AREA FORMULA FOR CENTERED HAUSDORFF MEASURES IN METRIC SPACES.

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Abstract. Motivated by an example in [Mag], we study, inside a separable metric space (X, d), the relations between centered and non centered m-dimensional densities of a Radon measure μ in X and their relations with spherical and centered spherical m-dimensional Hausdorff measures. Eventually we give an application to finite perimeter sets in Carnot groups.

1. Introduction

In a recent interesting note [Mag], Valentino Magnani observed the following fact. In a separable metric space (X, d), endowed with a Radon measure μ, absolutely continuous with respect to the m-dimensional spherical measure $\mathcal{S}^m$, the area formula for μ with respect to $\mathcal{S}^m$ i.e.

$$\mu(B) = \int_B \Theta_F^{*m}(\mu, x)\, d\mathcal{S}^m(x) \tag{1.1}$$

for any Borel set B may fail to be true if the m-dimensional Federer density $\Theta_F^{*m}(\mu, \cdot)$ is substituted by the (centered) m-dimensional density $\Theta^{*m}(\mu, \cdot)$ (see Definition 1.7 (i) and (ii)).

Indeed Magnani provides the following example: in the Heisenberg group $X = \mathbb{H}^1 \equiv \mathbb{R}^3$, equipped with its sub-Riemannian metric d, there is a Radon measure μ, a set $A \subset \mathbb{H}^1$ and two constants $0 < k_1 < k_2$ such that μ is absolutely continuous w.r.t. $\mathcal{S}^2$ and for all $x \in A$

$$\Theta^2(\mu, x) = k_1 < k_2 = \Theta_F^{*2}(\mu, x)$$

and for all $t \in (k_1, k_2)$

$$\mu(A) > t\,\mathcal{S}^2(A). \tag{1.2}$$

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Because of (1.2), given $A \subset X$ and $k > 0$, the implication

$$\Theta^m(\mu, x) = k \quad \forall x \in A \quad \Rightarrow \quad \mu \llcorner A = k\, \mathcal{S}^m \llcorner A \tag{1.3}$$

fails to be true in general.

Implication (1.3) was used by us to prove that the perimeter measure $|\partial E|_{\mathbb{G}}$ agrees, up to a multiplicative constant, with the $(Q-1)$-dimensional spherical Hausdorff measure $\mathcal{S}^{Q-1}$, in a step 2 Carnot group $\mathbb{G}$ of Hausdorff dimension Q. Hence a new proof of this result is in order.

Indeed, in [Mag2], Magnani himself provides an alternative proof of our result using his new notion of $(n-1)$ *vertical regular* distance (see Theorem 4.19 in this paper).

We take here a different approach to the same topic. From the preceding considerations it appears that Federer density $\Theta_F^{*m}(\mu, x)$ plays a privileged role in area formulas when the spherical measure $\mathcal{S}^m$ is used. On the other hand, non centered densities as $\Theta_F^{*m}(\mu, x)$ are often harder to compute than the corresponding centered densities $\Theta^{*m}(\mu, x)$. Therefore, motivated by Magnani's note, we looked for an area formula different from (1.1) in which the density $\Theta^{*m}(\mu, x)$ is used, but the measure $\mathcal{S}^m$ is replaced by an equivalent one.
Centered Hausdorff measures $\mathcal{C}^m$ (see Definition 2.1 (iii)) seem to be the right substitutes. Indeed we could prove the following theoretic area formula: if A is a Borel set in a metric space X, if $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m \llcorner A$ then for each for each Borel $B \subset A$,

$$\mu(B) = \int_B \Theta^{*m}(\mu, x)\, d\mathcal{C}^m(x), \tag{1.4}$$

see Theorem 3.1 and Corollary 3.14.

Centered Hausdorff measures $\mathcal{C}^m$ were introduced in [SRT] to estimate more efficiently the Hausdorff dimension of self-similar fractal sets (see also [LM]). Inside a general metric space a detailed study of centered Hausdorff measures has been carried on in [E].

Spherical and centered Hausdorff measures $\mathcal{S}^m$ and $\mathcal{C}^m$ may disagree (see [SRT]), even if they are equivalent, that is

$$\mathcal{S}^m \leq \mathcal{C}^m \leq 2^m\, \mathcal{S}^m\,.$$

However in the Euclidean case, i.e. when $X = \mathbb{R}^n$, they agree on *rectifiable* sets (see [SRT]). We show that this coincidence keeps being true inside Carnot groups at least for 1-codimensional *intrinsic rectifiable* sets (see Theorem 4.28).

Using area formula (1.4) a new proof of the previously mentioned representation result for the perimeter measure $|\partial E|_{\mathbb{G}}$ follows, so filling -in a different way- the gaps in [FSS1, FSS2, Mar].

Let us introduce some notation and notions. Throughout this paper (X, d) will be a separable metric space,

$$B(a,r) := \{x \in X : d(a,x) \le r\}$$

are the *closed ball* with centre a and radius $r > 0$. The *diameter* of a set $E \subset X$ is denoted as

$$\operatorname{diam}(E) := \sup \{d(x,y) : x, y \in E\} .$$

If μ is an outer measure in X and $A \subset X$ the *restriction* of μ to A is denoted as

$$\mu \llcorner A(E) = \mu(A \cap E) \quad \text{if } E \subset X.$$

We assume the following condition on the diameter of closed balls: *there are constants ρ_0, $0 < \rho_0 \le 2$ and $\delta_0 > 0$ such that, for all $r \in (0, \delta_0)$ and $x \in X$,*

$$\operatorname{diam}(B(x,r)) = \rho_0 \, r. \tag{1.5}$$

For $m > 0$, we denote

$$\alpha_m := \frac{\Gamma(\frac{1}{2})^m}{\Gamma(\frac{m}{2} + 1)}$$

being Γ the Euler function and

$$\beta_m := \alpha_m \, \rho_0^{-m} , \tag{1.6}$$

According to Federer's notation [F], we define a centered and a non centered density of an outer measure μ on X.

1.7. **Definition. (i)** The *upper and lower m-densities* of μ at $x \in X$ are

$$\Theta^{*m}(\mu, x) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{\alpha_m \, r^m}$$

and

$$\Theta^{m}_{*}(\mu, x) := \liminf_{r \to 0} \frac{\mu(B(x,r))}{\alpha_m \, r^m} .$$

If they agree their common value

$$\Theta^m(\mu, x) := \Theta^{*m}(\mu, x) = \Theta^m_*(\mu, x)$$

is called the *m-density* of μ at x.

(ii) The *m-Federer densities* of μ at $x \in X$ are

$$\Theta^{*m}_F(\mu, x) := \inf_{\epsilon > 0} \sup \left\{ \frac{\mu(B(y,r))}{\beta_m \operatorname{diam}(B(y,r))^m} : x \in B(y,r), \, \rho_0 \, r \le \epsilon \right\}$$

It is easy to see that

$$\Theta^{*m}(\mu, x) \leq \Theta_F^{*m}(\mu, x) \leq 2^m \Theta^{*m}(\mu, x) \quad \forall x \in X. \tag{1.8}$$

1.9. **Theorem.** [F, Theorems 2.10.17 (2) and 2.10.18 (1)] *Let (X, d) be a separable metric space, μ be an outer measure in X and $t > 0$.*

(i) *Let $A \subset X$, suppose that μ is Borel regular and*

$$\Theta_F^{*m}(\mu \llcorner A, x) < t \quad \forall x \in A.$$

Then

$$\mu(A) \leq t\, \mathcal{S}^m(A).$$

(ii) *Let $V \subset X$ be an open set, $B \subset V$ and suppose that*

$$\Theta_F^{*m}(\mu, x) > t \quad \forall x \in B.$$

Then

$$\mu(V) \geq t\, \mathcal{S}^m(B).$$

1.10. *Remark.* From (1.2), it follows that the conclusion of Theorem 1.9 (i) may fail to be true when replacing Federer density Θ_F^{*m} by the density Θ^{*m}.

The following (theoretic) area formula for the spherical Hausdorff measure $\mathcal{S}^m$ has been recently proved in [Mag] improving previous Federer's results.

1.11. **Theorem.** [Mag, Theorem 5] *Let μ be a Borel regular measure in X such that there exists a countable open covering of X whose elements have finite μ measure; let $A \subset X$ be a Borel set. Suppose that $\mathcal{S}^m(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^m \llcorner A$. Then*

$$\mu(B) = \int_B \Theta_F^{*m}(\mu, x)\, d\mathcal{S}^m(x) \tag{1.12}$$

for each Borel set $B \subset A$.

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2. Hausdorff measures and density: comparison with the centered Hausdorff measure

We begin repeating the well known definitions of Hausdorff measures to stress their differences with the less known notion of centered spherical Hausdorff measure.

2.1. **Definition.** Let $A \subset X$, $m \in [0,\infty)$, $\delta \in (0,\infty)$, and let β_m be the constant (1.6).

(i) The *m-dimensional Hausdorff measure* $\mathcal{H}^m$ is defined as

$$\mathcal{H}^m(A) := \lim_{\delta\to 0} \mathcal{H}^m_\delta(A)$$

where

$$\mathcal{H}^m_\delta(A) = \inf\left\{ \sum_i \beta_m \mathrm{diam}(E_i)^m :\ A \subset \bigcup_i E_i, \quad \mathrm{diam}(E_i) \le \delta \right\}.$$

(ii) The *m-dimensional spherical Hausdorff measure* $\mathcal{S}^m$ is defined as

$$\mathcal{S}^m(A) := \lim_{\delta\to 0} \mathcal{S}^m_\delta(A)$$

where

$$\mathcal{S}^m_\delta(A) = \inf\Big\{ \sum_i \beta_m \mathrm{diam}(B(x_i,r_i))^m :\ A \subset \bigcup_i B(x_i,r_i), \\ \mathrm{diam}(B(x_i,r_i)) \le \delta \Big\}$$

(iii) The *m-dimensional centered Hausdorff measure* $\mathcal{C}^m$ is defined as

$$\mathcal{C}^m(A) := \sup_{E\subseteq A} \mathcal{C}^m_0(E)\,.$$

where $\mathcal{C}^m_0(E) := \lim_{\delta\to 0^+} \mathcal{C}^m_\delta(E)$, and, in turn, $\mathcal{C}^m_\delta(E) = 0$ if $E = \emptyset$ and for $E \neq \emptyset$

$$\mathcal{C}^m_\delta(E) = \inf\Big\{ \sum_i \beta_m \mathrm{diam}(B(x_i,r_i))^m :\ E \subset \bigcup_i B(x_i,r_i), \\ x_i \in E, \quad \mathrm{diam}(B(x_i,r_i)) \le \delta \Big\}$$

Notice that the set function $\mathcal{C}^m_0$ is not necessarily monotone (see [SRT, Sect. 4]) while $\mathcal{C}^m$ is monotone.

For reader's convenience we collect a few results about the measures $\mathcal{C}^m$. Most of these results are taken from [E].
Let

$$\mathrm{dist}(E,F) := \inf\{d(x,y) :\ x \in E,\ y \in F\}$$

denote the *distance* between E and F. Recall that an outer measure μ on X is said to be *metric* if

$$\mu(A\cup B) = \mu(A) + \mu(B) \qquad \text{whenever } \mathrm{dist}(A,B) > 0\,.$$

Being obtained by Carathëodory's construction, $\mathcal{H}^m$ and $\mathcal{S}^m$ are metric (outer) measures (see [F, 2.10.1] or [M, Theorem 4.2]). Also the measures $\mathcal{C}^m$ are metric measures in any metric space, but this fact is not as immediate as for $\mathcal{H}^m$ and $\mathcal{S}^m$.

2.2. **Lemma.** [E, Proposition 4.1] $\mathcal{C}^m$ *is a metric outer measure.*

Observe also that Lemma 2.2, yields that $\mathcal{C}^m$ is a Borel regular outer measure.

2.3. *Remark.* The measures $\mathcal{H}^m$, $\mathcal{S}^m$ and $\mathcal{C}^m$ are all equivalent measures. Indeed, it is well known that (see, for instance, [F, 2.10.2])

$$\mathcal{H}^m \le \mathcal{S}^m \le 2^m \mathcal{H}^m$$

and, by definition,

$$\mathcal{H}^m \le \mathcal{S}^m \le \mathcal{C}^m \,. \tag{2.4}$$

The opposite inequality between $\mathcal{H}^m$ (or $\mathcal{S}^m$) and $\mathcal{C}^m$ is less immediate: it was proved in [SRT, Lemma 3.3] for the case $X = \mathbb{R}^n$. See also [Sch], but for a differently defined centered Hausdorff-type measure. The comparison in a general metric space is contained [E].

2.5. **Lemma.** [E, Proposition 4.2] $\mathcal{H}^m \le \mathcal{C}^m \le 2^m \mathcal{H}^m$.

By Lemma 2.5, it follows in particular that the metric dimension induced by $\mathcal{H}^m$ or $\mathcal{S}^m$ or $\mathcal{C}^m$ is the same one.

Let us begin to prove that the measures agree for the simplest example of 1-dimensional regular submanifold in a metric space, that is, for Lipschitz curves.

2.6. **Theorem.** *Suppose that $\gamma : [a,b] \to (X,d)$ is a Lipschitz curve and let $\Gamma = \gamma([a,b])$. Then*

$$\mathcal{H}^1(\Gamma) \le \mathcal{S}^1(\Gamma) \le \mathcal{C}^1(\Gamma) \le Var(\gamma) < +\infty \,,$$

and equality holds if γ is injective. Here

$$Var(\gamma) := \sup\{\sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i))\}$$

where the supremum is taken over all finite partitions of $[a,b]$ with $a = t_0 < t_1 < \cdots < t_N = b$.

Proof. We have only to show that

$$\mathcal{C}^1(\Gamma) \le Var(\gamma) \,. \tag{2.7}$$

Indeed, from [AT, Theorem 4.4.2], it follows that

$$\mathcal{H}^1(\Gamma) = Var(\gamma) \quad \text{if } \gamma \text{ is injective.}$$

Thus, from (2.4) and (2.7), we get the desired conclusion.

Let us prove (2.7). By a new parametrization of γ (see [AT, Theorem 4.2.1], we can assume that $a = 0$, $b = Var(\gamma)$ with metric derivative of $\gamma = 1$ a.e. on $[a, b]$. In particular observe that γ is 1-Lipschitz and

$$Var(\gamma, [t, s]) = s - t \quad \forall\, a \le t \le s \le b\,. \tag{2.8}$$

Given $\delta > 0$, choose $N \in \mathbb{N}$ such that $h := Var(\gamma)/N < \delta$. For $i = 0, \dots, N-1$ let $J_i := [i\,h, (i+1)\,h]$, $p_i := \gamma((2i+1)\,h/2)$ and $B_i := B(p_i, h/\varrho_0)$, where ϱ_0 is the constant in (1.5). Then, by (1.5),

$$\operatorname{diam}(B_i) = h \tag{2.9}$$

and

$$\Gamma \subseteq \cup_{i=1}^{N-1} B_i\,. \tag{2.10}$$

Indeed by (2.8) $\gamma(J_i) \subset B_i$, for $i = 0, \dots, N-1$ because

$$d\left(\gamma(t), \gamma\left(\frac{2i+1}{2}h\right)\right) \le \left|t - \frac{2i+1}{2}h\right| \le \frac{h}{2} \le \frac{h}{\varrho_0} \quad \forall\, t \in J_i\,.$$

From (2.10) and (2.9), we get that, for each $E \subseteq \Gamma$

$$\mathcal{C}^1_\delta(E) \le \mathcal{C}^1_\delta(\Gamma) \le \sum_{i=0}^{N-1} \operatorname{diam}(B_i) = Var(\gamma) \quad \forall\, \delta > 0\,. \tag{2.11}$$

Passing to the limit as $\delta \to 0$ in (2.11), we get for all $E \subseteq \Gamma$,

$$\mathcal{C}^1_0(E) \le Var(\gamma)$$

and, taking the supremum on all subsets $E \subseteq \Gamma$, (2.7) follows. □

The estimates needed to relate the m-dimensional density $\Theta^{*\,m}(\mu, \cdot)$ with the centered Hausdorff measure $\mathcal{C}^m$ are the following ones.

2.12. **Theorem.** [E, Theorem 4.15] *Let (X, d) be a separable metric space, let μ be a finite Borel outer measure in X and let $B \subset X$ be a Borel set. Then*

(i)
$$\mu(B) \le \sup_{x \in B} \Theta^{*\,m}(\mu, x)\, \mathcal{C}^m(B)$$
except when the product is $\infty \cdot 0$;

(ii)
$$\inf_{x \in B} \Theta^{*\,m}(\mu, x)\, \mathcal{C}^m(B) \le \mu(B)\,.$$

By easy modifications of the proof of Theorem 2.12, one gets the following density estimates involving $\Theta^{*\,m}(\mu, x)$ and $\mathcal{C}^m$. These estimates are analogous to Federer's ones involving $\Theta^{*\,m}_F(\mu, x)$ and $\mathcal{S}^m$ (see Theorem 1.9).

2.13. **Theorem.** *Let (X,d) be a separable metric space, let μ be an outer measure in X and $t>0$.*

(i) *If μ is Borel regular and*

$$\Theta^{*m}(\mu \llcorner A, x) < t, \qquad \forall x \in A \subset X$$

then

$$\mu(A) \le t\, \mathcal{C}^m(A).$$

(ii) *If $V \subset X$ is an open set and*

$$\Theta^{*m}(\mu, x) > t, \qquad \forall x \in B \subset V$$

then

$$\mu(V) \ge t\, \mathcal{C}^m(B).$$

2.14. *Remark.* If μ is supposed to be a Radon measure, approximating from above by open sets, we can strengthen the conclusion in Theorem 2.13 (ii) getting the inequality $\mu(B) \ge t\, \mathcal{C}^m(B)$.

3. Area formula for the centered Hausdorff measure

3.1. **Theorem.** *Let μ be a Borel regular measure in X such that there exists a countable open covering of X whose elements have μ finite measure; let $A \subset X$ be a Borel set. If $\mathcal{C}^m(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m \llcorner A$, then*

$$\Theta^{*m}(\mu, \cdot) : X \to [0,+\infty] \text{ is Borel measurable}$$

and, for each Borel set $B \subset A$,

$$\mu(B) = \int_B \Theta^{*m}(\mu, x)\, d\mathcal{C}^m(x). \tag{3.2}$$

3.3. *Remark.* Since $\mathcal{C}^m$ and $\mathcal{S}^m$ are equivalent, then $\mathcal{C}^m(A) < \infty$ if and only if $\mathcal{S}^m(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m$ if and only if $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^m$.

Proof. The proof is strongly inspired by [M, Theorem 2.12].

First we prove that $\Theta^{*m}(\mu, \cdot) : X \to [0,+\infty]$ is Borel measurable. For $r>0$, let $g_r : X \to [0,+\infty)$ be the function

$$g_r(x) := \frac{\mu(B(x,r))}{\beta_m \text{diam}(B(x,r))^m} = \frac{\mu(B(x,r))}{\alpha_m r^m}.$$

By Fatou's Lemma, $g_r : X \to [0,+\infty)$ is lower semicontinuous. Thus

$$\Theta^*(\mu, \cdot) = \limsup_{r \in \mathbb{Q},\, r \to 0} g_r(\cdot)$$

gives the desired conclusion.

To show (3.2), first we prove that

$$\mu(A \setminus S) = 0\,, \tag{3.4}$$

if

$$S := \{x \in A:\, 0 < \Theta^{*m}(\mu, x) < +\infty\} \tag{3.5}$$

Since

$$A \setminus S = N_1 \cup N_2\,,$$

$N_1 := \{x \in A:\, \Theta^{*m}(\mu, x) = +\infty\}$, $N_2 := \{x \in A:\, \Theta^{*m}(\mu, x) = 0\}$, let us prove that

$$\mu(N_i) = 0 \quad (i = 1, 2)\,. \tag{3.6}$$

Let $(U_h) \subset X$ be an increasing sequence of open sets such that

$$\cup_{h=1}^{\infty} U_h = X \quad \text{and} \quad \mu(U_h) < +\infty. \tag{3.7}$$

Let

$$N_{1,h} := N_1 \cap U_h.$$

For a given h, since

$$+\infty = \Theta^{*m}(\mu, x) > n \quad \forall\, n \in \mathbb{N} \text{ and } x \in N_{1,h}\,,$$

by Theorem 2.13 (i), it follow that

$$+\infty > \mu(U_h) \geq \mu(N_{1,h}) \geq n\, \mathcal{C}^m(N_{1,h}) \quad \forall\, n \in \mathbb{N}\,. \tag{3.8}$$

By (3.8), it follows that $\mathcal{C}^m(N_{1,h}) = 0$. Since $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^m$,

$$\mu(N_{1,h}) = 0 \quad \forall\, h\,. \tag{3.9}$$

Because $N_1 = \cup_{h=1}^{\infty} N_{1,h}$, (3.9) implies (3.6) for $i = 1$.

For $0 < \epsilon$, let

$$S^*_\epsilon := \{x \in A:\, \Theta^{*m}(\mu, x) \leq \epsilon\}\ . \tag{3.10}$$

By Theorem 2.13 (i), it follows that

$$\mu(N_2) \leq \mu(S^*_\epsilon) \leq \epsilon\, \mathcal{C}^m(S^*_\epsilon) \leq \epsilon\, \mathcal{C}^m(A) \quad \forall\, \epsilon > 0\,.$$

Since $\mathcal{C}^m(A) < +\infty$, letting $\epsilon \to 0$ in the previous inequality, we get

$$\mu(N_2) = 0\,,$$

which establishes (3.6) for $i = 2$.

Finally, the proof follows from the two inequalities

$$\int_B \Theta^{*m}(\mu, x)\, d\mathcal{C}^m(x) \leq \mu(B) \quad \text{for each Borel set } B \subseteq A\,, \tag{3.11}$$

$$\int_B \Theta^{*m}(\mu, x)\, d\mathcal{C}^m(x) \geq \mu(B) \quad \text{for each Borel set } B \subseteq A\,. \tag{3.12}$$

For $t \in (0,\infty)$ and $k \in \mathbb{Z}$, let

$$B_k := \left\{x \in B : t^k \le \Theta^{*m}(\mu,x) < t^{k+1}\right\}$$

and notice that, for $k \neq k'$,

$$B_k \cap B_{k'} = \emptyset \quad \text{and} \quad S \cap B = \cup_{k\in\mathbb{Z}} B_k \tag{3.13}$$

where S is the set defined in (3.5). By (3.4) and Theorem 2.13 (ii),

$$\begin{aligned}\int_B \Theta^{*m}(\mu,x)\,d\mathcal{C}^m(x) &= \int_{S\cap B} \Theta^{*m}(\mu,x)\,d\mathcal{C}^m(x)\\ &= \sum_{k=-\infty}^{\infty} \int_{B_k} \Theta^{*m}(\mu,x)\,d\mathcal{C}^m(x)\\ &\le \sum_{k=-\infty}^{\infty} t^{k+1}\,\mathcal{C}^m(B_k)\\ &\le t \sum_{k=-\infty}^{\infty} \mu(B_k) \le t\,\mu(B)\,.\end{aligned}$$

Letting $t \to 1$ in the previous inequality, we establish (3.11).

Now choose $0 < t < 1$, let

$$B_k := \left\{x \in B : t^{k+1} \le \Theta^{*m}(\mu,x) < t^k\right\} \quad k \in \mathbb{Z}\,,$$

and notice that (3.13) still holds. Arguing as before we get

$$\begin{aligned}\int_B \Theta^{*m}(\mu,x)\,d\mathcal{C}^m(x) &= \int_{S\cap B} \Theta^{*m}(\mu,x)\,d\mathcal{C}^m(x)\\ &= \sum_{k=-\infty}^{\infty} \int_{B_k} \Theta^{*m}(\mu,x)\,d\mathcal{C}^m(x)\\ &\ge \sum_{k=-\infty}^{\infty} t^{k+1}\,\mathcal{C}^m(B_k)\\ &\ge t \sum_{k=-\infty}^{\infty} \nu(B_k) = t\,\mu(S\cap B) = t\,\mu(B)\,,\end{aligned}$$

which establishes (3.12) after letting $t \to 1$. □

3.14. **Corollary.** *Under the same assumptions of Theorem 3.1, if there is $k > 0$ such that*

$$\Theta^{*m}(\mu,x) = k \qquad \forall x \in A \subseteq X$$

then

$$\mu \llcorner A = k\,\mathcal{C}^m \llcorner A\,.$$

4. An application to Carnot groups

Carnot groups are a relevant class of separable metric spaces (X, d) satisfying (1.5) (see Lemma 4.5 below). A detailed study of Carnot groups can be found in [BLU]. Here we will recall only a few of their properties using the notations of [FSS2] and we will refer to this last paper for notions and results.

A n-dimensional Carnot group $\mathbb{G}$ of *step* k is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ has dimension n and admits a step k stratification, i.e., there are linear subspaces V_1, $V_2, \dots$ of $\mathfrak{g}$ such that

$$\begin{aligned} &\mathfrak{g} = V_1 \oplus ... \oplus V_k, \\ &[V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \end{aligned} \tag{4.1}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the all the commutators $[X, Y]$, with $X \in V_1$ and $Y \in V_i$.

The integer

$$Q := \sum_{i=1}^{k} i \, \dim(V_i) \tag{4.2}$$

is the *homogeneous dimension* of $\mathbb{G}$.

By exponential coordinates, we can identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$ where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula ([FSS2, Proposition 2.1])

Any Carnot group $\mathbb{G}$ is equipped with a family of automorphisms, the *intrinsic dilations*

$$\delta_\lambda : \mathbb{G} \to \mathbb{G}, \quad \lambda > 0$$

4.3. **Definition.** A distance d in $\mathbb{G}$ is said to be *invariant* if

$$d(z \cdot x, z \cdot y) = d(x, y) \quad \text{and} \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda \, d(x, y) \tag{4.4}$$

for all $x, y, z \in \mathbb{G}$ and $\lambda > 0$.

The so-called sub-Riemannian distance or Carnot-Carathéodory distance ([FSS2, Definition 2.3]) is an example of an invariant distance in $\mathbb{G}$. On the other hand, several invariant distances, equivalent to the sub-Riemannian one and sometimes easier to compute, have been used in the literature. Notice that two invariant distances d and $\tilde{d}$ on $\mathbb{G}$ are always equivalent, that is there exists $C > 1$ such that for all $x, y \in \mathbb{G}$,

$$C^{-1} \, d(x, y) \leq \tilde{d}(x, y) \leq C \, d(x, y),$$

(see[BLU, Corollary 5.1.5]).

4.5. **Lemma.** *(see* [FSS2, Proposition 2.4]*) Let $\mathbb{G}$ be a Carnot group endowed with an invariant distance d. Then* (1.5) *holds with $\rho_0 = 2$.*

Observe also that the homogeneous dimension Q is the Hausdorff dimension of $\mathbb{G}$ with respect to any invariant distance on $\mathbb{G}$.

We want to compare the Hausdorff measures $\mathcal{H}^Q$, $\mathcal{S}^Q$ and $\mathcal{C}^Q$ on a Carnot group $\mathbb{G} = (\mathbb{R}^n, \cdot)$ of homogeneous dimension Q, endowed with an invariant metric d.
First of all observe that all of them are Radon measure on $\mathbb{G}$. In addition they are invariant under left-translations hence they are Haar measures of the group. On the other hand $\mathcal{L}^n$ too is a Haar measure. Thus (see [M, Theorem 3.4]) each one of them is a constant multiple of $\mathcal{L}^n$.

4.6. **Theorem.** *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group of homogeneous dimension Q endowed with an invariant distance d.*

(i) *(*[R, Proposition 2.1]*) $\mathcal{S}^Q(B) = \beta_Q \operatorname{diam}(B)^Q$ for each ball $B \subset \mathbb{G}$;*
(ii) *(*[R, Proposition 2.3]*) $\mathcal{S}^Q = C_d \mathcal{H}^Q$ where C_d is the* isodiametric constant*, that is*

$$C_d := \sup \left\{ \frac{\mathcal{S}^Q(A)}{\beta_Q \operatorname{diam}(A)^Q} : 0 < \operatorname{diam}(A) < +\infty \right\} ; \tag{4.7}$$

(iii) *$\mathcal{C}^Q(B) = \beta_Q \operatorname{diam}(B)^Q$ for each ball $B \subset \mathbb{G}$*

Proof. Let us prove (iii). From (i) and the definition of $\mathcal{C}^Q$, we get

$$\mathcal{S}^Q(B) = \beta_Q \operatorname{diam}(B)^Q \le \mathcal{C}^Q(B)$$

for each closed ball $B \subset \mathbb{G}$. Let μ be the normalized Haar measure on $\mathbb{G}$ such that $\mu(B) = \beta_Q \operatorname{diam}(B)^Q$ for each closed ball B. Then it is sufficient to prove that

$$\mathcal{C}^Q(U) \le \mu(U) , \tag{4.8}$$

for each open ball U. Indeed, since $\mathcal{C}^Q$ is a left-invariant Radon measure, $\operatorname{diam}(\overline{U}) = \operatorname{diam}(U)$, from Lemma 4.5, if $B = \overline{U}$ with U open ball, then

$$\mathcal{C}^Q(B) = \mathcal{C}^Q(U) ,$$

and, by (4.8), we get the desired inequality.

Let us prove (4.8). Given an open ball U, $E \subset U$ and $\delta > 0$, let

$$\mathcal{F} := \Big\{ B(x,r) : x \in E,\ B(x,r) \subset U, \operatorname{diam}(B(x,r)) < \delta \Big\} .$$

Since U is open, $\mathcal{F}$ is a fine covering of E, and, because $\mathcal{C}^Q$ is a doubling measure, by a Vitali-type covering lemma there is a countable, disjoint family of closed balls $(B_i)_i \subset \mathcal{F}$ such that

$$\mathcal{C}^Q(E \setminus \cup_{i=1}^{\infty} B_i) = 0\,. \tag{4.9}$$

Observe now that

$$\mathcal{C}^Q_\delta(\cup_{i=1}^{\infty} B_i) \le \sum_{i=1}^{\infty} \beta_Q \operatorname{diam}(B_i)^Q = \mu(\cup_{i=1}^{\infty} B_i) \le \mu(U)\,, \tag{4.10}$$

and, by (4.9),

$$\mathcal{C}^Q_\delta(E \setminus \cup_{i=1}^{\infty} B_i) \le \mathcal{C}^Q_0(E \setminus \cup_{i=1}^{\infty} B_i) \le \mathcal{C}^Q(E \setminus \cup_{i=1}^{\infty} B_i) = 0\,. \tag{4.11}$$

On the other hand, by (4.10) and (4.11), for all $E \subset U$ and $\delta > 0$

$$\mathcal{C}^Q_\delta(E) \le \mathcal{C}^Q_\delta(E \setminus \cup_{i=1}^{\infty} B_i) + \mathcal{C}^Q_\delta(\cup_{i=1}^{\infty} B_i) \le \mu(U). \tag{4.12}$$

Taking, first the limit as $\delta \to 0$ in (4.12) and then the supremum on all sets $E \subseteq U$, (4.8) follows.

□

4.13. **Corollary.** *If $\mathbb{G}$ is a Carnot group of homogeneous dimension Q and endowed with an invariant distance d then*

$$\mathcal{S}^Q = \mathcal{C}^Q.$$

Proof. Since there is $\beta > 0$ such that $\mathcal{S}^Q = \beta\, \mathcal{C}^Q$, and $\mathcal{S}^Q(B) = \mathcal{C}^Q(B)$ on any ball, then $\beta = 1$. □

Notice that, choosing $\beta_Q = 2^{-Q} \mathcal{L}^n(B_d(0,1))$ in Definition 2.1, it follows that $\mathcal{S}^Q = \mathcal{C}^Q = \mathcal{L}^n$.

4.14. *Remark.* The constant C_d in (4.7) is related to the so-called isodiametric problem. In the setting of Carnot groups, the isodiametric problem has been studied in [R] and [LRV]. More precisely in [R] it is studied whether - in a Carnot group endowed with a generic invariant distance - the sharp isodiametric inequality holds or, equivalently, if balls realize the supremum in the right-hand side of (4.7). If this were the case, by Theorem 4.6 (i) it would follow $C_d = 1$ and the *sharp isodiametric inequality*

$$\mathcal{S}^Q(A) \le \beta_Q \operatorname{diam}(A)^Q \quad \forall A \subset \mathbb{G} \tag{SII}$$

and, eventually, from Theorem 4.6 (ii)

$$\mathcal{S}^Q = \mathcal{H}^Q\,.$$

On the contrary, Rigot proves that in every non abelian Carnot group $\mathbb{G}$, there is an invariant distance for which the sharp isodiametric inequality (SII) fails to be true (see [R, Theorems 3.4, 3.5 and 3.6]).

Note that it is not difficult to prove the existence of sets achieving the supremum in (4.7), these sets are called *isodiametric sets* (see [R, Theorem 3.1]). Very little is known about these isodiametric sets in general Carnot groups. In [LRV], when the group $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^n$ equipped with its CC distance d_c, the regularity of the isodiametric sets as well as their characterization under some symmetry assumption is studied. The characterization of a general isodiametric set even inside Heisenberg groups is still an open problem.

In the following we are going to use frequently a particular invariant distance, the so called distance d_∞. It is defined in [FSS2, (2.8)], it can be explicitly computed and the balls constructed with it are convex sets with many rotational simmetries.

Several notions typical of geometric measure theory have been studied inside Carnot groups. Among them we mention the so called sets of *locally finite perimeter* (their definition in Carnot groups was given in [CDG]; see also [FSS1]). Let us recall that a measurable set $E \subset \mathbb{G}$ has locally finite $\mathbb{G}$-perimeter in an open set $\Omega \subset \mathbb{G}$ (or E is a $\mathbb{G}$-Caccioppoli set) if

$$|\partial E|_{\mathbb{G}}(V) < +\infty \quad \text{for all open sets } V \Subset \Omega\,,$$

where $|\partial E|_{\mathbb{G}}$, known as the $\mathbb{G}$*-perimeter measure*, is the Radon measure defined as in [FSS2, (2.18)]. In general, the measure $|\partial E|_{\mathbb{G}}$ is supported on a subset of the topological boundary of a $\mathbb{G}$-Caccioppoli set. In a large class of Carnot groups $\mathbb{G}$ this subset, known as the *reduced boundary* $\partial^* E$ of E is known to be a rectifiable subset of $\mathbb{G}$ (for a precise statement see Theorem 4.18). This result is the Carnot group counterpart of the celebrated De Giorgi's structure theorem for sets of finite perimeter in Euclidean spaces.

In order to state correctly Theorem 4.18, adapted intrinsic notions of regular hypersurfaces and of rectifiable sets in Carnot groups are needed ([FSS1, FSS2, FSS4]). Observe that this definition of rectifiable sets is different from the one in [AK].

4.15. **Definition. (i)** A set $S \subset \mathbb{G}$ is a $\mathbb{G}$*-regular hypersurface* if for each $x \in S$ there are a neighborhood U of x and a function $f \in \mathbf{C}^1_{\mathbb{G}}(U)$ such that

$$S \cap U = \{y \in U : f(y) = 0\}$$

and

$$\nabla_{\mathbb{G}} f(y) \neq 0 \quad \forall y \in U\,,$$

where $\mathbf{C}^1_{\mathbb{G}}(U)$ and $\nabla_{\mathbb{G}}$ denote, respectively, the space of functions defined in [FSS2, Definition 2.8] (see also [FSS2, Proposition 2.12]) and the *horizontal gradient* section ([FSS2, Proposition 2.11]). We denote by $T^g_{\mathbb{G}}S(x)$ the *tangent group* to S at x as in [FSS2, Section 2.4].

(ii) A set $\Gamma \subset \mathbb{G}$ is said to be $(Q-1)$-dimensional $\mathbb{G}$-*rectifiable* if there is a sequence of $\mathbb{G}$-regular hypersurfaces $(S_j)_j$ such that

$$\mathcal{H}^{Q-1}(\Gamma \setminus \cup_{j=1}^{\infty} S_j) = 0\,.$$

Here $\mathcal{H}^{Q-1}$ denotes the $(Q-1)$-dimensional Hausdorff measure defined as in Definition 2.1 (i) with $X = \mathbb{G}$ and d any invariant distance in $\mathbb{G}$.

4.16. *Remark.* Both the notions of $\mathbb{G}$-regular hypersurfaces and of $\mathbb{G}$-rectifiable sets are independent of the chosen invariant distance d. Indeed, from the equivalence of invariant distances and from [FSS2, Definition 2.8 and Proposition 2.11], if S is a $\mathbb{G}$-regular hypersurface with respect to an invariant distance d so it is with respect to any other invariant distance $\tilde{d}$. The same holds if S is a $\mathbb{G}$-rectifiable set.

The following structure result for sets of finite perimeter was proved inside step 2 Carnot groups in [FSS2] and recently extended to the much larger class of groups of type $\star$, in [Mar].

4.17. **Definition.** A stratified Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$ is of type $\star$ if there exists a basis $\{X_1, \dots, X_{m_1}\}$ of V_1 such that

$$[X_j,[X_j,X_i]] = 0, \qquad \text{for all } \; i,j = 1,\dots,m_1.$$

A Carnot group $\mathbb{G}$ is a group of type $\star$ if its Lie algebra $\mathfrak{g}$ is of type $\star$.

4.18. **Theorem.** *Let $\mathbb{G}$ be a Carnot group of type $\star$, endowed with the invariant distance d_∞. Let $E \subset \mathbb{G}$ be a $\mathbb{G}$-Caccioppoli set and let $\partial^*_{\mathbb{G}}E$ denote its reduced boundary (*[FSS2, Definition 2.25]*). Then*

(i) $|\partial E|_{\mathbb{G}} = |\partial E|_{\mathbb{G}} \llcorner \partial^*_{\mathbb{G}}E$;

(ii) *there exists $k = k(\mathbb{G}) > 0$ s.t.*

$$\Theta^{Q-1}(|\partial E|_{\mathbb{G}}, x) = k \qquad \forall x \in \partial^*_{\mathbb{G}}E;$$

where the $(Q-1)$-dimensional density $\Theta^{Q-1}(|\partial E|_{\mathbb{G}}, \cdot)$ must be understood according to Definition 1.7 (i) with $X = \mathbb{G}$ and $d = d_\infty$;

(iii) *$\partial^*_{\mathbb{G}}E$ is $(Q-1)$-dimensional $\mathbb{G}$-rectifiable;*

(iv) $|\partial E|_{\mathbb{G}} = k\,\mathcal{S}^{Q-1} \llcorner \partial^*_{\mathbb{G}}E$,
where $\mathcal{S}^{Q-1}$ denotes the $(Q-1)$-dimensional spherical Hausdorff measure defined according to Definition 2.1 (ii) with $X = \mathbb{G}$ and $d = d_\infty$.

We recall that Theorem 4.18 has been recently extended even to a class of sub-Riemannian manifolds (see [AGM]).

Magnani [Mag2] correctly observes that the example mentioned in the introduction shows that Theorem 4.18 (iv) - notwithstanding being true - is not an immediate consequence of Federer' results [F, 2.10. 17(2) and 2.10.19 (3)], as was claimed in [FSS2] (see also [FSS3, Theorem 3.4]). However Magnani himself has proved Theorem 4.18 (iv) in the following result.

4.19. **Theorem.** *Let $\mathbb{G}$ be a Carnot group endowed with the invariant distance d_∞.*
(i) [Mag2, Theorem 5.2] *Let $E \subset \mathbb{G}$. Assume that the topological boundary ∂E is a $\mathbb{G}$-regular hypersurface, then there is $k = k(\mathbb{G}) > 0$ such that*

$$\Theta^{Q-1}(|\partial E|_\mathbb{G}, x) = \Theta_F^{*\,Q-1}(|\partial E|_\mathbb{G}, x) = k \qquad \forall x \in \partial E;$$

(ii) [Mag2, Theorem 1.3] *Under the same assumptions of Theorem 4.18, statement (iv) holds.*

As anticipated in the introduction we are taking here a different approach. As a consequence of Corollary 3.14 and of Theorem 4.18(ii), we directly obtain the following version of Theorem 4.18 (iv), where the centered Hausdorff measure takes the role of the spherical measure $\mathcal{S}^{Q-1}$. Observe that no regularity of $\partial^*_\mathbb{G} E$ is assumed.

4.20. **Corollary.** *Let $\mathbb{G}$ be a Carnot group of type $\star$ of homogeneous dimension Q and endowed with the invariant distance d_∞. Let $E \subset \mathbb{G}$ be a $\mathbb{G}$-Caccioppoli set. Then there is $k = k(\mathbb{G}) > 0$ for which*

$$|\partial E|_\mathbb{G} = k\, \mathcal{C}^{Q-1} \llcorner \partial^*_\mathbb{G} E\,,$$

where $\mathcal{C}^{Q-1}$ denotes the $(Q-1)$-dimensional centered Hausdorff measure defined as in Definition 2.1 (iii) with $X = \mathbb{G}$ and $d = d_\infty$.

As an other application to Carnot groups of the area formula for centered Hausdorff measure, we provide an explicit characterization of the centered density $\Theta^{Q-1}(|\partial E|_\mathbb{G}, \cdot)$ for sets E whose boundary ∂E is a $\mathbb{G}$-regular surface. This is an emendated version of [FSS3, Theorem 3.5].

4.21. **Theorem.** *Let $\mathbb{G}$ be a Carnot group and d an invariant distance on it. Let $\Omega \subset \mathbb{G}$ be an open set, let $E \subset \mathbb{G}$ be such that $\partial E \cap \Omega = S \cap \Omega$, where S is a $\mathbb{G}$-regular hypersurface. Then, for all $x \in S$*

$$\Theta^{Q-1}(|\partial E|_\mathbb{G}, x) = \frac{\mathcal{L}^{n-1}\left(U_d(0,1) \cap T^g_\mathbb{G} S(x)\right)}{\alpha_{Q-1}}, \tag{4.22}$$

where $T^g_{\mathbb{G}}S(x)$ *denotes the tangent group to* S *at* x*, and* $U_d(0,1)$ *denotes the unit (open) ball induced by the distance* d*. In particular*

$$|\partial E|_{\mathbb{G}} \llcorner \Omega = \Theta^{Q-1}(|\partial E|_{\mathbb{G}}, \cdot)\, \mathcal{C}^{Q-1} \llcorner (S \cap \Omega). \tag{4.23}$$

where $\mathcal{C}^{Q-1}$ *denotes the* $(Q-1)$*-dimensional centered Hausdorff measure defined as in Definition 2.1 (iii) with* $X = \mathbb{G}$ *equipped by* d*.*
Moreover, there is $\beta_d > 1$*, depending only on* d*, such that*

$$0 < \frac{1}{\beta_d} \leq \Theta^{Q-1}(|\partial E|_{\mathbb{G}}, x) \leq \beta_d < \infty \qquad \forall x \in S. \tag{4.24}$$

4.25. *Remark.* If $S = \partial E$ is $\mathbb{G}$-regular the equality

$$\Theta^{Q-1}(|\partial E|_{\mathbb{G}}, x) = \Theta^{*Q-1}_F(|\partial E|_{\mathbb{G}}, x) \qquad \forall x \in S$$

is known to hold only for a restricted class of distances on $\mathbb{G}$. These distances are denoted by Magnani as $(n-1)$-*vertical regular* (see [Mag2, Definition 2.2]) and the distance d_∞ is one of them. In the general case, an explicit characterization of Federer density $\Theta^{*Q-1}_F(|\partial E|_{\mathbb{G}}, \cdot)$ - similar to the one in (4.22) - seems to be unknown. It seems to be unknown even for the Carnot-Carathédory distance d_c.

Proof of Theorem 4.21. In [FSS3, (64)] is proved that, when $0 \in S$,

$$\lim_{r\to 0} \frac{|\partial E|_{\mathbb{G}}(U_d(0,r))}{r^{Q-1}} = \mathcal{L}^{n-1}\left(U_d(0,1) \cap T^g_{\mathbb{G}}S(0)\right). \tag{4.26}$$

The $\mathbb{G}$-perimeter is left-invariant and $(Q-1)$-homogeneous, that is

$$|\partial E|_{\mathbb{G}}(U) = |\partial(x \cdot E)|_{\mathbb{G}}(x \cdot U), \quad |\partial E|_{\mathbb{G}}(U) = \lambda^{1-Q}|\partial \delta_\lambda(E)|_{\mathbb{G}}(\delta_\lambda(U))$$

for any open $U \subset \mathbb{G}$, $x \in \mathbb{G}$ and $\lambda > 0$. Hence from (4.26) it follows that for all $x \in S$

$$\lim_{r\to 0} \frac{|\partial E|_{\mathbb{G}}(U_d(x,r))}{r^{Q-1}} = \mathcal{L}^{n-1}\left(U_d(0,1) \cap T^g_{\mathbb{G}}S(x)\right)$$

that proves (4.22).

Now, also (4.23) follows from (4.22), Lemma 4.5 and Theorem 3.1.

To prove (4.24) recall that d and d_∞ are equivalent distances. Thus it suffices to show that there exists $\beta > 0$ such that

$$\mathcal{L}^{n-1}\left(U_{d_\infty}(0,1) \cap T^g_{\mathbb{G}}S(x)\right) = \beta \qquad \forall x \in S\,. \tag{4.27}$$

Since d_∞ is invariant under rotations of the first layer V_1 of the algebra, arguing as in [FSS2, Thrm.3.9], (4.27) follows. □

Eventually it holds that

4.28. **Theorem.** *Let $\mathbb{G}$ be a Carnot group endowed with the invariant distance d_∞. Let $\mathcal{S}^{Q-1}$ and $\mathcal{C}^{Q-1}$ denote the $(Q-1)$-dimensional spherical and centered Hausdorff measures, defined as in Definition 2.1 (ii) and (iii) with $X = \mathbb{G}$ and $d = d_\infty$. Then $\mathcal{S}^{Q-1}$ and $\mathcal{C}^{Q-1}$ agree on $\mathbb{G}$-rectifiable sets.*

Proof. Let $\Gamma \subset \mathbb{G}$ be a $(Q-1)$-dimensional $\mathbb{G}$-rectifiable set. By definition there are Borel sets Γ_i $(i = 0, 1, 2, \dots)$ and $\mathbb{G}$-regular hypersurfaces S_i $(i = 1, 2, \dots)$ such that

$$\Gamma = \Gamma_0 \cup (\cup_{i=1}^{\infty} \Gamma_i), \tag{4.29}$$

$$\mathcal{H}^{Q-1}(\Gamma_0) = 0, \tag{4.30}$$

and, for $i = 1, 2, \dots$

$$\Gamma_i \subset S_i. \tag{4.31}$$

By a well known argument, we can suppose that the sets Γ_i are pairwise disjoint. Moreover, from [FSS2, Theorem 2.35], we can also assume the existence of functions $f_i \in \mathbf{C}^1_{\mathbb{G}}(U_i)$ with $\nabla_{\mathbb{G}} f_i(x) \neq 0$ for each $x \in U_i$ such that, if $E_i := \{x \in U_i : f_i(x) < 0\}$,

$$S_i = \{x \in U_i : f_i(x) = 0\}, \quad \partial E_i \cap U_i = \partial^*_{\mathbb{G}} E_i \cap U_i = S_i, \tag{4.32}$$

$$|\partial E_i|_{\mathbb{G}}(U_i) < +\infty. \tag{4.33}$$

From [Mag2, Theorem 4.19], [Mag, Theorem 1.11], (4.31), (4.32) and (4.33), it follows that

$$|\partial E_i|_{\mathbb{G}}(\Gamma_i) = k\, \mathcal{S}^{Q-1}(\Gamma_i \cap S_i) = k\, \mathcal{S}^{Q-1}(\Gamma_i) \quad i = 1, 2, \dots. \tag{4.34}$$

From Corollary 4.20, (4.31), (4.32) and (4.33), it follows that

$$|\partial E_i|_{\mathbb{G}}(\Gamma_i) = k\, \mathcal{C}^{Q-1}(\Gamma_i \cap S_i) = k\, \mathcal{C}^{Q-1}(\Gamma_i) \quad i = 1, 2, \dots. \tag{4.35}$$

Thus, from (4.34) and (4.35), we deduce that for $i = 1, 2, \dots$

$$\mathcal{S}^{Q-1}(\Gamma_i) = \mathcal{C}^{Q-1}(\Gamma_i).$$

Since the sets Γ_i are disjoint, and $\mathcal{H}^{Q-1}$, $\mathcal{S}^{Q-1}$ and $\mathcal{C}^{Q-1}$ are equivalent measures, by (4.29) and (4.30), we get that

$$\mathcal{S}^{Q-1}(\Gamma) = \mathcal{C}^{Q-1}(\Gamma).$$

□

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