rightarrow -\infty}{\sim} e^{i\frac{\pi}{4}} (p-1)^{-1/2} (pct)^{-\frac{1}{2(p-1)}} \frac{(-x)^{\frac{2-p}{2(p-1)}}}{\sqrt{2\pi}} e^{i(-x)^{p/p-1} \frac{1-p}{p} (pct)^{-\frac{1}{p-1}}}.$$

Eventually these results give the boundedness of the function g_t^p . Furthermore, if p is even and $\operatorname{Re}(\alpha) < 0$ then g_t^p is even summable. \square

3. INFINITE DIMENSIONAL FRESNEL INTEGRALS WITH POLYNOMIAL PHASE

Classical oscillatory integrals on \mathbb{R}^n are objects of this form

$$\int_{\mathbb{R}^n} f(x) e^{i\Phi(x)} dx, \quad (15)$$

where Φ and f are complex Borel functions. The interesting case where the *phase function* Φ is real valued has been extensively studied in connection with the theory of Fourier integral operator [21]. If the function f is not summable the integral (15) is not defined in Lebesgue sense. In [21], Hörmander proposes and exploits an alternative definition which can handle the case where $f \notin L^1(\mathbb{R}^n)$. We present here a formulation of Hörmander's definition of oscillatory integral, which was applied to the mathematical construction of Feynman path integrals in [15, 1].

Definition 1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be Borel functions. Assuming that:*

- (1) *for any Schwartz test function $\phi \in S(\mathbb{R}^n)$ such that $\phi(0) = 1$ the function $g_\epsilon(x) := \phi(\epsilon x) f(x) e^{i\Phi(x)}$ is summable,*
- (2) *the limit $\lim_{\epsilon \rightarrow 0} \int g_\epsilon(x) dx$ exists and is independent of ϕ .*

Then the oscillatory integral $\int_{\mathbb{R}^n}^o f(x) e^{i\Phi(x)} dx$ is defined as:

$$\int_{\mathbb{R}^n}^o f(x) e^{i\Phi(x)} dx := \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \phi(\epsilon x) f(x) e^{i\Phi(x)} dx$$

In the case where $f \in L^1(\mathbb{R}^n)$ the oscillatory integral reduces to a Lebesgue integral, i.e. $\int_{\mathbb{R}^n}^o f(x)e^{i\Phi(x)}dx = \int_{\mathbb{R}^n} f(x)e^{i\Phi(x)}dx$.

Definition 1 gives sense to classical Fresnel integrals such as $\int_{\mathbb{R}^n} f(x)e^{\frac{i}{2}\|x\|^2}dx$ which are extensively applied in the theory of wave diffraction. In particular, for $f = 1$ definition (15) yields the equality $\int_{\mathbb{R}^n} e^{\frac{i}{2}\|x\|^2}dx = (2\pi i)^{n/2}$.

In [3] oscillatory integration is generalized to the case where \mathbb{R}^n is replaced by a real separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and the definition of *infinite dimensional Fresnel integral* is introduced. The construction relies upon a generalization of the Parseval equality

$$\int_{\mathbb{R}^n} \frac{e^{\frac{i}{2}\|x\|^2}}{(2\pi i)^{n/2}} f(x)dx = \int_{\mathbb{R}^n} e^{-\frac{i}{2}\|x\|^2} \hat{f}(x)dx, \quad (16)$$

(valid for Schwartz test functions $f \in S(\mathbb{R}^n)$, where $\hat{f}(x) = \int_{\mathbb{R}^n} e^{ixy} f(y)dy$). In fact (see [15]) equality (16) can be generalized to the case the function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is the Fourier transform of a complex bounded Borel measure μ_f on \mathbb{R}^n , giving the following Parseval equality for the oscillatory integral

$$\int_{\mathbb{R}^n}^o \frac{e^{\frac{i}{2}\|x\|^2}}{(2\pi i)^{n/2}} f(x)dx = \int_{\mathbb{R}^n} e^{-\frac{i}{2}\|x\|^2} d\mu_f(x), \quad (17)$$

with $f(x) = \int_{\mathbb{R}^n} e^{ixy} d\mu_f(y)$. Formula (19) is crucial for the extension of oscillatory integration theory to an infinite dimensional setting.

Let us introduce the Banach space $\mathcal{M}(\mathcal{H})$ of complex Borel measures on \mathcal{H} with finite total variation, endowed with the total variation norm $\|\mu\|_{\mathcal{M}(\mathcal{H})}$. $\mathcal{M}(\mathcal{H})$ is a commutative Banach algebra under convolution, the unit being the Dirac point measure at 0.

Let $\mathcal{F}(\mathcal{H})$ be the space of complex functions $f : \mathcal{H} \rightarrow \mathbb{C}$ the form:

$$f(x) = \int_{\mathcal{H}} e^{i\langle x, y \rangle} d\mu(y) \equiv \hat{\mu}(x), \quad x \in \mathcal{H} \quad (18)$$

for some $\mu \in \mathcal{M}(\mathcal{H})$. The map $\mathcal{F} : \mathcal{M}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H})$ sending a complex measure $\mu \in \mathcal{M}(\mathcal{H})$ to its Fourier transform $\hat{\mu}$ defined by Eq. (18) is linear and one to one. By endowing the space $\mathcal{F}(\mathcal{H})$ with the norm $\|f\|_{\mathcal{F}} := \|\mathcal{F}^{-1}(f)\|_{\mathcal{M}(\mathcal{H})}$, $\mathcal{F}(\mathcal{H})$ becomes a commutative Banach algebra of continuous functions and the map $\mathcal{F} : \mathcal{M}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H})$ is an isometry.

In [3, 2, 15] the Parseval equality (19) is generalized to the case where $f \in \mathcal{F}(\mathcal{H})$. The *infinite dimensional Fresnel integral* of a function

$f \in \mathcal{F}(\mathcal{H})$ is denoted by $\widetilde{\int} e^{\frac{i}{2}\|x\|^2} f(x) dx$ and defined as

$$\widetilde{\int} e^{\frac{i}{2}\|x\|^2} f(x) dx := \int_{\mathcal{H}} e^{-\frac{i}{2}\|x\|^2} d\mu(x), \quad (19)$$

where $f(x) = \int_{\mathcal{H}} e^{i\langle x, y \rangle} d\mu(y)$ and the right hand side of (19) is a well defined (absolutely convergent) Lebesgue integral.

Infinite dimensional Fresnel integrals have been successfully applied to the representation of the solution of Schrödinger equation (3) (see i.e. [3, 30] and references therein). Let us denote with \mathcal{H}_t the real Hilbert space of absolutely continuous paths $\gamma : [0, t] \rightarrow \mathbb{R}^d$, such that $\int_0^t \dot{\gamma}(s)^2 ds < \infty$ and $\gamma(t) = 0$. The inner product in \mathcal{H}_t is defined as $\langle \gamma, \eta \rangle = \int_0^t \dot{\gamma}(s) \dot{\eta}(s) ds$. By assuming that the initial datum u_0 and the potential V in Eq. (3) belong to $\mathcal{F}(\mathbb{R}^d)$, it is possible to prove that the function on \mathcal{H}_t :

$$\gamma \mapsto u_0(\gamma(0) + x) e^{-i \int_0^t V(\gamma(s) + x) ds}, \quad \gamma \in \mathcal{H}_t, \quad x \in \mathbb{R}^d,$$

belongs to $\mathcal{F}(\mathcal{H}_t)$. Further the infinite dimensional Fresnel integral

$$\widetilde{\int} e^{\frac{i}{2}\|\gamma\|^2} e^{-i \int_0^t V(\gamma(s) + x) ds} u_0(\gamma(0) + x) d\gamma$$

provides a functional integral representation of the solution to the Schrödinger equation (3).

A partial generalization of the definition of infinite dimensional Fresnel integrals and of formula (19) was developed in [5], where the quadratic phase function $\Phi(x) = \frac{i}{2}\|x\|^2$ was replaced with a fourth order polynomial. This new functional integral allows the mathematical definition of the Feynman path integrals for the Schrödinger equation with a quartic-oscillator potential [5, 6, 29].

In the following we are going to generalize the definition in (19) to polynomial phase functions of any order and apply these *generalized Fresnel integrals* to the construction of a Feynman-Kac formula for the solution of high-order heat-type equations (7).

Let us consider a real separable Banach space $(\mathcal{B}, \|\cdot\|)$. Let $\mathcal{M}(\mathcal{B})$ be the space of complex bounded variation measures on \mathcal{B} , endowed with the total variation norm. As remarked above, $\mathcal{M}(\mathcal{B})$ is a Banach algebra under convolution. Let \mathcal{B}^* be the topological dual of \mathcal{B} and $\mathcal{F}(\mathcal{B})$ the Banach algebra of complex-valued functions $f : \mathcal{B}^* \rightarrow \mathbb{C}$ of the form

$$f(x) = \int_{\mathcal{B}} e^{i\langle x, y \rangle} d\mu(y) \equiv \hat{\mu}(x), \quad x \in \mathcal{B}^*, \quad \mu \in \mathcal{M}(\mathcal{B}), \quad (20)$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between \mathcal{B} and \mathcal{B}^* . The space $\mathcal{F}(\mathcal{B})$ endowed with the norm $\|\hat{\mu}\|_{\mathcal{F}} := \|\mu\|_{\mathcal{M}(\mathcal{B})}$ and the pointwise multiplication is a Banach algebra of functions.

In the following we are going to define a class of linear continuous functionals on $\mathcal{F}(\mathcal{B})$, by generalizing the construction of infinite dimensional Fresnel integrals defined by Eq. (19).

Definition 2. Let $\Phi : \mathcal{B} \rightarrow \mathbb{C}$ be a continuous map such that $\text{Re}(\Phi_p(x)) \leq 0$ for all $x \in \mathcal{B}$. The infinite dimensional Fresnel integral on \mathcal{B}^* with phase function Φ is the functional $I_{\Phi} : \mathcal{F}(\mathcal{B}) \rightarrow \mathbb{C}$, given by

$$I_{\Phi}(f) := \int_{\mathcal{B}} e^{\Phi(x)} d\mu(x), \quad f \in \mathcal{F}(\mathcal{B}), f = \hat{\mu}. \quad (21)$$

By construction, the functional I_{Φ} is linear and continuous, indeed:

$$|I_{\Phi_p}(f)| \leq \int_{\mathcal{B}} |e^{\Phi_p}| d|\mu|(x) \leq \|\mu\| = \|f\|_{\mathcal{F}}$$

Further I_{Φ} is normalized, i.e., $I_{\Phi_p}(1) = 1$. We summarize these properties in the following proposition.

Proposition 1. The space $\mathcal{F}(\mathcal{B})$ of Fresnel integrable functions is a Banach function algebra in the norm $\|\cdot\|_{\mathcal{F}}$. The infinite dimensional Fresnel integral with phase function Φ is a continuous bounded linear functional $I_{\Phi} : \mathcal{F}(\mathcal{B}) \rightarrow \mathbb{C}$ such that $|I_{\Phi_p}(f)| \leq \|f\|_{\mathcal{F}}$ and $I_{\Phi_p}(1) = 1$.

We can now present an interesting example of infinite dimensional Fresnel integral with polynomial phase function.

Fixed a $p \in \mathbb{N}$, with $p \geq 2$, let us consider the Banach space \mathcal{B}_p of absolutely continuous maps $\gamma : [0, t] \rightarrow \mathbb{R}$, with $\gamma(t) = 0$ and a weak derivative $\dot{\gamma}$ belonging to $L^p([0, t])$, endowed with the norm:

$$\|\gamma\|_{\mathcal{B}_p} = \left(\int_0^t |\dot{\gamma}(s)|^p ds \right)^{1/p}.$$

The application $T : \mathcal{B}_p \rightarrow L^p([0, t])$ mapping an element $\gamma \in \mathcal{B}_p$ to its weak derivative $\dot{\gamma} \in L^p([0, t])$ is an isomorphism and its inverse $T^{-1} : L^p([0, t]) \rightarrow \mathcal{B}_p$ is given by:

$$T^{-1}(v)(s) = - \int_s^t v(u) du \quad v \in L^p([0, t]). \quad (22)$$

Analogously the dual space \mathcal{B}_p^* is isomorphic to $L^q([0, t]) = (L^p([0, t]))^*$, with $\frac{1}{p} + \frac{1}{q} = 1$, and the pairing $\langle \eta, \gamma \rangle$ between $\eta \in \mathcal{B}_p^*$ and $\gamma \in \mathcal{B}_p$ can be written in the following form:

$$\langle \eta, \gamma \rangle = \int_0^t \dot{\eta}(s) \dot{\gamma}(s) ds \quad \dot{\eta} \in L^q([0, t]), \gamma \in \mathcal{B}_p.$$

Further \mathcal{B}_p^* is isomorphic to \mathcal{B}_q .

Let us consider the space $\mathcal{F}(\mathcal{B}_q)$ of functions $f : \mathcal{B}_q \rightarrow \mathbb{C}$ of the form

$$f(\eta) = \int_{\mathcal{B}_p} e^{i \int_0^t \dot{\eta}(s) \dot{\gamma}(s) ds} d\mu_f(\gamma), \quad \eta \in \mathcal{B}_q, \mu_f \in \mathcal{M}(\mathcal{B}_p).$$

Let $\Phi_p : \mathcal{B}_p \rightarrow \mathbb{C}$ be the phase function defined as

$$\Phi_p(\gamma) := (-1)^p \alpha \int_0^t \dot{\gamma}(s)^p ds,$$

where $\alpha \in \mathbb{C}$ is a complex constant such that

- $\operatorname{Re}(\alpha) \leq 0$ if p is even,
- $\operatorname{Re}(\alpha) = 0$ if p is odd.

The infinite dimensional Fresnel integral on B_q with phase function Φ_p is the functional $I_{\Phi_p} : \mathcal{F}(\mathcal{B}_q) \rightarrow \mathbb{C}$ given by

$$I_{\Phi_p}(f) = \int_{\mathcal{B}_p} e^{(-1)^p \alpha \int_0^t \dot{\gamma}(s)^p ds} d\mu_f(\gamma), \quad f \in \mathcal{F}(\mathcal{B}_q), f = \hat{\mu}_f. \quad (23)$$

The following lemma states an interesting connection between the functional (23) and the high-order PDE (8).

Lemma 3. *Let $f : \mathcal{B}_q \rightarrow \mathbb{C}$ be a cylinder function of the following form:*

$$f(\eta) = F(\eta(t_1), \eta(t_2), \dots, \eta(t_n)), \quad \eta \in \mathcal{B}_q,$$

with $0 \leq t_1 < t_2 < \dots < t_n < t$ and $F : \mathbb{R}^n \rightarrow \mathbb{C}$, $F \in \mathcal{F}(\mathbb{R}^n)$:

$$F(x_1, x_2, \dots, x_n) = \int_{\mathbb{R}^n} e^{i \sum_{k=1}^n y_k x_k} d\nu_F(y_1, \dots, y_n), \quad \nu_F \in \mathcal{M}(\mathbb{R}^n).$$

Then $f \in \mathcal{F}(\mathcal{B}_p)$ and its infinite dimensional Fresnel integral with phase function Φ_p is given by

$$I_{\Phi_p}(f) = \int_{\mathbb{R}^n} F(x_1, x_2, \dots, x_n) \prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}, x_k) dx_1 \dots dx_n, \quad (24)$$

where $x_{n+1} \equiv 0$, $t_{n+1} \equiv t$, G_s^p is the fundamental solution (9) of the high order heat-type equation (8) and the integral on the right hand side of (24) is an oscillatory integral in the sense of definition 1.

Remark 3. *In the case p is even and $\operatorname{Re}(\alpha) < 0$ the integral (24) is an absolutely convergent Lebesgue integral because of the boundedness of the function $F \in \mathcal{F}(\mathbb{R}^n)$ and the summability of the function g_t^p stated in lemma 2.*

Proof of lemma 3. The proof that $f \in \mathcal{F}(\mathcal{B}_p)$ follows from the explicit form of the function f

$$f(\eta) = F(\eta(t_1), \eta(t_2), \dots, \eta(t_n)) = \int_{\mathbb{R}^n} e^{i \sum_{k=1}^n y_k \eta(t_k)} d\nu_F(y_1, \dots, y_n), \quad \eta \in \mathcal{B}_q.$$

and the identity

$$e^{iy\eta(s)} = \int_{\mathcal{B}_p} e^{i\langle \eta, \gamma \rangle} \delta_{y v_s}(\gamma),$$

where $v_s \in \mathcal{B}_p$ is the vector of \mathcal{B}_p defined by

$$\langle \eta, v_s \rangle = \eta(s), \quad \forall \eta \in \mathcal{B}_q,$$

which can be explicitly written as

$$v_s(\tau) = \chi_{[0,s]}(t-s) + \chi_{(s,t]}(t-\tau)s.$$

By the definition of the functional I_{Φ_p} we have

$$\begin{aligned} I_{\Phi_p}(f) &= \int_{\mathbb{R}^n} e^{(-1)^p \alpha \int_0^t (\sum_{k=1}^n y_k \dot{v}_{t_k}(\tau))^p d\tau} d\nu_F(y_1, \dots, y_n) \\ &= \int_{\mathbb{R}^n} e^{\alpha \int_0^t (\sum_{k=1}^n y_k \chi_{(t_k, t]}(\tau))^p d\tau} d\nu_F(y_1, \dots, y_n) \\ &= \int_{\mathbb{R}^n} e^{\alpha \int_0^t (\sum_{k=1}^n \chi_{(t_k, t_{k+1}]}(\tau) \sum_{j=1}^k y_j)^p d\tau} d\nu_F(y_1, \dots, y_n) \\ &= \int_{\mathbb{R}^n} e^{\alpha \sum_{k=1}^n (\sum_{j=1}^k y_j)^p (t_{k+1} - t_k)} d\nu_F(y_1, \dots, y_n) \end{aligned} \quad (25)$$

On the other hand the last line of Eq. (25) coincides with the oscillatory integral

$$\int_{\mathbb{R}^n}^o F(x_1, x_2, \dots, x_n) \prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}, x_k) dx_1 \dots dx_n. \quad (26)$$

Indeed, taken an arbitrary test function $\phi \in S(\mathbb{R}^n)$ such that $\phi(0) = 1$, the the function $F_\epsilon : \mathbb{R}^n \rightarrow \mathbb{C}$

$$F_\epsilon(x_1, x_2, \dots, x_n) \equiv F(x_1, x_2, \dots, x_n) \phi(\epsilon x_1, \epsilon x_2, \dots, \epsilon x_n) \prod_{k=1}^n G_{t_{k+1}-t_k}^p(x_{k+1}, x_k)$$

is summable because of the boundedness of $F \in \mathcal{F}(\mathbb{R}^n)$ and the decaying properties at infinity stated in lemma 2. Further a change of variable argument and Fubini theorem yield:

$$\int_{\mathbb{R}^n} F_\epsilon(x) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{\alpha \sum_{k=1}^n (t_{k+1}-t_k) (\sum_{j=1}^k y_j + \epsilon \xi_j)^p} \hat{\phi}(\xi) d\xi \right) d\nu_F(y),$$

where $\phi(x) = \int_{\mathbb{R}^n} e^{ix\xi} \hat{\phi}(\xi) d\xi$. By dominated convergence theorem and the condition $\phi(0) = \int_{\mathbb{R}^n} \hat{\phi}(\xi) d\xi = 1$, we eventually obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} F_\epsilon(x) dx = \int_{\mathbb{R}^n} e^{\alpha \sum_{k=1}^n (t_{k+1} - t_k) (\sum_{j=1}^k y_j)^p} d\nu_F(y),$$

□

Corollary 1. *Let $u_0 \in \mathcal{F}(\mathbb{R})$. Then the cylinder function $f_0 : \mathcal{B}_q \rightarrow \mathbb{C}$ defined by*

$$f_0(\eta) := u_0(x + \eta(0)), \quad x \in \mathbb{R}, \eta \in \mathcal{B}_q,$$

belongs to $\mathcal{F}(\mathcal{B}_q)$ and its infinite dimensional Fresnel integral with phase function Φ_p provides a representation for the solution of the Cauchy problem (8), in the sense that the function $u(t, x) := I_{\Phi_p}(f_0)$ has the form

$$u(t, x) = \int_{\mathbb{R}} G_t(x, y) u_0(y) dy. \quad (27)$$

In the case p is even and $\operatorname{Re}(\alpha) < 0$ then the integral (27) is absolutely convergent, while in the general case it is meant in the oscillatory sense of definition 1.

4. A GENERALIZED FEYNMAN-KAC FORMULA

In the present section, we consider a Cauchy problem of the form

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t, x) + V(x) u(t, x) \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, t \in [0, +\infty) \end{cases} \quad (28)$$

where $p \in \mathbb{N}$, $p \geq 2$, and $\alpha \in \mathbb{C}$ is a complex constant such that $|e^{\alpha t x^p}| \leq 1$ for all $x \in \mathbb{R}, t \in [0, +\infty)$, while $V : \mathbb{R} \rightarrow \mathbb{C}$ is a bounded continuous function. Under these assumption the Cauchy problem (28) is well posed in $L^2(\mathbb{R})$. Indeed the operator $\mathcal{D}_p : D(\mathcal{D}_p) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined by

$$\begin{aligned} D(\mathcal{D}_p) &:= H^p = \{u \in L^2(\mathbb{R}), k \mapsto k^p \hat{u}(k) \in L^2(\mathbb{R})\}, \\ \widehat{\mathcal{D}_p u}(k) &:= k^p \hat{u}(k), \quad u \in D(\mathcal{D}_p), \end{aligned}$$

(\hat{u} denoting the Fourier transform of u) is self-adjoint. For $\alpha \in \mathbb{C}$, with $|e^{\alpha t x^p}| \leq 1$ for all $x \in \mathbb{R}, t \in [0, +\infty)$, one has that the operator $A := \alpha \mathcal{D}_p$ generates a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ on $L^2(\mathbb{R})$. By denoting with $B : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ the bounded multiplication operator defined by

$$Bu(x) = V(x)u(x), \quad u \in L^2(\mathbb{R}),$$

one has that the operator sum $A + B : D(A) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $L^2(\mathbb{R})$. Moreover, given a $u \in L^2(\mathbb{R})$, the vector $T(t)u$ can be computed by means of the convergent (in the $L^2(\mathbb{R})$ -norm) Dyson series (see [18], Th. 13.4.1):

$$T(t)u = \sum_{n=0}^{\infty} S_n(t)u, \quad (29)$$

where $S_0(t)u = e^{tA}u$ and $S_n(t)u = \int_0^t e^{(t-s)A} V S_{n-1}(s) u ds$. By passing to a subsequence, the series above converges also a.e. in $x \in \mathbb{R}$ giving

$$\begin{aligned} T(t)u(x) &= \\ &= \sum_{n=0}^{\infty} \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \int_{\mathbb{R}^{n+1}} V(x_1) \dots V(x_n) G_{t-s_n}(x, x_n) G_{s_n-s_{n-1}}(x_n, x_{n-1}) \\ &\quad \dots G_{s_1}(x_1, x_0) u_0(x_0) dx_0 \dots dx_n ds_1 \dots ds_n, \quad a.e. x \in \mathbb{R}. \end{aligned} \quad (30)$$

Under suitable assumptions on the initial datum u_0 and the potential V , we are going to construct a representation of the solution of equation (28) in $L^2(\mathbb{R})$ in terms of an infinite dimensional oscillatory integral with polynomial phase.

Theorem 1. *Let $u_0 \in \mathcal{F}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $V \in \mathcal{F}(\mathbb{R})$, with $u_0(x) = \int_{\mathbb{R}} e^{ixy} d\mu_0(y)$ and $V(x) = \int_{\mathbb{R}} e^{ixy} d\nu(y)$, $\mu_0, \nu \in \mathcal{M}(\mathbb{R})$. Then the functional $f_{t,x} : \mathcal{B}_q \rightarrow \mathbb{C}$ defined by*

$$f_{t,x}(\eta) := u_0(x + \eta(0)) e^{\int_0^t V(x + \eta(s)) ds}, \quad x \in \mathbb{R}, \eta \in \mathcal{B}_q, \quad (31)$$

belongs to $\mathcal{F}(\mathcal{B}_q)$ and its infinite dimensional Fresnel integral with phase function Φ_p provides a representation for the solution of the Cauchy problem (28).

Remark 4. *By Plancherel's theorem the assumption that $u_0 \in \mathcal{F}(\mathbb{R}) \cap L^2(\mathbb{R})$ is equivalent to the fact that u_0 is the Fourier transform of a function $\hat{u}_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

Proof. Let $\mu_V \in \mathcal{M}(\mathcal{B}_p)$ be the measure defined by

$$\int_{\mathcal{B}_p} f(\gamma) d\mu_V(\gamma) = \int_0^t \int_{\mathbb{R}} e^{ixy} f(y v_s) d\nu(y) ds, \quad f \in C_b(\mathcal{B}_p),$$

where $v_s \in \mathcal{B}_p$ is the function $v_s(\tau) = \chi_{[0,s]}(\tau)(t-s) + \chi_{(s,t]}(\tau)(t-\tau)s$. One can easily verify that $\|\mu_V\|_{\mathcal{M}(\mathcal{B}_p)} \leq t\|\nu\|_{\mathcal{M}(\mathbb{R})}$ and the map $\eta \in \mathcal{B}_q \mapsto \int_0^t V(x + \eta(s)) ds$ is the Fourier transform of μ_V . Analogously the map $\eta \in \mathcal{B}_q \mapsto \exp(\int_0^t V(x + \eta(s)) ds)$ is the Fourier transform of the measure $\nu_V \in \mathcal{M}(\mathcal{B}_p)$ given by $\nu_V = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_V^{*n}$, where μ_V^{*n} denotes

the n -fold convolution of μ_V with itself. The series is convergent in the $\mathcal{M}(\mathcal{B}_p)$ -norm and one has $\|\nu_V\|_{\mathcal{M}(\mathcal{B}_p)} \leq e^{t\|\nu\|_{\mathcal{M}(\mathbb{R})}}$. Further, by lemma 3 the cylinder function $\eta \mapsto u_0(x + \eta(0))$, $\eta \in \mathcal{B}_q$, is an element of $\mathcal{F}(\mathcal{B}_q)$. More precisely, it is the Fourier transform of the measure ν_{u_0} defined by

$$\int_{\mathcal{B}_p} f(\gamma) d\nu_{u_0}(\gamma) = \int_{\mathbb{R}} e^{ixy} f(y v_0) d\mu_0(y), \quad f \in C_b(\mathcal{B}_p).$$

We can then conclude that the map $f_{t,x} : \mathcal{B}_q \rightarrow \mathbb{C}$ defined by (31) belongs to $\mathcal{F}(\mathcal{B}_q)$ and its infinite dimensional Fresnel integral $I_{\Phi_p}(f_{t,x})$ with phase function Φ_p is given by

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{B}_p} e^{(-1)^p \alpha \int_0^y \dot{\gamma}(s)^p ds} d\nu_{u_0} * \mu_V * \dots * \mu_V = \\ & \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t \dots \int_0^t I_{\Phi_p}(u_0(x + \eta(0))V(x + \eta(s_1)) \dots V(x + \eta(s_n))) ds_1 \dots ds_n \end{aligned}$$

By the symmetry of the integrand the latter is equal to

$$\sum_{n=0}^{\infty} \int \dots \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} I_{\Phi_p}(u_0(x + \eta(0))V(x + \eta(s_1)) \dots V(x + \eta(s_n))) ds_1 \dots ds_n$$

By lemma 3 we eventually obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \int \dots \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \int_{\mathbb{R}^{n+1}} u_0(x + x_0)V(x + x_1) \dots V(x + x_n) G_{s_1}(x_1, x_0) \\ & \quad G_{s_2 - s_1}(x_2, x_1) \dots G_{t - s_n}(0, x_n) dx_0 dx_1 \dots dx_n ds_1 \dots ds_n, \end{aligned}$$

that coincides with the Dyson series (30) for the solution of the high-order PDE (28), as one can easily verify by means of a change of variables argument. \square

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REFERENCES

- [1] S. Albeverio and Z. Brzeźniak. Finite-dimensional approximation approach to oscillatory integrals and stationary phase in infinite dimensions. *J. Funct. Anal.*, 113(1): 177-244, 1993.

- [2] S. Albeverio and R. Høegh-Krohn. Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics. *Invent. Math.*, 40(1):59-106, 1977.
- [3] S. Albeverio, R. Hoegh-Krohn, S. Mazzucchi, Mathematical theory of Feynman path integrals - An Introduction. 2nd corrected and enlarged edition. Lecture Notes in Mathematics, Vol. 523. Springer, Berlin, (2008).
- [4] S. Albeverio, S. Mazzucchi, Generalized Fresnel integrals. *Bull. Sci. Math.* 129 (2005), no. 1, 123.
- [5] S. Albeverio, S. Mazzucchi, Feynman path integrals for polynomially growing potentials, *J. Funct. Anal.* 221 no.1, 83–121 (2005).
- [6] S. Albeverio, S. Mazzucchi, The time-dependent quartic oscillator. A Feynman path integral approach. *J. Funct. Anal.* 238 (2006), no. 2, 471488.
- [7] L. Beghin, K. Hochberg, E. Orsingher. Conditional maximal distributions of processes related to higher-order heat-type equations. *Stochastic Process. Appl.* 85 (2000), no. 2, 209–223.
- [8] S. Bonaccorsi, S. Mazzucchi. High order heat-type equations and random walks on the complex plane. *Stochastic processes and their Applications* Volume 125, Issue 2, February 2015, Pages 797-818.
- [9] S. Bochner. Harmonic analysis and the theory of probability. University of California Press, Berkeley and Los Angeles, (1955).
- [10] K. Burdzy. Some path properties of iterated Brownian motion. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 67–87. Birkhäuser Boston, Boston, MA, 1993.
- [11] K. Burdzy and A. Mađrecki. An asymptotically 4-stable process. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993)*, number Special Issue, pages 97–117, 1995.
- [12] K. Burdzy, A. Madrecki. Ito formula for an asymptotically 4-stable process. *Ann. Appl. Probab.* 6 (1996), no. 1, 200–217.
- [13] R.H. Cameron. A family of integrals serving to connect the Wiener and Feynman integrals, *J. Math. and Phys.* 39, 126–140 (1960).
- [14] E. B. Dynkin. Theory of Markov processes. Dover Publications, Inc., Mineola, NY, 2006.
- [15] D. Elworthy and A. Truman. Feynman maps, Cameron-Martin formulae and anharmonic oscillators. *Ann. Inst. H. Poincaré Phys. Théor.*, 41(2):115–142, 1984.
- [16] T. Funaki. Probabilistic construction of the solution of some higher order parabolic differential equation. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(5):176–179, 1979.
- [17] T. Hida, H.H. Kuo, J. Potthoff, L. Streit. White Noise. Kluwer, Dordrecht (1995).
- [18] E. Hille, R. S: Phillips. Functional analysis and semi-groups. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.
- [19] K. J. Hochberg. A signed measure on path space related to Wiener measure. *Ann. Probab.*, 6(3):433–458, 1978.
- [20] K. Hochberg, E. Orsingher. Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations. *J. Theoret. Probab.* 9 (1996), no. 2, 511–532.

- [21] L. Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Reprint of the second (1990) edition. *Classics in Mathematics*. Springer-Verlag, Berlin, 2003.
- [22] K. Itô. Wiener integral and Feynman integral. *Proc. Fourth Berkeley Symposium on Mathematical Statistics and Probability*. Vol 2, pp. 227-238, California Univ. Press, Berkeley, 1961.
- [23] K. Itô. Generalized uniform complex measures in the hilbertian metric space with their applications to the Feynman path integral. *Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability*. Vol 2, part 1, pp. 145-161, California Univ. Press, Berkeley, 1967.
- [24] I. Karatzas, S.E. Shreve. *Brownian motion and stochastic calculus*. Springer-Verlag, New York, 1991
- [25] V. J. Krylov. Some properties of the distribution corresponding to the equation $\partial u / \partial t = (-1)^{q+1} \partial^{2q} u / \partial x^{2q}$. *Soviet Math. Dokl.*, 1:760-763, 1960.
- [26] R. Léandre. Stochastic analysis without probability: study of some basic tools. *J. Pseudo-Differ. Oper. Appl.* 1 (2010), no. 4, 389-400.
- [27] D. Levin, T. Lyons. A signed measure on rough paths associated to a PDE of high order: results and conjectures. *Rev. Mat. Iberoam.* 25 (2009), no. 3, 971-994.
- [28] A. Madrecki, M. Rybaczuk. New Feynman-Kac type formula. *Rep. Math. Phys.* 32 (1993), no. 3, 301-327.
- [29] S. Mazzucchi, Feynman path integrals for the inverse quartic oscillator. *J. Math. Phys.* 49 (2008), no. 9, 093502, 15 pp.
- [30] S. Mazzucchi. *Mathematical Feynman Path Integrals and Applications*. World Scientific Publishing, Singapore (2009)
- [31] J. D. Murray, *Asymptotic analysis*. Clarendon Press, Oxford, 1974.
- [32] E. Orsingher, X. Zhao. Iterated processes and their applications to higher order differential equations. *Acta Math. Sin. (Engl. Ser.)* 15 (1999), no. 2, 173-180.
- [33] P. Sainy. Construction of a complex-valued fractional Brownian motion of order N . *J. Math. Phys.*, 33(9):3128-3149, 1992.
- [34] B. Simon, *Functional integration and quantum physics*. Second edition. AMS Chelsea Publishing, Providence, RI, 2005.
- [35] E. Thomas, Projective limits of complex measures and martingale convergence. *Probab. Theory Related Fields* 119 (2001), no. 4, 579588.

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