Poincaré-type inequality for Lipschitz continuous vector fields

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Abstract

The scope of this paper is to prove a Poincaré type inequality for a family of non linear vector fields, whose coefficients are only Lipschitz continuous with respect to the distance induced by the vector fields themselves.

Keywords: Poincaré inequalities, nonlinear vector fields, Lie groups.
2010 MSC: 46T30, 43A80.

1. Introduction and statement of the result

The Poincaré inequality is one of the main tools in the proof of regularity of solutions of PDEs in divergence form. Indeed, as proved by Saloff-Coste in [1] and Grigor'yan in [2] (see also [3]), it is equivalent to the Harnack inequality and to Hölder continuity for solutions. Thus, to prove regularity of solutions, it suffices to establish a suitable Poincaré inequality.

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Preprint submitted to Journal de Mathématiques Pures et Appliquées *May 27, 2017*

The Poincaré inequality for smooth Hörmander vector fields is well known and was proved by Jerison [4]. We recall that a Hörmander family of vector fields in $\mathbb{R}^n$, is defined by $m \leq n$ smooth vector fields, say $\nabla = (\nabla_1, \dots, \nabla_m)$, such that the generated Lie algebra has maximum rank at every point.
Denote by $B_r(x) \subset \mathbb{R}^n$ the metric ball of center x and radius $r > 0$ associated to the CC-distance defined in terms of the family ∇. The Poincaré inequality proved in [4] is:

$$\int_{B_r(x)} |\psi(y) - \psi_r(y)| \, \mathrm{d}\mathcal{L}^n(y) \leq C\, r \int_{B_r(x)} |\nabla\psi(y)| \, \mathrm{d}\mathcal{L}^n(y) \quad \forall \psi \in C^\infty(\overline{B_r(x)})$$

where, as usual, $\psi_r(y) := 1/\mathcal{L}^n(B_r(y)) \int_{B_r(y)} \psi$ and $C > 0$. The previous inequality can be also stated using balls defined with respect to different (but equivalent) distances. We mention here the ball box distance (see [5]) and the frozen distance defined by Rothschild and Stein in [6].

The Poincaré inequality for non smooth vector fields was first considered in [7]. The later works on related questions include the papers by Biroli and Mosco [8], Capogna, Danielli and Garofalo [9, 10], Chernikov and Vodopyanov [11], Danielli, Garofalo, Nhieu [12], Franchi, Gallot and Wheeden [13], Franchi, Lu and Wheeden [14] and Lu [15, 16]. More recently, in [17], the authors proved a general Poincaré inequality which was applied in [18], [19], [20] to families of Lipschitz vector fields with different regularity conditions, and different assumptions on the rank of the generated Lie algebra.

In [21], the authors studied the relationship between the validity of the Poincaré inequality and the existence of representation formulas for functions as (fractional) integral transforms of first-order vector fields. They showed that the Poincaré inequality leads to (and in fact it is often equivalent to) a suitable representation formula. This approach was later developed in [9], in which another proof has been given of the representation formula relying on the Poincaré inequality proved by Jerison. Finally, in [22], a general representation formula is proved in terms of the fundamental solution of a Hörmander type sublaplacian.

Unfortunately, all these results are expressed in terms of vector fields with Lipschitz continuous coefficients with respect to the Euclidean distance. On the other hand, in order to study partial differential equations with non linear vector fields, this assumption is no longer natural. A typical differential equation of this type can be of the form

$$\sum_{i,j=1}^{m} \nabla_i^{\phi}(a_{ij}(\phi)\ \nabla_j^{\phi}\phi) = f \tag{1.1}$$

where (a_{ij}) is a smooth, symmetric, uniformly positive definite matrix, f is a fixed function and the coefficients of the vector fields ∇^{ϕ} depend on the solution ϕ. Equations of this type naturally arise while studying curvature equations [23], Monge-Ampére equation [24, 25, 26], mathematical finance [27, 28, 29] or intrinsic minimal graphs in the Heisenberg group (see for instance [30, 31, 32, 33, 34, 35]).

A particular, but very interesting instance of (1.1), is the so-called minimal surface equation for intrinsic graphs in the Heisenberg group (see also [36, 35, 37] for the case of T-graphs). In the $n-$dimensional Heisenberg group $\mathbb{H}^n$, such graphs are described as follows (see [38, 39]):

$$M = \{(\phi(x_1, \ldots, x_{2n}), x_2, \ldots, x_{2n}, x_{2n}+2x_n\phi(x_1, \ldots, x_{2n})),\ (x_1, \ldots, x_{2n}) \in \omega\}$$

where $\omega \subset \mathbb{R}^{2n}_{(x_1,\ldots,x_{2n})}$ is an open set and $\phi : \omega \to \mathbb{R}$ is a continuous function satisfying suitable regularity properties. Intrinsic graphs have been extensively studied in connection with the notion of rectifiable sets in $\mathbb{H}^n$ (see for instance [40, 41, 42, 43]), the regularity problem for minimal surfaces (see for instance [31, 32, 44, 45, 46, 35]) and the Bernstein problem in $\mathbb{H}^n$ ([30, 33, 47, 48, 49]). In particular, in [38] it is proved that the so-called horizontal perimeter of M can be expressed by

$$P_{\mathbb{H}}(M) = \int_{\omega} \sqrt{1 + |\nabla^{\phi}\phi|^2} \mathrm{d}\mathcal{L}^{2n}$$

where $\nabla^{\phi} = (\nabla_1^{\phi}, \ldots, \nabla_{2n-1}^{\phi})$ is the family of non linear vector fields defined by:

$$\nabla_i^{\phi} = \partial_{x_i} - x_{i+n}\partial_{x_{2n}},\ \nabla_n^{\phi} = \partial_{x_n} + 2\phi\partial_{x_{2n}}, \nabla_{i+n}^{\phi} = \partial_{x_{i+n}} + x_i\partial_{x_{2n}}, \tag{1.2}$$

where $i = 1, \ldots, n-1$. Moreover, as pointed out in [38, 33], one can show that the condition that the intrinsic surface M be $\mathbb{H}-$minimal is expressed by the non linear equation of type (1.1):

$$\nabla^{\phi} \cdot \frac{\nabla^{\phi}\phi}{\sqrt{1 + |\nabla^{\phi}\phi|^2}} = 0.$$

Notice that the regularity of the solution ϕ of (1.1) can be obtained only in the Hölder spaces defined in terms of the distance naturally associated to the family ∇^ϕ. As a consequence, the coefficients of the equation, (which depend on the solution itself), are not expected to be Lipschitz with respect to the Euclidean distance. Moreover, this lack of regularity of the coefficients implies that the equivalence of the different definitions of distances cannot be deduced using [5]. To overcome this problem, a distance modeled on the frozen distance of Rothschild and Stein was defined in [23] while studying the Levi equation. In [38, 39], an analogous frozen distance d_ϕ has been proposed for the vector fields in (1.2), as the symmetrized distance associated to the frozen vector fields

$$\nabla_i^{\phi(x_0)} := \nabla_i^\phi \text{ for } i = 1, \dots, 2n-1, i \neq n, \quad \nabla_n^{\phi(x_0)} := \partial_{x_n} + 2\phi(x_0)\partial_{x_{2n}}, \tag{1.3}$$

where $x_0 \in \omega$ is fixed. The main advantage of working with this family of vector fields relies on the fact that they have C^∞ coefficients and they can be considered as a zero order approximation of the family ∇^ϕ. We point out that the equivalence between the frozen distance d_ϕ and the ball box distance defined in [5] was proved in [39]. Moreover, the equivalence between d_ϕ and the CC-distance generated by ∇^ϕ can be found in [50].

Motivated by the discussion above, in this paper we prove a Poincaré type inequality for the model vector fields in (1.2), under the assumption that the coefficients are Lipschitz continuous with respect to d_ϕ and for functions which belong to an intrinsic Sobolev space:

Definition 1.1. *Let $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be an intrinsic Lipschitz continuous function, in the sense of definition 2.1 below. We say that a function $\psi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ belongs to the space $W_\phi(\omega)$ if there exist sequences $\{\psi_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ in $C^\infty(\omega)$ such that*

(i) $\psi_k \to \psi$ *in* $L^1_{loc}(\omega)$ *as* $k \to +\infty$;

(ii) $\phi_k \to \phi$ *uniformly in* ω *as* $k \to +\infty$;

(iii) $|\nabla^{\phi_k}\psi_k(x)| \leq M$ $\forall x \in \omega$ *and* k *and for some positive constant* M;

(iv) $\nabla^{\phi_k}\psi_k \rightharpoonup^* \nabla^\phi\psi$ *as* $k \to +\infty$.

Then, our main result is the following:

Theorem 1.2. *Let ω be a bounded and open subset of $\mathbb{R}^{2n}$ with $n \geq 2$, and let $p \geq 1$. Let $\phi : \omega \to \mathbb{R}$ be an intrinsic Lipschitz function and $\psi \in W_\phi(\omega)$. Then there exist positive constants C_1, C_2 with $C_2 > 1$ (depending continuously on the Lipschitz constant L_ϕ of ϕ) such that*

$$\int_{U_\phi(\bar{x},r)} |\psi(y) - \psi_{U_\phi(\bar{x},r)}|^p \, \mathrm{d}\mathcal{L}^{2n}(y) \leq C_1 \, r^p \int_{U_\phi(\bar{x},C_2\, r)} |\nabla^\phi \psi(y)|^p \, \mathrm{d}\mathcal{L}^{2n}(y), \tag{1.4}$$

for every $U_\phi(\bar{x}, C_2\, r) \subset \omega$, where

$$U_\phi(x,r) := \{y \in \omega : \mathrm{d}_\phi(x,y) < r\}\,. \tag{1.5}$$

Here $\psi_{U_\phi(\bar{x},r)}$ denotes the mean of ψ on the ball $U_\phi(\bar{x},r)$ with respect to the Lebesgue measure, i.e.

$$\psi_{U_\phi(\bar{x},r)} := \frac{1}{\mathcal{L}^{2n}(U_\phi(\bar{x},r))} \int_{U_\phi(\bar{x},r)} \psi(y) \, \mathrm{d}\mathcal{L}^{2n}(y)\,. \tag{1.6}$$

Corollary 1.3. *If $\phi : \omega \to \mathbb{R}$ is an intrinsic Lipschitz function, then there exist positive constants C_1, C_2 with $C_2 > 1$ (depending continuously on the Lipschitz constant L_ϕ of ϕ) such that*

$$\int_{U_\phi(\bar{x},r)} |\phi(y) - \phi_{U_\phi(\bar{x},r)}| \, \mathrm{d}\mathcal{L}^{2n}(y) \leq C_1 \, r \int_{U_\phi(\bar{x},C_2\, r)} |\nabla^\phi \phi(y)| \, \mathrm{d}\mathcal{L}^{2n}(y), \tag{1.7}$$

for every $U_\phi(\bar{x}, C_2\, r) \subset \omega$.

We briefly describe our approach. Since the coefficients of the vector fields ∇^ϕ are only Lipschitz continuous (with respect to d_ϕ), we cannot consider the Lie algebra generated by the vector fields. Nevertheless, the explicit expression of the vector fields $\nabla^\phi_1, \dots, \nabla^\phi_{2n-1}$ ensures that

$$[\nabla^\phi_i, \nabla^\phi_{n+i}] = 2\, \partial_{x_{2n}} \quad \forall\, i = 1, \dots, n-1,$$

so that the vector fields and their commutators span the whole space at every point, which can be interpreted as a Hörmander condition for non regular vector fields. This approach can be considered a version of the Rothschild and Stein method for non smooth vector fields, and has been used in [23] in a different setting. In particular, every $\psi \in C^\infty(\omega)$ can be represented by means of a suitable representation formula (proved in [22]) in terms of the vector fields $\nabla_i^{\phi(x_0)}$, the fundamental solution Γ_{x_0} of the Laplacian operator

$$\mathcal{L}_{\phi(x_0)} := \sum_{i=1}^{2n-1} (\nabla_i^{\phi(x_0)})^2$$

and the super level sets $\Omega_{\phi(x_0)}(x_0, r)$ of Γ_{x_0}, which are equivalent to the balls $U_\phi(x_0, r)$.

In order to prove Theorem 1.2, in section 3 we will first modify the aforementioned representation formula to obtain another representation formula in terms of the family ∇^ϕ. Subsequently, using an approximation result for intrinsic Lipschitz functions contained in [45], (see also [50] for a refinement) we prove that the representation formula proved in Section 3 still holds for intrinsic Lipschitz functions. Finally, in Section 4 we will provide the proof of Theorem 1.2.

2. Preliminaries

2.1. The intrinsic distance.

Fix $n \geq 2$. Let $\omega \subset \mathbb{R}^{2n}$ be an open and bounded set and $\phi : \omega \longrightarrow \mathbb{R}$ be a continuous function. The Lie algebra generated by the family $\nabla^\phi := (\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi)$ defined in (1.2) has maximum rank at every point, hence it is possible to define on ω the exponential ball box distance and the CC-distance, see [5]. These distances are not explicitly computable, therefore, it is convenient to introduce an equivalent, explicitly computable, quasi-distance on ω. To do this, we use the freezing method developed in [6] and successively refined in [39] (see also [38]). Precisely, let us fix $x_0 \in \omega$ and consider the family of smooth vector fields $\nabla^{\phi(x_0)}$ defined in (1.3) and the new family of frozen vector fields

$$\hat{\nabla}^{\phi(x_0)} = (\nabla^{\phi(x_0)}, \partial_{x_{2n}}) \,. \tag{2.8}$$

Let us now introduce the Lie algebra $\mathcal{G}$ generated by the family of vector fields $\nabla^{\phi(x_0)}$. Notice that, since the only non-vanishing commutator is $[\nabla_i^{\phi(x_0)}, \nabla_{n+i}^{\phi(x_0)}] = 2\nabla_{2n}^{\phi(x_0)}$, for each $i = 1, \dots, n-1$, $\mathcal{G}$ is isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$, as Carnot groups, where $\mathbb{H}^{n-1} \equiv \mathbb{R}^{2n-1}$ denotes the $(n-1)$-dimensional Heisenberg group. Any element $\tilde{x} \in \mathcal{G}$ can be identified by its coordinates with respect to the basis $\hat\nabla^{\phi(x_0)}$, that is $\tilde x \equiv (\tilde x_1, \dots, \tilde x_{2n})$, if $\tilde x = \sum_{i=1}^{2n} \tilde x_i \hat\nabla_i^{\phi(x_0)}$. We can also induce a norm on $\mathcal{G}$ by defining

$$\|(\tilde x_1, \dots, \tilde x_{2n})\| := \max\{|(\tilde x_1, \dots, \tilde x_{2n-1})|_{\mathbb{R}^{2n-1}}, |\tilde x_{2n}|^{\frac{1}{2}}\} \tag{2.9}$$

for each $\mathcal{G}$. The exponential map associated to the family of vector fields $\hat\nabla^{\phi(x_0)}$ is well defined. Precisely, for each $x \in \mathbb{R}^{2n}$:

$$Exp_{\phi(x_0),x} : \mathcal{G} \longrightarrow \mathbb{R}^{2n}, \quad Exp_{\phi(x_0),x}(\tilde y) := \exp\Big(\sum_{i=1}^{2n} \tilde y_i \hat\nabla_i^{\phi(x_0)}\Big)(x)\,.$$

In coordinates we get

$$Exp_{\phi(x_0),x}(\tilde y) = \Big(x_1 + \tilde y_1, \dots, x_{2n-1} + \tilde y_{2n-1}, x_{2n} + \tilde y_{2n} + 2\tilde y_n \phi(x_0) - \sigma(\tilde y, x)\Big) \tag{2.10}$$

where

$$\sigma(x, y) := \sum_{i=1}^{n-1} (y_{i+n} x_i - x_{i+n} y_i). \tag{2.11}$$

The inverse mapping of $Exp_{\phi(x_0),x}$ will be denoted by $Log_{\phi(x_0),x} : \mathbb{R}^{2n} \longrightarrow \mathcal{G}$, and the ball-box exponential distance associated to the vector fields $\hat\nabla^{\phi(x_0)}$ is defined for every $x, y \in \mathbb{R}^{2n}$

$$\mathrm{d}_{\phi(x_0)}(x, y) := \|Log_{\phi(x_0),x}(y)\|.$$

In particular, a simple computation gives:

$$Log_{\phi(x_0),x}(y) = \Big(y_1 - x_1, \dots, y_{2n-1} - x_{2n-1}, y_{2n} - x_{2n} - 2\phi(x_0)(y_n - x_n) - \sigma(x, y)\Big) \tag{2.12}$$

and therefore using (2.9) we get:

$$\mathrm{d}_{\phi(x_0)}(x, y) = \max\Big\{|\hat x - \hat y|_{\mathbb{R}^{2n-1}}, \sigma_{\phi,x_0}(x, y)\Big\}, \tag{2.13}$$

where for every $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ we have denoted $\hat x := (x_1, \dots, x_{2n-1}) \in \mathbb{R}^{2n-1}$ and

$$\sigma_{\phi,x_0}(x, y) := |y_{2n} - x_{2n} - 2\phi(x_0)(y_n - x_n) - \sigma(x, y)|^{1/2} \quad x_0 \in \omega,\ x,\ y \in \mathbb{R}^{2n}. \tag{2.14}$$

Moreover we will simply denote

$$\sigma_\phi(x,y) := \sigma_{\phi,x}(x,y) \quad x,\, y \in \omega. \tag{2.15}$$

Finally, we define the following symmetric function:

$$\mathrm{d}_\phi(x,y) = \frac{1}{2}\Big(\mathrm{d}_{\phi(x)}(x,y) + \mathrm{d}_{\phi(y)}(y,x)\Big) \quad \forall x, y \in \omega. \tag{2.16}$$

Definition 2.1. *We say that $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is an intrinsic Lipschitz continuous function in ω and we write $\phi \in Lip(\omega)$, if there is a constant $L > 0$ such that:*

$$|\phi(x) - \phi(y)| \le L\, \mathrm{d}_\phi(x,y) \qquad \forall x, y \in \omega. \tag{2.17}$$

The Lipschitz constant of ϕ in ω is the infimum of the numbers L such that (2.17) holds and we write $L_{\phi,\omega}$ (or simply L_ϕ) to denote it. We also say that ϕ is a locally intrinsic Lipschitz function, and we write $\phi \in Lip_{loc}(\omega)$ if $\phi \in Lip(\omega')$ for every $\omega' \Subset \omega$.

Remark 2.2. *It immediately follows from the explicit expression of d_ϕ (see also [23]) that, if $\phi \in Lip(\omega)$ then d_ϕ is a quasi-distance on ω. Precisely,*

$$\mathrm{d}_\phi(x,y) = 0 \Longleftrightarrow x = y;$$
$$\mathrm{d}_\phi(x,y) = \mathrm{d}_\phi(y,x);$$

and for each $x, y, z \in \omega$:

$$\mathrm{d}_\phi(x,y) \le \tag{2.18}$$
$$\le \mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(y,z) + |\phi(x) - \phi(z)|^{1/2}|x_n - z_n|^{1/2} + |\phi(y) - \phi(z)|^{1/2}|y_n - z_n|^{1/2}$$

so that

$$\mathrm{d}_\phi(x,y) \le (1 + L_\phi)^{1/2}(\mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(y,z))\,.$$

Remark 2.3. *It is easy to see that, if $\phi \in Lip(\omega)$, then*

$$\sigma_\phi(y,x) \le \sigma_\phi(x,y) + |\phi(x) - \phi(y)|^{1/2}|x_n - y_n|^{1/2} \quad \forall x, y \in \omega$$

whence, by (2.16),

$$\mathrm{d}_\phi(x,y) \le |\hat{x} - \hat{y}|_{\mathbb{R}^{2n-1}} + \sigma_\phi(x,y) + |\phi(x) - \phi(y)|^{1/2}|x_n - y_n|^{1/2} \quad \forall x, y \in \omega. \tag{2.19}$$

Remark 2.4. *Moreover, by a simple calculation, we obtain that there exist* $C_1 > 1$ *depending only on* L_ϕ *such that for each* $x, y \in \omega$

$$\frac{1}{C_1} \mathrm{d}_{\phi(y)}(y,x) \leq \mathrm{d}_\phi(x,y) \leq C_1 \mathrm{d}_{\phi(y)}(y,x), \tag{2.20}$$

$$\frac{1}{C_1} \mathrm{d}_{\phi(x)}(x,y) \leq \mathrm{d}_\phi(x,y) \leq C_1 \mathrm{d}_{\phi(x)}(x,y). \tag{2.21}$$

Besides, there exists a positive constant $C_2 = C_2(L_\phi)$ *such that for each* $x, y, z \in \omega$

$$\mathrm{d}_{\phi(x)}(x,y) \leq C_2 \Big(\mathrm{d}_{\phi(x)}(x,z) + \mathrm{d}_{\phi(z)}(z,y) \Big). \tag{2.22}$$

2.2. Lipschitz continuous functions with respect to non linear vector fields.

A detailed analysis of $Lip(\omega)$ can be found in [50, 40], here we recall only those properties that we will use in the proof of Theorem 1.2.

Notice that $Lip(\omega)$ is not a vector space (see [35, Remark 4.2]). Nevertheless, the intrinsic Lipschitz functions amount to a thick class of functions. Indeed, it holds that

$$Lip_E(\omega) \subsetneq Lip_{loc}(\omega) \subsetneq C^{1/2}_{loc}(\omega)\,, \tag{2.23}$$

where, $Lip_E(\omega)$ and $C^{1/2}_{loc}(\omega)$ denote the classes of real-valued Euclidean Lipschitz and locally 1/2-Euclidean-Hölder continuous functions on ω respectively, see [40, Propositions 4.8 and 4.11].

Theorem 2.5. *([40]) If* $\phi \in Lip(\omega)$ *then* ϕ *is* ∇^ϕ*-differentiable for* $\mathcal{L}^{2n}$*-a.e* $x \in \omega$*, in the sense defined in [38]. Besides, for* $\mathcal{L}^{2n}$*-a.e* $x \in \omega$ *there is a unique vector* $\nabla^\phi \phi(x) \in \mathbb{R}^{2n-1}$ *called* ∇^ϕ*−gradient of* ϕ *such that*

$$\phi(y) = \phi(x) + \big\langle \nabla^\phi \phi(x), \tilde{\pi}(y) \big\rangle + o(\mathrm{d}_\phi(x,y)) \quad \textit{as } y \to x$$

where $\langle \cdot, \cdot \rangle$ *denotes the Euclidean scalar product in* $\mathbb{R}^{2n-1}$ *and* $\tilde{\pi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$, $\tilde{\pi}(x_1, \dots, x_{2n-1}, x_{2n}) := (x_1, \dots, x_{2n-1})$, $\forall x \in \mathbb{R}^{2n-1}$.

In [50] the following estimates for L_ϕ are proved. Precisely, for each $\bar{x} \in \omega$ and each $r > 0$ sufficiently small there is $C_1 > 0$ depending only on $\|\nabla^\phi \phi\|_{L^\infty(\omega)}$ such that

$$L_{\phi, U_\phi(\bar{x}, r)} \leq C_1 \|\nabla^\phi \phi\|_{L^\infty(\omega)},$$

and there is $C_2 = C_2(n) > 0$ such that

$$\|\nabla^{\phi}\phi\|_{L^\infty(\omega)} \leq C_2 L_\phi (L_\phi + 1)$$

where $U_\phi(x, r)$ is defined in (1.5).

Moreover, the following approximation result for intrinsic Lipschitz functions it has been recently proved in [45]:

Theorem 2.6. *Let $\omega \subset \mathbb{R}^{2n}$ be a bounded open set and let $\phi \in Lip(\omega)$. Then there exists a sequence $\{\phi_k\}$ with $\phi_k \in C^\infty(\omega)$ such that*

(i) $\phi_k \to \phi$ uniformly in ω as $k \to \infty$,

(ii) $|\nabla^{\phi_k}\phi_k(x)| \leq \|\nabla^\phi \phi\|_{L^\infty(\omega)}$ $\forall x \in \omega$.

We also quote the paper [50] where we proved that every $\phi \in Lip(\omega)$ can be approximated by a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ of smooth functions satisfying (i), (ii) and also

$$\nabla^{\phi_k}\phi_k(x) \to \nabla^\phi \phi(x) \quad \mathcal{L}^{2n} - \text{a.e. in } \omega.$$

2.3. Sub-Laplacian and fundamental solution

In order to study the dependence of the vector fields $\hat{\nabla}^{\phi(x_0)}$ (defined in (2.8)) on the variable x_0 we recognize that the map

$$Log_{\phi(x_0),x_0} : \mathbb{R}^{2n} \longrightarrow \mathcal{G}$$

changes the families $\nabla^{\phi(x_0)}$ and $\hat{\nabla}^{\phi(x_0)}$ into the family ∇ and $\hat{\nabla}$ respectively, where:

$$\nabla_i := \nabla_i^{\phi(x_0)}, \quad \hat{\nabla}_j := \hat{\nabla}_j^{\phi(x_0)} \text{ for } i \in \{1, \dots, 2n-1\}, j \in \{1, \dots, 2n\}, i, j \neq n, \tag{2.24}$$

$$\nabla_n := \partial_{x_n}, \quad \hat{\nabla}_n = \partial_{x_n}.$$

Precisely, for each $\psi \in C^\infty(\mathbb{R}^{2n})$, if we define

$$\tilde{\psi}(\tilde{x}) := \psi(Log^{-1}_{\phi(x_0),x_0}(\tilde{x})), \tag{2.25}$$

then

$$\hat{\nabla}_i^{\phi(x_0)}\psi(x) = \hat{\nabla}_i \tilde{\psi}(Log_{\phi(x_0),x_0}(x)), \qquad \forall i \in \{1, \dots, 2n\}.$$

We can define a metric $\tilde{\mathrm{d}}$ on $\mathcal{G}$ associated to the vector fields $\hat{\nabla}$, independent of x_0. Namely, given $x \in \mathbb{R}^{2n}$ let

$$Exp_{\hat\nabla,x} : \mathcal{G} \to \mathbb{R}^{2n}, \quad Exp_{\hat\nabla,x}(\tilde y) := \exp\Big(\sum_{i=1}^{2n} \tilde y_i \hat\nabla_i\Big)\Big(x\Big) \text{ if } \tilde y = \sum_{i=1}^{2n} \tilde y_i \hat\nabla_i\,.$$

We can also identify $\mathcal{G}$ with $\mathbb{R}^{2n}$, by identifying an element of $\mathcal{G}$ with its coordinates with respect to basis $\hat\nabla$. In such a way, we can define

$$\tilde{\mathrm{d}}(\tilde x, \tilde y) := \|Exp^{-1}_{\hat\nabla,\tilde x}(\tilde y)\| \tag{2.26}$$

where $\|\cdot\|$ denotes the norm in (2.9). In particular it holds that

$$\begin{aligned} &\tilde{\mathrm{d}}(0, \tilde x) = \|\tilde x\|, \quad \forall \tilde x \in \mathcal{G} \equiv \mathbb{R}^{2n}, \\ &\mathrm{d}_{\phi(x_0)}(x, y) = \tilde{\mathrm{d}}(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)) \qquad \forall x, y, x_0 \in \omega. \end{aligned} \tag{2.27}$$

Moreover it follows that $\tilde{\mathrm{d}}$ turns out to be a homogeneous norm on $\mathcal{G} \equiv \mathbb{H}^{n-1} \times \mathbb{R}$.

Let us call sub-Laplacian the second order differential operator defined as

$$\mathcal{L}_{\phi(x_0)} := \sum_{i=1}^{2n-1} (\nabla_i^{\phi(x_0)})^2. \tag{2.28}$$

It is well known that $\mathcal{L}_{\phi(x_0)}$ admits a fundamental solution which we will denote by $\Gamma_{\phi(x_0)}$ (see for [51] for the details). This operator is changed by the map $Log_{\phi(x_0),x_0}$ into the sub-Laplacian operator

$$\mathcal{L} := \sum_{i=1}^{2n-1} (\nabla_i)^2.$$

That is, for each $\psi \in C^\infty(\mathbb{R}^{2n})$

$$(\mathcal{L}_{\phi(x_0)}\psi)(x) = (\mathcal{L}\tilde\psi)(Log_{\phi(x_0),x_0}(x)) \quad \forall\ x \in \mathbb{R}^{2n},$$

where $\tilde\psi$ is defined in (2.25).

Clearly, the operator $\mathcal{L}$ has a fundamental solution Γ of class C^∞ far from the pole $\tilde x = \tilde y$, which is homogeneous of degree $2 - \mathcal{Q}$ with respect to the dilation family naturally associated to $\mathcal{G}$, where $\mathcal{Q}$ is the homogeneous dimension of $\mathbb{H}^{n-1} \times \mathbb{R}$ (see [51, Section 5.3] and the references therein). This means that there exist positive constants C_1, C_2 such that for every $\tilde{x}$ and $\tilde{y}$ in $\mathbb{R}^{2n}$, $\tilde{x} \neq \tilde{y}$

$$\begin{aligned} \frac{C_1}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}-2}} \leq \Gamma(\tilde{x},\tilde{y}) &\leq \frac{C_2}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}-2}}; \\ |\nabla_i \Gamma(\tilde{x},\tilde{y})| &\leq \frac{C_2}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}-1}}; \\ |\nabla_j \nabla_i \Gamma(\tilde{x},\tilde{y})| &\leq \frac{C_2}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}}}, \end{aligned} \tag{2.29}$$

for every $i, j = 1, \ldots, 2n-1$ (see [52] and [51, Section 5.4]). Besides, the fundamental solution $\Gamma_{\phi(x_0)}$ of $\mathcal{L}_{\phi(x_0)}$ can be explicitly written in terms of Γ as

$$\Gamma_{\phi(x_0)}(x,y) = \Gamma(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)), \tag{2.30}$$

and

$$\nabla_i^{\phi(x_0)} \Gamma_{\phi(x_0)}(x,y) = \nabla_i \Gamma(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)),$$

for $i = 1, \ldots, 2n-1$. It follows that the inequalities in (2.29) are satisfied also for $\Gamma_{\phi(x_0)}(x,y)$ and $\mathrm{d}_{\phi(x_0)}(x,y)$ with the same constants. In particular, it is clear that these constants are independent of x_0. Using the estimates for $\Gamma_{\phi(x_0)}$ it follows that the spheres in the metric $\mathrm{d}_{\phi(x_0)}$ are equivalent to the super levels of the fundamental solution $\Gamma_{\phi(x_0)}$:

$$\Omega_{\phi(x_0)}(x,r) = \left\{ y \in \mathbb{R}^{2n} \mid \Gamma_{\phi(x_0)}(x,y) > r^{2-\mathcal{Q}} \right\}, \ r > 0. \tag{2.31}$$

Moreover, for every fixed $x_0 \in \omega$, the set $\Omega_{\phi(x_0)}(x_0, r)$ has regular boundary (see [22]). In particular, from (2.21), (2.27) and (2.29), there exist $r_0, \alpha > 0$ with $\alpha = \alpha(L_\phi)$ such that for any $x_0 \in \omega$ and $r \leq r_0$

$$\Omega_{\phi(x_0)}(x_0, r/\alpha) \subset U_\phi(x_0, r) \subset \Omega_{\phi(x_0)}(x_0, \alpha\, r), \tag{2.32}$$

where $U_\phi(x_0, r)$ is the ball defined in (1.5). By (2.30) we have that

$$\Omega_{\phi(x_0)}(x,r) = \left\{ y \in \mathbb{R}^{2n} \mid \Gamma(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)) > r^{2-Q} \right\}, \tag{2.33}$$

in particular the sets $\Omega_{\phi(x_0)}(x_0, r)$ can be expressed in terms of the super levels of the fundamental solution Γ as follows:

$$\begin{aligned} \Omega_{\phi(x_0)}(x_0,r) &= \left\{ y \in \mathbb{R}^{2n} \mid \Gamma(0, Log_{\phi(x_0),x_0}(y)) > r^{2-Q} \right\} \\ &= Exp_{\phi(x_0),x_0}(\tilde{\Omega}(0,r)), \end{aligned} \tag{2.34}$$

where

$$\tilde{\Omega}(0,r) := \left\{\tilde{y} \in \mathbb{R}^{2n} \mid \Gamma(0,\tilde{y}) > r^{2-\mathcal{Q}}\right\}. \tag{2.35}$$

We will also denote

$$K(\tilde{y}) := \Gamma^{-\frac{1}{(\mathcal{Q}-2)}}(0,\tilde{y}),\ \tilde{y} \in \mathbb{R}^{2n}, \tag{2.36}$$

so that, we can rewrite $\tilde{\Omega}(0,r)$ as:

$$\tilde{\Omega}(0,r) = \left\{\tilde{y} \in \mathbb{R}^{2n} \mid K(\tilde{y}) < r\right\}. \tag{2.37}$$

3. A representation formula in terms of the intrinsic gradient

This section is organized in two subsections. In the first one, we fix a smooth function $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ and we introduce the notion of integral mean $\bar{m}$ of another regular function $\psi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ on the super levels $\Omega_{\phi(x_0)}(x_0,r)$ of the fundamental solution $\Gamma_{\phi(x_0)}$. Then, we prove a representation formula for any regular function ψ in terms of its integral mean and its intrinsic gradient.
The second subsection is quite technical, and is devoted to establish some properties of the integral mean of ψ.

3.1. Representation formula

In this section, we fix $\omega \subset \mathbb{R}^{2n}$ open and bounded, $n \geq 2$ and $\phi, \psi \in C^\infty(\omega)$. We prove a representation formula for ψ on the super levels $\Omega_{\phi(x_0)}(x_0,r)$ of $\Gamma_{\phi(x_0)}$. A similar representation formula has been proved in [51, 22] for the approximated vector fields $\nabla^{\phi(x_0)}$. From this formula, we will deduce a new and intrinsic representation formula for ψ, expressed in terms of its intrinsic gradient $\nabla^\phi\psi$. In the case under consideration the result of [51, 22] can be stated as follow:

Proposition 3.1. *For every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0,R) \subset \omega$ and for every $\psi \in C^\infty(\omega)$ we have*

$$\psi(x_0) = \frac{\mathcal{Q}}{(\mathcal{Q}-2)(1-\frac{1}{2^{\mathcal{Q}}})R^{\mathcal{Q}}} \int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}\,\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y) \tag{3.38}$$

$$+ \frac{\mathcal{Q}}{(1-\frac{1}{2^{\mathcal{Q}}})R^{\mathcal{Q}}} \int_{\frac{R}{2}}^{R} r^{\mathcal{Q}-1} \int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi(x_0)}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^{2n-1}$.

Remark 3.2. *We explicitly note that, if we choose* $\psi \equiv 1$*, then from* (3.38) *we get:*

$$1 = \frac{C(\mathcal{Q})}{R^{\mathcal{Q}}} \int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)} \, \mathrm{d}\mathcal{L}^{2n}(y) \tag{3.39}$$

where $C(\mathcal{Q}) := \frac{\mathcal{Q}}{(\mathcal{Q}-2)(1-\frac{1}{2\mathcal{Q}})}$.

This remark allows to say that (3.38) represents a function ψ as the sum of its mean on a suitable set and the gradient $\nabla^{\phi(x_0)}\psi$. Hence, it seems natural to give the following definition

Definition 3.3. *Let* $\psi \in L^1_{loc}(\omega)$*. For every* $x_0 \in \omega$ *and* $R > 0$ *such that* $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ *we define the following mean of* ψ*, on the set* $\Omega_{\phi(x_0)}(x_0, R) \setminus \Omega_{\phi(x_0)}(x_0, \frac{R}{2})$*, in terms of the fundamental solution* $\Gamma_{\phi(x_0)}$*:*

$$\bar{m}(\psi, \phi, R)(x_0) := \frac{C(\mathcal{Q})}{R^{\mathcal{Q}}} \int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)} \, \psi(y) \, \mathrm{d}\mathcal{L}^{2n}(y).$$

In the sequel we will need another mean of ψ *on the same set* $\Omega_{\phi(x_0)}(x_0, R) \setminus \Omega_{\phi(x_0)}(x_0, \frac{R}{2})$*. Precisely, we denote:*

$$m(\psi, \phi, R)(x_0) := \frac{2}{R} \int_{\frac{R}{2}}^{R} \bar{m}(\psi, \phi, r)(x_0) \, \mathrm{d}r. \tag{3.40}$$

The following remark, which will be very useful later on, provides an integration formula by parts for the derivative ∂_{2n}.

Remark 3.4. *Let* $g \in C^1(\mathbb{R}^{2n})$*,* $r > 0$ *and* $c_1, c_2 > 0$ *we define:*

$$A_{r,c_1,c_2} := \{y \in \mathbb{R}^{2n} : c_1 r < g(y) < c_2 r\}.$$

Then, for every $f, \psi \in C^1(\mathbb{R}^{2n})$ *and* $R_1, R_2 \in \mathbb{R}$ *with* $R_1 < R_2$*, using the fact* *that* $\partial_{2n} = \frac{1}{2}(\nabla_1^{\phi(x_0)}\nabla_{n+1}^{\phi} - \nabla_{n+1}^{\phi(x_0)}\nabla_1^{\phi})$ *and integrating by part we have:*

$$\begin{aligned}
&\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} f(y)\partial_{2n}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)dr = \\
&=\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_2=r\}} f(y)\nabla_{n+1}^{\phi}\psi(y)\frac{\nabla_1^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\,\mathrm{d}r \\
&\quad-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_1=r\}} f(y)\nabla_{n+1}^{\phi}\psi(y)\frac{\nabla_1^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r \\
&\quad-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_2=r\}} f(y)\nabla_1^{\phi}\psi(y)\frac{\nabla_{n+1}^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r \\
&\quad+\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_1=r\}} f(y)\nabla_1^{\phi}\psi(y)\frac{\nabla_{n+1}^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r \\
&\quad-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla_1^{\phi(x_0)}f(y)\nabla_{n+1}^{\phi}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r \\
&\quad+\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla_{n+1}^{\phi(x_0)}f(y)\nabla_1^{\phi}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r,
\end{aligned}$$

where ∇_E *denotes the Euclidean gradient. By the coarea formula we infer* *that:*

$$\begin{aligned}\int_{R_1}^{R_2} r^{\mathcal{Q}-1} \int_{A_{r,c_1,c_2}} & f(y)\partial_{2n}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)dr = \\ =&\frac{1}{2}\int_{A_{r,c_2R_1,c_2R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_2^{\mathcal{Q}-1}} f(y)\nabla^{\phi}_{n+1}\psi(y)\nabla^{\phi(x_0)}_1 g(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\\ &-\frac{1}{2}\int_{A_{r,c_1R_1,c_1R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_1^{\mathcal{Q}-1}} f(y)\nabla^{\phi}_{n+1}\psi(y)\nabla^{\phi(x_0)}_1 g(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\\ &-\frac{1}{2}\int_{A_{r,c_2R_1,c_2R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_2^{\mathcal{Q}-1}} f(y)\nabla^{\phi}_{1}\psi(y)\nabla^{\phi(x_0)}_{n+1} g(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\\ &+\frac{1}{2}\int_{A_{r,c_1R_1,c_1R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_1^{\mathcal{Q}-1}} f(y)\nabla^{\phi}_{1}\psi(y)\nabla^{\phi(x_0)}_{n+1} g(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\\ &-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla^{\phi(x_0)}_1 f(y)\nabla^{\phi}_{n+1}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r\\ &+\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla^{\phi(x_0)}_{n+1} f(y)\nabla^{\phi}_{1}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.\end{aligned}$$

If in addition $c_1 = 0$ then the integrals on A_{r,c_1R_1,c_1R_2} are not present.

In the following proposition we will slightly modify the representation formula proved in Proposition 3.1 in order to obtain a mean representation formula containing only derivatives with respect to the vector fields ∇^{ϕ}.

Proposition 3.5. *Let $\phi, \psi \in C^{\infty}(\omega)$. For every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we have*

$$\psi(x_0) - m(\psi, \phi, R)(x_0) = I_R(x_0)$$

where

$$\begin{aligned}I_R(x_0) =& \frac{2}{R}\int_{\frac{R}{4}}^{R} f_1\left(\frac{r}{R}\right)\int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle K_1(x_0,y), \nabla^{\phi}\psi(y)\right\rangle\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r\\ &+\frac{2}{R}\int_{\frac{R}{2}}^{R}\int_{\Omega_{\phi(x_0)}(x_0,r)\setminus\Omega_{\phi(x_0)}(x_0,\frac{r}{2})} \left\langle K_2(x_0,y,r), \nabla^{\phi}\psi(y)\right\rangle\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.\end{aligned}$$

Here, $f_1 \in C^0([\frac{1}{4}, 1])$ and the vector valued functions K_1 and K_2 are defined in (3.44) and (3.45) respectively. Moreover,

$$|K_1(x_0, y)| \le \tilde{C}_1(L_{\phi,\Omega_{\phi(x_0)}(x_0,R)} + 1)^2 \mathrm{d}^{1-\mathcal{Q}}_{\phi(x_0)}(x_0, y) \quad \forall y \in \Omega_{\phi(x_0)}(x_0, R) \quad (3.41)$$

and

$$|K_2(x_0,y,r)| \leq \tilde{C}_2(L_\phi+1)^2 \mathrm{d}^{1-\mathcal{Q}}_{\phi(x_0)}(x_0,y), \ \ \forall y \in \Omega_{\phi(x_0)}(x_0,R) \setminus \Omega_{\phi(x_0)}(x_0,\frac{R}{2}), r \in \Big(\frac{R}{2},R\Big) \tag{3.42}$$

where L_ϕ *means* $L_{\phi,\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,R/2)}$ *and* $\tilde{C}_1,\tilde{C}_2>0$ *are suitable constants depending only on the homogeneous dimension* $\mathcal{Q}$ *and on the structure constants* C_1 *and* C_2 *in* (2.29).

Proof. We will always denote by C a positive constant depending only on $\mathcal{Q}$ which can be different from line to line. By Proposition 3.1 for all $r \in (0,R)$

$$\begin{aligned}
&\psi(x_0) - \bar{m}(\psi,\phi,r)(x_0) = \\
&= \frac{C}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi(x_0)}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s \\
&= \frac{C}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s \\
&+ \frac{C}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \nabla^{\phi(x_0)}_n\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\partial_{2n}\psi(y)\, \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s.
\end{aligned}$$

Using Remark 3.4 with $g(y) := \Gamma^{\frac{1}{2-\mathcal{Q}}}_{\phi(x_0)}(x_0,y)$ we obtain:

$$\begin{aligned}
\psi(x_0) - \bar{m}(\psi,\phi,r)(x_0) =& \frac{1}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \left\langle K_1(x_0,y), \nabla^{\phi}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s \\
&+ \int_{\Omega_{\phi(x_0)}(x_0,r)\setminus\Omega_{\phi(x_0)}(x_0,\frac{r}{2})} \left\langle K_2(x_0,y,r)\nabla^{\phi}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)
\end{aligned} \tag{3.43}$$

where

$$\begin{aligned}
K_1(x_0,y) :=& C\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y) \\
&- C\nabla^{\phi(x_0)}_1\nabla^{\phi(x_0)}_n\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))e_{n+1} \\
&+ C\nabla^{\phi(x_0)}_{n+1}\nabla^{\phi(x_0)}_n\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))e_1,
\end{aligned} \tag{3.44}$$

and

$$K_2(x_0,y,r) := \frac{C}{r^{\mathcal{Q}}}\frac{\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}(\phi(x_0)-\phi(y))\nabla_{n+1}^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)e_1$$

$$- \frac{C}{r^{\mathcal{Q}}}\frac{\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}(\phi(x_0)-\phi(y))\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)e_{n+1}, \tag{3.45}$$

where e_i is the i-th element of the canonical basis of $\mathbb{R}^{2n-1}$.

Integrating (3.43) from $\frac{R}{2}$ to R we get

$$\begin{aligned}\psi(x_0) - m(\psi,\phi,R)(x_0) =&\\ =&\frac{2}{R}\int_{\frac{R}{2}}^{R}\frac{1}{\rho^{\mathcal{Q}}}\int_{\frac{\rho}{2}}^{\rho} r^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,r)}\left\langle K_1(x_0,y),\nabla^{\phi}\psi(y)\right\rangle \,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r\mathrm{d}\rho\\ &+\frac{2}{R}\int_{\frac{R}{2}}^{R}\int_{\Omega_{\phi(x_0)}(x_0,\rho)\setminus\Omega_{\phi(x_0)}(x_0,\frac{\rho}{2})}\left\langle K_2(x_0,y,\rho),\nabla^{\phi}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}\rho.\end{aligned}$$

Exchanging the order of integration in the first integral and setting:

$$f_1(t) := \frac{2^{1-\mathcal{Q}}-(2t)^{\mathcal{Q}-1}}{1-\mathcal{Q}} \text{ if } t\in[1/4,1/2], \quad f_1(t) := \frac{t^{\mathcal{Q}-1}-1}{1-\mathcal{Q}} \text{ if } t\in[1/2,1],$$

we get the thesis. Finally, the estimates on K_1 and K_2 are direct consequences of (2.29). □

3.2. Some properties of the integral mean

In this section we collect some properties of the integral mean $\bar{m}(\psi,\phi,r)$ to be used in the next section. We will see that in order to conclude the proof of the Poincaré inequality we will need a detailed estimate of the difference

$$\bar{m}(\psi,\phi,r)(x) - \bar{m}(\psi,\phi,r)(y) \tag{3.46}$$

at two different points $x,y\in\omega$ with $x\neq y$. This difference is estimated in Proposition 3.11, and it is based on some technical lemmas.

Lemma 3.6. *Let $\psi\in C^{\infty}(\omega)$. For each $x_0\in\omega$ and each $R>0$ such that $\Omega_{\phi(x_0)}(x_0,R)\subset\omega$ the mean, $\bar{m}(\psi,\phi,R)(x_0)$, of a smooth function ϕ can be expressed as follows:*

$$\bar{m}(\psi,\phi,R)(x_0) = \frac{1}{R^{\mathcal{Q}}}\int_{\tilde{\Omega}(0,R)\setminus\tilde{\Omega}(0,\frac{R}{2})} K_3(0,\tilde{y})\psi(Exp_{\phi(x_0),x_0}(\tilde{y}))\,\mathrm{d}\mathcal{L}^{2n}(\tilde{y}),$$

where $\tilde{\Omega}(0,R)$ *is defined in (2.35) and*

$$K_3(0,\tilde{y}) := \frac{\mathcal{Q}}{(\mathcal{Q}-2)(1-\frac{1}{2^{\mathcal{Q}}})}\frac{|\nabla\Gamma(0,\tilde{y})|^2}{\Gamma(0,\tilde{y})^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}}.$$

Moreover, there exist constants $\tilde{C}_3, \tilde{C}_4$ *depending only* $\mathcal{Q}$ *and on the structure constants* C_1 *and* C_2 *in (2.29) such that*

$$|K_3(0,\tilde{y})| \leq \tilde{C}_3, \qquad |\nabla K_3(0,\tilde{y})| \leq \frac{\tilde{C}_4}{\|\tilde{y}\|} \quad \forall \tilde{y} \in \tilde{\Omega}(0,R) \setminus \tilde{\Omega}\Big(0,\frac{R}{2}\Big). \tag{3.47}$$

Proof. By (2.34) we have that

$$\Omega_{\phi(x_0)}(x_0,R) = Exp_{\phi(x_0),x_0}(\tilde{\Omega}(0,R)).$$

So that, by Definition 3.3 and (2.30) we have:

$$\begin{aligned}
&\bar{m}(\psi,\phi,R)(x_0) = \\
&= \frac{C(\mathcal{Q})}{(\mathcal{Q}-2)R^{\mathcal{Q}}}\int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}\,\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y) \\
&= \frac{C(\mathcal{Q})}{(\mathcal{Q}-2)R^{\mathcal{Q}}}\int_{\tilde{\Omega}(0,R)\setminus\tilde{\Omega}(0,\frac{R}{2})} \frac{|\nabla\Gamma(0,\tilde{y})|^2}{\Gamma^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(0,\tilde{y})}\psi(Exp_{\phi(x_0),x_0}(\tilde{y}))\,\mathrm{d}\mathcal{L}^{2n}(\tilde{y}),
\end{aligned}$$

where in the last equality we have applied a change of variables and the fact that the determinant of the Jacobian matrix of $Exp_{\phi(x_0),x_0}$ is equal to 1. Finally, we observe that (3.47) follows directly from the estimates on Γ in (2.29). □

In the next proposition we will start studying properties of the difference $\psi(Exp_{\phi(x),x}(\tilde{y})) - \psi(Exp_{\phi(x_0),x_0}(\tilde{y}))$, which, thanks to the previous lemma, can be considered the first step in the proof of (3.46).

Proposition 3.7. *For every* $\bar{x} \in \omega$ *there exist a constants* $C_0 > 0$*, such that for every* $R > 0$ *with* $\Omega_{\phi(\bar{x})}(\bar{x}, C_0R) \Subset \omega$*, every* $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x},R)$ *and every* $\tilde{y} \in \tilde{\Omega}(0,R)$ *there is an integral curve of the vector fields* $\hat{\nabla}^{\phi(x_0)}$, $\gamma_{\tilde{y}} : [0,1] \to \omega$ *joining* $Exp_{\phi(x),x}(\tilde{y})$ *and* $Exp_{\phi(x_0),x_0}(\tilde{y})$*. Moreover,* $\gamma_{\tilde{y}}$ *can be explicitly written as:*

$$\gamma_{\tilde{y}}(t) := \exp\Big(t\tilde{h}\hat{\nabla}^{\phi(x_0)}\Big)\Big(\exp(\tilde{y}\hat{\nabla}^{\phi(x_0)})(x_0)\Big) \quad t \in [0,1], \tag{3.48}$$

where

$$\tilde{h} = Log_{\phi(x_0),Exp_{\phi(x_0),x_0}(\tilde{y})}\Big(Exp_{\phi(x),x}(\tilde{y})\Big). \tag{3.49}$$

Proof. Let us fix $\bar{x} \in \omega$, and a sphere $\Omega_{\phi(\bar{x})}(\bar{x}, CR)$ subset of ω. The constant $C = C(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},R)}) > 0$ will be chosen at the end. We first note that, there exists

$$C = C(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},\bar{R})}) > 0 \tag{3.50}$$

such that for every $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$, $\tilde{y} \in \tilde{\Omega}(0, R)$, the points $Exp_{\phi(x),x}(\tilde{y})$ and $Exp_{\phi(x_0),x_0}(\tilde{y})$ belong to $\Omega_{\phi(\bar{x})}(\bar{x}, CR)$. By (2.10) we get

$$Exp_{\phi(x),x}(\tilde{y}) = \Big(x_1 + \tilde{y}_1, \dots, x_{2n-1} + \tilde{y}_{2n-1}, x_{2n} + \tilde{y}_{2n} + 2\tilde{y}_n\phi(x) - \sigma(\tilde{y}, x)\Big),$$

$$Exp_{\phi(x_0),x_0}(\tilde{y}) = \Big(x_{0,1} + \tilde{y}_1, \dots, x_{0,2n-1} + \tilde{y}_{2n-1}, x_{0,2n} + \tilde{y}_{2n} + 2\tilde{y}_n\phi(x_0) - \sigma(\tilde{y}, x_0)\Big).$$

Then, using (3.49) and (2.12), we obtain

$$\tilde{h}_i = (x - x_0)_i \quad i = 1, \dots, 2n-1, \tag{3.51}$$
$$\tilde{h}_{2n} = (x - x_0)_{2n} - 2\phi(x_0)(x - x_0)_n + 2\tilde{y}_n(\phi(x) - \phi(x_0)) - 2\sigma(\tilde{y}, x - x_0) + \sigma(x, x_0)$$

and calling

$$\tilde{x} := Log_{\phi(x_0),x_0}(x), \tag{3.52}$$

we realize that

$$\tilde{h} = \tilde{x} + \Big(2\tilde{y}_n(\phi(x) - \phi(x_0)) - 2\sigma(\tilde{y}, \tilde{x})\Big)e_{2n}. \tag{3.53}$$

By (3.48) and the Baker-Campbell-Hausdorff formula we have

$$\begin{aligned}\gamma_{\tilde{y}}(t) &= \exp\Big(t\tilde{h}\hat{\nabla}^{\phi(x_0)}\Big)\Big(\exp\Big(\tilde{y}\hat{\nabla}^{\phi(x_0)}\Big)(x_0)\Big)\\ &= \exp\Big(2t\tilde{y}_n(\phi(x) - \phi(x_0))\partial_{2n} + 2t\sigma(\tilde{y}, \tilde{x})\partial_{2n} - t\sigma(\tilde{y}, \tilde{x})\partial_{2n} + (t\tilde{x} + \tilde{y})\hat{\nabla}^{\phi(x_0)}\Big)(x_0).\end{aligned}$$

From this and using (2.10) we get

$$\begin{aligned}(\gamma_{\tilde{y}}(t))_i &= t(x - x_0)_i + (\tilde{y} + x_0)_i \quad i = 1, \dots, 2n-1 \\ (\gamma_{\tilde{y}}(t))_{2n} &= t(x - x_0)_{2n} + (\tilde{y} + x_0)_{2n} + 2t\tilde{y}_n(\phi(x) - \phi(x_0)) \\ &\quad + 2\phi(x_0)\tilde{y}_n + \sigma(t(x - x_0) + x_0, \tilde{y}).\end{aligned} \tag{3.54}$$

Therefore, the following estimate holds

$$\mathrm{d}_{\phi(x_0)}(x_0, \gamma_{\tilde{y}}(t)) \leq \|\tilde{y}\| + \|\tilde{x}\| + \sqrt{\|\tilde{y}\|\|\tilde{x}\|} + \sqrt{\|\tilde{y}\||\phi(x) - \phi(x_0)|}, \quad (3.55)$$

where $\tilde{x}$ is as in (3.52). Indeed, using (2.13) and (3.54) we get

$$\mathrm{d}_{\phi(x_0)}(x_0, \gamma_{\tilde{y}}(t)) \leq \Big|(t\tilde{x}_1 + \tilde{y}_1, \dots, t\tilde{x}_{2n-1} + \tilde{y}_{2n-1})\Big|_{\mathbb{R}^{2n-1}} + \\ + \Big|t\tilde{x}_{2n} + \tilde{y}_{2n} + 2t\tilde{y}_n(\phi(x) - \phi(x_0)) + t\sigma(\tilde{x}, \tilde{y})\Big|^{\frac{1}{2}}$$

and (3.55) follows using the triangle inequality. Since, $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ and $\|\tilde{y}\| \leq R$ then by (3.52) and (2.22) we get

$$\|\tilde{x}\| \leq CR, \quad \text{and} \quad \mathrm{d}_{\phi}(x, x_0) \leq CR \quad (3.56)$$

for some constant $C = C(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},\bar{R})}) > 0$. Finally, by (3.55), (3.56) and (2.29) we conclude that

$$\gamma_{\tilde{y}}(t) \in \Omega_{\phi(x_0)}(x_0, C_1 R) \quad \forall t \in [0, 1]$$

for some $C_1 = C_1(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},R)}) > 0$ and the thesis follows with $C_0 = max(C, C_1)$.

□

Proposition 3.8. *Let C_0 be as in in Proposition 3.7. For every $\bar{x} \in \omega$ such that $\Omega_{\phi(\bar{x})}(\bar{x}, C_0 R) \Subset \omega$ and for every $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ and for every $\tilde{y} \in \tilde{\Omega}(0, R)$ we have*

$$\psi(Exp_{\phi(x),x}(\tilde{y})) - \psi(Exp_{\phi(x_0),x_0}(\tilde{y})) = \quad (3.57)$$
$$= \int_0^1 \sum_{i=1}^{2n} (Log_{\phi(x_0),x_0}(x))_i \hat{\nabla}_i^{\phi(x_0)} \psi(\gamma_{\tilde{y}}(t)) dt + K_4(x, x_0, \tilde{y}) \int_0^1 \partial_{2n} \psi(\gamma_{\tilde{y}}(t)) \mathrm{d}t$$

where

$$K_4(x, x_0, \tilde{y}) := 2(\phi(x) - \phi(x_0))\tilde{y}_n - 2\sigma(\tilde{y}, x - x_0). \quad (3.58)$$

The kernel K_4 is of class C^∞ with respect to $\tilde{y}$ and the following estimates hold:

$$|K_4(x, x_0, \tilde{y})| \leq 2(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},R)} + 1)\mathrm{d}_{\phi}(x, x_0)\|\tilde{y}\|, \quad (3.59)$$
$$|\nabla K_4(x, x_0, \tilde{y})| \leq 2(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},R)} + 1)\mathrm{d}_{\phi}(x, x_0). \quad (3.60)$$

Proof. Since $\psi \in C^\infty(\omega)$ and $\gamma_{\tilde{y}}$ is horizontal with respect to the family of vector fields $\{\hat{\nabla}^{\phi(x_0)}\}$, we obtain

$$\begin{aligned}\psi(Exp_{\phi(x),x}(\tilde{y})) - \psi(Exp_{\phi(x_0),x_0}(\tilde{y})) &= \int_0^1 (\psi \circ \gamma_{\tilde{y}})'(t)\mathrm{d}t \\ &= \sum_{i=1}^{2n} \int_0^1 \tilde{h}_i \hat{\nabla}_i^{\phi(x_0)} \psi(\gamma_{\tilde{y}}(t))\mathrm{d}t,\end{aligned}$$

so that (3.57) immediately follows using (3.51). In order to prove (3.59) it suffices to observe that $\sigma(x - x_0, \tilde{y}) \le \mathrm{d}_\phi(x, x_0)\|\tilde{y}\|$. Moreover, since $\partial_{\tilde{y}_{2n}} K_4(x, x_0, \tilde{y}) = 0$ it follows that to prove (3.60) it is enough to estimate the Euclidean gradient of K_4 (with respect to the variable $\tilde{y}$). By a direct computation and using the expression of K_4 in (3.58) we obtain

$$\begin{aligned}&\partial_{\tilde{y}_i} K_4(x, x_0, \tilde{y}) = -2(x - x_0)_{n+i} \quad \text{if } \ i = 1, \dots, n-1,\\ &\partial_{\tilde{y}_n} K_4(x, x_0, \tilde{y}) = 2(\phi(x) - \phi(x_0)),\\ &\partial_{\tilde{y}_i} K_4(x, x_0, \tilde{y}) = 2(x - x_0)_i \quad \text{if } \ i = n+1, \dots, 2n-1.\end{aligned}$$

Hence $|\nabla K_4(x, x_0, \tilde{y})| \le 2(L_{\phi, \Omega_{\phi(\bar{x})}(\bar{x}, R)} + 1)\mathrm{d}_\phi(x, x_0)$, which is the thesis. □

There is a natural change of variables, naturally associated to the curve $\gamma_{\tilde{y}}$ defined in Proposition 3.7. Indeed the following lemma holds:

Lemma 3.9. *Let $\bar{x} \in \omega$, $C_0 > 0$ and $\gamma_{\tilde{y}}$ be as Proposition 3.7. For each $R > 0$, such that $\Omega_{\phi(\bar{x})}(\bar{x}, C_0 R) \subset \omega$ $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ and $\tilde{y} \in \tilde{\Omega}(0, R)$. Then the function*

$$\begin{aligned}H : [0,1] \times \tilde{\Omega}(0, R) &\longrightarrow [0,1] \times \omega\\ (t, \tilde{y}) &\mapsto (t, \gamma_{\tilde{y}}(t))\end{aligned}$$

has inverse function $(t, \tilde{F}(z, t))$, the map $z \to (t, \tilde{F}(z, t))$ is C^∞ and its Jacobian matrix has determinant equal to 1.

Proof. Using (3.54), (2.10) and setting $(t, z) := (t, \gamma_{\tilde{y}}(t))$, $\tilde{F}$ can be expressed as

$$\begin{aligned}\tilde{F}_i(z,t) =&(z - x_0)_i - t(x - x_0)_i \quad i = 1, \dots, 2n-1, \qquad (3.61)\\ \tilde{F}_{2n}(z,t) =&(z - x_0)_{2n} - t(x - x_0)_{2n} - 2t((z - x_0)_n - t(x - x_0)_n)(\phi(x) - \phi(x_0)) +\\ &- 2\phi(x_0)((z - x_0)_n - t(x - x_0)_n) + \sigma(z, t(x - x_0) + x_0).\end{aligned}$$

In particular it is clear from (3.61) that $\tilde{F}$ is of class C^∞ as a function of the variable z and that the Jacobian determinant of $z \to \tilde{F}(z,t)$ is equal to 1 for each $t \in [0,1]$. $\square$

Lemma 3.10. *Let $g \in C^\infty(\mathbb{R}^n)$ and $\tilde{F}(z,t)$ as in Lemma 3.9 then*

$$\nabla_{z_i}^{\phi(x_0)}(g(\tilde{F}(z,t)) = \begin{cases} (\nabla_{\tilde{y}_i} g)(\tilde{F}(z,t)) & i = 1, \dots, n-1, \\ (\nabla_{\tilde{y}_n} g)(\tilde{F}(z,t)) - 2t(\phi(x) - \phi(x_0))(\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)) & i = n, \\ (\nabla_{\tilde{y}_i} g)(\tilde{F}(z,t)) & i = n+1, \dots, 2n-1, \end{cases} \tag{3.62}$$

where $(\nabla_1, \dots, \nabla_{2n-1})$ is the family of vector fields defined in (2.24).

Proof. Let us start computing $\nabla_{z_i}^{\phi(x_0)}(g(\tilde{F}(z,t)))$ with $i = 1, \dots, n-1$, that is

$$\Big(\partial_{z_i} - z_{i+n}\partial_{z_{2n}}\Big)(g(\tilde{F}(z,t))). \tag{3.63}$$

To this end, we calculate

$$\partial_{z_i}(g(\tilde{F}(z,t))) \quad \text{and} \quad \partial_{z_{2n}}(g(\tilde{F}(z,t))).$$

By the explicit expression of $\tilde{F}(z,t)$ we obtain:

$$\partial_{z_i}(g(\tilde{F}(z,t))) = (\partial_{\tilde{y}_i} g)(\tilde{F}(z,t)) + (\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t))\partial_{z_i}\tilde{F}_{2n}(z,t), \tag{3.64}$$

$$\partial_{z_{2n}}(g(\tilde{F}(z,t))) = (\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)), \tag{3.65}$$

hence by (3.63), (3.64) and (3.65) we get:

$$\begin{aligned} \nabla_{z_i}^{\phi(x_0)}(g(\tilde{F}(z,t))) =& \Big(\partial_{\tilde{y}_i} g - \tilde{F}_{i+n}(z,t)\partial_{\tilde{y}_{2n}} g\Big)(\tilde{F}(z,t)) + \\ &+ \Big(\tilde{F}_{i+n}(z,t) - z_{i+n} + \partial_{z_i}\tilde{F}_{2n}(z,t)\Big)\partial_{\tilde{y}_{2n}} g(\tilde{F}(z,t)). \end{aligned}$$

Since

$$\begin{aligned} \tilde{F}_i(z,t) =& (z-x_0)_i - t(x-x_0)_i \quad i = 1, \dots, 2n-1 \qquad (3.66) \\ \tilde{F}_{2n}(z,t) =& (z-x_0)_{2n} - t(x-x_0)_{2n} - 2t((z-x_0)_n - t(x-x_0)_n)(\phi(x) - \phi(x_0)) + \\ &- 2\phi(x_0)((z-x_0)_n - t(x-x_0)_n) + \sigma(z, t(x-x_0)+x_0) \end{aligned}$$

this implies

$$\nabla_{z_i}^{\phi(x_0)}(g(\tilde{F}(z,t))) = (\nabla_{\tilde{y}_i} g)(\tilde{F}(z,t)).$$

The computations for $\nabla_{z_i}^{\phi(x_0)}(g(\tilde{F}(z,t)))$ when $i = n+1, \dots, 2n-1$ are similar.

Finally, let us compute $\nabla_{z_n}^{\phi(x_0)}(g(\tilde{F}(z,t)))$. By definition:

$$\nabla_{z_n}^{\phi(x_0)}(g(\tilde{F}(z,t))) = \Big(\partial_{z_n} + 2\phi(x_0)\partial_{z_{2n}}\Big)(g(\tilde{F}(z,t))) \tag{3.67}$$

and since

$$\partial_{z_n}(g(\tilde{F}(z,t))) = (\partial_{\tilde{y}_n} g)(\tilde{F}(z,t)) - 2[t(\phi(x) - \phi(x_0)) + \phi(x_0)](\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)) \tag{3.68}$$

by (3.67), (3.65) and (3.68) we get:

$$\nabla_{z_n}^{\phi(x_0)} g(\tilde{F}(z,t)) = (\nabla_{\tilde{y}_n} g)(\tilde{F}(z,t)) - 2t(\phi(x) - \phi(x_0))(\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)).$$

□

Proposition 3.11. *For every $t \in [0,1], c_1, c_2 > 0$ and $r > 0$ let us define*

$$D_{t,c_1,c_2,r} := \Big\{ z \in \mathbb{R}^{2n} : (z,t) \in \tilde{F}^{-1}\Big(\tilde{\Omega}(0, c_2 r) - \tilde{\Omega}(0, c_1 r)\Big)\Big\}. \tag{3.69}$$

Let $\bar{x} \in \omega$ and $C_0 > 0$ be as in Proposition 3.7, then for every $0 < R$ such that $\Omega_{\phi(\bar{x})}(\bar{x}, C_0 R) \subset \omega$ and $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ with $x \neq x_0$ it holds:

$$\begin{aligned} m(\psi,\phi,R)(x) - m(\psi,\phi,R)(x_0) =& \frac{2}{R}\int_0^1 \int_{\frac{R}{2}}^R \int_{D_{t,\frac{1}{2},1,r}} < K_5(x,x_0,t,z,r), \nabla^\phi \psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t \\ &+ \int_0^1 \int_{D_{t,\frac{1}{2},1,R}} < K_6(x,x_0,t,z,R), \nabla^\phi \psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t \\ &- \int_0^1 \int_{D_{t,\frac{1}{4},\frac{1}{2},R}} < K_7(x,x_0,t,z,R), \nabla^\phi \psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t \end{aligned}$$

for suitable kernels K_5, K_6, K_7 defined in (3.74), (3.75) and (3.76) respectively. Moreover, there are positive constants $\tilde{C}_5$, $\tilde{C}_6$ independent of L_ϕ such *that*

$$|K_5(x,x_0,t,z,r)| \le \tilde{C}_5(L_\phi+1)^2 \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{r^{\mathcal{Q}}} \qquad on\ D_{t,\frac{1}{2},1,r}, \forall t \in [0,1], \tag{3.70}$$

$$|K_6(x,x_0,t,z,R)| \le \tilde{C}_6(L_\phi+1)^2 \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{R\|\tilde{F}(z,t)\|^{\mathcal{Q}-1}} \qquad on\ D_{t,\frac{1}{2},1,R}, \forall t \in [0,1], \tag{3.71}$$

$$|K_7(x,x_0,t,z,R)| \le \tilde{C}_6(L_\phi+1)^2 \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{R\|\tilde{F}(z,t)\|^{\mathcal{Q}-1}} \qquad on\ D_{t,\frac{1}{4},\frac{1}{2},R}, \forall t \in [0,1]. \tag{3.72}$$

Proof. By Lemma 3.6 for every $0 < r < R_0$ such that $\Omega_{\phi(x)}(x,r), \Omega_{\phi(x_0)}(x_0,r) \Subset \omega$, we have

$$\begin{aligned}&\bar{m}(\psi,\phi,r)(x) - \bar{m}(\psi,\phi,r)(x_0) = \\ &= \frac{1}{r^{\mathcal{Q}}} \int_{\tilde{\Omega}(0,r)\setminus\tilde{\Omega}(0,\frac{r}{2})} K_3(0,\tilde{y}) \Big(\psi(Exp_{\phi(x),x}(\tilde{y})) - \psi(Exp_{\phi(x_0),x_0}(\tilde{y}))\Big) \mathrm{d}\mathcal{L}^{2n}(\tilde{y})\end{aligned}$$

by Proposition 3.7

$$\begin{aligned}&= \frac{1}{r^{\mathcal{Q}}} \int_{\tilde{\Omega}(0,r)\setminus\tilde{\Omega}(0,\frac{r}{2})} K_3(0,\tilde{y}) \int_0^1 < Log_{\phi(x_0),x_0}(x), \hat{\nabla}^{\phi(x_0)}\psi(\gamma_{\tilde{y}}(t)) > \mathrm{d}t\mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \\ &+ \frac{1}{r^{\mathcal{Q}}} \int_{\tilde{\Omega}(0,r)\setminus\tilde{\Omega}(0,\frac{r}{2})} K_3(0,\tilde{y}) \int_0^1 K_4(x,x_0,\tilde{y})\partial_{2n}\psi(\gamma_{\tilde{y}}(t))\mathrm{d}t\mathrm{d}\mathcal{L}^{2n}(\tilde{y}).\end{aligned}$$

The change of variables $z = \gamma_{\tilde{y}}(t)$, changes $\tilde{\Omega}(0,r) \setminus \tilde{\Omega}(0,\frac{r}{2})$ in the set $D_{t,\frac{1}{2},1,r}$ and the inverse mapping has Jacobian determinant equal to 1. Hence:

$$\begin{aligned}&m(\psi,\phi,R)(x) - m(\psi,\phi,R)(x_0) = \\ &= \frac{2}{R}\int_0^1 \int_{\frac{R}{2}}^R \frac{1}{r^{\mathcal{Q}}} \int_{D_{t,\frac{1}{2},1,r}} K_3(0,\tilde{F}(z,t)) < Log_{\phi(x_0),x_0}(x), \hat{\nabla}^{\phi(x_0)}\psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t + \\ &+ \frac{2}{R}\int_0^1 \int_{\frac{R}{2}}^R \frac{1}{r^{\mathcal{Q}}} \int_{D_{t,\frac{1}{2},1,r}} K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t))\partial_{2n}\psi(z)\mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t.\end{aligned} \tag{3.73}$$

Now applying Remark 3.4 we get the thesis calling:

$$K_5(x,x_0,t,z,r) := \frac{1}{r^{\mathcal{Q}}} K_3(0,\tilde F(z,t)) Log_{\phi(x_0),x_0}(x) + \\ + \frac{1}{2r^{\mathcal{Q}}} \nabla_1^{\phi(x_0)} \Big(K_3(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) \Big) e_{n+1} + \\ + \frac{1}{2r^{\mathcal{Q}}} \nabla_{n+1}^{\phi(x_0)} \Big(K_3(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) \Big) e_1; \tag{3.74}$$

$$K_6(x,x_0,t,z,R) := \frac{1}{R} \frac{\nabla_1^{\phi(x_0)} K(\tilde F(z,t))}{K^{\mathcal{Q}}(\tilde F(z,t))} K_3(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) e_{n+1} - \\ - \frac{1}{R} \frac{\nabla_{n+1}^{\phi(x_0)} K(\tilde F(z,t))}{K^{\mathcal{Q}}(\tilde F(z,t))} K_3(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) e_1; \tag{3.75}$$

$$K_7(x,x_0,t,z,R) := - \frac{1}{R} \frac{\nabla_1^{\phi(x_0)} K(\tilde F(z,t))}{2^{\mathcal{Q}} K^{\mathcal{Q}}(\tilde F(z,t))} K_3(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) e_{n+1} \\ + \frac{1}{R} \frac{\nabla_{n+1}^{\phi(x_0)} K(\tilde F(z,t))}{2^{\mathcal{Q}} K^{\mathcal{Q}}(\tilde F(z,t))} K_3(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) e_1, \tag{3.76}$$

where as usual e_i denotes the i-th element of the canonical basis of $\mathbb{R}^{2n-1}$. To prove (3.70) we observe that by Lemma 3.10

$$K_5(x,x_0,t,z,r) = \frac{1}{r^{Q}} K_3(0,\tilde F(z,t)) Log_{\phi(x_0),x_0}(x) \\ + \frac{1}{2r^{Q}} \Big((\nabla_1 K_3)(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) + (\nabla_1 K_4)(x,x_0,\tilde F(z,t)) K_3(0,\tilde F(z,t)) \Big) e_{n+1} \\ + \frac{1}{2r^{Q}} \Big((\nabla_{n+1} K_3)(0,\tilde F(z,t)) K_4(x,x_0,\tilde F(z,t)) + (\nabla_{n+1} K_4)(x,x_0,\tilde F(z,t)) K_3(0,\tilde F(z,t)) \Big) e_1$$

hence using (3.47), (3.59) and (3.60) we get

$$|K_5(x,x_0,t,z,r)| \le \frac{\tilde C_3}{r^{\mathcal{Q}}} \mathrm{d}_{\phi(x_0)}(x_0,x) + 2\frac{\tilde C_4 (L_\phi+1) \mathrm{d}_\phi(x,x_0)}{r^{\mathcal{Q}}} + 2\frac{\tilde C_3 (L_\phi+1) \mathrm{d}_\phi(x,x_0)}{r^{\mathcal{Q}}}$$

and the conclusion follows using (2.20). Finally, (3.71) and (3.72) are direct consequences of (2.29),(3.47), (3.59) and Lemma 3.10. □

4. Poincaré inequality

The scope of this section is to prove Theorem 1.2. The Poincaré type inequality proved here is partially inspired to the Sobolev type inequality for vector fields with non regular coefficients contained in [23] and successively extended to a more general class of vector fields in [53]. The key point in our strategy, is to establish a representation formula for intrinsic Lipschitz continuous functions. To this end we use Theorem 2.6 and the representation formula proved in Proposition 3.11 for C^∞ functions.

Throughout this section we denote by ω an open and bounded subset of $\mathbb{R}^{2n}$ with $n \geq 2$ and by ϕ an intrinsic Lipschitz function defined on ω with Lipschitz constant equal to L_ϕ.

Let $\psi \in W_\phi$ and let $\{\psi_k\}_{k\in\mathbb{N}}$, $\{\phi_k\}_{k\in\mathbb{N}}$ smooth functions on ω which satisfy conditions $(i)-(iv)$ in Definition 1.1. We denote by $\mathrm{d}_{\phi_k(x_0)}$ the distance introduced in (2.13), by $\Gamma_{\phi_k(x_0)}$ the fundamental solution of the operator $\mathcal{L}_{\phi_k(x_0)}$ defined in (2.30) and by $\Omega_{\phi_k(x_0)}(x_0, r)$ the super level set of $\Gamma_{\phi_k(x_0)}$ defined in (2.31).

We start proving that the average $\bar{m}(\psi, \phi, R)(x_0)$ can be approximated by means of the regular sequence $\bar{m}(\psi_k, \phi_k, R)(x_0)$. Precisely:

Lemma 4.1. *Let $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$. Then*

(i) $\chi_{\Omega_{\phi_k(x_0)}(x_0,R)} \to \chi_{\Omega_{\phi(x_0)}(x_0,R)}$ *uniformly in ω as $k \to +\infty$;*

(ii) $\bar{m}(\psi_k, \phi_k, R)(x_0) \to \bar{m}(\psi, \phi, R)(x_0)$ *uniformly in $R > 0$ as $k \to +\infty$.*

Here χ_A denotes the characteristic function of A.

Proof. We recall that

$$\tilde{\Omega}(0, R) = \left\{ \tilde{y} \in \mathbb{R}^{2n} \mid \Gamma(0, \tilde{y}) > R^{2-Q} \right\},$$

then, by (2.34), for each $k \in \mathbb{N}$ we have:

$$\Omega_{\phi_k(x_0)}(x_0, R) = Exp_{\phi_k(x_0),x_0}(\tilde{\Omega}(0, R)).$$

Using the explicit form of $Exp_{\phi_k(x_0),x_0}$ and $Exp_{\phi(x_0),x_0}$ stated in (2.10) we easily conclude that $(Exp_{\phi_k(x_0),x_0})_{k\in\mathbb{N}}$ uniformly converges to $Exp_{\phi(x_0),x_0}$ in

ω as $k \to +\infty$. In order to prove (i) we observe that it is sufficient to prove that for all $\epsilon > 0$ there exists $\bar{k} = \bar{k}(\epsilon) > 0$ such that for all $k > \bar{k}$

$$\Omega_{\phi(x_0)}(x_0, R) \subseteq (\Omega_{\phi_k(x_0)}(x_0, R))_\epsilon \tag{4.77}$$

where

$$(\Omega_{\phi_k(x_0)}(x_0, R))_\epsilon := \{y \in \omega \mid \mathrm{d}_{\phi_k(x_0)}(\partial\Omega_{\phi_k(x_0)}(x_0, R), y) < \epsilon\}. \tag{4.78}$$

For simplifying the notation we define

$$E_k(\tilde{\Omega}(0, R)) := Exp_{\phi_k(x_0), x_0}(\tilde{\Omega}(0, R)), \quad E(\tilde{\Omega}(0, R)) := Exp_{\phi(x_0), x_0}(\tilde{\Omega}(0, R)).$$

Suppose by contradiction that there exists $\epsilon > 0$ such that for every $\bar{k}$ there are $k > \bar{k}$ and $y_k \in E(\tilde{\Omega}(0, R))$ such that $y_k \notin E_k(\tilde{\Omega}(0, R))_\epsilon$. Then, there exist $(k_j)_j$, $k_j \to +\infty$ as $j \to +\infty$ and $(x_{k_j})_j$ in $\tilde{\Omega}(0, R)$ such that $E(x_{k_j}) \notin E_{k_j}(\tilde{\Omega}(0, R))_\epsilon$. So that, the distance between $E(x_{k_j})$ and $E_{k_j}(x_{k_j})$ is greater than ϵ and this is absurd being E_k uniformly convergent to E. Then, (4.77) follows and hence (i).

To prove (ii) we observe that by Definition 3.3:

$$\lim_{k \to +\infty} \bar{m}(\psi_k, \phi_k, R)(x_0) =$$
$$= \lim_{k \to +\infty} \frac{C(\mathcal{Q})}{\mathcal{Q} - 2} \frac{1}{R^{\mathcal{Q}}} \int_\omega \frac{|\nabla^{\phi_k(x_0)} \Gamma_{\phi_k(x_0)}(x_0, y)|^2}{\Gamma_{\phi_k(x_0)}(x_0, y)^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}} \psi_k(y)\, \chi_{\Omega_{\phi_k(x_0)}(x_0, R) \setminus \Omega_{\phi_k(x_0)}(x_0, \frac{R}{2})}(y)\, \mathrm{d}\mathcal{L}^{2n}(y).$$

By (2.29) and (2.30)

$$\lim_{k \to +\infty} \frac{|\nabla^{\phi_k(x_0)} \Gamma_{\phi_k(x_0)}(x_0, y)|^2}{\Gamma_{\phi_k(x_0)}(x_0, y)^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}} = \frac{|\nabla^{\phi(x_0)} \Gamma_{\phi(x_0)}(x_0, y)|^2}{\Gamma_{\phi(x_0)}(x_0, y)^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}} \leq C \quad \forall y \neq x_0 \tag{4.79}$$

therefore, (ii) follows from (i) and the fact that $\psi_k \to \psi$ in $L^1_{loc}(\omega)$. □

In what follows we prove that the representation formulas obtained in Proposition 3.5 and in Proposition 3.11 for C^∞ functions still hold if ϕ is intrinsic Lipschitz and $\psi \in W_\phi(\omega)$.

Lemma 4.2. *Let ϕ be a Lipschitz continuous function and $\psi \in W_\phi(\omega)$. For each $x_0 \in \omega$ and each $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$, the following formula holds:*

$$\psi(x_0) = m(\psi, \phi, R)(x_0) + I_R(x_0)$$

where $m(\psi, \phi, R)(x_0)$ is as in (3.40) and

$$\begin{aligned} I_R(x_0) :=& \frac{2}{R}\int_{\frac{R}{4}}^{R} f_1\left(\frac{r}{R}\right)\int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle K_1(x_0,y), \nabla^\phi\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r \\ &+\frac{2}{R}\int_{\frac{R}{2}}^{R}\int_{\Omega_{\phi(x_0)}(x_0,r)\setminus\Omega_{\phi(x_0)}(x_0,\frac{r}{2})} \left\langle K_2(x_0,y,r), \nabla^\phi\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r, \end{aligned} \tag{4.80}$$

where K_1, K_2 are as in Proposition 3.5. Let $\bar{x} \in \omega$ and $C_0 > 0$ be as in Proposition 3.7, then for every $0 < R$ such that $\Omega_{\phi(\bar{x})}(\bar{x}, C_0R) \subset \omega$ and $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ with $x \neq x_0$ it holds:

$$\begin{aligned} m(\psi,\phi,R)(x) - m(\psi,\phi,R)(x_0) =& \\ =&\frac{2}{R}\int_0^1\int_{\frac{R}{2}}^{R}\int_{D_{t,\frac{1}{2},1,r}} < K_5(x,x_0,t,z,r), \nabla^\phi\psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t \\ &+\int_0^1\int_{D_{t,\frac{1}{2},1,R}} < K_6(x,x_0,t,z,R), \nabla^\phi\psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t \\ &-\int_0^1\int_{D_{t,\frac{1}{4},\frac{1}{2},R}} < K_7(x,x_0,t,z,R), \nabla^\phi\psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t, \end{aligned} \tag{4.81}$$

where $D_{t,c_1,c_2,r}$ is as in (3.69) *and K_5, K_6, K_7 are as in* (3.74),(3.75) *and* (3.76) *respectively and they satisfy the same estimates proved in Proposition 3.11 with possibly different constants.*

Proof. By definition of 1.1 there are $\{\psi_k\}_{k\in\mathbb{N}}$, $\{\phi_k\}_{k\in\mathbb{N}}$ sequences of smooth functions defined on ω satisfying conditions $(i) - (iv)$ of Definition 1.1. By Proposition 3.5 and 3.11, the thesis is true for every ϕ_k, ψ_k as above. Passing to the limit as in the previous proposition, it holds true also for the limit functions ϕ and ψ. □

It is well known (see for example [14], [21]) that the key step in the proof of the Poincaré inequality is a representation formula as the one proved in Lemma 4.2, which is indeed equivalent to the Poincaré inequality itself. For further applications, we note that we can obtain the representation formula on any family of balls, equivalent to the super levels $\Omega_{\phi(\bar{x})}(\bar{x}, R)$, which can be $\Omega_{\phi(\bar{x})}(\bar{x}, R)$ or $U_\phi(\bar{x}, R)$, defined respectively in (2.31) and (1.5).

Let us denote by $B_\phi(\bar{x}, R)$ a family of spheres centered at $\bar{x}$ and radius R, equivalent to the family $\Omega_{\phi(\bar{x})}(\bar{x}, R)$. Let us denote by $\psi_{B_\phi(\bar{x},R)}$ the mean of ψ on the set $B_\phi(\bar{x}, R)$ with respect to the Lebesgue measure, i.e.

$$\psi_{B_\phi(\bar{x},R)} := \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x}, R))} \int_{B_\phi(\bar{x},R)} \psi(x) \, \mathrm{d}\mathcal{L}^{2n}(x) \tag{4.82}$$

we will prove the following result:

We use our representation formula to prove an upper bound for the quantity $|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}|$, precisely:

Proposition 4.3. *Let $\phi \in Lip(\omega)$ and $\psi \in W_\phi(\omega)$. Let $\bar{x} \in \omega$ and $C_0 > 0$ be as in Proposition 3.7. There are $\tilde{C}_0 > C_0$, $\tilde{C}_1, \tilde{C}_2 > 0$, depending only on $L_{\phi,\omega}, \mathcal{Q}$ and the structure constants in (2.29), such that if $B_\phi(\bar{x}, \tilde{C}_0 R) \Subset \omega$ and $x_0 \in B_\phi(\bar{x}, R)$ then*

$$\begin{aligned} &|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}| \leq \\ &\quad \leq \tilde{C}_1 \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0, y) |\nabla^\phi \psi(y)| \mathrm{d}\mathcal{L}^{2n}(y) + \\ &\quad + \frac{\tilde{C}_2}{\mathcal{L}^{2n}(B_\phi(\bar{x}, R))} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x, y) |\nabla^\phi \psi(y)| \mathrm{d}\mathcal{L}^{2n}(y) \mathrm{d}\mathcal{L}^{2n}(x). \end{aligned} \tag{4.83}$$

Proof. Let $R > 0$ such that $B_\phi(\bar{x}, R) \subset \Omega_{\phi(\bar{x})}(\bar{x}, C_0 R) \subset B_\phi(\bar{x}, \tilde{C}_0 R) \Subset \omega$. By Lemma 4.2, for each $x, x_0 \in B_\phi(\bar{x}, R)$ we have:

$$\begin{aligned} \psi(x_0) &= m(\psi, \phi, R)(x_0) + I_R(x_0), \\ \psi(x) &= m(\psi, \phi, R)(x) + I_R(x), \end{aligned}$$

hence

$$\psi(x_0) - \psi(x) = m(\psi, \phi, R)(x_0) - m(\psi, \phi, R)(x) + I_R(x_0) - I_R(x). \tag{4.84}$$

Integrating equation (4.84) with respect to the variable x on the sphere $B_\phi(\bar{x}, R)$ and recalling the definition of $\psi_{B_\phi(\bar{x},R)}$ we get:

$$\begin{aligned} \psi(x_0) - \psi_{B_\phi(\bar{x},R)} =& \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x}, R))} \int_{B_\phi(\bar{x},R)} m(\psi, \phi, R)(x_0) - m(\psi, \phi, R)(x) \, \mathrm{d}\mathcal{L}^{2n}(x) \\ &+ \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x}, R))} \int_{B_\phi(\bar{x},R)} I_R(x_0) - I_R(x) \, \mathrm{d}\mathcal{L}^{2n}(x). \end{aligned}$$

Hence:

$$|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}| \leq \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))} \int_{B_\phi(\bar{x},R)} \Big| m(\psi,\phi,R)(x_0) - m(\psi,\phi,R)(x) \Big| \, \mathrm{d}\mathcal{L}^{2n}(x) \tag{4.85}$$
$$+ |I_R(x_0)| + \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))} \int_{B_\phi(\bar{x},R)} |I_R(x)| \, \mathrm{d}\mathcal{L}^{2n}(x).$$

Now, by Lemma 4.2, we have:

$$\begin{aligned}|m(\psi,\phi,R)(x_0) - & m(\psi,\phi,R)(x)| \\ & \leq \frac{2}{R} \int_0^1 \int_{\frac{R}{2}}^R \int_{D_{t,\frac{1}{2},1,r}} | < K_5(x,x_0,t,z,r), \nabla^\phi \psi(z) > | \mathrm{d}\mathcal{L}^{2n}(z) \mathrm{d}r \mathrm{d}t \\ & + \int_0^1 \int_{D_{t,\frac{1}{2},1,R}} | < K_6(x,x_0,t,z,R), \nabla^\phi \psi(z) > | \mathrm{d}\mathcal{L}^{2n}(z) \mathrm{d}t \\ & + \int_0^1 \int_{D_{t,\frac{1}{4},\frac{1}{2},R}} | < K_7(x,x_0,t,z,R), \nabla^\phi \psi(z) > | \mathrm{d}\mathcal{L}^{2n}(z) \mathrm{d}t.\end{aligned}$$

We claim that there exists $C = C(L_\phi) > 0$ such that for all $r \in (R/2, R)$, $t \in [0,1]$ it holds

$$D_{t,\frac{1}{2},1,r} \subseteq \Omega_{\phi(x_0)}(x_0, CR) \subset B_\phi(\bar{x}, \tilde{C}_0 R). \tag{4.86}$$

To this end let us fix $t \in [0,1]$ and $r \in (R/2, R)$ then for each $\tilde{y} \in \tilde{\Omega}(0,r) \setminus \tilde{\Omega}(0, r/2)$ we have

$$\frac{r}{2} < \|\tilde{y}\| < r \leq R \tag{4.87}$$

and, by (3.55), it also holds

$$\mathrm{d}_{\phi(x_0)}(x_0, \gamma_{\tilde{y}}(t)) \leq \|\tilde{y}\| + \|\tilde{x}\| + \sqrt{\|\tilde{y}\| \|\tilde{x}\|} + \sqrt{\|\tilde{y}\| |\phi(x) - \phi(x_0)|}. \tag{4.88}$$

Since $x, x_0 \in B_\phi(\bar{x}, R)$ by (2.20) and (2.29) we have

$$\mathrm{d}_\phi(x, x_0) \leq CR \quad \text{and} \quad \|\tilde{x}\| \leq CR \tag{4.89}$$

for some $C = C(L_\phi) > 0$. Using (4.87), (4.88) and (4.89) we immediately get (4.86) with possibly smaller R.

By Lemma 4.2 we know that the estimates for K_5, K_6, K_7 proved in Proposition 3.11 also hold for $\phi \in Lip(\omega)$. Hence, by (3.70) and (4.87) for each $z \in D_{t,\frac{1}{2},1,r}$ and $t \in [0,1]$ we have

$$|K_5(x, x_0, t, z, r)| \leq \tilde{C} \frac{\mathrm{d}_{\phi(x_0)}(x_0, x)}{r^{\mathcal{Q}}} \leq \tilde{C} \frac{\mathrm{d}_{\phi(x_0)}(x_0, x)}{\|\tilde{F}(z,t)\|^{\mathcal{Q}}}$$

for some $\tilde{C} = \tilde{C}(L_\phi) > 0$. Using (4.86) and (4.89) we get

$$\mathrm{d}_{\phi(x_0)}(x_0, x) = \|\tilde{x}\| \leq CR \leq 2Cr \leq C\|\tilde{F}(z,t)\|$$

for a suitable constant $C = C(L_\phi) > 0$. Then

$$|K_5(x, x_0, t, z, r)| \leq C \frac{1}{\|\tilde{F}(z,t)\|^{\mathcal{Q}-1}}.$$

Moreover, by (4.86), $z \in \Omega_{\phi(x_0)}(x_0, CR)$ and

$$0 < \mathrm{d}_{\phi(x_0)}(x_0, z) \leq CR \leq 2Cr \leq C\|\tilde{F}(z,t)\|.$$

So that

$$|K_5(x, x_0, t, z, r)| \leq C \mathrm{d}_{\phi(x_0)}(x_0, z)^{1-\mathcal{Q}}. \tag{4.90}$$

Analogously, we can prove that there exists $C = C(L_\phi) > 0$ such that

$$|K_6(x, x_0, t, z, r)|, |K_7(x, x_0, t, z, r)| \leq C \mathrm{d}_{\phi(x_0)}(x_0, z)^{1-\mathcal{Q}}. \tag{4.91}$$

In conclusion we proved that:

$$\begin{aligned} &|m(\psi, \phi, R)(x_0) - m(\psi, \phi, R)(x)| \leq \\ &\qquad \leq \tilde{C}_1 \int_{\Omega_{\phi(x_0)}\left(x_0, CR\right)} \mathrm{d}^{1-\mathcal{Q}}_{\phi(x_0)}(x_0, z) |\nabla^\phi \psi(z)| \mathrm{d}\mathcal{L}^{2n}(z). \end{aligned} \tag{4.92}$$

Furthermore, by Lemma 4.2, (3.41) and (3.42) we have:

$$|I_R(x_0)| \leq \tilde{C}_2 L_\phi \int_{\Omega_{\phi(x_0)}(x_0, R)} \mathrm{d}^{1-\mathcal{Q}}_{\phi(x_0)}(x_0, y) |\nabla^\phi \psi(y)| \mathrm{d}\mathcal{L}^{2n}(y), \tag{4.93}$$

$$|I_R(x)| \leq \tilde{C}_2 L_\phi \int_{\Omega_{\phi(x)}(x, R)} \mathrm{d}^{1-\mathcal{Q}}_{\phi(x)}(x, y) |\nabla^\phi \psi(y)| \mathrm{d}\mathcal{L}^{2n}(y). \tag{4.94}$$

Finally, since the integrals can be extended on the sphere $B_\phi(\bar{x}, \tilde{C}_0 R)$ and by Remark 2.4 we can replace $\mathrm{d}^{1-\mathcal{Q}}_{\phi(x)}(x, y)$ with $\mathrm{d}^{1-\mathcal{Q}}_{\phi}(x, y)$, then the thesis follows by (4.85), (4.92), (4.93) and (4.94). □

The proof of Theorem 1.2 follows from Proposition 4.3. We will prove Theorem 1.2 for any family of spheres equivalent to $\Omega_{\phi(x)}$. This will easily imply Theorem 1.2 using the spheres U_ϕ.

Proof of Theorem 1.2. As in the previous proof we denote by $C, \tilde{C}$ positive constants depending only on L_ϕ, $\mathcal{Q}$ and the constants defined in (2.29) which could be different from line to line. Firstly let us assume that $p > 1$. From (4.83), if we denote p' the conjugate exponent of p we get:

$$\begin{aligned}
&|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}| \leq \qquad (4.95)\\
&\qquad \leq \tilde{C}_1 \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{(1-\mathcal{Q})/p'}(x_0,y)\mathrm{d}_\phi^{(1-\mathcal{Q})/p}(x_0,y)|\nabla^\phi \psi(y)| \mathrm{d}\mathcal{L}^{2n}(y) + \\
&\qquad + \frac{\tilde{C}_2}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{(1-\mathcal{Q})/p'}(x,y)\mathrm{d}_\phi^{(1-\mathcal{Q})/p}(x,y)|\nabla^\phi \psi(y)| \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}\mathcal{L}^{2n}(x).
\end{aligned}$$

Hence, applying Hölder inequality we have:

$$\begin{aligned}
&|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}|^p \leq \qquad (4.96)\\
&\qquad \leq \tilde{C}_1 \Big(\int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0,y)\mathrm{d}\mathcal{L}^{2n}(y) \Big)^{p/p'} \Big(\int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0,y)|\nabla^\phi \psi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \Big) + \\
&\qquad + \frac{\tilde{C}_2}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \Big(\int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x,y) \Big)^{p/p'} \\
&\qquad\qquad \Big(\int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x,y)|\nabla^\phi \psi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \Big) \mathrm{d}\mathcal{L}^{2n}(x). \qquad (4.97)
\end{aligned}$$

If $d_\phi(x_0, \bar{x}) \leq R$ we have

$$\int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0,y)\, \mathrm{d}\mathcal{L}^{2n}(y) \leq \int_{B_\phi(x_0,(C+1)R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0,y)\, \mathrm{d}\mathcal{L}^{2n}(y)\,. \qquad (4.98)$$

By (2.21), it follows that we can consider as admissible family of balls also the the one defined by $B_\phi(x_0,r) := \{y \in \omega : \mathrm{d}_{\phi(x_0)}(x_0,y) < r\}$ where $\mathrm{d}_{\phi(\cdot)}$ is defined in (2.13) and we can replace $\mathrm{d}_\phi^{1-\mathcal{Q}}(x_0,y)$ with $\mathrm{d}_{\phi(x_0)}^{1-\mathcal{Q}}(x_0,y)$ in the previous integral. Let us now recall that $\mathcal{G}$ is isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^{2n}$, meant as a Carnot group, with homogeneous dimension $\mathcal{Q}$. Denote

$\tilde{U}(\tilde{x}_0, r) := \{\tilde{y} \in \mathcal{G} : \tilde{\mathrm{d}}(\tilde{x}_0, \tilde{y}) < r\}$ with $\tilde{\mathrm{d}}$ homogeneous distance on $\mathcal{G}$ defined in (2.26). By (2.27), for a given $x_0 \in \omega$ and $r > 0$,

$$\tilde{U}(F(x_0), r) \cap F(\omega) = F(B_\phi(x_0, r))$$

where $F : \mathbb{R}^{2n} \to \mathcal{G} \equiv \mathbb{R}^{2n}$ is the diffeomorphism $F(y) := Log_{\phi(x_0),x_0}(y)$ (see (2.12)). Moreover, since it is easy to see that the Jacobian determinant of F is 1, it follows that

$$\begin{aligned} \int_{B_\phi(x_0,(C+1)R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0, y)\, \mathrm{d}\mathcal{L}^{2n}(y) &\leq \tilde{C}_3 \int_{B_\phi(x_0,(C+1)R)} \mathrm{d}_{\phi(x_0)}^{1-\mathcal{Q}}(x_0, y)\, \mathrm{d}\mathcal{L}^{2n}(y) \\ &\leq \tilde{C}_3 \int_{\tilde{U}(F(x_0),(C+1)R)} \tilde{\mathrm{d}}^{1-\mathcal{Q}}(F(x_0), z)\, \mathrm{d}\mathcal{L}^{2n}(z) \leq \tilde{C}_4\, R\,. \end{aligned} \tag{4.99}$$

Thus, from (4.98)and (4.99), we get that, if $d_\phi(x_0, \bar{x}) \leq R$, then

$$\int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0, y)\, \mathrm{d}\mathcal{L}^{2n}(y) \leq \tilde{C}_4\, R\,. \tag{4.100}$$

Inserting this in the previous expression, we immediately get:

$$\begin{aligned} &|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}|^p \leq \\ &\quad \leq \tilde{C}_1 R^{p-1} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0, y)|\nabla^\phi \psi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) + \\ &\quad + \frac{\tilde{C}_2 R^{p-1}}{\mathcal{L}^{2n}(B_\phi(\bar{x}, R))} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \int_{B_\phi(\bar{x},\tilde{C}_0 R)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x, y)|\nabla^\phi \psi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \mathrm{d}\mathcal{L}^{2n}(x). \end{aligned} \tag{4.101}$$

Integrating on $B_\phi(\bar{x}, R)$ we get

$$\begin{aligned} &\int_{B_\phi(\bar{x},R)} |\psi(x_0) - \psi_{B_\phi(\bar{x},R)}|^p \mathrm{d}\mathcal{L}^{2n}(x_0) \leq \\ &\leq \tilde{C}_1 R^{p-1} \int_{B_\phi(\bar{x},CR)} \int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0, y)|\nabla^\phi \psi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \mathrm{d}\mathcal{L}^{2n}(x_0) \\ &+ \tilde{C}_2 R^{p-1} \int_{B_\phi(\bar{x},CR)} \int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x, y)|\nabla^\phi \psi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \mathrm{d}\mathcal{L}^{2n}(x). \end{aligned} \tag{4.102}$$

This implies:

$$\int_{B_\phi(\bar{x},R)} |\psi(x_0) - \psi_{B_\phi(\bar{x},R)}|^p \mathrm{d}\mathcal{L}^{2n}(x_0) \leq \tag{4.103}$$
$$\leq CR^{p-1} \int_{B_\phi(\bar{x},CR)} |\nabla^\phi \psi(y)|^p \Big(\int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(y, x_0) \mathrm{d}\mathcal{L}^{2n}(x_0) \Big) \mathrm{d}\mathcal{L}^{2n}(y).$$

Finally, using again (4.100) we get the thesis. If $p = 1$, we can directly integrate (4.83) on $B_\phi(\bar{x}, R)$ and we get (4.103). Eventually, since we can choose as $B_\phi(\bar{x}, R)$ any family of balls equivalent to $\Omega_{\phi(\bar{x})}(\bar{x}, R)$, by (2.32), it follows we can also select $B_\phi(\bar{x}, R) = U_\phi(\bar{x}, R)$ and we get the desired conclusion. □

By the approximation result in Theorem 2.6 we can choose $\psi = \phi$ and get the proof of Corollary 1.3.

Acknowledgments

G. C. and M. M. are partially supported by MAnET Marie Curie Initial Training Networks (ITN). A. P. was supported by the Progetto CaRiPaRo "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems" and now is supported by ERC ADG GeMeThNES and GNAMPA of INDAM. F. S.C. is supported by MIUR, Italy, GNAMPA of INDAM, University of Trento and MAnET Marie Curie Initial Training Networks (ITN).
The authors wish to thank the anonymous referee for his useful comments and for his constructive criticism about the previous version of this paper.

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