MAXIMAL RANK OF SPACE CURVES IN THE RANGE A

EDOARDO BALLICO, PHILIPPE ELLIA, CLAUDIO FONTANARI

Abstract. We prove the following statement, which has been conjectured since 1985: *There exists a constant K such that for all natural numbers d, g with $g \leq Kd^{3/2}$ there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element is a smooth, connected curve of degree d and genus g of maximal rank.*

1. Introduction

The postulation of algebraic space curves has been the object of wide interest in the last thirty years (see for instance [1], [2], [28], [22], [27]). In particular, the following Conjecture was stated in 1985 in [2], p. 2 (see also [3], §6, Problem 4):

Conjecture 1. *There exists a constant K such that for all natural numbers d, g with $g \leq Kd^{3/2}$ there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element is a smooth, connected curve of degree d and genus g of maximal rank.*

We recall that a space curve C is of *maximal rank* if the natural maps $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_C(m))$ are either injective or surjective for every m.

Here we consider smooth and connected curves X with $h^1(\mathcal{I}_X(m)) = 0$, $h^0(\mathcal{I}_X(m-1)) = 0$, $\deg(X) = d$, $g(X) = g$ and $h^1(\mathcal{O}_X(m-2)) = 0$ (hence of maximal rank by Castelnuovo-Mumford regularity). Since $h^1(\mathcal{I}_X(m)) = 0$ and $h^1(\mathcal{O}_X(m)) = 0$, we have

$$1 + md - g \leq \binom{m+3}{3} \tag{1}$$

Let $d(m,g)_{\max}$ be the maximal integer d such that (1) is satisfied, i.e. set $d(m,g)_{\max} := \lfloor \binom{m+3}{3} + g - 1)/m \rfloor$. Since $h^0(\mathcal{I}_X(m-1)) = 0$ and $h^1(\mathcal{O}_X(m-1)) = 0$, we have

$$1 + (m-1)d - g \geq \binom{m+2}{3} \tag{2}$$

Let $d(m,g)_{\min}$ be the minimal integer d such that (2) is satisfied, i.e. set $d(m,g)_{\min} := \lceil \binom{m+2}{3} + g - 1)/(m-1) \rceil$.

2010 *Mathematics Subject Classification.* Primary: 14H50. Secondary: 14N05.

Key words and phrases. Space curve. Postulation. Hilbert function. Hilbert scheme.

This research was partially supported by PRIN 2012 "Geometria delle varietà algebriche", by FIRB 2012 "Moduli spaces and Applications", and by GNSAGA of INdAM (Italy).

For every integer $s > 0$ define the number $p_a(C_s) := s(s+1)(2s-5)/6+1$ (which is going to to be the genus of the curve C_s to be introduced later in Section 2). For all positive integers $m \geq 3$ set

$$\begin{aligned}\varphi(m) &= p_a(C_{\lfloor m/\sqrt{20}\rfloor -4}) + p_a(C_{\lfloor m/\sqrt{20}\rfloor -5}) \\ &= \frac{(\lfloor m/\sqrt{20}\rfloor - 4)(\lfloor m/\sqrt{20}\rfloor - 3)(2\lfloor m/\sqrt{20}\rfloor - 13)}{6} + 1 \\ &\quad + \frac{(\lfloor m/\sqrt{20}\rfloor - 5)(\lfloor m/\sqrt{20}\rfloor - 4)(2\lfloor m/\sqrt{20}\rfloor - 15)}{6} + 1.\end{aligned}$$

For any smooth curve $X \subset \mathbb{P}^3$ let N_X denote the normal bundle of X in $\mathbb{P}^3$. If $h^1(N_X) = 0$, then X is a smooth point of the Hilbert scheme of $\mathbb{P}^3$ and this Hilbert scheme has the expected dimension $h^0(N_X)$ at X.

Our main result is the following:

Theorem 1. *For every integer $m \geq 3$ and every (d, g) with $17052 \leq g \leq \varphi(m)$ and $d(m, g)_{\min} \leq d \leq d(m, g)_{\max}$ there exists a component of the Hilbert scheme of curves in $\mathbb{P}^3$ of genus g and degree d, whose general element X is smooth and satisfies $h^0(\mathcal{I}_X(m-1)) = 0$, $h^1(\mathcal{I}_X(m)) = 0$, $h^1(\mathcal{O}_X(m-2)) = 0$, and $h^1(N_X(-1)) = 0$.*

As an application of Theorem 1 we prove Conjecture 1. Indeed, if $g = 0$ we have just to quote [19]. Next, if $0 < g < 17052$ we may choose $K > 0$ such that $g \geq K(g+3)^{3/2}$. Hence from $K(g+3)^{3/2} \leq g \leq Kd^{3/2}$ we get $d \geq g+3$ and we are done by [1]. Finally, if $g \geq 17052$ we have the following:

Corollary 1. *Let $K = \frac{2}{3}\left(\frac{1}{10}\right)^{3/2}$ and $\varepsilon = \frac{11}{20} + 4\left(\frac{1}{20}\right)^{3/2}$. If $17052 \leq g \leq Kd^{3/2} - 6\varepsilon d$ then there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element X is a smooth, connected curve of degree d and genus g of maximal rank and with $h^1(N_X(-1)) = 0$.*

The constant K in Corollary 1 is certainly not optimal, but the exponent $d^{3/2}$ is sharp among the curves with $h^1(N_X) = 0$ (see [12], [29, Corollaire 5.18] and [20, II.3.6] for the condition $h^1(N_X(-2)) = 0$, [20, II.3.7] and [31] for the condition $h^1(N_X(-1)) = 0$, and [20, II.3.8] for the condition $h^1(N_X) = 0$).

If X is as in Theorem 1, then by Castelnuovo-Mumford regularity we have $h^1(\mathcal{I}_X(t)) = 0$ for all $t > m$ and the homogeneous ideal of X is generated by forms of degree m and degree $m+1$. A smooth curve $Y \subset \mathbb{P}^3$ with $h^0(\mathcal{I}_Y(m-1)) = 0$, $\frac{m^2+4m+6}{6} \leq \deg(Y) < \frac{m^2+4m+6}{3}$ and maximal genus among the curves with $h^0(\mathcal{I}_Y(m-1)) = 0$ satisfies $h^1(\mathcal{O}_Y(m-1)) = 0$ ([16, proof of Theorem 3.3 at p. 97]). In the statement of Theorem 1 we claim one shift more, namely, $h^1(\mathcal{O}_X(m-2)) = 0$, in order to apply Castelnuovo-Mumford regularity to X.

We describe here one of the main differences with respect to [19, 1, 2]. Fix integers d, g as in Theorem 1 or Corollary 1. Suppose that we have constructed two irreducible and generically smooth components W_1, W_2 of the Hilbert scheme of smooth space curves of degree d and genus g. Suppose also that we have proved the existence of $Y_1 \in W_1$ and $Y_2 \in W_2$ with $h^0(\mathcal{I}_{Y_2}(m-1)) = 0$, $h^1(\mathcal{I}_{Y_1}(m)) = 0$ and $h^1(N_{Y_i}) = h^1(\mathcal{O}_{Y_i}(m-3)) = 0$, $i = 1, 2$. If $W_1 = W_2$, then by the semicontinuity theorem for cohomology and Castelnuovo-Mumford regularity a general $X \in W_1$ satisfies $h^0(\mathcal{I}_X(m-1)) = 0$, $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq m$ and $h^1(N_X) = 0$. In particular a general element of W_1 has maximal rank. But we need to know that $W_1 = W_2$. If $d \geq g+3$ it was not known at that time that the Hilbert scheme of smooth space curves of degree d and genus g is irreducible ([7]), but it was obvious since at least Castelnuovo that its part parametrizing the non-special curves is irreducible (modulo the irreducibility of the moduli scheme $\mathcal{M}_g$ of genus g smooth curves). When $d < g+3$, the Hilbert scheme of smooth space curves of degree d and genus g is often reducible, even in ranges with d/g not small ([6, 21, 23, 24, 25, 26]). In [2] when $d \ge (g+2)/2$ we defined a certain irreducible component $Z(d,g)$ of the Hilbert scheme of smooth space curves of degree d and genus g and (under far stronger assumptions on d, g) we were able find Y_1 and Y_2 with $W_1 = W_2 = Z(d,g)$. Several pages of Section 5 are devoted to solve this problem.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

We thank the anonymous referee for useful comments.

2. Preliminaries

2.1. **The curves $C_{t,k}$.** For each locally Cohen-Macaulay curve $C \subset \mathbb{P}^3$ the index of speciality $e(C)$ of C is the maximal integer e such that $h^1(\mathcal{O}_C(e)) \ne 0$.

Fix an integer $s > 0$. Let $C_s \subset \mathbb{P}^3$ be any curve fitting in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-s-1) \to (s+1)\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_{C_s} \to 0 \tag{3}$$

Each C_s is arithmetically Cohen-Macaulay and in particular $h^0(\mathcal{O}_{C_s}) = 1$. By taking the Hilbert function in (3) we get $\deg(C_s) = s(s+1)/2$, $p_a(C_s) = s(s+1)(2s-5)/6+1$ and $e(C_s) = s-3$. Hence $h^i(\mathcal{I}_{C_s}(s-1)) = 0$, $i = 0,1,2$. By taking $d := \deg(C_s)$ we get $p_a(C_s) = 1 + d(s-1) - \binom{s+2}{3} = G_A(d,s)$. The set of all curves fitting in (3) is an irreducible variety and its general member is smooth and connected. Among them there are the stick-figures called $\mathbf{K}_s$ in [13], [14] and [4]. We have $h^1(N_{C_s}(-2)) = 0$ for all C_s ([11, Lemme 1], see also [10]). Unless otherwise stated we only use smooth C_s.

For any t, k let $C_{t,k} := C_t \sqcup C_k$ be the union of a smooth C_t and a smooth C_k with the only restriction that they are disjoint. By definition each $C_{t,k}$ is smooth. Let $d_{t,k} := \deg(C_{t,k}) = t(t+1)/2 + k(k+1)/2$ and $g_{t,k} := h^1(\mathcal{O}_{C_{t,k}}) = 2 + t(t+1)(2t-5)/6 + k(k+1)(2k-5)/6$ for $t \ge k > 0$. If $t \ge k > 0$ then we have

$$(t+k-1)d_{t,k} + 2 - g_{t,k} = \binom{t+k+2}{3} \tag{4}$$

Since each connected component A of $C_{t,k}$ satisfies $h^i(N_A(-2)) = 0$, $i = 0,1$, we have $h^i(N_{C_{t,k}}(-2)) = 0$, $i = 0,1$.

Lemma 1. *We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1,2$.*

Proof. Since $C_t \cap C_k = \emptyset$, we have $Tor^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_{C_t}, \mathcal{I}_{C_k}) = 0$ and $\mathcal{I}_{C_t} \otimes \mathcal{I}_{C_k} = \mathcal{I}_{C_{t,k}}$. Therefore tensoring (3) with $s := t$ by $\mathcal{I}_{C_k}(t+k-1)$ we get

$$0 \to t\mathcal{I}_{C_k}(k-2) \to (t+1)\mathcal{I}_{C_k}(k-1) \to \mathcal{I}_{C_{t,k}}(t+k-1) \to 0 \tag{5}$$

We have $h^2(\mathcal{I}_{C_k}(k-2)) = h^1(\mathcal{O}_{C_k}(k-2))$ and the latter integer is zero, because $e(C_k) = k-3 < k-2$. We have $h^1(\mathcal{I}_{C_k}(k-1)) = 0$, because C_k is arithmetically Cohen-Macaulay. We have $h^0(\mathcal{I}_{C_k}(k-1)) = 0$, by the case $s = k$ of (3). Hence $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1,2$. □

Remark 1. In this paper we only need $k \in \{t-1, t\}$.

Remark 2. We have $e(C_{t,k}) = \max\{e(C_t), e(C_k)\} = \max\{t-3, k-3\} \le t+k-4$. Recall that $d_{t,k} = \deg(C_{t,k})$. If $s := t+k$, then $d_{s-1,1} = (s^2 - s + 2)/2 \ge d_{t,k}$. If s is even then $d_{t,k} \ge s(s+2)/4 = d_{\frac{s}{2},\frac{s}{2}}$. If s is odd, then $d_{t,k} \ge (s+1)^2/4 = d_{\frac{s+1}{2},\frac{s-1}{2}}$.

Remark 3. Let X be a general smooth curve of genus g and degree $d \ge g+3$ such that $h^1(\mathcal{O}_X(1)) = 0$; if either $g \ge 26$ ([29, p. 67, inequality $D_P(g) \le g+3$]) or $g \le 25$ and $d \ge g+14$ ([29, p. 67]), then $h^1(N_X(-2)) = 0$ ([29] uses the case $g = 0$ done in [9]).

2.2. **Smoothing.** We are going to apply standard smoothing techniques (see for instance [18] and [30]).

Lemma 2. *Fix $A \sqcup B$ with $A = C_t$ and $B = C_k$. Let X be a nodal curve with $X = A \cup B \cup Y$, Y a smooth curve of degree $d' \ge 2$ and genus g', $\sharp(A \cap Y) = 1$, $\sharp(B \cap Y) = 1$, $h^1(\mathcal{O}_Y(1)) = 0$ and $h^1(N_Y(-2)) = 0$. Then $h^1(N_X(-1)) = 0$ and X is smoothable.*

Proof. Set $C := A \cup B$. Write $\{p_1\} = A \cap Y$ and $\{p_2\} = B \cap Y$. We have an exact sequence

$$0 \to N_X(-1) \to N_X(-1)|_C \oplus N_X(-1)|_Y \to N_X(-1)|_{\{p_1,p_2\}} \to 0 \tag{6}$$

Since $N_X(-1)|_C$ is obtained from $N_C(-1)$ by making two positive elementary transformations and $h^1(N_C(-1)) = 0$, we have $h^1(N_X(-1)|_C) = 0$. Since $N_X(-2)|_Y$ is obtained from $N_Y(-2)$ by making two positive elementary transformations and $h^1(N_Y(-2)) = 0$, we have $h^1(N_X(-2)|_Y) = 0$. Let $H \subset \mathbb{P}^3$ be a general plane containing $\{p_1, p_2\}$. Since Y is not a line, $Y \cap H$ is a zero-dimensional scheme. Since $h^1(N_X(-2)|_Y) = 0$, the restriction map

$$H^0(Y, N_X(-1)|_Y) \to H^0(Y \cap H, N_X(-1)|_{H \cap Y})$$

is surjective. Since $\{p_1, p_2\} \subseteq Y \cap H$, the restriction map $H^0(Y \cap H, N_X(-1)|_{H\cap Y}) \to H^0(\{p_1, p_2\}, N_X(-1)|_{\{p_1,p_2\}})$ is surjective. Hence the restriction map

$$H^0(Y, N_X(-1)|_Y) \to H^0(\{p_1, p_2\}, N_X(-1)|_{\{p_1,p_2\}})$$

is surjective. From (6) we get $h^1(N_X(-1)) = 0$.

Since $h^1(N_X(-1)) = 0$, X is smoothable ([13, Corollary 1.2]). □

Call $U(t, k, d', g')$ the set of all curves $X = A \cup B \cup Y$ appearing in Lemma 2. For all integer $y \ge 0$ and $x \ge y+3$ the Hilbert scheme of smooth space curves of degree x and genus y is irreducible ([7, 8]). By Lemma 2 there is a unique irreducible component $W(t, k, d', g')$ of the Hilbert scheme of $\mathbb{P}^3$ containing the curve X of Lemma 2. A general $C \in W(t, k, d', g')$ is smooth and $h^1(N_C(-1)) = 0$. We have $\deg(C) = d' + \deg(C_t) + \deg(C_k) = d' + t(t+1)/2 + k(k+1)/2$ and genus $g(C) = g' + p_a(C_t) + p_a(C_k) = g' - 2 + t(t+1)(2t-5)/6 + k(k+1)(2k-5)/6$.

3. Assertion $M(s, t, k)$, $k \in \{t-1, t\}$

For any $t \ge 27$, set $c(2t+1, t, t) = t+3$, $d(2t+1, t, t) = 0$, $c(2t, t, t-1) = t+2$ and $d(2t, t, t-1) = t-1$. Set $g(t+k+1, t, k) := c(t+k+1, t, k) - 3$. Note that if $k \in \{t-1, t\}$ we have

$$t(t+1) + k(k+1) + d(t+k+1, t, k) = (t+k)(t+k+4 - c(t+k+1, t, k)) \tag{7}$$

Now fix an integer $s \ge t+k+3$ with $s-t-k-1 \equiv 0 \pmod 2$ and define the integers $c(s,t,k)$, $g(s,t,k)$ and $d(s,t,k)$ in the following way. Set

$$c(s,t,k) := \lfloor \frac{\binom{s+3}{3} - sd_{t,k} - 6 - 3(s-t-k-1)/2 + g_{t,k}}{s-1} \rfloor,$$

$$d(s,t,k) := \binom{s+3}{3} - sd_{t,k} - 6 - 3(s-t-k-1)/2 + g_{t,k} - (s-1)c(s,t,k),$$

and $g(s,t,k) := c(s,t,k) - 3 - 3(s-t-k-1)/2$. Note that

$$s(d_{t,k}+c(s,t,k))+3-g_{t,k}-g(s,t,k)+d(s,t,k) = \binom{s+3}{3},\ 0 \le d(s,t,k) \le s-2 \tag{8}$$

and (8) holds even if $s = t+k+1$. From (8) for the integers $s+2$ and s and the equality $g(s+2,t,k) - g(s,t,k) = c(s+2,t,k) - c(s,t,k) - 3$ we get

$$\begin{aligned} &2d_{t,k} + 2c(s,t,k) + (s+1)(c(s+2,t,k) - c(s,t,k))+ \\ &d(s+2,t,k) - d(s,t,k) + 3 = (s+3)^2 \end{aligned} \tag{9}$$

Remark 4. We have $c(2t+1,t,t) = t+3$, $d(2t+1,t,t) = 0$, $c(2t,t,t-1) = t+2$, $d(2t,t,t-1) = t-1$, $c(2t+2,t,t-1) = 2t+6$, $d(2t+2,t,t-1) = 2t-3$, $c(2t+3,t,t) = 2t+7$, $d(2t+3,t,t) = 2t-1$.

Remark 5. We explain here the main reason for the assumption $t \ge 27$ made in this section. Fix an integer $s \ge t+k+1$ with $s \equiv t+k+1 \pmod 2$. We work with a curve $X = C_{t,k} \sqcup A$ with A a general smooth curve of degree $c(s,t,k)$ and genus $g(s,t,k)$ and we need $h^1(N_X(-2)) = 0$, i.e. we need $h^1(N_A(-2)) = 0$. We have $c(s,t,k) \ge g(s,t,k)+3$. By Lemma 4 below we have $g(s,t,k) \ge g(t+k+1,t,k)$. We have $g(2t+1,t,t) = t \ge 27$ and $g(2t,t,t-1) = t-1 \ge 26$. Since $g(s,t,k) \ge 26$, Remark 3 gives $h^1(N_A(-2)) = 0$.

Lemma 3. *For all integer $s \ge t+k+1$ with $s \equiv t+k-1 \pmod 2$ and $t \ge 27$ we have $g_{\lceil (s+1)/2 \rceil, \lfloor (s+1)/2 \rfloor} > g_{t,k} + g(s,t,k)$.*

Proof. The lemma is true if $s = t+k+1$ by the explicit value of $g(t+k+1,t,k) = c(t+k+1,t,k) - 3$ (Remark 4). Now let $s \ge t+k+3$ and assume that the lemma is true for the integer $s-2$. Since $s-2 \ge t+k+1$ the inductive assumption gives $g_{\lceil (s-1)/2 \rceil, \lfloor (s-1)/2 \rfloor} > g_{t,k} + g(s-2,t,k)$. Thus it is sufficient to check that $g_{\lceil (s+1)/2 \rceil, \lfloor (s+1)/2 \rfloor} - g_{\lceil (s-1)/2 \rceil, \lfloor (s-1)/2 \rfloor} \ge g(s,t,k) - g(s-2,t,k) = c(s,t,k) - c(s-2,t,k) - 3$. An elementary numerical computation shows that this inequality holds for any $s > t \ge 27$: indeed, the key point is that the difference $g_{\lceil (s+1)/2 \rceil, \lfloor (s+1)/2 \rfloor} - g_{\lceil (s-1)/2 \rceil, \lfloor (s-1)/2 \rfloor}$ is quadratic in s by definition of $g_{t,k}$, while the difference $c(s,t,k) - c(s-2,t,k)$ is linear in s by (9). □

Lemma 4. *For each $s \ge t+k+1$ with $s \equiv t+k-1 \pmod 2$ we have $2(c(s+2,t,k) - c(s,t,k)) \ge s+4$.*

Proof. Since $g_{t,k} + g(s,t,k) < g_{\lceil (s+1)/2 \rceil, \lfloor (s+1)/2 \rfloor}$ (Lemma 3), (8) for s,t,k and (1) for $t' = \lceil (s+1)/2 \rceil$ and $k' = \lfloor (s+1)/2 \rfloor$ imply $d_{t',k'} \ge c(s,t,k) + d_{t,k}$. Remark 4 gives $c(s+2,t',k') = k'+3$. Since $0 \le d(s+2,t,k) \le s$ and $0 \le d(s,t,k) \le s-2$, (9) and the difference between (8) for $s' := s+2$ and (4) for t', k' imply $c(s+2,t,k) - c(s,t,k) \ge -1 + c(s+2,t',k') = \lfloor (s+1)/2 \rfloor + 2$. □

Let $Q := \mathbb{P}^1 \times \mathbb{P}^1$. The elements of $|\mathcal{O}_Q(0,1)|$ are the fibers of the projection $\pi_2 : Q \to \mathbb{P}^1$, so that each $D \in |\mathcal{O}_Q(1,0)|$ contains exactly one point of each fiber of π_2.

Assertion $M(s,t,k)$, $k \in \{t-1,t\}$, $s \geq t+k+1$, $s \equiv t+k+1 \pmod 2$: Set $e = 1$ if $0 \leq d(s,t,k) \leq c(s+2,t,k) - c(s,t,k) - 3$ and $e = 2$ if $d(s,t,k) > c(s+2,t,k) - c(s,t,k) - 3$. There is a 6-tuple $(X, Q, D_1, D_2, S_1, S_2)$ such that

(a) Q is a smooth quadric surface, $X = C_{t,k} \sqcup Y$, Y is a smooth curve of degree $c(s,t,k)$ and genus $g(s,t,k)$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;

(b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, each of them containing one point of $Y \cap Q$, $S_i \subset D_i \setminus D_i \cap Y$, $1 \leq i \leq 2$, and $\sharp(S_1) + \sharp(S_2) = d(s,t,k)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$; $S_2 = \emptyset$ and $\pi_2(S_1) \subseteq \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$ if $e = 1$, $\sharp(S_2) = d(s,t,k) - c(s+2,t,k) + c(s,t,k) + 3$ and $\pi_2(S_2) \subseteq \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$ if $e = 2$;

(c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(s)) = 0$, $i = 0, 1$.

Remark 6. Fix lines $L, R \subset \mathbb{P}^3$ such that $L \cap R = \emptyset$ and $o \in \mathbb{P}^3 \setminus (L \cup R)$. Let $\ell : \mathbb{P}^3 \setminus \{o\} \to \mathbb{P}^2$ denote the linear projection from o. We have $\sharp(\ell(L) \cap \ell(R)) = 1$, i.e. there is a unique line $D(L,R,o) \subset \mathbb{P}^3$ such that $o \in D(L,R,o)$, $D(L,R,o) \cap L \neq \emptyset$ and $D(L,R,o) \cap R \neq \emptyset$. We have $\sharp(D(L,R,o) \cap L) = \sharp(D(L,R,o) \cap R) = 1$. The function $(L,R,o) \mapsto D(L,R,o)$ is regular.

Remark 7. For any $o \in \mathbb{P}^3$ let $\chi(o)$ denote the first infinitesimal neighbourhood of o in $\mathbb{P}^3$, i.e. the closed subscheme of $\mathbb{P}^3$ with $\mathcal{I}_o^2$ as its ideal sheaf. For any surface $F \subset \mathbb{P}^3$ and any scheme $B \subset \mathbb{P}^3$ let $\mathrm{Res}_F(B)$ denote the closed subscheme of $\mathbb{P}^3$ with $\mathcal{I}_B : \mathcal{I}_F$ as its ideal sheaf. We have $\mathrm{Res}_F(B) \subseteq B$. If B is the disjoint union of closed subschemes B_1 and B_2 then $\mathrm{Res}_F(B) = \mathrm{Res}_F(B_1) \cup \mathrm{Res}_F(B_2)$. If B is reduced then $\mathrm{Res}_F(B)$ is the union of the irreducible components of B not contained in F. If $o \notin F$ then $\mathrm{Res}_F(\chi(o)) = \chi(o)$. If $o \in F$ and F is smooth at o then $\mathrm{Res}_F(\chi(o)) = \{o\}$.

Lemma 5. *For all $t \geq 27$ and $k \in \{t-1,t\}$ assertion $M(t+k+1,t,k)$ is true.*

Proof. Fix $C_{t,k}$ intersecting Q at $2d_{t,k}$ general points ([29]).

(a) Assume $k = t$. We have $c(2t+1,t,t) = t+3$ and $d(2t+1,t,t) = 0$ and so we take $e = 1$ with $S_1 = S_2 = \emptyset$. Take any $A \in |\mathcal{O}_Q(2,t+1)|$ with $A \cap C_{t,k} = \emptyset$. We have $\mathrm{Res}_Q(C_{t,t} \cup A) = C_{t,t}$ and thus $h^i(\mathcal{I}_{\mathrm{Res}_Q(C_{t,t} \cup A)}(2t-1)) = 0$, $i = 0,1$. We have $h^i(Q, \mathcal{I}_{Q \cap (C \cap A)}(2t+1, 2t+1)) = h^i(Q, \mathcal{I}_{C_{t,t} \cap Q}(2t-1,t)) = 0$, $i = 0,1$, by (7) and the generality of $C_{t,k} \cap Q$. Hence $h^i(\mathcal{I}_{C_{t,k} \cup A}(2t+1)) = 0$, $i = 0,1$.

We deform A to a curve Y of degree $t+3$ and genus t with $Y \cap C_{t,k} = \emptyset$, Y intersecting transversally Q and with no line of Q containing ≥ 2 points of $Q \cap (C_{t,k} \cup Y)$. By the semicontinuity theorem for cohomology ([15, III.8.8]), for a general Y we have $h^i(\mathcal{I}_{C_{t,k} \cup Y}(2t+1)) = 0$, $i = 0,1$. Set $X := C_{t,k} \cup Y$, $S_1 = S_2 = \emptyset$ and take as D_1 and D_2 any two different elements of $|\mathcal{O}_Q(1,0)|$, each of them containing one point of $Y \cap Q$.

(b) Assume $k = t-1$. We have $c(2t,t,t-1) = t+2$, $d(2t,t,t-1) = t-1$ and $c(2t+2,t,t-1) - c(2t,t,t-1) = t+4$ (Remark 4). Hence $e = 1$. However, in the proof of $M(t+k+1,t,k)$ we will exchange the two rulings (as we will do below for the general proof that $M(s,t,k) \Longrightarrow M(s+2,t,k)$), so that $D_1, D_2 \in |\mathcal{O}_Q(0,1)|$. Take lines $L_1, L_2 \in |\mathcal{O}_Q(1,0)|$ such that $L_1 \neq L_2$ and $C_{t,t-1} \cap (L_1 \cup L_2) = \emptyset$, and

t different lines $R_j \in |\mathcal{O}_Q(0,1)|$, $1 \le j \le t$, none of them containing a point of $C_{t,t-1} \cap Q$. Fix $D_1, D_2 \in |\mathcal{O}_Q(0,1)|$ containing no point of $C_{t,t-1} \cap Q$ and with $D_h \neq R_j$ for all h, j. Set $u_h := L_1 \cap D_h$, $h = 1,2$. Fix $E_1 \subset D_1$ with $\sharp(E_1) = t-1$ and $E_1 \cap (L_1 \cup L_2) = \emptyset$. We have $h^1(Q, \mathcal{I}_{E_1}(2t-2,t)) = 0$. Since $C_{t,k} \cap Q$ is a general subset of Q with cardinality $2d_{t,k}$, we have $h^i(Q, \mathcal{I}_{Q\cap(C\cap A)\cup E_1}(2t,2t)) = h^i(Q, \mathcal{I}_{(C_{t,t}\cap Q)\cup E_1}(2t-2,t)) = 0$, $i = 0,1$, by (7). The residual sequence of Q gives $h^i(\mathcal{I}_{C_{t,k}\cup A\cup E_1}(2t)) = 0$, $i = 0,1$.

Take an ordering $\{o_1, \dots, o_{t-1}\}$ of E_1 and let M_i the only element of $|\mathcal{O}_Q(1,0)|$ with $o_i \in M_i$. Set $w_i := R_i \cap M_i$, $1 \le i \le t-1$. We fix a deformation $\{L_h(\lambda)\}_{\lambda\in\Lambda}$, $h = 1,2$, of L_h with the following properties: Λ is a connected and affine smooth curve, $o \in \Lambda$, $L_h(o) = L_h$, $u_h \in L_h(\lambda)$ for all λ, $L_1(\lambda) \cap L_2(\lambda) = \emptyset$ for all λ and $L_h(\lambda)$ is transversal to Q for all $\lambda \neq o$. For each i with $1 \le i \le t-1$ there is a unique line $R_i(\lambda)$ containing w_i and intersecting both $L_1(\lambda)$ and $L_2(\lambda)$ (Remark 6). There is a deformation $\{R_t(\lambda)\}_{\lambda\in\Lambda}$ of R_t with $R_t(o) = R_t$, $R_t(\lambda)$ intersecting both $L_1(\lambda)$ and $L_2(\lambda)$. Taking instead of Λ a smaller neighborhood of o we may assume $R_i(\lambda) \cap R_j(\lambda) = \emptyset$ for all $i \neq j$ and all λ so that $A(\lambda) := L_1(\lambda) \cup L_2(\lambda) \cup R_1(\lambda) \cup \cdots \cup R_t(\lambda)$ is a connected nodal curve of degree $t+2$ and arithmetic genus $t-1$. By semicontinuity (restricting if necessary Λ to a neighborhood of o) we have $h^i(\mathcal{I}_{C_{t,k}\cup A(\lambda)\cup E_1}(2t)) = 0$, $i = 0,1$, for all $\lambda \in \Lambda$. Fix $\lambda_0 \in \Lambda \setminus \{o\}$. Let $\{B_\delta\}_{\delta\in\Delta}$ be a smoothing of $A(\lambda_0)$ fixing u_1 and u_2, i.e. take a smooth and connected affine curve Δ and $a \in \Delta$ with $B_a = A(\lambda_0)$, B_δ a smooth curve of degree $t+2$ and genus $t-1$ and $\{u_1, u_2\} \subset B_\delta$ for all δ. Restricting if necessary Δ we may assume that B_δ is transversal to Q and disjoint from $C_{t,k} \cup E_1$ for all $\delta \in \Delta$ and (by semicontinuity) that $h^i(\mathcal{I}_{C_{t,k}\cup B_\delta\cup E_1}(2t)) = 0$, $i = 0,1$. Since $A(\lambda_0)$ is transversal to Q, we may (up to a finite covering of Δ) find $t-1$ sections $s_1, \dots, s_{t-1}$ of the family $\{B_\delta \cap Q\}_{\delta\in\Delta}$ of $2t+4$ ordered points of Q with $s_i(a) = w_i$, $i = 1, \dots, t-1$. Let $M_j(\delta)$, $\delta \in \Delta$, be the only element of $|\mathcal{O}_Q(1,0)|$ with $w_i \in M_i(\delta)$. Set $o_i(\delta) := L_1 \cap M_i(\delta)$ and $E_1(\delta) := \{o_1(\delta), \dots, o_{t-1}(\delta)\}$. By semicontinuity for a general $\delta \in \Delta \setminus \{a\}$ we have $h^i(\mathcal{I}_{C_{t,k}\cup B_\delta\cup E_1(\delta)}(2t)) = 0$. We fix such a δ and set $X := C_{t,k} \cup B_\delta$, $S_1 := E_1(\delta)$, $S_2 := \emptyset$. For $M(2t,t,t-1)$ we use the lines D_1, D_2 and $M_j(\delta)$, $1 \le j \le t-1$. □

Lemma 6. *For each integer $s \ge t+k+1$ such that $s \equiv t+k+1 \pmod 2$ we have $2c(s,t,k) \ge s+4$ and $2c(s,t,k) \ge s+6$ is $s \ge t+k+3$.*

Proof. The case $s = t+k+1$ is true by Remark 4. The general case follows by induction $s-2 \Longrightarrow s$ by Lemma 4. □

We need the following auxiliary result, proved in [5, Lemma 2.5] and [17, bottom of page 176].

Lemma 7. *Fix lines $D, L \subset \mathbb{P}^3$ such that $D \cap L$ is a point o and $q \in L \setminus \{o\}$. Then there is a family $\{L_\lambda\}_{\lambda\in\mathbb{K}}$ of lines of $\mathbb{P}^3$ such that $L_0 = L$, $L_\lambda \cap D = \emptyset$ for all $t \neq 0$, $D \cup L \cup \chi(o)$ is a flat limit of the family $\{D \cup L_\lambda\}_{\lambda\in\mathbb{K}\setminus\{0\}}$.*

Proof. Take homogenous coordinates x_0, x_1, x_2, x_3 such that $o = (1:0:0:0)$, $D = \{x_1 = x_2 = 0\}$, $L = \{x_1 = x_3 = 0\}$ and $q = (0:0:1:0)$. Take $L_\lambda = \{x_1 + \lambda x_0 = x_3 = 0\}$. Note that $L_0 = L$ and that $L \cap D = \emptyset$ for all $\lambda \neq 0$. Set $Y_\lambda := D \cup L_\lambda$. For $\lambda \neq 0$ the ideal sheaf of scheme Y_t is generated by the quadrics $x_1(x_1 + \lambda x_0)$, $x_2(x_1 + \lambda x_0)$, x_1x_3, x_2x_3, while the ideal sheaf of the scheme Y_0 is determined by the quadrics x_1^2, x_1x_2, x_1x_3 and x_2x_3. This algebraic family of projective schemes is flat because it has constant Hilbert polynomial [15, III.9.8.4]. □

Lemma 8. *Assume $t \ge 27$ and $k \in \{t-1, t\}$. Fix an integer $s \ge t+k+1$ such that $s \equiv t+k+1 \pmod 2$. If $M(s,t,k)$ is true, then $M(s+2,t,k)$ is true.*

Proof. Let $e \in \{1,2\}$ be the integer arising in $M(s,t,k)$ and $f \in \{1,2\}$ the corresponding integer for $M(s+2,t,k)$. Take $(X, Q, D_1, D_2, S_1, S_2)$ satisfying $M(s,t,k)$ with $X = C_{t,k} \sqcup Y$ and $D_1, D_2 \in |\mathcal{O}_Q(1,0)|$. The 6-tuple $(X', Q, D_1', D_2', S_1', S_2')$ will be a solution after exchanging the two rulings of Q, i.e. we will take $D_1', D_2' \in |\mathcal{O}_Q(0,1)|$ and we use π_1 instead of π_2. In each step with $d(s,t,k) \neq 0$ we obtain X' smoothing a curve W union of X, $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$, $e+1$ elements $|\mathcal{O}_Q(1,0)|$ and $c(s+2,t,k) - c(s,t,k) - e - 1$ elements of $|\mathcal{O}_Q(0,1)|$. See step (c) for the easier case $d(s,t,k) = 0$ (here to get W we add to X a line $D_0 \in |\mathcal{O}_Q(1,0)|$ and $c(s+2,t,k) - c(s,t,k) - 1$ elements of $|\mathcal{O}_Q(0,1)|$).

(a) Assume $e = 2$ and set $z := d(s,t,k) + 3 - c(s+2,t,k) + c(s,t,k)$. Since $d(s,t,k) \le s-2$, Lemma 4 gives $d(s,t,k) \le 2(c(s+2,t,k) - c(s,t,k) - 3)$, i.e. $z \le c(s+2,t,k) - c(s,t,k) - 3$. By assumption there is $E \subset Y \cap (Q \setminus (D_1 \cup D_2))$ such that $\sharp(E) = z$ and $\pi_2(E) = \pi_2(S_2) \subseteq \pi_2(S_1)$. Take a line $D_0 \in |\mathcal{O}_Q(1,0)|$ different from D_1, D_2, with $D_0 \cap E = \emptyset$, $D_0 \cap C_{t,k} \cap Q = \emptyset$ and $D_0 \cap Y \cap Q \neq \emptyset$; we use that $2c(s,t,k) \ge 3 + z$ (Lemma 6). Take distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le c(s+2,t,k) - c(s,t,k) - 3$, such that $L_i \cap Y \neq \emptyset$ if and only if $i \le z$, $X \cap (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} L_i) = E$, $L_i \cap (C_{t,k} \cap Q) = \emptyset$ for all i. Set $J := (D_0 \cup D_1 \cup D_2) \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} L_i)$. We fix f general lines $R_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le f$, and $A_i \subset R_i$, $1 \le i \le f$, with the conditions $\sum_{i=1}^{f} \sharp(A_i) = b(s+2,t,k)$, $\pi_1(A_f) \subseteq \pi_1(A_1)$ and $\pi_1(A_f) \subseteq \pi_1(Y \cap (Q \setminus J))$. Set $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$, $A := A_1 \cup A_2$ and $W := X \cup J \cup \chi$.

Claim 1: There is an affine connected smooth curve Δ such that the scheme $Y \cup J \cup \chi$ is a flat degeneration of a family of unions of $Y \cup D_0 \cup D_1 \cup D_2$ and $c(s+2,t,k) - c(s,t,k) - 3$ lines $L_{i\lambda}$, $\lambda \in \Delta$, such that $L_{i\lambda} \cap Y \neq \emptyset$ if and only if $i \le z$, $D_0 \cap L_{i\lambda} = D_0 \cap L_i$ for all i, $D_1 \cap L_{i\lambda} = \emptyset$ for all $\lambda \in \Delta$ and all i, and $D_2 \cap L_{i\lambda} \neq \emptyset$ (and it is a point) if and only if $z+1 \le i \le c(s+2,t,k) - c(s,t,k) - 3$.

Proof of Claim 1: As Δ we take a suitable curve contained in the affine manifold $\prod_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} \Delta_i$, which we are now going to define (in the use of Claim 1 we only need that Δ is irreducible, so we could use a similar claim, but with this connected manifold instead of the irreducible curve Δ). Each Δ_i is a smooth connected curve, so $\prod_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} \Delta_i$ is irreducible. We describe each $L_{i\lambda}$ with L_i as a limit separately for each i; Δ_i is the parameter space for the line $L_{i\lambda}$. We need to modify the proof of Lemma 7 in the following way. First assume $z < i \le c(s+2,t,k) - c(s,t,k) - 3$. In this case we fix the point $q_i := D_0 \cap L_i$ and use as Δ_i a Zariski open neighborhood of $D_2 \cap L_i$; for each $q \in D_2$ there is a unique line $L(q_i, q)$ containing $\{q_i, q\}$; when q goes to $D_2 \cap L_i$ the line $L(q_i, q)$ goes to the line; we need to restrict Δ_i to avoid the points q such that $L(q_i, q) \cap (Y \cup C_{t,k} \cup D_1) \neq \emptyset$. Now assume $i \le z$. We fix the point $q_i := Y \cap L_i$ and take as Δ_i a Zariski neighborhood of q_i in Y; since $q_i \notin D_0$ for each $q \in D_0 \setminus Y \cap D_0$ there is a unique line $L(q_i, q)$ containing $\{q_i, q\}$; we need to restrict Δ_i to avoid the points q such that $L(q_i, q) \cap (Y \cup C_{t,k}) \neq \{q\}$. We restrict $\Delta_1 \times \cdots \times \Delta_{c(s+2,t,k)-c(s,t,k)-3}$ to a non-empty Zariski open subset U such that for all $\lambda \in U$ the union U' of $Y \cup D_0 \cup D_1 \cup D_2$ and the line $L_{i\lambda}$, $1 \le i \le c(s+2,t,k) - c(s,t,k) - 3$ is nodal and it has no singular point which is not prescribed by the construction. Note that U' is connected and $p_a(U') = g(s+2,t,k)$.

Claim 2: W is a flat degeneration of a disjoint union of $C_{t,k}$ and a smooth curve of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$.

Proof of Claim 2: Since $C_{t,k} \cap (J \cap \chi) = C_{t,k} \cap Y = \emptyset$, it is sufficient to prove that $Y \cup J \cup \chi$ is a flat degeneration of a family of smooth curves of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$. By Claim 1, $Y \cup J \cup \chi$ is a flat degeneration of a family of unions of $Y \cup D_0 \cup D_1 \cup D_2$ and $c(s+2,t,k) - c(s,t,k) - 3$ disjoint lines, none of them intersecting $D_1 \cup D_2$ and each of them intersecting $Y \cup D_0 \cup D_1 \cup D_2$ quasi transversely at exactly two points. We first prove that $Y \cup J \cup \chi$ is a flat degeneration of a family of unions of $Y \cup D_0 \cup D_1 \cup D_2$ and $c(s+2,t,k) - c(s,t,k) - 3$ lines $L_{i\lambda}$, $\lambda \in \mathbb{K} \setminus \{0\}$, such that $L_{\cap i\lambda} \cap Y \neq \emptyset$ if and only if $i \le z$, $D_0 \cap L_{i\lambda} = D_0 \cap L_i$ for all i and $D_1 \cap L_{i\lambda} = D_2 \cap L_{i\lambda} = \emptyset$ for all $\lambda \in \mathbb{K} \setminus \{0\}$. We may do this smoothing separately, first for D_0, D_1, D_2 and then for each line $L_{i\lambda}$ quoting each time Lemma 2 and following the deformation with a family of lines with special fiber $L_{j\lambda}$, $j \neq i$, because any two points of $\mathbb{P}^3$ uniquely determine a line and the line depends regularly if we move regularly the two points (see the Side Remark below), but we may do all the smoothings simultaneously just choosing the appropriate references from [18] or [30] or other sources, e.g. Lemma 2 or [18, Corollary 5.2].

Side Remark: As the reader may have noticed in the proof Claim 1 we only used Lemma 7 (i.e. a known result) and the fact that two different points q, q' of $\mathbb{P}^r$, $r \ge 3$, uniquely determine a line $L(q,q') \subset \mathbb{P}^3$ and the regularity of the map $(q,q') \to L(q,q')$ from $\mathbb{P}^r \times \mathbb{P}^r \setminus \Delta_{\mathbb{P}^r}$, where $\Delta_{\mathbb{P}^r}$ is the diagonal, to the Grassmaniann $G(1,r)$. Lemma 7 is true in a more general situation, as the flat limit of a two smooth germs of curves colliding to an ordinary node; call o this nodal point. Instead of the lines D, L we take the tangent lines to the two smooth germs of curves. Their linear span determines an element of $G(3,r)$ and we consider the first infinitesimal neighborhood $\chi(o)$ of o in a 3-dimensional projective space which is a limit of these elements of $G(3,r)$. In the literature Claims 1 and 2 are often used, but without separating them. For the algebraic geometers of our generation the first instance of this flat limit with a nilpotent was [15, III.9.8.4 and figure 11 at page 260].

To obtain a smoothing of W as in the Claim 2, but compatible with the data A_1, A_2, see steps (a1) and (a2). We have $\mathrm{Res}_Q(W \cup A) = X \cup S_1 \cup S_2$ and so $h^i(\mathcal{I}_{\mathrm{Res}_Q(W\cup A)}(s)) = 0$, $i = 0,1$. We have $h^i(Q, \mathcal{I}_{(W\cap Q)\cup A}(s+2,s+2)) = h^i(Q, \mathcal{I}_{(X\cap(Q\setminus J)\cup A}(s-1, s+5+c(s,t,k)-c(s+2,t,k)))$. We have $\sharp((X \cap (Q \setminus J)) \cup A) = h^0(Q, \mathcal{O}_Q(s-1, s+5+c(s,t,k)-c(s+2,t,k))$. We have $h^1(Q, \mathcal{I}_A(s-1, s+5+c(s,t,k)-c(s+2,t,k))) = 0$, because $s+5+c(s,t,k)-c(s+2,t,k) > 0$, $f \le 2$ and $\sharp(A_1) \le s$; this is a key reason for our definition of $M(s+2,t,k)$. Therefore to prove that $h^i(Q, \mathcal{I}_{(X\cap(Q\setminus J)\cup A}(s-1, s+5+c(s,t,k)-c(s+2,t,k))) = 0$, $i = 0,1$, it is sufficient to prove that we may take as $X \cap (Q \setminus J)$ a general subset of Q with its prescribed cardinality. By Remark 5 we have $h^1(N_X(-2)) = 0$. Since $h^1(N_X(-2)) = 0$, we may deform X keeping fixed E so that the other points are general in Q.

(a1) We have just proved that $h^i(\mathcal{I}_{W\cup A}(s+2)) = 0$, $i = 0,1$. If $d(s+2,t,k) = 0$, then $M(s+2,t,k)$ is proved for $e = 2$. Now assume $d(s+2,t,k) > 0$. To prove $M(s+2,t,k)$ when $e = 2$ we need to deform W to a smooth $X' = C_{t,k} \sqcup Y'$ intersecting transversally Q and (perhaps moving A) to obtain condition (b) of

$M(s+2,t,k)$. Set $P_i := Y \cap D_i$, $i = 0,1,2$. Let $\{D_i(\lambda)\}_{\lambda\in\Lambda}$ be a deformation of D_i with Λ a smooth and connected affine curve, $o \in \Lambda$, $D_i(o) = D_i$, $D_i(\lambda)$, $\lambda \in \Lambda\setminus\{o\}$, a line of $\mathbb{P}^3$ transversal to Q and containing P_i. Fix $i \in \{1,\dots,z\}$. By Remark 6 for each $\lambda \in \Lambda$ there is a unique line $L_i(\lambda) \subset \mathbb{P}^3$ such that $D_0 \cap L_i \in L_i(\lambda)$, $L_i(\lambda) \cap D_1(\lambda) \neq \emptyset$ and $L_i(\lambda) \cap D_2(\lambda) \neq \emptyset$; restricting if necessary Λ we may assume that all $L_i(\lambda)$, $\lambda \neq o$, are transversal to Q. Fix an integer i with $z < i \leq c(s+2,t,k) - c(s,t,k) - 3$ and fix a general $m_i \in L_i$. By Remark 6 there is a unique line $L_i(\lambda)$ such that $m_i \in L_i(\lambda)$, $L_i(\lambda) \cap D_1(\lambda) \neq \emptyset$ and $L_i(\lambda) \cap D_2(\lambda) \neq \emptyset$; restricting if necessary Λ we may assume that all $L_i(\lambda)$, $\lambda \neq o$, are transversal to Q. Restricting if necessary Λ to a smaller neighborhood of o in Λ we may assume that $L_i(\lambda) \cap L_j(\lambda) = \emptyset$ for all $i \neq j$, that $C_{t,k} \cap L_i(\lambda) = \emptyset$ for all i and all λ, that $L_i(\lambda) \cap D_0 \neq \emptyset$ if and only if $i \leq z$. Fix a general $\lambda \in \Lambda$ and set $J(\lambda) := D_0(\lambda) \cup D_1(\lambda) \cup D_2(\lambda) \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} L_i(\lambda))$. Let $\chi(\lambda)$ be the union of all $\chi(q)$ with either $q \in D_1(\lambda) \cap L_i(\lambda)$, $1 \leq i \leq c(s+2,t,k) - c(s,t,k) - 3$ or $q \in D_2(\lambda) \cap L_i(\lambda)$, $1 \leq i \leq z$. Set $W(\lambda) := X \cup J(\lambda) \cup \chi(\lambda)$. $W(\lambda)$ is the disjoint union of $C_{t,k}$ and of a degeneration of a flat family of smooth and connected curves of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$. As in the first part of step (a), restricting if necessary Λ, by semicontinuity we get $h^i(\mathcal{I}_{W(\lambda)\cup A}(s+2)) = 0$, $i = 0,1$.

(a2) To prove $M(s+2,t,k)$ we need to prove that there is a set like A (call it A') satisfying both $h^i(\mathcal{I}_{W(\lambda)\cup A'}(s+2)) = 0$, $i = 0,1$, and condition (b) of $M(s+2,t,k)$. First of all, instead of P_i, $0 \leq i \leq 2$, we take a family $\{P_i(\lambda)\}_{\lambda\in\Lambda}$ of points of Y with $P_i(o) = P_i$ and $P_i(\lambda) \in Y \setminus Y \cap Q$ for all $\lambda \in \Lambda\setminus\{o\}$. Assume for the moment $f = 2$. We modify the definition of $D_i(\lambda)$, because we impose that $P_i(\lambda) \in D_i(\lambda)$ (instead of $P_i \in D_i$), but we also impose that $D_1(\lambda) \cap R_1 \neq \emptyset$ and $D_2(\lambda) \cap R_2 \neq \emptyset$ (this is possible by Remark 6). Then we construct $L_i(\lambda)$ as above. With this new definition R_1 and R_2 are secant lines of $W(\lambda) \setminus (C_{t,k} \cup Y)$, $Y \subset W(\lambda)$, $\pi_1(A_2) \subseteq \pi_1(A_1)$ and $\pi_1(A_f) \subseteq \pi_1(Q \cap (Y \setminus J(\lambda) \cap Y))$; call $m_1, \dots, m_x$, $x = \sharp(A_f)$, the points of $Y \cap Q$ whose image is $\pi_1(A_f)$. We fix $\lambda \in \Lambda \setminus \{o\}$. Let $\{B_\delta\}_{\delta\in\Delta}$ be a smoothing of $W(\lambda)$ with Δ an affine and connected smooth curve, $a \in \Delta$, and $B_a = W(\lambda)$. Set $A(a) := A$. Since Y is transversal to Q, up to a finite covering of Δ we may find $x+2$ sections $s_1,\dots,s_x,z_1,z_2$ of the total space of $\{B_\delta\}_{\delta\in\Delta}$ with $s_i(a) = m_i$, $z_1(a) = R_1 \cap D_1(\lambda)$, $z_2(a) = R_2 \cap D_2(\lambda)$, $s_i(\delta) \in B_\delta \cap Q$, $z_1(\delta) \in B_\delta \cap Q$ and $z_2(\delta) \in B_\delta \cap Q$ for all Δ. Let $R_h(\delta)$, $h = 1,2$, be the only element of $|\mathcal{O}_Q(0,1)|$ containing $z_h(\delta)$. For each $\delta \in \Delta \setminus \{a\}$ and $i \in \{1,\dots,x\}$ let $M_i(\delta) \in |\mathcal{O}_Q(1,0)|$ be the only line of this ruling of Q containing $s_i(\delta)$. Set $A_1(\delta) := \cup_{i=1}^{x}(R_1(\delta) \cap M_i(\delta))$ and $A_2(\delta) := \cup_{i=1}^{d(s+2,t,k)-x}(R_2(\delta) \cap M_i(\delta))$. Set $X_\delta := C_{t,k} \cup B_\delta$. By construction $(X_\delta, Q, R_1, R_2, A_1(\delta), A_2(\delta))$ satisfies condition (b) of $M(s+2,t,k)$, exchanging the two rulings of Q. By semicontinuity we have $h^i(\mathcal{I}_{B_\delta\cup A(\delta)}(s+2)) = 0$, $i = 0,1$, for a general $\delta \in \Delta$.

Now assume $f = 1$. In this case we only impose that $D_i(\lambda)$ meets R_1; we have $\pi_1(A_1) \subset \pi_1(Q \cap (Y \setminus J(\lambda) \cap Y))$ and $x = \sharp(A_1) = b(s+2,t,k)$.

(b) Assume $e = 1$ and $d(s,t,k) > 0$, i.e. assume $0 < d(s,t,k) \leq c(s+2,t,k) - c(s,t,k) - 3$. We set $S_2 := 0$ and ignore D_2. We fix $o \in S_1$. Take a line $D_0 \neq D_1$ meeting $Y \cap Q$ and $c(s+2,t,k) - c(s,t,k) - 2$ distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, with $L_i \cap (C_{t,k} \cap Q) = \emptyset$ for all i, $L_i \cap (Y \cap Q) \neq \emptyset$ if and only if $1 \leq i \leq d(s,t,k) - 1$ and $S_1 \setminus \{o\} = D_1 \cap (L_1 \cup \dots \cup L_{d(s,t,k)-1})$. Set $J := (D_0 \cup D_1) \cup (\bigcup_{i=1}^{c(s+2,t,k)+c(s,t,k)-2} L_i)$

and $\chi := \cup_{o\in S_1}\chi(o)$. Note that $\chi(X\cup J\cup\chi)-\chi(X) = c(s,t,k)-c(s+2,t,k)+3$. To modify step (a2) we impose that $D_1(\lambda)\cap R_1 \ne \emptyset$ and $D_0(\lambda)\cap R_2 \ne \emptyset$.

(c) Assume $d(s,t,k) = 0$. Hence $S_1 = S_2 = \emptyset$. Take a line $D_0 \in |\mathcal{O}_Q(1,0)|$ different from D_1, D_2 and with $D_0\cap Y\cap Q \ne \emptyset$. Take $c(s+2,t,k)-c(s,t,k)-1$ lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le c(s+2,t,k)-c(s,t,k)-1$, such that $L_i \cap (C_{t,k}\cap Q) = \emptyset$ for all i and $L_i \cap (Y\cap Q) \ne \emptyset$ if and only if $1 \le i \le c(s+2,t,k)-c(s,t,k)-3$. Set $J := D_0 \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(m,t,k)-1} L_i)$, $Y' := Y\cup J$ and $W := X\cup J$. Note that $\chi(W)-\chi(X) = c(s,t,k)-c(s+2,t,k)+3$. The union Y' is a connected nodal curve, which is a flat degeneration of a family of smooth curves of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$ not intersecting $C_{t,k}$. As in step (a) we get $h^1(\mathcal{I}_W(s+2)) = 0$ and $h^0(\mathcal{I}_W(s+2)) = d(s+2,t,k)$. If $d(s+2,t,k) = 0$, then we are done, because $A = \emptyset$ and so condition (b) of $M(s+2,t,k)$ is trivially true. Now assume $d(s+2,t,k) > 0$.

First assume $f = 2$. As in step (a) we prove $M(s+2,t,k)$ interchanging the rulings of Q and set $x := c(s+4,t,k)-c(s+2,t,k)-3$. We fix general lines $R_1, R_2 \in |\mathcal{O}_Q(0,1)|$ and take $A_i \subset R_i$ such that $\pi_1(A_2) \subseteq \pi_1(A_1)\cap\pi_1(Q\cap(Y\setminus J\cap Y))$. Set $A := A_1\cup A_2$. For a general X we have $h^i(\mathcal{I}_{W\cup A}(s+2)) = 0$, $i = 0,1$. Set $q := D_0\cap Y$. By Remark 6 there is a family $\{D_0(\lambda)\}_{\lambda\in\Lambda}$ of lines of $\mathbb{P}^3$ and $o \in \Lambda$ with $D_0(o) = D_0$, $\sharp(D_0(\lambda)\cap Y) = 1$ for all λ, $D_0(\lambda)\cap Y \notin Q$ if $\lambda \ne 0$, $D_0(\lambda)\cap R_1 \ne \emptyset$ and $D_0(\lambda)\cap R_2 \ne \emptyset$. Up to a finite covering of Λ we may also find families $\{L_i(\lambda)\}_{\lambda\in\Lambda}$, $1 \le i \le c(s+2,t,k)-c(s,t,k)-1$. Set $J(\lambda) = D_0(\lambda) := D_0 \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-1} L_i(\lambda))$. We do the smoothing of $Y\cup J(\lambda)$ as in step (a2).

Finally, if $f = 1$ we only need $D_0(\lambda)\cap R_1 \ne \emptyset$ for all λ. □

4. With a constant genus g

We fix an integer $t \ge 27$ and take $k \in \{t-1, t\}$. We fix an integer $g \ge g_{t,k} + g(t+k+5,t,k)$. Let y be the maximal integer $\ge t+k+5$ such that $y \equiv t+k-1 \pmod 2$ and $g_{t,k}+g(y,t,k) \le g$ (y exists, because $\lim_{u\to+\infty} g(t+k+1+2u,t,k) = +\infty$). By the definition of y we have $y \ge t+k+5$ and $y \equiv t+k-1 \pmod 2$. For all integers $x \ge y+2$ with $x \equiv y \pmod 2$ define the integers $a(x,t,k,y)$ and $b(x,t,k,y)$ by the relation

$$xd_{t,k} + 3 - g + xa(x,t,k,y) + b(x,t,k,y) = \binom{x+3}{3},\ 0 \le b(x,t,k,y) \le x-1 \tag{10}$$

If $x \ge y+4$, by taking the difference between equation (10) and the same equation for the integer $x' := x-2$ we get

$$\begin{aligned}&2d_{t,k} + 2a(x,t,k,y) + (x+2)(a(x+2,t,k,y)-a(x,t,k,y))\\ &\qquad +b(x+2,t,k,y)-b(x,t,k,y) = (x+3)^2\end{aligned} \tag{11}$$

Lemma 9. *For each $x \ge y+2$ with $x \equiv y \pmod 2$ we have $2(a(x+2,t,k,y) - a(x,t,k,y)) \ge x+5$.*

Proof. Assume by contradiction $2(a(x+2,t,k,y)-a(x,t,k,y)) \le x+4$. Recall that for all $u \ge v > 0$ we have

$$(u+v-1)d_{u,v} + 2 - g_{u,v} = \binom{u+v+2}{3} \tag{12}$$

Claim 1: We have $g_{\lceil (y+3)/2\rceil, \lfloor (y+3)/2\rfloor} > g$.

Proof of Claim 1: By the definition of y we have $g(y+2,t,k)+g_{t,k} > g$. Thus to prove Claim 1 it is sufficient to use that $g(y+2,t,k)+g_{t,k} \le g_{\lceil (y+3)/2\rceil,\lfloor (y+3)/2\rfloor}$ (Lemma 3).

First assume x odd, i.e. $k=t$. Since $g_{(x+1)/2,(x+1)/2} \ge g_{(y+3)/2,(y+3)/2} > g$ by Claim 1, (12) and (10) give $d_{(x+1)/2,(x+1)/2} \ge d_{t,k}+a(x,t,k,y)$. Since $b(x+2,t,k,y) \le x+1$ and $b(x,t,k,y) \ge 0$ (11) gives

$$(x+1)(x+3)/2+(x+2)(x+4)/2+x+1 \ge (x+3)^2,$$

which is false. Now assume x even, i.e. $k=t-1$. Since $g_{(x+2)/2,x/2} \ge g_{(y+4)/2,(y+2)/2} > g$ by Claim 1, (12) and (10) gives $d_{(x+2)/2,x/2} \ge d_{t,k}+a(x,t,k,y)$. Since $b(x+2,t,k,y) \le x+1$ and $b(x,t,k,y) \ge 0$ (11) gives

$$(x+2)^2/2+(x+2)(x+4)/2+x+1 \ge (x+3)^2,$$

which is false. □

Lemma 10. *We have* $2(a(y+2,t,k,y)-c(y,t,k)) \ge y+5$.

Proof. Define the integers w,z by the relations

$$(y+2)(w+d_{t,k})+3-g_{t,k}-g(y,t,k)+z=\binom{y+5}{3},\ 0 \le z \le y+1 \tag{13}$$

Since $g \ge g_{t,k}+g(y,t,k)$, we have $w \le a(y+2,t,k)$. Hence it is sufficient to prove that $2(w-c(y,t,k)) \ge y+5$. Taking the difference between (13) and the case $s=y$ of (8) we get

$$2d_{t,k}+2c(y,t,k)+(y+2)(w-c(y,t,k))+z-d(y,t,k)=(y+3)^2$$

Then we continue as in the proof of Lemma 9 with $y+2$ instead of $x+2$. □

The next lemma follows at once by induction on x, the inequality $2c(y,t,k) \ge y+6$ and Lemmas 9 and 10.

Lemma 11. *We have* $2a(x,t,k,y) \ge x+6$ *for all integers* $x \ge y+2$ *with* $x \equiv y \pmod 2$.

Lemma 12. *For each* $x \ge y+2$ *with* $x \equiv y \pmod 2$ *we have* $a(x,t,k,y) \ge g-g_{t,k}+3$.

Proof. First assume $x=y+2$. By the definition of the integer y we have $g_{t,k}+g(y,t,k) \le g \le \tau := g_{t,k}+g(y+2,t,k)-1$. The integers $a(y+2,t,k,y)$ and $b(y+2,t,k,y)$ depend on the choice of g and (only for this proof) we call them $a(y+2,t,k,y)_g$ and $a(y+2,t,k,y)_g$. Fix integers q,q' such that $g_{t,k}+g(y,t,k) \le q \le q' \le \tau$. From (10) or (11) for q and q' we get

$$\begin{aligned}&(y+2)(a(y+2,t,k,y)_{q'}-a(y+2,t,k,y)_q)=\\ &b(y+2,t,k,y)_q-b(y+2,t,k,y)_{q'}+q'-q\end{aligned} \tag{14}$$

Since $0 \le b(y+2,t,k,y)_q \le y+1$ and $0 \le b(y+2,t,k,y)_{q'} \le y+1$, (14) implies $a(y+2,t,k,y)_q \le a(y+2,t,k,y)_{q'} \le a(y+2,t,k,y)_q+q'-q$. Thus to prove the lemma for $x=y+2$ it is sufficient to prove it for the genus τ. We have, by (8) and (10),

$$\begin{aligned}&(y+2)(d_{t,k}+c(y+2,t,k))+3-g_{t,k}-g(y+2,t,k)+d(y+2,t,k)=\\ &(y+2)(d_{t,k}+a(y+2,t,k,y)_\tau)+b(y+2,t,k,y)_\tau+3-\tau\end{aligned}$$

Hence

$$(15)\quad (y+2)(c(y+2,t,k) - a(y+2,t,k,y)_\tau) = -d(y+2,t,k) + b(y+2,t,k,y)_\tau + 1$$

From (15) we get $a(y+2,t,k,y)_\tau \ge c(y+2,t,k) - 1$. Since $c_{y+2,t,k} \ge g(y+2,t,k) + 3 = \tau - g_{t,k} + 4$, we get $a(y+2,t,k,y)_\tau \ge \tau - g_{t,k} + 3$.

Now assume $x \ge y+4$. By Lemma 9 we have $a(x,t,k,y) \ge a(y+2,t,k,y)$. □

By Lemma 12 there is a non-special curve of degree $a(x,t,k,y)$ and genus $g - g_{t,k}$. We need this observation in the next statement.

Assertion $N(x,t,k,y)$, $x \ge y$, $x \equiv y \pmod 2$: Set $e = 1$ if $0 \le b(x,t,k,y) \le a(x+2,t,k,y) - a(x,t,k,y) - 1$ and $e = 2$ if $b(x,t,k,y) \ge a(x+2,t,k,y) - a(x,t,k,y)$. There is a 6-tuple $(X, Q, D_1, D_2, S_1, S_2)$ such that

(a) Q is a smooth quadric surface, $X = C_{t,k} \sqcup Y$, Y is a smooth non-special curve of degree $a(x,t,k,y)$ and genus $g - g_{t,k}$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;

(b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, each of them containing one point of $Y \cap Q$, $S_i \subset D_i \setminus D_i \cap Y$, $1 \le i \le 2$, and $\sharp(S_1) + \sharp(S_2) = b(x,t,k,y)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$ and $\pi_2(S_e) \subset \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$; $S_2 = \emptyset$ if $e = 1$, $\sharp(S_2) = b(x+2,t,k,y) - a(x+2,t,k,y) + a(x,t,k,y) + 2$ if $e = 2$;

(c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(x)) = 0$, $i = 0, 1$.

Lemma 13. *If $N(x,t,k,y)$ is true, then $N(x+2,t,k,y)$ is true.*

Proof. We outline the modifications of the proof of Lemma 8 needed to get Lemma 13. Let $e \in \{1,2\}$ (resp. $f \in \{1,2\}$) be the integer arising in $N(x,t,k,y)$ (resp. $N(x+2,t,k,y)$). Take $(X, Q, D_1, D_2, S_1, S_2)$ satisfying $N(x,t,k,y)$. Set $w := a(x+2,t,k,y) - a(x,t,k,y)$.

(a) Assume $e = 2$. Set $z := b(x+2,t,k,y) + 2 - w$. Since $b(x+2,t,k,y) \le x+1$, Lemma 9 gives $z \le w-2$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le w-2$, be the lines such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{w-2})$ and $S_2 = D_2 \cap (L_1 \cup \cdots \cup L_z)$. Set $J := D_1 \cup D_2 \cup (\bigcup_{i=1}^{w-2} L_i)$ and $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$. Condition (b) gives $\sharp(L_i \cap Y) = 1$ for all i. Condition (a) gives $C_{t,k} \cap J = \emptyset$. Hence $W := X \cup J \cup \chi$ is a smoothable curve of degree $a(x+2,t,k,y)$ with $h^1(\mathcal{O}_W) = g$.

(b) Assume $e = 1$, i.e. assume $d(x+2,t,k,y) \le w-1$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le b(x,t,k,y)$, be the lines such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{b(x,t,k,y)})$. Take general lines $L_j \in |\mathcal{O}_Q(0,1)|$, $b(x,t,k,y) < j \le w-1$. Set $J := D_1 \cup (\bigcup_{i=1}^{w-1} L_i)$ and $\chi := \cup_{o \in S_1} \chi(o)$. Condition (a) gives $C_{t,k} \cap J = \emptyset$. Hence $W := X \cup J \cup \chi$ is a smoothable curve of degree $a(x+2,t,k,y)$ with $h^1(\mathcal{O}_W) = g$. □

Lemma 14. *$N(y+2,t,k,y)$ is true.*

Proof. Use the proof of Lemma 8 and Lemma 13 starting with $(X, Q, D_1, D_2, S_1, S_2)$ satisfying $M(y,t,k)$ and quoting Lemma 10 instead of Lemma 9. □

5. Proving Conjecture 1

In order to prove Theorem 1 and Corollary 1, first of all we notice that from the previous section we could deduce with a small effort the following two facts, but that (as explained at the end of the introduction) they would not prove Theorem 1 and Corollary 1.

For each integer d such that $g-3 \le d \le d(m,g)_{\max}$ there exists a smooth and connected curve $X_1 \subset \mathbb{P}^3$ such that $\deg(X_1) = d$, $g(X) = g$, $h^1(\mathcal{O}_{X_1}(m-2)) = 0$, $h^1(\mathcal{I}_{X_1}(m)) = 0$ and $h^1(N_{X_1}(-1)) = 0$.

For each integer $d \ge d(m,g)_{\min}$ there exists a smooth and connected curve $X_2 \subset \mathbb{P}^3$ such that $\deg(X_2) = d$, $g(X) = g$, $h^1(\mathcal{O}_{X_2}(m-2)) = 0$, $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

Now fix an integer d such that $d(m,g)_{\min} \le d \le d(m,g)_{\max}$. To prove Theorem 1 for the pair (d,g) it is sufficient to prove that we may find X_1, X_2 as above and with the additional condition that X_1 and X_2 are in the same irreducible component, Γ, of $\mathrm{Hilb}(\mathbb{P}^3)$. If we prove this statement, then by the semicontinuity theorem for cohomology ([15, III.8.8]) we get $h^1(\mathcal{I}_X(m)) = 0$ and $h^0(\mathcal{I}_X(m-1)) = 0$, hence we would conclude the proof for the pair (d,g). To get X_1 and X_2 in the same irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ we need to rewrite the proofs of the previous section with a few improvements. But first we need to distinguish between the case in which d is very near to $d(m,g)_{\min}$ and the case in which d is very near to $d(m,g)_{\max}$. In the first case (say $d(m,g)_{\min} \le d \le d'$) we will modify the proof of the existence of X_2 with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ to get (for the same curve X_2) also $h^1(\mathcal{I}_{X_2}(m)) = 0$. If d is very near to $d(m,g)_{\max}$ (say $d'' \le d \le d(m,g)_{\max}$) we will modify the proof of the existence of the curve X_1 to get a curve X_1 with $h^1(\mathcal{I}_{X_1}(m)) = 0$ and $h^0(\mathcal{I}_{X_1}(m-1)) = 0$. We use that $N(x,t,k,y)$ are true for $x = m-5, m-4, m-3, m-2$ (Lemma 15).

Set $\varepsilon := 0$ if m is odd and $\varepsilon := 1$ if m is even.

5.0.1. *Near* $d(m,g)_{\min}$. In this range the most difficult part is the proof of the existence of X_2. It is the construction of X_2 which says in which $W(t',k',d',b')$ we will try to find X_1. Recall that to get a curve X_2 with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ we started with a curve $C_{t,t-\varepsilon}$ with $h^i(\mathcal{I}_{C_{t,t-\varepsilon}}(2t-1-\varepsilon)) = 0$, where t is the maximal integer $t > 0$ such that such that $g_{t,t-\varepsilon} + g(2t+5-\varepsilon, t, t-\varepsilon) \le g$. Set $k := t-\varepsilon$. Recall that an element W of $U(t,k,a_d,b)$ has degree d and $h^1(\mathcal{O}_W) = g$ if and only if $b = g - g_{t,k}$ and $a_d = d - d_{t,k}$. The component $W(t',k',d',b')$ is the component $W(t,k,a_d,b)$, where $b = g - g_{t,k}$ and $a_d = d - d_{t,k}$. The curve T satisfying $N(m-1,t,k,y)$ has $h^1(\mathcal{O}_T) = g$, 3 connected components, $h^0(\mathcal{I}_T(m-1)) = b(m-1,t,k,y)$ and $h^1(\mathcal{I}_T(m-1)) = 0$, hence $d > a(m-1,t,k,y) + d_{t,k}$. The minimum integer $d(m,g)_{\min}$ is $a(m-1,t,k,y) + d_{t,k} + 1$, unless $b(m-1,t,k,y) \in \{m-2, m-1\}$ (in the latter case we have $d(m,g)_{\min} = a(m-1,t,k,y) + d_{t,k} + 2$).

(a) We make the construction of Section 4 for the integer $m' := m-1 \equiv t+k-1 \pmod 2$ and the integer g (note that the numerology for g in Theorem 1 is such that we may do the construction of Section 4 for $m' := m-1$ and the integer g). We get an integer $y \le m'-4 = m-5$ with $y \equiv t+k-1 \equiv 0 \pmod 2$. Then for all integers $x \ge y+2$ with $x \equiv y \pmod 2$ we proved $N(x,t,k,y)$. Hence $N(m-5,t,k,y)$ and $N(m-3,t,k,y)$ are true (Lemma 15). Since $d \ge d(m,g)_{\min}$, we have $d > a(m-1,t,k,y) + d_{t,k}$, hence we want to add in a smooth quadric Q a certain union of $d - a(m-3,t,k,y) - d_{t,k}$ lines. We write $C_t \cup C'_k$ for a general (but fixed in this construction) $C_{t,k}$, because we need to distinguish the two connected components of $C_{t,k}$, even when $k = t$.

(a1) Assume $d = d(m,g)_{\min} = a(m-1,t,k,y) + d_{t,k} + 1$. Set $z := d - a(m-3,t,k,y) - d_{t,k} = 1 + a(m-1,t,k,y) - a(m-3,t,k,y)$. We need to modify $N(m-3,t,k,y)$ in the following way.

Assertion $N'(m-3,t,k,y)$, $m-3\equiv y \pmod 2$: Set $e=1$ if $b(m-3,t,k,y)\le z-3$ and $e=2$ if $b(m-3,t,k,y)\ge z-2$. There is a 6-tuple (X,Q,D_1,D_2,S_1,S_2) such that

(a) Q is a smooth quadric surface, $X=C_t\sqcup C'_k\sqcup Y$, Y is a smooth curve of degree $a(m-3,t,k,y)$ and genus $g-g_{t,k}$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X\cap Q$;
(b) D_1,D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, $D_1\cap C_t\ne\emptyset$, $D_2\cap C_k\ne\emptyset$, $S_i\subset D_i\setminus D_i\cap(C_t\cup C'_k)$, $1\le i\le 2$, and $\sharp(S_1)+\sharp(S_2)=b(m-3,t,k,y)$; $\pi_2(S_2)\subseteq\pi_2(S_1)$, $\pi_2(S_e)\subset\pi_2(Y\cap(Q\setminus(D_1\cup D_2)))$; $S_2=\emptyset$ if $e=1$, $\sharp(S_2)=b(m-3,t,k,y)-z+3$ if $e=2$;
(c) $h^i(\mathcal{I}_{X\cup S_1\cup S_2}(m-3))=0$, $i=0,1$.

As in the proof of Lemma 8 and Lemma 13 we get (X,Q,D_1,D_2,S_1,S_2), $X=C_t\sqcup C'_k\sqcup Y$ satisfying $N'(m-3,t,k,y)$; in the proof of Lemma 8 we take R_1 containing a point of $C_t\cap Q$ instead of a point of $Y\cap Q$ and R_2 containing a point of $C'_k\cap Q$ instead of a point of $Y\cap Q$.

(a1.1) Assume $b(m-3,t,k,y)=0$. Take $D_0\in|\mathcal{O}_Q(1,0)|$ containing one point of $Y\cap Q$, $L_1\in|\mathcal{O}_Q(0,1)|$ containing a point of C_t, $L_2\in|\mathcal{O}_Q(0,1)|$ containing a point of C'_k and general $L_i\in|\mathcal{O}_Q(0,1)|$, $3\le i\le z-1$. Set $J:=D_0\cup(\bigcup_{i=1}^{z-1}L_i)$. Since $X\cap(Q\setminus J)$ is a general subset of Q with cardinality $2d_{t,k}+2a(m-3,t,k,y)-3$, we have $h^0(Q,\mathcal{I}_{Q\cap(X\cup J)}(m-1))=h^0(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-2,m-z))=0$ (use (10) for $x=m-3$, that $z=1+a(m-1,t,k,y)-a(m-3,t,k,y)$ and that $b(m-1,t,k,y)\le m-2$). Since $\mathrm{Res}_Q(X\cup Y)=X$ and $h^0(\mathcal{I}_X(m-3))=0$, we have $h^0(\mathcal{I}_{X\cup J}(m-1))=0$. The union $X\cup J$ is a nodal and connected smoothable curve of degree d and arithmetic genus g and $Y\cup J$ is a connected smoothable curve of degree $d-d_{t,k}$ and arithmetic genus $g-g_{t,k}-2\ge 26$. We may smooth $Y\cup J$ in a family of curves, all of them containing the two points $(C_t\cup C'_k)\cap J$. Call E a general element of this smoothing. Since $\mathrm{Aut}(\mathbb{P}^3)$ is 2-transitive, we may see E as a general non-special space curve of its degree and its genus ≥ 26. By construction and Lemma 2 we have $C_t\cup C'_k\cup E\in U(t,k,a_d,b)$ and $h^1(N_{C_t\cup C'_k\cup E}(-1))=0$. By semicontinuity there is a smooth $X_2\in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1))=0$ and $h^1(N_{X_2}(-1))=0$.

(a1.2) Assume $0<b(m-3,t,k,y)\le z-3$. Hence $S_2=\emptyset$. We take D_1 and call $L_i\in|\mathcal{O}_Q(0,1)|$, $1\le i\le b(m-3,t,k,y)$, the elements of $|\mathcal{O}_Q(0,1)|$ such that $S_1=D_1\cap(L_1\cup\cdots\cup L_{b(m-3,t,k,y)})$; note that each line L_i contains a point of $Y\cap Q$. Take any $L_{b(m-3,t,k,y)+1}\in|\mathcal{O}_Q(0,1)|$ with $C'_k\cap L_{b(m-3,t,k,y)+1}\ne\emptyset$, any $L_{b(m-3,t,k,y)+2}\in|\mathcal{O}_Q(0,1)|$ with $Y\cap L_{b(m-3,t,k,y)+2}\ne\emptyset$, $L_{b(m-3,t,k,y)+2}\ne L_i$ for $i\le b(m-3,t,k,y)$ and (if $b(m-3,t,k,y)<z-3$) take general $L_j\in|\mathcal{O}_Q(0,1)|$, $b(m-3,t,k,y)+3\le j\le z-1$. Set $J:=D_1\cup(\bigcup_{i=1}^{z-1}L_i)$, $\chi:=\cup_{o\in S_1}\chi(o)$ and $W:=X\cup J\cup\chi$. We have $\mathrm{Res}_Q(W)=X\cup S_1$ and thus $h^0(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-3))=0$. Since $W\cap Q$ is the union of J and $2d_{t,k}+2a(m-3,t,k,y)-b(m-3,t,k,y)-3$ general points of Q and $b(m-1,t,k,y)\le m-1$, (11) gives $h^0(Q,\mathcal{I}_{W\cap Q}(m-1))=h^0(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-2,m-z))=0$. Thus $h^0(\mathcal{I}_W(m-1))=0$. We first deform W to the union F of $C_t\cup C'_k\cup D_1\cup Y\cup(\bigcup_{i=b(m-3,t,k,y)+1}^{z-1}L_i)$ and $b(m-3,t,k,y)$ disjoint lines $M_1,\dots,M_{b(m-3,t,k,y)}$, each of them containing one point of Y. The union F is a nodal and connected curve. Write $F=C_t\cup C'_k\cup G$. We have $\sharp(G\cap C_t)=\sharp(G\cap C'_k)=1$. Let G' be a general smoothing of G fixing the 2 points of $(C_t\cup C'_k)\cap G$. $C_t\cup C'_k\cup G'\in U(t,k,a_d,b)$. By Lemma 2 and semicontinuity there is a smooth $X_2\in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1))=0$ and $h^1(N_{X_2}(-1))=0$.

(a1.3) Assume $b(m-3,t,k,y) \ge z-2$. Since $z = a(m-1,t,k,y) - a(m-3,t,k,y)+1$ and $b(m-3,t,k)) \le m-4$, Lemma 9 gives $2(z-3) \ge b(m-3,t,k,y)$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le z-3$, be the lines such that $S_1 = D_1 \cap (\bigcup_{i=1}^{z-3} L_i)$ and $S_2 := D_2 \cap (\bigcup_{i=1}^{w} L_i)$. Take $L_{z-2} \in |\mathcal{O}_Q(0,1)|$ containing one point of $Y \cap Q$ and different from the other lines L_i, $i \le z-3$. Set $J := D_1 \cup D_2 \cup (\bigcup_{i=1}^{z-2} L_i)$, $\chi := \cup_{o \in S_1} \chi(o)$ and $W := X \cup J \cup \chi$. We have $\mathrm{Res}_Q(W) = X \cup S_1 \cup S_2$ and thus $h^0(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-3)) = 0$. Since $W \cap Q$ is the union of J and $2d_{t,k}+2a(m,t,k,y)-w-3$ general points of Q and $b(m-1,t,k,y) \le m-1$ (11) gives $h^0(Q, \mathcal{I}_{W\cap Q}(m-1)) = h^0(Q, \mathcal{I}_{X \cap (Q\setminus J)}(m-2, m-z)) = 0$. Thus $h^0(\mathcal{I}_W(m-1)) = 0$. We first deform W to the union F of $C_t \cup C'_k \cup D_1 \cup D_2 \cup Y \cup (\bigcup_{i=w+1}^{z-2} L_i)$ and w disjoint lines $M_1, \dots, M_w$, each of them containing one point of Y. The union F is a nodal and connected curve. Write $F = C_t \cup C'_k \cup G$. We have $\sharp(G \cap C_t) = \sharp(G \cap C'_k) = 1$. Let G' be a general smoothing of G fixing the 2 points of $(C_t \cup C'_k) \cap G$. We have $C_t \cup C'_k \cup G' \in U(t,k,a_d,b)$. By Lemma 2 and semicontinuity there is a smooth $X_2 \in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

(a1.4) Assume $d(m,g)_{\min} = a(m-1,t,k,y) + d_{t,k} + 2$. We are in the set-up of step (a1.3) with the integer $z' := a(m-1,t,k,y) - a(m-3,t,k,y) + 2$ instead of the integer $z := a(m-1,t,k,y) - a(m-3,t,k,y) + 1$.

(a2) Assume $d > d(m,g)_{\min}$ and set $w := d - d(m,g)_{\min}$. By step (a1) there is a nodal curve $E = C_t \cup C'_k \cup F \in U(t,k,a_d-w,b)$ with $\sharp(C_t \cap F) = \sharp(C'_k \cap F) = 1$, $C_t \cap D'_k = \emptyset$, F and $h^0(\mathcal{I}_E(m-1)) = 0$. Take a general union G of F and w lines, each of them meeting F at exactly one point and quasi-transversally. By construction $E' := C_t \cup C'_k \cup G$ is nodal and $C_t \cap G = C_t \cap F$, $C'_k \cap G = C'_k \cap F$. Since $h^0(\mathcal{I}_E(m-1)) = 0$ and $E' \supset E$, we have $h^0(\mathcal{I}_{E'}(m-1)) = 0$. We may smooth G keeping fixed the points $C_t \cap F$ and $C'_k \cap F$, because $\mathrm{Aut}(\mathbb{P}^3)$ is 2-transitive. Hence there is a non-special smooth curve G'' of degree $d - d_{t,k}$ and genus $g - g_{t,k}$ with $C_t \cap G'' = C_t \cap F$, $C'_k \cap G'' = C'_k \cap F$ and which is a general member of a family with F' as its special member and with $C_t \cup C'_k \cup G''$ nodal. By semicontinuity we have $h^0(\mathcal{I}_{C_t \cup C'_k \cup G''}(m-1)) = 0$. We have $C_t \cup C'_k \cup G'' \in U(t,k,a_d,b)$.

(b) Set $\alpha := t(t-2)$ if $k = t$ and $\alpha := t^2 - 3t + 1$ if $k = t-1$. Fix a plane H, a smooth conic $D \subset H$ and general $C_{t,k}$. We have $D \cap C_{t,k} = \emptyset$ and $C_{t,k} \cap H$ is a general subset of H with cardinality $d_{t,k}$. Hence $h^0(H, \mathcal{I}_{H\cap(C_{t,k}\cup D)}(t+k)) = h^0(H, \mathcal{I}_{C_{t,k}\cap H}(t+k-1)) = \binom{t+k+1}{2} - d_{t,k} = \alpha$ and $h^1(H, \mathcal{I}_{H\cap(C_{t,k}\cup D)}(t+k)) = 0$. Then we continue the construction from the critical value $t+k$ to the critical value $t+k+2$, then to the critical value $t+k+4$, and so on up to the critical value $m-2$; in each step, say to arrive at the critical value x from a curve A' and a set S' with $h^1(\mathcal{I}_{A'\cup S'}(x-2)) = 0$ and $h^0(\mathcal{I}_{A'\cup S'}(x-2)) = \alpha$ and $0 \le \sharp(S') \le x-3$ (and so $\sharp(S') = \binom{x+1}{3} - (x-2)\deg(A') - 3 + g - \alpha$; we have bijectivity inside Q and get a curve A'' and a set S'' with $h^1(\mathcal{I}_{A''\cup S''}(x)) = 0$ and $h^0(\mathcal{I}_{A''\cup S''}(x)) \le \alpha$. In the last step we also need to connect the connected components of the curve and get an element $B \in U(t,k,a',b)$ for some a'; we need to check that at each step the numerical conditions are satisfied. Call $(X, Q, D_1, D_2, S_1, S_2)$ the curve we get for $\mathcal{O}_{\mathbb{P}^3}(m-2)$ and either $e = 1$ or $e = 2$. Set $S := S_1 \cup S_2$ and $\alpha' := \sharp(S)$. We have $0 \le \alpha' \le m-3$. Since S is a union of connected components of $X \cup S$, the restriction map $H^0(\mathcal{O}_{X\cup S}(m-2)) \to H^0(\mathcal{O}_X(m-2))$ is surjective and its kernel has dimension $\sharp(S)$. Since $h^1(\mathcal{I}_{X\cup S}(m-2)) = 0$, we have $h^1(\mathcal{I}_X(m-2)) = 0$ and $h^0(\mathcal{I}_X(m-2) = \alpha + \alpha' \le \alpha + m - 3$. We cover in this way the integers

d such that $\binom{m+3}{3} + g - 1 - dm \ge \alpha + m - 3$. Hence we cover all d such that $d(m,g)_{\max} - d \ge 1 + \lfloor \alpha/m \rfloor$. If $t \le m/4$ we have $\alpha/m \le m/4$.

5.0.2. *Near* $d(m,g)_{\max}$. In this range the most difficult part is the existence of X_1 with $h^1(\mathcal{I}_{X_1}(m)) = 0$ and it is this part which dictates the component $W(t',k',a',b')$ in which we will find both X_1 and X_2. We stress that the integers t,k introduced in this subsection are not the same as in the previous one and hence also y may be different.

(a) In this step we prove the existence of X_1. We start with the maximal integer k such that $g_{k+1-\varepsilon,k} + g(2k+6-\varepsilon, k+1-\varepsilon, k) \le g$ and set $t := k+1-\varepsilon$. We use $N(x,t,k,y)$. In particular we have $N(m-4,t,k,y)$ and $N(m-2,t,k,y)$. Set $a_d := d - d_{t,k}$ and $b := g - g_{t,k}$. In this step we prove the existence of $A \in U(t,k,a_d,b)$ with $h^1(\mathcal{I}_A(m)) = 0$, hence by semicontinuity the existence of $X_1 \in W(t,t-1,a_d,b)$ with $h^1(\mathcal{I}_{X_1}(m)) = 0$. Set $z := d - a(m-2,t,k,y) - d_{t,k}$. We write $C_t \cup C'_k$ for a general (but fixed in this construction) $C_{t,k}$, because we need to distinguish the two connected components, even when $k = t$. Recall that we have (1).

(a1) Assume $d = d(m,g)_{\max}$. Let T be any curve satisfying $N(m,t,k,y)$. We have $\deg(T) = d_{t,k} + a(m,t,k,y)$, $h^1(\mathcal{O}_T) = g$, $h^1(\mathcal{O}_T(m)) = 0$, T has 3 connected components, $h^1(\mathcal{I}_T(m)) = 0$ and $h^0(\mathcal{I}_T(m)) = b(m,t,k,y)$. By (1) we have $d = a(m,t,k,y) + d_{t,k}$ if $b(m,t,k,y) \le m-3$ and $d = a(m,t,k,y) + d_{t,k} + 1$ if $m-2 \le b(m,t,k,y) \le m-1$. Hence $a(m,t,k,y) - a(m-2,t,k) \le z \le a(m,t,k,y) - a(m-2,t,k,y) + 1$. Call η the difference between the right hand side and the left hand side of (1).

Assertion $N''(m-2,t,k,y)$, $m \equiv y \pmod 2$: Set $e = 1$ if $b(m-2,t,k,y) \le z-3$ and $e = 2$ if $b(x,t,k,y) \ge z-2$. There is a 6-tuple (X,Q,D_1,D_2,S_1,S_2) such that

(a) Q is a smooth quadric surface, $X = C_t \sqcup C'_k \sqcup Y$, Y is a smooth curve of degree $a(m-2,t,k,y)$ and genus $g - g_{t,k}$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;

(b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, $D_1 \cap C_t \ne \emptyset$, $D_2 \cap C'_k \ne \emptyset$, $S_i \subset D_i \setminus D_i \cap (C_t \cup C'_k)$, $1 \le i \le 2$, and $\sharp(S_1) + \sharp(S_2) = b(x,t,k,y)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$ and $\pi_2(S_e) \subset \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$; $S_2 = \emptyset$ if $e = 1$, $\sharp(S_2) = b(m-2,t,k,y) - z + 2$ if $e = 2$;

(c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(x)) = 0$, $i = 0,1$.

As in the proof of Lemma 8 and Lemma 13 we get (X,Q,D_1,D_2,S_1,S_2), $X = C_t \sqcup C'_k \sqcup Y$ satisfying $N''(m-2,t,k,y)$; in the proof of Lemma 8 we take R_1 containing a point of $C_t \cap Q$ instead of a point of $Y \cap Q$ and R_2 containing a point of $C'_k \cap Q$ instead of a point of $Y \cap Q$.

(a1.1) Assume $b(m-2,t,k,y) = 0$. Take $z-1$ distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le z-1$, such that $L_i \cap C_t = \emptyset$ for all i, $L_i \cap C'_k \ne \emptyset$ if and only if $i = 1$ and $L_i \cap Y \ne \emptyset$ if and only if $i = 2$. Set $J := D_1 \cup (\bigcup_{i=1}^{z-1} L_i)$. Since $X \cap (Q \setminus J)$ is a general subset of Q with cardinality $2d_{t,k} + 2a(m-3,t,k,y) - 3$, we have $h^1(Q, \mathcal{I}_{Q \cap (X \cup J)}(m)) = h^1(Q, \mathcal{I}_{X \cap (Q \setminus J)}(m-1, m+1-z)) = 0$ (use the generality of $X \cap (Q \setminus J)$ and the difference between (1) and the case $x := m-2$ of (10), which gives an upper bound for $\sharp(X \cap (Q \setminus J))$; we get an equality if and only if $\eta = 0$, i.e. $b(m,t,k,y) = m-2$ and $d = a(m,t,k,y) + d_{t,k} + 1$). Since $\mathrm{Res}_Q(X \cup J) = X$ and $h^1(\mathcal{I}_X(m-2)) = 0$, we have $h^1(\mathcal{I}_{X \cup J}(m)) = 0$. The union $X \cup J$ is a nodal and connected smoothable curve of degree d and arithmetic genus g and $Y \cup J$ is a smooth and connected curve of degree $d-d_{t,k}$ and arithmetic genus $g-g_{t,k}-2 \ge 26$. We may smooth $Y \cup J$ in a family of curves, all of them containing the two points $(C_t \cup C'_k) \cap J$. Call E a general element of this smoothing. Since $\mathrm{Aut}(\mathbb{P}^3)$ is 2-transitive, we may see E as a general non-special space curve of its degree and its genus ≥ 26. By construction and Lemma 2 we have $C_t \cup C'_k \cup E \in U(t,k,a_d,b)$ and $h^1(N_{C_t \cup C'_k \cup E}(-1)) = 0$. By semicontinuity there is a smooth $X_1 \in W(t,k,a_d,b)$ with $h^1(\mathcal{I}_{X_1}(m)) = 0$ and $h^1(N_{X_1}(-1)) = 0$.

(a1.2) Assume $0 < b(m-2,t,k,y) \le z-3$. Hence $S_2 = \emptyset$. We take D_1 and call $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le b(m-2,t,k,y)$, the elements of $|\mathcal{O}_Q(0,1)|$ such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{b(m-2,t,k,y)})$; note that each line L_i contains a point of $Y \cap Q$. Take any $L_{b(m-2,t,k,y)+1} \in |\mathcal{O}_Q(0,1)|$ with $C'_k \cap L_{b(m-2,t,k,y)+1} \ne \emptyset$, any $L_{b(m-2,t,k,y)+2} \in |\mathcal{O}_Q(0,1)|$ with $Y \cap L_{b(m-2,t,k,y)+2} \ne \emptyset$, $L_{b(m-2,t,k,y)+2} \ne L_i$ for $i \le b(m-2,t,k,y)$ and (if $b(m-2,t,k,y) < z-3$) take general $L_j \in |\mathcal{O}_Q(0,1)|$, $b(m-2,t,k,y)+3 \le j \le z-1$. Set $J := D_1 \cup (\bigcup_{i=1}^{z-1} L_i)$, $\chi := \cup_{o \in S_1} \chi(o)$ and $W := X \cup J \cup \chi$. We have $\mathrm{Res}_Q(W) = X \cup S_1$ and thus $h^1(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-2)) = 0$. Since $\eta \ge 0$, (1) and the case $x = m-2$ of (11) give $2d_{t,k} + 2a(m,t,k,y) - b(m-2,t,k,y) - 3 = m(m+3-z) - \eta \le h^0(Q, \mathcal{O}_Q(m-2, m+2-z))$. Since $W \cap Q$ is the union of J and $2d_{t,k} + 2a(m,t,k,y) - b(m-2,t,k,y) - 3$ general points of Q, we have $h^1(Q, \mathcal{I}_{W \cap Q}(m)) = h^1(Q, \mathcal{I}_{X \cap (Q \setminus J)}(m-1, m+1-z)) = 0$. Thus $h^1(\mathcal{I}_W(m)) = 0$. We first deform W to the union F of $C_t \cup C'_k \cup D_1 \cup Y \cup (\bigcup_{i=b(m-3,t,k,y)+1}^{z-1} L_i)$ and $b(m-3,t,k,y)$ disjoint lines $M_1, \dots, M_{b(m-3,t,k,y)}$, each of them containing one point of Y. The union F is a nodal and connected curve. Write $F = C_t \cup C'_k \cup G$. We have $\sharp(G \cap C_t) = \sharp(G \cap C'_k) = 1$. Let G' be a general smoothing of G fixing the 2 points of $(C_t \cup C'_k) \cap G$. $C_t \cup C'_k \cup G' \in U(t,k,a_d,b)$. By Lemma 2 and semicontinuity there is a smooth $X_2 \in W(t,k,a_d,b)$ with $h^1(\mathcal{I}_{X_2}(m)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

(a1.3) Assume $b(m-2,t,k,y) \ge z-2$. Since $z \ge a(m,t,k,y) - a(m-2,t,k)$ and $b(m-2,t,k,y) \le m-3$, the case $x = m-2$ of Lemma 9 gives $2(z-3) \ge b(m-2,t,k,y)$. Set $w := b(m-2,t,k) - z + 3$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le z-3$, be the line such that $S_1 = D_1(\bigcup_{i=1}^{z-3} L_i)$ and $S_2 := D_2 \cap (\bigcup_{i=1}^{w} L_i)$. Let $L_{z-2} \in |\mathcal{O}_Q(0,1)|$ be a line with $L_{z-2} \ne L_i$ for any $i \ne z-2$ and $L_{z-2} \cap Y \ne \emptyset$. Note that $L_j \cap Y \ne \emptyset$ if and only if either $j \le w$ or $j = z-2$. Set $J := D_1 \cup D_2 \cup (\bigcup_{i=1}^{z-2} L_i)$, $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$ and $W := X \cup J \cup \chi$ and continue as in the last step.

(a2) Assume $d < d(m,g)_{\max}$ We have $\eta \ge m(d(m,g)_{\max} - d) \ge m$ and in particular $\eta \ge m \ge b(m-2,t,k,y) + 2$. To prove the existence of X_1 in this component we only need that $z \ge 3$, i.e. that $d \ge a_{m-2,t,k,y} + d_{t,k} + 3$, which is true because $1 + (m-1)d - g \ge \binom{m+2}{3}$ and $(m-1)(a(m-2,t,k,y) + d_{t,k}) + 3 - g = \binom{m+1}{2} - a(m-2,t,k) - d_{t,k} + b(m-2,t,k,y) \ge 3m$. Take $(X, Q, D_1, D_2, S_1, S_2)$ satisfying $N(m-2,t,k,y)$ with $X = C_t \sqcup C'_k \sqcup Y$ and throw away D_1, D_2, S_1 and S_2. Fix $D \in |\mathcal{O}_Q(1,0)|$ containing one point of $Y \cap Q$ and $z-1$ distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$ with $L_i \cap Y = \emptyset$ for all i, $L_i \cap C_t \ne \emptyset$ if and only if $i = 1$ and $L_i \cap C'_k \ne \emptyset$ if and only if $i = 2$. Set $J := D \cup (\bigcup_{i=1}^{z-1} L_i)$ and $W := X \cup J$. As in the previous steps it is sufficient to prove that $h^1(\mathcal{I}_W(m)) = 0$. We have $\mathrm{Res}_Q(W) = X$ and thus $h^1(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-2)) = 0$. Hence it is sufficient to prove that $h^1(Q, \mathcal{I}_{W \cap Q}(m)) = 0$. We have $h^1(Q, \mathcal{I}_{Q \cap W}(m)) = h^1(Q, \mathcal{I}_{X \cap (Q \setminus J)}(m-1, m+1-z))$. Since $X \cap Q$ is general in Q, it is sufficient to prove that $\sharp(X \cap (Q \setminus J)) \le m(m+2-z)$. We have $\sharp(X \cap (Q \setminus J)) = 2d_{t,k} + 2a(m-2,t,k,y) - 3$. By the definition of η and (10) for

$x=m-2$ we have $2d_{t,k}+2a(m-2,t,k,y)-3 = m(m+2-z)+b(m-2,t,k,y)+2-\eta \le m(m+2-z)$.

(b) In this part we get the existence of $A \in U(t,k,a_d,b)$ with $h^0(\mathcal{I}_A(m-1)) = 0$, $\deg(A) = d$ and $p_a(A) = g$, hence by semicontinuity the existence of $X_2 \in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$. We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1$ and $m-1 \equiv t+k \pmod 2$. Fix a plane H. Let c be the maximal integer such that $\binom{t+k+2-c}{2} \le d_{t,k}$. Let $E \subset H$ be a general linear projection of a general smooth and rational degree c curve $E' \subset \mathbb{P}^3$. The curve E is nodal and it has $(c-1)(c-2)/2$ singular points. Set $\chi := \cup_{p\in \mathrm{Sing}(E)}\chi(p)$. The union $E\cup\chi$ is the flat limit of a family of degree c smooth rational curves in $\mathbb{P}^3$ ([15, Fig. 11 at p. 260]. Hence to prove that a general union of some $C_{t,k}$ and a smooth rational curve of degree c is contained in no surface of degree $t+k$ it is sufficient to prove that $h^0(\mathcal{I}_{C_{t,k}\cup E\cup\chi}(t+k) = 0$ for a general $C_{t,k}$. Thus it is sufficient to prove that $h^0(\mathcal{I}_{C_{t,k}\cup E}(t+k)) = 0$ for a general $C_{t,k}$. For a general $C_{t,k}$ we have $C_{t,k}\cap E = \emptyset$ and $C_{t,k}\cap H$ is a general subset of H with cardinality $d_{t,k}$. By definition c is the minimal positive integer such that $h^0(H,\mathcal{I}_{C_{t,k}\cap H}(t+k-c)) = 0$. Set $\beta = h^0(\mathcal{O}_{C_{t,k}\cup E\cup\chi}(t+k)) - \binom{t+k+3}{3}$. Since $\binom{t+k+2-c}{2} - \binom{t+k-1}{2} = t+k+1-c$, we have $\beta \le (c-1)(c-2)/2+t+k+1-c$. Then we continue from the critical value $t+k$ to the critical value $t+k+2$ and so on.

At the end we obtain some $B \in U(t,k,a_d,b)$ with $h^0(\mathcal{I}_B(m-1)) = 0$ if $1 + d(m-1)-g \ge \binom{m+2}{3}+\beta$. In particular it is sufficient to assume $d \ge d(m,g)_{\min} + \lceil \beta/(m-1)\rceil$. We have $c \sim \sqrt{2}t$, because $\deg(C_{t,k}) \sim t^2$ and $\binom{t+k+2}{2} \sim 2t^2$. Hence $\beta \sim (c-1)(c-2)/2 \sim t^2$. Since $t \le m/4$, it is sufficient to have roughly $d \ge d(m,g)_{\min} + m/4$.

Lemma 15. *Fix t and $k \in \{t-1,t\}$ such that $y \equiv t+k-1 \pmod 2$ and let $g_{t,k}+g(t+k+5,t,k) \le g \le -1+g_{t+1,k+1}+g(t+k+7,t+1,k+1)$. Then we have $y \le \sqrt{20}t-1$. In particular, if $t \ge \lfloor m/\sqrt{20}\rfloor - 5$ then $y \le m-6$.*

Proof. We have $g_{t+1,k+1} - g_{t,k} = 2t^2-2$ if $k = t$ and $g_{t+1,k+1} - g_{t,k} = 2t^2 - 2t-1$ if $k = t-1$. For all integer $x \ge t+k+1$ such that $x \equiv t+k+1 \pmod 2$ we have $c(x,t,k) - c(x-2,t,k)) \ge (x+2)/2$ (Lemma 4). Remark 4 gives $c(t+k+1,t,k) = k+3$. By the definition of y, we have $y \ge k+t+5$ and $g \ge g_{t,k}+g(y,t,k) = g_{t,k}+c(y,t,k)-3(y-t-k-1)/2-3 \ge g_{t,k}-3(y-t-k-1)/2+k+\sum_{i=1}^{(y-t-k-1)/2}(c(t+k+1+2i,t,k)-c(t+k+1+2i-2,t,k)) \ge g_{t,k}-3(y-t-k-1)/2+k+(t+k+y+7)(y-t-k-1)/8$. On the other hand, we have $g \le -1+g_{t+1,k+1}+g(t+k+7,t+1,k+1) \le -1+g_{t+1,k+1}+3(t+k+7)$. Hence we get $(t+k+y+7)(y-t-k-1)/8 \le g_{t+1,k+1}-g_{t,k}+3(y-t-k-1)/2-k-1+3(t+k+7)$ and in particular $(y+1)^2 \le 20t^2$. □

Proof of Theorem 1: We fix the integer g and we perform the above construction in both the odd and the even case, by taking either $k = t$ or $k = t-1$. We have $h^1(\mathcal{O}(C_{t,k}(t-1) = 0$, hence we get $h^1(\mathcal{O}(C_X(t-1) = 0$ by a repeated application of Mayer-Vietoris and semicontinuity. For every $t \ge 27$ such that $g \ge g_{t+3,k+3} \ge g_{t,k}+g(t+k+5,t,k)$ we get an integer $y \equiv t+k-1$ such that the statement of Theorem 1 holds for every $m \ge y+6$ with $m \equiv y \pmod 2$. By Lemma 15, the condition $m \ge y+6$ is satisfied for every $t \ge \lfloor m/\sqrt{20}\rfloor - 5$, hence we obtain our statement for every g with $2g_{30} = 17052 \le g \le \varphi(m)$. □

Proof of Corollary 1: Let m be the minimal non-negative integer such that

$$md + 1 - g \leq \binom{m+3}{3} \tag{16}$$

The minimality of m gives

$$(m-1)d + 1 - g > \binom{m+2}{3}, \tag{17}$$

in particular $d \geq \frac{(m+2)(m+1)m}{6(m-1)} \geq \frac{m^2}{6}$. From (16) and (17) we get $d \leq \binom{m+2}{2}$. Since $g \leq Kd^{3/2} - 6\varepsilon d$, we have

$$\begin{aligned} g &\leq \frac{2}{3}\left(\frac{1}{10}\right)^{3/2} \binom{m+2}{2}^{3/2} - 6\varepsilon d \\ &\leq \frac{2}{3}\left(\frac{1}{10}\right)^{3/2} \left(\frac{(m+2)^2}{2}\right)^{3/2} - 6\varepsilon d \\ &\leq \frac{2}{3}\left(\frac{1}{20}\right)^{3/2} (m+2)^3 - \varepsilon m^2 \leq \varphi(m) \end{aligned}$$

(notice that the coefficients of m^3 are controlled by our choice of K and the coefficients of m^2 are controlled by our choice of ε). Since $g \leq \varphi(m)$, Theorem 1 covers all degrees d_0 in the interval $d(m,g)_{\min} \leq d_0 \leq d(m,g)_{\max}$. In order to check that d is in this interval, just notice that $d \geq d(m,g)_{\min}$ by (17) and $d \leq d(m,g)_{\max}$ by (16). □

References

[1] E. Ballico and Ph. Ellia, The maximal rank conjecture for non-special curves in $\mathbf{P}^3$. Invent. Math. 79 (1985), 541–555.

[2] E. Ballico and Ph. Ellia, Beyond the maximal rank conjecture for curves in $\mathbb{P}^3$, in: Space Curves, Proceedings Rocca di Papa, pp. 1–23, Lecture Notes in Math. 1266, Springer, Berlin, 1985.

[3] E. Ballico and Ph. Ellia, A program for space curves. Conference on algebraic varieties of small dimension (Turin, 1985). Rend. Sem. Mat. Univ. Politec. Torino 1986, Special Issue, 25–42 (1987).

[4] E. Ballico, G. Bolondi, Ph. Ellia and R. M. Mirò-Roig, Curves of maximum genus in the range A and stick-figures, Trans. Amer. Math. Soc. 349 (1997), 4589–4608.

[5] E. Carlini, M. V. Catalisano and A. Geramita, Bipolynomial Hilbert functions, J. Algebra 324 (2010), 758–781.

[6] K. Dasaratha, The reducibility and dimension of Hilbert schemes of complex projective curves, Undergraduate Thesis, Harvard University, Department of Mathematics, available at http://www.math.harvard.edu/theses/senior/dasaratha/dasaratha.

[7] L. Ein, Hilbert scheme of smooth space curves. Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 4, 469–478.

[8] L. Ein, The irreducibility of the Hilbert scheme of smooth space curves, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 83–87, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[9] D. Eisenbud and A. Van de Ven, On the normal bundle of smooth rational space curves, Math. Ann. 256 (1981), no. 4, 453–463.

[10] Ph. Ellia, Exemples de courbes de $\mathbb{P}^3$ à fibré normal semi-stable, stable, Math. Ann. 264 (1983), 389–396.

[11] Ph. Ellia, Sur le genre maximal des courbes gauches de degré d non sur une surface de degré $s-1$, J. Reine Angew. Math. 413 (1991), 77–87.

[12] G. Ellingsrud and A. Hirschowitz, Sur le fibré normal des courbes gauches, C. R. Acad. Sci. Paris 299 (1984), 245–248.

[13] G. Fløystad, Construction of space curves with good properties, Math. Ann. 289 (1991), 33–54.
[14] G. Fløystad, On space curves with good cohomological properties, Math. Ann. 291 (1991), 505–549.
[15] R. Hartshorne, Algebraic Geometry, Springer-Verlag, Berlin–Heidelberg–New York, 1977.
[16] R. Hartshorne, On the classification of algebraic space curves, in: Vector bundles and differential equations (Nice 1979), p. 82–112, Progress in Math. 7, Birkhäuser, Boston 1980.
[17] R. Hartshorne and A. Hirschowitz, Droites en position générale dans l'espace projectif. Algebraic geometry (La Rábida, 1981), 169–188, Lecture Notes in Math., 961, Springer, Berlin, 1982.
[18] R. Hartshorne and A. Hirschowitz, Smoothing algebraic space curves, Algebraic Geometry, Sitges 1983, 98–131, Lecture Notes in Math. 1124, Springer, Berlin, 1985.
[19] A. Hirschowitz, Sur la postulation générique des courbes rationnelles. Acta Math. 146 (1981), 209–230.
[20] A. Hirschowitz, Sections planes et multisécantes pour les courbes gauches génériques principales, Space curves (Rocca di Papa, 1985), 124–155, Lecture Notes in Math. 1266, Springer, Berlin, 1987.
[21] H. Iliev, On the irreducibility of the Hilbert scheme of space curves, Proc. Amer. Math. Soc. 134 (2006), 2823–2832.
[22] D. Jensen and S. Payne, Tropical independence II: The maximal rank conjecture for quadrics, Algebra Number Theory 10 (2016), no. 8, 1601–1640.
[23] C. Keem, Reducible Hilbert scheme of smooth curves with positive Brill-Noether number, Proc. Amer. Math. Soc. 122 (1994), 349–354.
[24] C. Keem and Y.-H. Kim, Irreducibility of the Hilbert scheme of smooth curves in $\mathbb{P}^3$ of degree g and genus g, Arch. Math. 108 (2017), 593–600.
[25] C. Keem, Y.-H. Kim, and A.F. Lopez, Irreducibility and components rigid in moduli of the Hilbert Scheme of smooth curves, arXiv:1605.00297 (2016).
[26] C. Keem and S. Kim, Irreducibility of a subscheme of the Hilbert scheme of complex space curves, J. Algebra 145 (1992), 240–248.
[27] E. Larsen, The Maximal Rank Conjecture, arXiv: 1711:04906 (2017).
[28] T. Le Tat, Space curves with the expected postulation, J. Pure Appl. Algebra 77 (2003), 253–285.
[29] D. Perrin, Courbes passant par m points généraux de $\mathbb{P}^3$, Bull. Soc. Math. France, Mémoire 28/29 (1987).
[30] E. Sernesi, On the existence of certain families of curves, Invent. Math. 75 (1984), 25–57.
[31] Ch. H. Walter, Curves in $\mathbb{P}^3$ with the expected monad, J. Algebraic Geom. 4 (1995), 301–320.

(Edoardo Ballico, Claudio Fontanari) Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123 Povo (Trento), Italy.
E-mail address: edoardo.ballico@unitn.it, claudio.fontanari@unitn.it

(Philippe Ellia) Dipartimento di Matematica e Informatica, Università degli Studi di Ferrara, Via Machiavelli 30, 44121 Ferrara, Italy.
E-mail address: phe@unife.it