TWISTOR LINES ON ALGEBRAIC SURFACES

A. ALTAVILLA[‡] AND E. BALLICO[†]

ABSTRACT. We give quantitative and qualitative results on the family of surfaces in \mathbb{CP}^3 containing finitely many twistor lines. We start by analyzing the ideal sheaf of a finite set of disjoint lines E. We prove that its general element is a smooth surface containing E and no other line. Afterwards we prove that twistor lines are Zariski dense in the Grassmannian Gr(2, 4). Then, for any degree $d \ge 4$, we give lower bounds on the maximum number of twistor lines contained in a degree d surface. The smooth and singular cases are studied as well as the j-invariant one.

1. INTRODUCTION AND MAIN RESULTS

Given a four dimensional Riemannian manifold (M^4, g) , its twistor space Z(M) is the total space of a bundle containing all the complex structures that can be defined on M and are compatible with g. If (M, g) is anti-self-dual, then Z(M) is a complex manifold of (complex) dimension 3. Moreover, a complex 3-manifold Z is the twistor space of some 4-dimensional Riemannian manifold M if and only if it admits a fixed-point-free anti-holomorphic involution $j: Z \to Z$ and a foliation by j-invariant rational curves \mathbb{CP}^1 each of which has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (see e.g. [16]).

Since any complex structure on M compatible with g is as well compatible with any conformal equivalent metric $e^{f}g$, the whole theory is invariant under conformal transformations of M.

The first interesting example is given by the 4-sphere \mathbb{S}^4 , identified with the *left* quaternionic projective line \mathbb{HP}^1 , whose twistor space is given by \mathbb{CP}^3 with fibers \mathbb{CP}^1 , i.e.:

 $\mathbb{CP}^1 \to \mathbb{CP}^3 \xrightarrow{\pi} \mathbb{HP}^1$,

where π is the real analytic submersion defined by

$$\pi[z_0, z_1, z_2, z_3] = [z_0 + z_1 j, z_2 + z_3 j].$$

The fibration π is the main object of this paper and, from now on, we focus our attention only on it. On the affine subset $\{[p,q] \in \mathbb{HP}^1 | p \neq 0\}$, if $q = q_1 + q_2 j$, we have that the fibers of π are explicitly given by

$$\begin{cases} z_2 = z_0 q_1 - z_1 \overline{q}_2 \\ z_3 = z_0 q_2 + z_1 \overline{q}_1. \end{cases}$$

In this case the fibers are identified with projective lines l such that j(l) = l, where $j : \mathbb{CP}^3 \to \mathbb{CP}^3$ is the fixed-point-free anti-holomorphic involution given by

$$j[z_0, z_1, z_2, z_3] \mapsto [-\overline{z}_1, \overline{z}_0, -\overline{z}_3, \overline{z}_2].$$

Notice that the map j coincides with the map induced, via π^{-1} , by quaternionic left multiplication by j (see e.g. [12, Formula (5.8)]). We now state a formal definition for our main object.

Definition 1.1. A projective line $l \subset \mathbb{CP}^3$ is said to be a *twistor line* if j(l) = l, i.e.: if l is a fibre for π .

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In this setting an important theme is the analysis of complex surfaces in \mathbb{CP}^3 transverse to fibers. This because, any surface $\mathcal{S} \subset \mathbb{CP}^3$, that can be considered a graph for π^{-1} , produces a complex structure compatible with the standard round metric on $\pi(\mathcal{S}) \subset \mathbb{S}^4$. In fact, the map π restricted to any degree d algebraic surface Y can be considered as a d-fold branched covering over \mathbb{S}^4 and the set of twistor lines contained in \mathcal{S} is contained in the ramification locus. In particular, in this framework, the number of twistor lines contained in a surface plays an important role being an invariant under conformal transformations of \mathbb{S}^4 (for more details see, e.g., [5]).

Motivated by some recent result on the geometry of twistor lines in particular cases (degree 1 [1, 23], degree 2 [11, 21], degree 3 [1, 3, 4, 5, 6] and degree 4 [12], see also [21, arXiv version v1]), in this paper we give some general algebro-geometric result related to them in the same spirit of the papers [2, 7].

After the introduction, this paper has 2 sections. Section 2 contains preparatory material and standard tools from algebraic-geometry. The only main result is Theorem 1.3 below which regards surfaces containing a fixed set of lines. Before state it we set the following standard notation: for any union of lines E let \mathcal{I}_E be the ideal sheaf of E (see Section 2 for definitions and properties). We denote by $|\mathcal{I}_E(d)|$ the projective space associated to the vector space $H^0(\mathcal{I}_E(d))$.

Remark 1.2. Let E be a union of s disjoint lines. Recall that, for $d \geq s > 0$, the set $|\mathcal{I}_E(d)|$ is a projective space of dimension $\binom{d+3}{3} - s(d+1) - 1$. In fact, it is well known that for integers $t \ge s-1$ such that t > 0 and $s \ge 0$, if $E \subset \mathbb{CP}^3$ is a union of s disjoint lines, then $h^1(\mathcal{I}_E(t)) = 0$ and $h^0(\mathcal{I}_E(t)) = \binom{t+3}{3} - s(t+1)$ (see e.g. [24, Corollary 1.10], where "d-regular" means exactly that $h^1 = 0$).

Theorem 1.3. Fix integers $d \ge k > 0$ with $d \ge 4$. Let $E \subset \mathbb{CP}^3$ be a union of k disjoint lines. A general $Y \in |\mathcal{I}_E(d)|$ is a smooth degree d hypersurface containing E and containing no line $L \subset \mathbb{CP}^3$ with $L \cap E = \emptyset$.

The proof of the previous theorem passes through a number of lemmas analyzing the first two cohomology numbers of $\mathcal{I}_E(d)$ and of $\mathcal{I}_{2q\cup E}(d)$, for some fat point 2q.

In Section 3, we add the hypothesis that the set E of Theorem 1.3 is composed by twistor lines. Therefore we have firstly to explain what a general set of twistor lines is. Let Gr(2,4) be the Grassmannian of lines in the complex projective space and $\Lambda \subset Gr(2,4)$ denote the set of all twistor lines. Topologically we have $\Lambda \cong \mathbb{S}^4$. In the beginning of the Section, by means of what we call *Density Lemma* 3.2, we give meaning to the words "k general twistor lines" for any integer k > 0. A first consequence of the Density Lemma is Corollary 3.4 stating that, for any s greater or equal to 5, there are s twistor lines L_1, \ldots, L_s such that no other line intersects all of them. We recall that in [5] the authors give conditions in order to establish whether 5 twistor lines lie on a cubic. This property implies that there exist other two distinct lines intersecting all the given five twistor lines.

Now, before stating the main results of the third section, for any integer d > 0 we set the following quantities that will be our lower bounds:

- $\nu(d) := \lfloor (\binom{d+3}{3} 1)/(d+1) \rfloor;$ $\nu_n(d) := \nu(d-1) \text{ if } d \ge 2 \text{ and } \nu_n(d) := 0 \text{ if } d \le 1;$ $\nu_s(d) := \nu(d-3) \text{ if } d \ge 4 \text{ and } \nu_s(d) := 0 \text{ if } d \le 3;$
- $\nu_i(d) := \nu_n(d-8) = \nu(d-9)$ if $d \ge 9$ and $\nu_i(d) := 0$ if $d \le 8$.

The 3 subscripts for ν stand for normal, smooth and *j*-invariant as the following results will suggest. More explicitly,

(1)
$$\begin{cases} \nu(d) = (d^2 + 5d)/6 & \text{if } d \equiv 0, 1 \pmod{3} \\ \nu(d) = (d^2 + 5d + 4)/6 & \text{if } d \equiv 2 \pmod{3}, \end{cases}$$

and, if $d \geq 3$ we have

$$\begin{cases} \nu_s(d) = (d-3)(d+2)/6 & \text{if } d \equiv 0,1 \pmod{3} \\ \nu_s(d) = (d^2 - d - 2)/6 & \text{if } d \equiv 2 \pmod{3}. \end{cases}$$

Notice that, for $d \ge 2$, $\nu(d) < d^2$ and that $\nu(d) \sim \nu_s(d) \sim \nu_n(d) \sim \nu_i(d) \sim d^2/6$ for $d \to \infty$.

Recalling that a surface is said to be *integral* if it is reduced and irreducible, we can state our first main result.

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Theorem 1.4. Fix an integer $d \ge 4$.

- (1) There is an irreducible degree d surface $Y \subset \mathbb{CP}^3$ containing $\nu(d)$ general twistor lines.
- (2) There is an irreducible degree d surface $Y \subset \mathbb{CP}^3$ containing $\nu_n(d)$ twistor lines, none of them intersecting $\operatorname{Sing}(Y)$, and with $\operatorname{Sing}(Y)$ finite. (3) There is a smooth degree d surface $Y \subset \mathbb{CP}^3$ containing $\nu_s(d)$ general twistor lines.

The three cases are proven separately. Firstly, we prove the existence of a complex projective space of dimension $\binom{d+3}{3} - \nu(d)(d+1) - 1$ parametrizing a family of integral degree d projective surfaces, all of them containing a general (but fixed) union $E \subset \mathbb{CP}^3$ of $\nu(d)$ general twistor lines, namely $|\mathcal{I}_E(d)|$ (see also Corollary 3.5 and Proposition 3.6). Then, in the following step we prove that, reducing the number of lines from $\nu(d)$ to $\nu_n(d)$, then the general element Y of $|\mathcal{I}_E(d)|$ is such that Sinq(Y) is finite and does not intersect E. Afterwards, in the last step we prove that, if the number of twistor lines is reduced to $\nu_s(d)$, then the general element of $|\mathcal{I}_E(d)|$ is smooth.

Thanks to the Density Lemma, Theorem 1.4 is true for twistor lines L_1, \ldots, L_k belonging to a non-empty open subset of Λ^k .

In case (2) of Theorem 1.4 we obviously allow the case $\operatorname{Sing}(Y) = \emptyset$. Since any surface $Y \subset \mathbb{CP}^3$ is a locally complete intersection, Sing(Y) is finite if and only if Y is normal ([14, II.8.22A]).

The previous theorem can be specialized as follows.

Theorem 1.5. Fix integers $d \ge 10$ and k such that $0 \le k \le \nu_i(d)$. Then there is a smooth degree d surface $Y \subset \mathbb{CP}^3$ containing k general twistor lines and no other no other line L not intersecting any of the k twistor lines.

A surface Y in \mathbb{CP}^3 is said to be j-invariant, if j(Y) = Y. This concept will be better explained at the end of Section 3. Examples of j-invariant surfaces are the so-called *real quadrics* in [21] and the smooth cubic analyzed in [5, 6]. With the help of some observation on the topology of *j*-invariant surfaces in $|\mathcal{I}_E(d)|$ we are able to improve Theorem 1.5 as follows.

Theorem 1.6. Take d, $\nu_i(d)$ and $k \leq \nu_i(d)$ as in Theorem 1.5 with d even; if $d \equiv 2 \pmod{4}$ assume k odd; if $d \equiv 0 \pmod{4}$ assume k even. Then there exists a degree d smooth j-invariant surface $Y \subset \mathbb{CP}^3$ such that Y contains exactly k twistor lines.

Remark 1.7. General results on the maximum number N_d of lines on a smooth degree d projective surface are given in [8]. In this remark we recall some of them. It is well known that any smooth cubic surface contains exactly 27 lines and, from the work of Segre [22], that $N_4 = 64$ and $N_d \leq$ (d-2)(11d-6). More recently it was proved in [9] that $N_d \geq 3d^2$. Moreover, thanks to [8, 9], we have $N_6 \ge 180$, $N_8 \ge 352$ $N_{12} \ge 864$ and $N_{20} \ge 1600$.

Since twistor lines are skew, it is interesting to look at the maximum number S_d of skew lines contained in a smooth degree d projective surface. When d = 3 this number is exactly $S_3 = 6$. Thanks to [19], $S_4 = 16$. If $d \ge 4$, $S_d \le 2d(d-2)$ [18]. Concerning lower bounds, $S_d \ge d(d-2) + d(d-2)$ 2 [20], but if $d \ge 7$ is an odd number then [8] $S_d \ge d(d-2) + 4$.

In the particular case of twistor lines these number can be improved as follows: any cubic surface contains at most 5 twistor lines [5], while it was observed in [21, arXiv version v1] that a smooth degree d projective surface contains at most d^2 twistor lines and that there exists a quartic containing exactly 8 twistor lines. In Theorem 1.4 case (3) we give a first general lower bound, for $d \geq 4$, on the maximal number of twistor lines T_d lying on a smooth degree d surface, obtaining $d^2/6 \simeq \nu_i(d) \leq T_d \leq d^2$. However, this bound is not optimal due to the quartic with 8 twistor lines in [21, arXiv version v1].

In the next remark we give a possible interpretation in terms of OCS's of our results.

Remark 1.8. Thanks to Theorems 1.5 and 1.6, for any general set of k points $P = \{p_1, \ldots, p_k\} \subset$ \mathbb{S}^4 , there is an infinite number of conformally inequivalent OCS's, induced by different smooth surfaces in \mathbb{CP}^3 , that are singular at P and cannot be extended at any point of P. Moreover, for any fixed degree d > 10, there are at least $\nu_i(d)$ different conformal classes of surfaces of degree d. Even if this fact was already known, our results give new information on the minimum degree d_{\min} of a smooth surface $\mathcal{S} \subset \mathbb{CP}^3$, such that it contains $\{\pi^{-1}(p_1), \ldots, \pi^{-1}(p_k)\}$ and no other twistor line.

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2. Preliminaries and Proof of Theorem 1.3

In this section, by means of some technical lemma, we prove Theorem 1.3. In this part of the paper the geometry of the twistor projection is not involved and all the ingredients come from basic geometric constructions (we use as main references the books [13, 14]). We recall the main tools.

For each closed subscheme $A \subset \mathbb{CP}^3$ let $\mathcal{I}_A \subseteq \mathcal{O}_{\mathbb{CP}^3}$ denote its *ideal sheaf*, i.e.: \mathcal{I}_A is defined to be the kernel of the morphism $\mathcal{O}_{\mathbb{CP}^3} \to i_*\mathcal{O}_A$, *i* being the inclusion map (see [14, p. 115]).

If $A \subset F$, let $\mathcal{I}_{A,F} \subseteq \mathcal{O}_F$ be the ideal sheaf of A in F. For any closed subscheme $A \subset \mathbb{CP}^3$ we denote by $\operatorname{Res}_F(A)$ the residual scheme of A with respect to F, i.e. the closed subscheme of \mathbb{CP}^3 with the conductor $\mathcal{I}_A : \mathcal{I}_F$ as its ideal sheaf. We always have $\operatorname{Res}_F(A) \subseteq A$ and if A is a reduced algebraic set, then $\operatorname{Res}_F(A)$ is the closure in \mathbb{CP}^3 of $A \setminus (A \cap F)$, i.e. the union of the irreducible components of A not contained in F. Assuming $\operatorname{deg}(F) = f$, for any $t \in \mathbb{Z}$ we have a natural exact sequence of coherent sheaves on \mathbb{CP}^3 :

(2)
$$0 \to \mathcal{I}_{\operatorname{Res}_F(A)}(t-f) \to \mathcal{I}_A(t) \to \mathcal{I}_{A\cap F,F}(t) \to 0.$$

For any scheme $A \subset F$ and every curve $C \subset F$ let $\operatorname{Res}_{C,F}(A)$ be the closed subscheme of F with $\mathcal{I}_{A,F} : \mathcal{I}_{C,F}$ as its ideal sheaf. If F is smooth, then C is an effective Cartier divisor of F. We get the following residual exact sequence of coherent sheaves on F:

(3)
$$0 \to \mathcal{I}_{\operatorname{Res}_{C,F}(A)}(t)(-C) \to \mathcal{I}_{A,F}(t) \to \mathcal{I}_{A\cap C,C}(t) \to 0.$$

If A is the disjoint union of two closed subschemes of \mathbb{CP}^3 , say $A = A_1 \sqcup A_2$ with each A_i closed in \mathbb{CP}^3 , then $\operatorname{Res}_F(A) = \operatorname{Res}_F(A_1) \cup \operatorname{Res}_F(A_2)$ and $F \cap A = (F \cap A_1) \cup (F \cap A_2)$. Since $\operatorname{Res}_F(A_i) \subseteq A_i$, we have $\operatorname{Res}_F(A_1) \cap \operatorname{Res}_F(A_2) = \emptyset$.

When dealing with singularities it is useful to exploit the so-called first infinitesimal neighborhood of a point (also called fat point). We recall some of its features. For any $q \in \mathbb{CP}^3$ let 2q be the first infinitesimal neighborhood of q in \mathbb{CP}^3 , i.e. the closed subscheme of \mathbb{CP}^3 with $(\mathcal{I}_q)^2$ as its ideal sheaf. The scheme 2q is a zero-dimensional scheme with reduced scheme $(2q)_{\text{red}} = \{q\}$ and if q is contained in an affine or projective *n*-dimensional space, then $\deg(2q) = \binom{2+n-1}{n}$ (and hence, for n = 3, 2, 1, we have $\deg(2q) = 4, 3, 2$, respectively). If $q \notin F$, we have $2q \cap F = \emptyset$ and $\operatorname{Res}_T(2q) = 2q$. If $q \in F$ let (2q, F) be the closed subscheme of F with $(\mathcal{I}_{q,F})^2$ as its ideal sheaf. We have that $(2q, F) = 2q \cap F$ (scheme-theoretic intersection) and the scheme (2q, F) is a zerodimensional subscheme of F with $(2q, F)_{\text{red}} = \{q\}$ and if F is smooth at q then $\deg((2q, F)) = 3$ and $\operatorname{Res}_F(2q) = \{q\}$.

Remark 2.1. Take now as F a plane $H \subset \mathbb{CP}^3$ such that $q \in H$ and as C a line $L \subset H$ such that $q \in L$ (and so with f = 1). Hence $\deg((2q, H) \cap L) = 2$ and so $\mathcal{I}_{(2q,H)\cap L,L}(t)$ is a line bundle on $L \cong \mathbb{CP}^1$ with degree t-2. The cohomology of line bundles on \mathbb{CP}^1 gives $h^1(L, \mathcal{I}_{(2q,H)\cap L,L}(t)) = 0$ for all t > 0. We have $\operatorname{Res}_{L,H}((2q, H)) = \{q\}$ and hence, recalling that $\mathcal{O}_H(t)(-L) = \mathcal{O}_H(t-1)$, the exact sequence (3), can be written as,

(4)
$$0 \to \mathcal{I}_{q,H}(t-1) \to \mathcal{I}_{(2q,H),H}(t) \to \mathcal{I}_{(2q,H)\cap L,L}(t) \to 0.$$

We have $h^1(H, \mathcal{I}_{q,H}(x)) = 0$ for all $x \ge 0$ (see [17, Corollary 3.13 pag 150]). Hence the long cohomology exact sequence of (4) gives $h^1(H, \mathcal{I}_{(2q,H),H}(t)) = 0$ for all t > 0.

Recall now that $\operatorname{Res}_H(2q) = \{q\}$ and $h^1(\mathcal{I}_q(t)) = 0$ for all $t \ge 0$. Thus the exact sequence (2) for F = H and f = 1 can be written as

$$0 \to \mathcal{I}_q(t-1) \to \mathcal{I}_{2q}(t) \to \mathcal{I}_{(2q,H),H}(t) \to 0$$

and gives $h^1(\mathcal{I}_{2q}(t)) = 0$ and $h^0(\mathcal{I}_{2q}(t)) = h^0(\mathcal{O}_{\mathbb{CP}^3}(t)) - \deg(2q) = \binom{t+3}{3} - 4$ for all t > 0. The projective space $|\mathcal{I}_{2q}(t)|$ parametrizes all degree t surfaces passing through q and singular at q (e.g. $|\mathcal{I}_{2q}(1)| = \emptyset$ and $|\mathcal{I}_{2q}(2)|$ is the set of all quadric cones with vertex containing q).

The proof of Theorem 1.3 relies on the computation of the first two cohomology numbers of $\mathcal{I}_E(t)$ and the analysis of such numbers in the presence of fat points. The following results will enable us to perform such analysis.

In the following proposition, given a finite set of points S in a plane H, we compute, for any $x \ge |S| - 2$ the number $h^1(H, \mathcal{I}_{S,H}(x))$. It turns out that this is zero for $x \ge |S| - 1$ while, for x = |S| - 2 is equal to 1 if and only if S is contained in a line.

Proposition 2.2. Let $H \subset \mathbb{CP}^3$ be a plane and $S \subset H$ a finite set. Set s := |S|.

- (1) We have $h^1(H, \mathcal{I}_{S,H}(t)) = 0$ for all $t \ge s 1$.
- (2) If $S \neq \emptyset$ and S is not contained in a line, then $h^1(H, \mathcal{I}_{S,H}(s-2)) = 0$.
- (3) If $S \neq \emptyset$ and there is a line $L \subset H$ such that $S \subset L$, then $h^1(H, \mathcal{I}_{S,H}(s-2)) = 1$.

Proof. Note that if $s \leq 2$, then the set S is always contained in a line (unique if and only if s = 2), while for $s \geq 3$ the set S is contained in at most one line.

The proposition is true if s = 0, because $h^1(H, \mathcal{O}_H(t)) = 0$ for all t. It is also true for s = 1, because $h^1(H, \mathcal{I}_{p,H}(t)) = 0$ and $h^1(H, \mathcal{I}_{p,H}(-1)) = s = 1$ for all $p \in H$ and all $t \geq 0$. Thus we may assume $s \geq 2$ and use induction on the integer s.

Assume that there is a line L such that $S \subset L$. We have an exact sequence

(5)
$$0 \to \mathcal{O}_H(s-3) \to \mathcal{I}_{S,H}(s-2) \to \mathcal{I}_{S,L}(s-2) \to 0.$$

Since |S| = s the sheaf $\mathcal{I}_{S,L}(s-2)$ is the degree -2 line bundle on $L \cong \mathbb{CP}^1$ and so $h^1(H, \mathcal{I}_{S,L}(s-2))$ 2)) = 1 (see [17, Corollary 3.13 pag 150]). We have $h^1(H, \mathcal{O}_H(s-3)) = 0$ and, since s = |S| > 0, then $h^1(L, \mathcal{I}_{S,L}(s-2)) = 1$ and $h^2(H, \mathcal{O}_H(s-3)) = 0$. To have that $h^1(H, \mathcal{I}_{S,H}(s-2)) = 1$, use the long cohomology exact sequence of (5).

We now pass to the case in which S is not contained in any line. Assume $s \ge 3$. Fix $p \in S$ and set $A := S \setminus \{p\}$. Fix a general line $L \subset H$ such that $p \in L$. We have $L \cap S = \{p\}$. Consider the residual exact sequence of L in H

(6)
$$0 \to \mathcal{I}_{A,H}(t-1) \to \mathcal{I}_{S,H}(t) \to \mathcal{I}_{p,L}(t) \to 0$$

Since $\mathcal{I}_{p,L}(t)$ is the degree t-1 line bundle on $L \cong \mathbb{CP}^1$, we have $h^1(L, \mathcal{I}_{p,L}(x)) = 0$ for all x > 0. Hence, if t > 0 and $h^1(\mathcal{I}_{A,H}(t-1)) = 0$, then the cohomology exact sequence of (6) gives $h^1(\mathcal{I}_{S,H}(t)) = 0$. By the inductive assumption this is always the case if either $t \geq s-1$ or t=s-2and A is not contained in a line. Therefore we may assume that A is contained in a line R. We have $\{p\} \in S \setminus S \cap R$ and hence we have a residual exact sequence

(7)
$$0 \to \mathcal{I}_{p,H}(s-3) \to \mathcal{I}_{S,H}(s-2) \to \mathcal{I}_{A,R}(s-2) \to 0.$$

Since $s-3 \ge 0$, we have $h^1(H, \mathcal{I}_{p,H}(s-3)) = 0$. Since $\mathcal{I}_{A,R}(s-2)$ is the degree -1 line bundle on $R \cong \mathbb{CP}^1$, we have $h^1(R, \mathcal{I}_{A,R}(s-2)) = 0$ (see [14, Example IV.1.3.5] or the Riemann-Roch formula [13, page 245] or [14, IV.1.3]). To conclude the proof it is sufficient to use the long cohomology exact sequence of (7).

The next two results are a consequence of Proposition 2.2 in the case in which a point of S is replaced with its first infinitesimal neighborhood. We need to recall the following remark.

Remark 2.3. Take any surface $F \subset \mathbb{CP}^3$ and any zero-dimensional scheme $Z \subset F$. Since Z is zero-dimensional, we have $h^i(Z, S) = 0$ for every i > 0 and every coherent sheaf S on Z. Thus the exact sequence

$$0 \to \mathcal{I}_{Z,F}(t) \to \mathcal{O}_F(t) \to \mathcal{O}_Z(t) \to 0$$

gives $h^2(F, \mathcal{I}_{Z,F}(t)) = h^2(F, \mathcal{O}_F(t))$. Moreover, we have $h^2(\mathcal{O}_F(t)) = 0$ if and only if $t \ge -\deg(F)$.

Lemma 2.4. Fix a plane $H \subset \mathbb{CP}^3$, $p \in H$ and a set $S \subset H \setminus \{p\}$. Set s := |S| and $Z := (2p, H) \cup S$. (1) We have $h^1(H, \mathcal{I}_{Z,H}(t)) = 0$ for all $t \ge s + 1$.

- (2) If $S \cup \{p\}$ is not contained in a line, then $h^1(H, \mathcal{I}_{Z,H}(s)) = 0$.
- (3) If there is a line containing $S \cup \{p\}$, then $h^1(H, \mathcal{I}_{Z,H}(s)) = 1$.

Proof. Since any two points of H are collinear, if $s \leq 1$, then $S \cup \{p\}$ is always contained in a line. First assume the existence of a line $R \subset H$ such that $S \cup \{p\} \subset R$. Note that $Z \cap R$ is the disjoint union of S and the degree 2 scheme (2p, R) with p as its support. Hence $\deg(Z \cap R) = s + 2$ and so $\mathcal{I}_{Z\cap R,R}(t)$ is a line bundle on $R \cong \mathbb{CP}^1$ with degree t-s-2. The cohomology of line bundles on \mathbb{CP}^1 gives $h^1(R, \mathcal{I}_{Z \cap R, R}(t)) = 0$ if $t \geq s+1$ and $h^1(R, \mathcal{I}_{Z \cap R, R}(t)) = 1$ otherwise. We have $\operatorname{Res}_{R,H}(Z) = \{p\}$ and hence a residual exact sequence

(8)
$$0 \to \mathcal{I}_{p,H}(t-1) \to \mathcal{I}_{Z,H}(t) \to \mathcal{I}_{Z\cap R,R}(t) \to 0.$$

We have $h^1(H, \mathcal{I}_{p,H}(x)) = 0$ for all $x \ge 0$ (see [17, Corollary 3.13 pag 150]). Remark 2.3 gives $h^{2}(H, \mathcal{I}_{p,H}(t-1)) = 0$. Hence the long cohomology exact sequence of (8) gives $h^{1}(H, \mathcal{I}_{Z,H}(t)) = 0$ for all $t \geq s+1$ and $h^1(H, \mathcal{I}_{Z,H}(s)) = 1$.

Now assume that $S \cup \{p\}$ is not contained in any line and take a line $L \subset \mathbb{CP}^3$ such that $p \in L$, L contains at least one point of S and with $|L \cap S|$ minimal among all lines through p intersecting S. Since there are at least 2 lines through p meeting S, we have $|L \cap S| \leq \lfloor s/2 \rfloor$. We get a residual exact sequence

(9)
$$0 \to \mathcal{I}_{\operatorname{Res}_{L,H}(Z)}(t-1) \to \mathcal{I}_{Z,H}(t) \to \mathcal{I}_{Z\cap L,L}(t) \to 0.$$

The scheme $\operatorname{Res}_{L,H}(Z)$ is the union of p and $S \setminus S \cap L$. Thus $\operatorname{Res}_{L,H}(Z)$ is a finite set with cardinality at most $\lfloor s/2 \rfloor + 1$. Proposition 2.2 gives $h^1(H, \mathcal{I}_{\operatorname{Res}_{L,H}(Z)}(t-1)) = 0$ for all $t \geq \lfloor s/2 \rfloor + 1$. Since $|S \cap L| \leq s - 1$, we have $\deg(L \cap Z) = 2 + |L \cap S| \leq s + 1$, the cohomology of line bundles on $R \cong \mathbb{CP}^1$ gives $h^1(R, \mathcal{I}_{Z \cap L,L}(t)) = 0$ for all $t \geq s$. To conclude the proof, use the long cohomology exact sequence of (9).

We now pass to the case of lines.

Lemma 2.5. Fix an integer k > 0. Let $E \subset \mathbb{CP}^3$ be a disjoint union of k lines. Fix $q \in \mathbb{CP}^3 \setminus E$. Then $h^1(\mathcal{I}_{\{q\}\cup E}(t)) = 0$ for all $t \geq k$.

Proof. Fix a line $L \subseteq E$ and set $A := E \setminus L$. Let H be the plane spanned by q and L. Since the lines of E are pairwise disjoints, H contains no line of A. Thus $S := A \cap H$ is a finite set of cardinality k - 1 and $\operatorname{Res}_H(\{q\} \cup E) = A$. We have $H \cap (E \cup \{q\}) = S \cup \{q\} \cup L$ (as schemes) and hence we have $h^1(H, \mathcal{I}_{E \cup \{q\}}) \cap H(t)) = h^1(H, \mathcal{I}_{S \cup \{q\} \cup L}(t)) = h^1(H, \mathcal{I}_{S \cup \{q\}}(t-1))$. By Proposition 2.2 we have $h^1(H, \mathcal{I}_{\{q\} \cup S}(x)) = 0$ for all $x \ge k - 1$. Thus $h^1(H, \mathcal{I}_{E \cup \{q\}}) \cap H(t) = 0$ for all $t \ge k$.

Consider now the following residual exact sequence

(10)
$$0 \to \mathcal{I}_A(t-1) \to \mathcal{I}_{\{q\} \cup E}(t) \to \mathcal{I}_{(\{q\} \cup E) \cap H}(t) \to 0.$$

Remark 1.2 gives $h^1(\mathcal{I}_A(x)) = 0$ for all $x \ge k - 2$, hence, thanks to the long cohomology exact sequence of (10) we get $h^1(\mathcal{I}_{\{q\}\cup E}(t)) = 0$ for all $t \ge k$.

The following two lemmas deal with the interplay between a set of lines E and a fat point. They will be used in the proof of Theorem 1.3 when dealing with singularities.

Lemma 2.6. Fix an integer k > 0. Let $E \subset \mathbb{CP}^3$ be a disjoint union of k lines. Fix $q \in \mathbb{CP}^3 \setminus E$.

- (1) We have $h^1(\mathcal{I}_{2q\cup E}(t)) = 0$ for all $t \ge k+1$.
- (2) If there is no line containing q and intersecting all lines of E, then $h^1(\mathcal{I}_{2q\cup E}(k)) = 0$.
- (3) If there is a line R containing q and intersecting all lines of E, then $h^1(\mathcal{I}_{2q\cup E}(k)) = 1$.

Proof. Fix a line $L \subseteq E$ and call H the plane spanned by $L \cup \{q\}$. Set $A := E \setminus L$ and $S := A \cap H$. We have $\operatorname{Res}_H(2q \cup E) = \{q\} \cup A$. By Lemma 2.4 and 2.5 we have $h^1(\mathcal{I}_{\{q\}\cup A}(t-1)) = 0$ for all $t \geq k$ and if t = k - 1 and there is no line R through q containing q and meeting all lines of E, while $h^1(\mathcal{I}_{\{q\}\cup A}(k-2)) = 1$ if there is such R. We have $H \cap (2q \cup E) = L \cup S \cup (2q, H)$ and hence $h^1(H, \mathcal{I}_{H \cap (2q \cup E)}(t)) = h^1(H, \mathcal{I}_{L \cup S \cup (2q, H), H}(t)) = h^1(H, \mathcal{I}_{S \cup (2q, H), H}(t-1))$. Since |S| = k - 1, Lemma 2.4 gives $h^1(H, \mathcal{I}_{S \cup (2q, H), H}(t-1)) = 0$ if either t > k or t = k and there is no line through q containing A and $h^1(H, \mathcal{I}_{S \cup (2q, H), H}(k-1)) = 1$ if there is a line $R \subset H$ containing $\{q\} \cup A$ and hence meeting each line of A; since $L, R \subset H$, we get $L \cap R \neq \emptyset$. Thus the residual exact sequence of H:

(11)
$$0 \to \mathcal{I}_{\{q\}\cup A}(t-1) \to \mathcal{I}_{2q\cup E}(t) \to \mathcal{I}_{L\cup(2q,H),H}(t) \to 0$$

gives the lemma, except that in the set-up of (3) it only gives $h^1(\mathcal{I}_{2q\cup E}(k)) \leq 1$. Assume then $h^1(\mathcal{I}_{2q\cup E}(k)) = 0$, i.e. assume $h^0(\mathcal{I}_{2q\cup E}(k)) = h^0(\mathcal{I}_{\{q\}\cup E}(k)) - 3$. Since $\deg(R \cap (\{q\} \cup E)) = k+1$, the line R is in the base locus of $|\mathcal{I}_{\{q\}\cup E}(k)|$. Thus the degree 2 scheme $R \cap 2q$ is in the base locus of $|\mathcal{I}_{\{q\}\cup E}(k)|$ and hence $h^0(\mathcal{I}_{2q\cup E}(k)) \geq h^0(\mathcal{I}_{\{q\}\cup E}(k)) - 2$.

We now examine the case in which $q \in E$.

Lemma 2.7. Fix an integer k > 0, a union $E \subset \mathbb{CP}^3$ of k disjoint lines and $q \in E$. Then $h^1(\mathcal{I}_{2q \cup E}(x)) = 0$ for all $x \geq k$.

Proof. Call L the line of E containing q and set $A := E \setminus L$. Let $H \subset \mathbb{CP}^3$ be a plane containing L. Set $S := A \cap H$. Since any two lines of H meet while the lines of E are disjoint, S is a finite set, |S| = k - 1 and $S \cap L = \emptyset$. Since $q \in H$, we have $\operatorname{Res}_H(2q) = \{q\}$. Since L is the only irreducible component of E contained in H, we have $\operatorname{Res}_H(E) = A$. It is easy to check (if you prefer use local

coordinates around q) that $\operatorname{Res}_A(2q \cup E) = \{q\} \cup A$. Since A is a union of k-1 disjoint lines, Lemma 2.5 gives $h^1(\mathcal{I}_{\operatorname{Res}_A(2q \cup E)}(t-1)) = 0$ for all $t \ge k$.

We have $H \cap (2q \cup E) = L \cup S \cup (2q, H)$. Note that $L \cap (2q, H) = (2q, L)$. Thus $\{q\}$ is the residual scheme $\operatorname{Res}_L(L \cup (2q, H))$ of H with respect to L (seen as a divisor in H). Thus

$$h^{1}(H, \mathcal{I}_{L\cup S\cup(2q,H)}(t)) = h^{1}(H, \mathcal{I}_{\{q\}\cup S}(t-1)).$$

Since $k = |\{q\} \cup S|$, Proposition 2.2 gives $h^1(H, \mathcal{I}_{\{q\} \cup S}(t-1)) = 0$ for all $t \ge k$. To conclude it is sufficient to look at the long cohomology exact sequence of the residual exact sequence (11) of H in \mathbb{CP}^3 .

We are now in position to prove our first main result, but first we recall the following fact.

Remark 2.8. The set Δ of all degree d surfaces of \mathbb{CP}^3 (allowing also the reducible ones and those with multiple components), is a projective space of dimension $\binom{d+3}{3} - 1$. The set of all singular $Y \in \Delta$ is a non-empty hypersurface Σ of Δ . Every complex closed subset of positive dimension of Δ meets Σ . Thus there is no compact complex family of positive dimension parametrizing different smooth degree d surfaces of \mathbb{CP}^3 . Allowing singularities (but not allowing decomposable surfaces), we may get complex positive-dimensional compact families of solutions.

Proof of Theorem 1.3. To prove Theorem 1.3 it is sufficient to prove that a general $Y \in |\mathcal{I}_E(d)|$ is smooth and contains no line L with $L \cap E = \emptyset$. We use several times that a finite intersection of non-empty Zariski open subsets of $|\mathcal{I}_E(d)|$ is non-empty and obviously open and hence Zariski-dense in $|\mathcal{I}_E(d)|$ to check separately the smoothness condition and the condition on lines.

Fix any line $L \subset \mathbb{CP}^3$ such that $L \cap E = \emptyset$. By Remark 1.2 for the integer s = k + 1 the set $|\mathcal{I}_{E \cup L}(d)|$ is a linear subspace of $|\mathcal{I}_E(d)|$ with codimension d + 1. Since the set Gr(2, 4) of all lines of \mathbb{CP}^3 has dimension 4, the set of all $Y \in |\mathcal{I}_E(d)|$ containing at least one line L with $L \cap E = \emptyset$ is contained in an algebraic subvariety of $|\mathcal{I}_E(d)|$ with codimension at least d + 1 - 4 > 0. Thus a general $Y \in |\mathcal{I}_E(d)|$ contains no line L with $L \cap E = \emptyset$.

We now deal with the smoothness property of a general $Y \in |\mathcal{I}_E(d)|$. For any $q \in \mathbb{CP}^3$ set $\Sigma_q := |\mathcal{I}_{2q \cup E}|$, i.e. Σ_q is the set of all $Y \in |\mathcal{I}_E(d)|$ singular at q. Thanks to Remark 2.8, it is sufficient to prove that $\cup_{q \in \mathbb{CP}^3} \Sigma_q$ is contained in a proper closed algebraic subvariety of $|\mathcal{I}_E(d)|$. We have dim E = 1 and, for each $q \in E$, Lemma 2.7 gives dim $\Sigma_q = \dim |\mathcal{I}_E(d)| - 2$. Thus $\cup_{q \in E} \Sigma_q$ is contained in a proper closed subvariety of $|\mathcal{I}_E(d)|$. Let Δ be the set of all $q \in \mathbb{CP}^3 \setminus E$ such that $h^1(\mathcal{I}_{2q \cup E}(d)) > 0$. If $q \in \mathbb{CP}^3 \setminus (\Delta \cup E)$, then dim $\Sigma_q = \dim |\mathcal{I}_E(d)| - 4$. Since dim $\mathbb{CP}^3 = 3$, the set $\cup_{q \in \mathbb{CP}^3 \setminus (\Delta \cup E)} \Sigma_q$ is contained in a proper closed subvariety of $|\mathcal{I}_E(d)| - 4$. Since dim $\mathbb{CP}^3 = 3$, the set $\cup_{q \in \mathbb{CP}^3 \setminus (\Delta \cup E)} \Sigma_q$ is contained in a proper closed subvariety of $|\mathcal{I}_E(d)| - 4$. Since dim $\mathbb{CP}^3 = 3$, the set $\cup_{q \in \mathbb{CP}^3 \setminus (\Delta \cup E)} \Sigma_q$ is contained in a proper closed subvariety of $|\mathcal{I}_E(d)| - 4$. Since dim $\mathbb{CP}^3 = 3$, the set $\cup_{q \in \mathbb{CP}^3 \setminus (\Delta \cup E)} \Sigma_q$ is contained in a proper closed subvariety of $|\mathcal{I}_E(d)| - 4$. Since dim $\mathbb{CP}^3 = 3$, the set $\cup_{q \in \mathbb{CP}^3 \setminus (\Delta \cup E)} \Sigma_q$ is contained in a proper closed subvariety of $|\mathcal{I}_E(d)| - 4$. Since dim $\mathbb{CP}^3 \setminus E$ of the union of the lines R meeting each line of E. Since dim $\Sigma_q = \dim |\mathcal{I}_E(d)| - 3$ for each $q \in \Delta$, it is sufficient to prove that dim $\Delta \leq 2$. Let Q be any quadric surface containing 3 of the lines of E. Bezout theorem implies that each line R meeting each line of E is contained in Q. Thus $\Delta \subseteq Q$ and so we get the thesis.

3. FINITELY MANY TWISTOR LINES AND PROOF OF THEOREMS 1.4, 1.5 AND 1.6

This section is devoted to the proof of Theorems 1.4, 1.5 and 1.6, therefore we start to deal with twistor lines. At first we prove the Density Lemma 3.2 which allows us to prove a number of results on surfaces containing twistor lines only using already known material. Then, we will use the map j to prove the more precise Theorem 1.6. We recall that the Grassmannian Gr(2, 4) can be identified with the Klein quadric $\mathcal{K} = \{t_1t_6 - t_2t_3 + t_4t_5 = 0\} \subset \mathbb{CP}^5$ via Plücker embedding. The map j induces, then, a map on \mathbb{CP}^5 (that will be also denoted by j) defined as

$$j([t_1:t_2:t_3:t_4:t_5:t_6]) = [\overline{t}_1:\overline{t}_5:-\overline{t}_4:-\overline{t}_3:\overline{t}_2:\overline{t}_6],$$

(see [1, 12]). This new map j identifies twistor lines in Λ and it is an anti-holomorphic involution.

Let k be a positive integer, then we have $\Lambda^k \subset Gr(2,4)^k$ and both sets are compact with their euclidean topology. $Gr(2,4)^k$ also has the Zariski topology, which obviously, if $k \geq 2$, is strictly finer than the product topology.

We now introduce a formalism that allows us to state simultaneously a finite number of properties that may be true or false for a Zariski open dense set of twistor fibers. In particular, thanks to this result we give meaning to the words "general sets of twistor lines". First of all fix an integer k > 0 and a property \wp which may be true or false for any set of k different twistor lines. If $k \ge 2$ we extend \wp to all $\mathbb{L} = (L_1, \ldots, L_k) \in \Lambda^k$ saying that \wp is false at \mathbb{L} if $L_m = L_n$ for some $m \ne n$, while, if $L_m \ne L_n$ for all $m \ne n$, \wp is true at \mathbb{L} if and only if it is true at $\{L_1, \ldots, L_k\}$.

Definition 3.1. Fix an integer k > 0 and a property \wp that is is either *true* or *false* for any k distinct twistor lines. We say that \wp is true for k general twistor lines or that k general twistor lines satisfy (the property) \wp or that \wp is true for a general union of k twistor lines if there is a non-empty Zariski open subset U of $Gr(2,4)^k$ such that \wp is true for all $\mathbb{L} = (L_1, \ldots, L_k) \in U \cap \Lambda^k$.

Note that the union of the big diagonals of $Gr(2,4)^k$, $k \ge 2$, is Zariski closed in $Gr(2,4)^k$ and it is different from $Gr(2,4)^k$. Thus in Definition 3.1 it is not restrictive to assume that all $(R_1,\ldots,R_k) \in U$ have $R_m \ne R_n$ for all $m \ne n$. This is the main reason for our extension of any \wp as false when the k twistor lines are not distinct.

Lemma 3.2 (Density Lemma). For any k > 0 the following statements hold true.

(a) Λ^k is dense in $Gr(2,4)^k$ in which the latter has the Zariski topology.

(b) Fix finitely many properties φ_i , $1 \leq i \leq s$, about k distinct twistor lines and which are true for k general twistor lines. Then there are k distinct twistor lines for which all φ_i , $1 \leq i \leq s$, are true.

Proof. Part (a) is true if k = 1 because, as said before, Λ is the set fixed by j and we recall that the fixed locus of an anti-holomorphic involution which acts on a connected smooth algebraic manifold M, is either empty or Zariski dense in M.

Now assume $k \geq 2$. Every non-empty Zariski open subset of $Gr(2,4)^k$ is dense in $Gr(2,4)^k$. The Zariski topology of $Gr(2,4)^k$ is finer than the product topology of its factors. Thus if U_i , $1 \leq i \leq s$, are non-empty Zariski open subsets of Gr(2,4), then $U_1 \times \cdots \times U_k$ is a non-empty and dense Zariski open subset of $Gr(2,4)^k$.

Since a finite intersection of non-empty Zariski open subsets of $Gr(2,4)^k$ is non-empty, open and dense in $Gr(2,4)^k$, we get the thesis of part (a) and, as a direct consequence, also part (b).

Thanks to the previous lemma, we are able to prove the following two results. The first is interesting by itself and states that for any number $s \ge 4$ there are s twistor lines such that no other line intersects all of them. In the second corollary we compute first two cohomology numbers of $\mathcal{I}_E(t)$, E being a general union of twistor lines.

Definition 3.3. Consider a set of k disjoint lines L_1, \ldots, L_k . These are said to be *collinear* if there is another line R intersecting all of them.

Thanks to [5, Proposition 2], if five lines are collinear, then they lie on a cubic surface and if five skew lines lie on a cubic surface, then they are collinear. In particular, five twistor lines that lie on a cubic surface are collinear.

Corollary 3.4. For any integer $s \ge 5$, there are s twistor lines which are not collinear.

Proof. Of course, it is sufficient to do the case s = 5. Take any 4 distinct twistor lines L_1, L_2, L_3, L_4 not all of them contained in a smooth quadric surface (e.g. take 4 of the twistor lines contained in the Fermat cubic (see [6]). There is a unique quadric surface T containing L_1, L_2, L_3 , this quadric surface is smooth and L_1, L_2, L_3 are in the same ruling of T, say $L_i \in |\mathcal{O}_T(1,0)|$, i = 1, 2, 3 (see [13, page 478]). By Bezout theorem each line $R \subset \mathbb{CP}^3$ intersecting each L_i , i = 1, 2, 3, is contained in T. Since any two lines in the same ruling of T are disjoint, R must be an element of $|\mathcal{O}_T(0,1)|$. Since $L_4 \notin T$, the set $L_4 \cap T$ has either cardinality 2 (case L_4 transversal to T) or cardinality 1 (case L_4 tangent to T). Set $e := |T \cap L_4|$ and let $D_i, 1 \leq i \leq e$ be the line in $|\mathcal{O}_T(0,1)|$ containing one of the points of $L_4 \cap T$. These e lines D_i are the only lines of \mathbb{CP}^3 meeting each of the lines L_1, L_2, L_3, L_4 . It is sufficient to prove the existence of a twistor line not meeting each of the e lines D_i . Let $\Gamma_i, 1 \leq i \leq e$, denote the set of all lines of \mathbb{CP}^3 meeting D_i . Each Γ_i is a hypersurface of Gr(2, 4) of degree 2, but then, to conclude, it is sufficient to invoke the Density Lemma.

Corollary 3.5. Fix an integer k > 0. Let $E \subset \mathbb{CP}^3$ be a general union of k twistor lines. Then $h^0(\mathcal{I}_E(t)) = \max\{0, \binom{t+3}{3} - k(t+1)\}$ and $h^1(\mathcal{I}_E(t)) = \max\{0, k(t+1) - \binom{t+3}{3}\}$ for all $t \in \mathbb{N}$.

Proof. Note that for a fixed t and any union F of k disjoint lines we have $h^0(\mathcal{I}_F(t)) = \max\{0, \binom{t+3}{3} - k(t+1)\} \Leftrightarrow h^1(\mathcal{I}_F(t)) = \max\{0, k(t+1) - \binom{t+3}{3}\} \Leftrightarrow h^0(\mathcal{I}_F(t)) \cdot h^1(\mathcal{I}_F(t)) = 0$. By Remark 1.2 we

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have $h^1(\mathcal{I}_F(t)) = 0$ for all $t \ge k-1$. Thus for a fixed k we only need to check finitely many integers t, i.e. $0 \le t \le k-2$. For each t the condition $h^0(\mathcal{I}_F(t)) \cdot h^1(\mathcal{I}_F(t)) = 0$ is an open condition for the Zariski topology of Gr(2,4) by the semicontinuity theorem for cohomology [14, III.12.8]. By [15, Theorem 0.1] or [10] these conditions are satisfied in a non-empty Zariski open subset of $Gr(2,4)^k$. Using the Density Lemma 3.2 we get the final thesis.

We now pass to the proof of the first part of Theorem 1.4.

Let $E \subset \mathbb{CP}^3$ be the union of $\nu(d)$ general lines. Notice that $\nu(d)$ is the maximal integer y such that $y(d+1) < \binom{d+3}{3}$. Therefore, by Corollary 3.5 we have $h^1(\mathcal{I}_E(d)) = 0$ and thanks to Theorem 1.3, dim $|\mathcal{I}_E(d)| = d$ if $d \equiv 0, 1 \mod 3$ and dim $|\mathcal{I}_E(d)| = (d-2)/3$ if $d \equiv 2 \mod 3$. In particular, if d = 2, then $\nu(d) = 3$ and dim $|\mathcal{I}_E(2)| = 0$. We start by proving the following general fact.

Proposition 3.6. Let $E \subset \mathbb{CP}^3$ be the union of $\nu(d)$ general twistor lines. If $d \ge 5$, then every $Y \in |\mathcal{I}_E(d)|$ is integral, i.e. it is reduced and irreducible. If $d \le 4$, then a general $Y \in |\mathcal{I}_E(d)|$ is integral.

Proof. Thanks to Corollary 3.5, we have that $h^0(\mathcal{I}_E(d-1)) = 0$ therefore, since $E \subset Y_{\text{red}}$, then each $Y \in |\mathcal{I}_E(d)|$ has no multiple component, i.e.: all Y are reduced.

Assume now that $|\mathcal{I}_E(d)| \ni Y = Y_1 \cup Y_2$ with Y_1 a surface of degree y, Y_2 a surface of degree d - y and 0 < y < d (we do not assume that some of the Y_i are irreducible). Let E_i , i = 1, 2, be the set of all lines $L \subset E \cap Y_i$. We have $E = E_1 \cup E_2$. Since E is a union of general twistor lines, every union of some of the lines of E is a union of general twistors lines. Moreover if $x_i := \deg(E_i)$, we have $x_1 + x_2 \ge \nu(d)$.

Set now $d \leq 4$ and recall that $\nu(1) = 1$, $\nu(2) = 3$, $\nu(3) = 4$, $\nu(4) = 6$. Thus (assuming $y \leq d-y$), we only have to check d = 3 and y = 1 or d = 4 and y = 2 = d - y. In both cases d - y = 2, hence, since dim $|\mathcal{I}_{E_2}(2)| = 0$ (and dim $|\mathcal{I}_L(1)| = 1$, for $L \in E$), then we have $|\mathcal{I}_{E_2}(2)| = \{Y_2\}$. But dim $|\mathcal{I}_E(3)| = 4$ and dim $|\mathcal{I}_E(4)| = 5$ and so a general $Y \in |\mathcal{I}_E(d)|$, d = 3, 4, is irreducible.

Now assume $d \ge 5$. By Corollary 3.5 we have

$$\binom{y+3}{3} > x_1(y+1), \qquad \binom{d-y+3}{3} > x_2(d-y+1),$$

that is $x_1 \leq \nu(y)$ and $x_2 \leq \nu(d-y)$. Thus $\nu(y) + \nu(d-y) \geq \nu(d)$. Recall from Equation (1), that, for all $t \geq 4$ we have

$$\frac{t^2 + 5t}{6} \le \nu(t) \le \frac{t^2 + 5t + 4}{6}.$$

Thus,

$$d^{2} + 5d \le y^{2} + 5y + 4 + (d - y)^{2} + 5(d - y) + 4,$$

that is $y(d-y) \le 4$. Since $1 \le y \le d-1$ and $d \ge 5$, the only possibility is d = 5 and either y = 1 or y = 4. But have $\nu(1) = 1$, $\nu(4) = 6$ and $\nu(5) = 9 > \nu(1) + \nu(4)$, obtaining then a contradiction. \Box

We can now perform the following proof.

Proof of Theorem 1.4 case (1): Consider a general set E of $\nu(d)$ twistor lines with $d \ge 4$. Thanks to Corollary 3.5 we have dim $|\mathcal{I}_E(d)| > 0$. Therefore, Proposition 3.6 implies that, for $d \ge 4$, there is a positive dimensional family of irreducible projective surfaces containing $\nu(d)$ lines. \Box

Before proving the second part of Theorem 1.4, we need a technical result. In particular, the following lemma will be used to control the singular locus of a generic surface $Y \in |\mathcal{I}_E(d)|$.

Remark 3.7. Fix integers $a \ge x \ge 0$. Let $S \subset \mathbb{CP}^1$ be a finite set with |S| = x. Since $h^1(\mathcal{O}_{\mathbb{CP}^1}(a-x)) = 0$ and the line bundle $\mathcal{O}_{\mathbb{CP}^1}(a-x)$ is base-point-free, S is the scheme-theoretic base locus of the linear system $|\mathcal{I}_S(a)|$ on \mathbb{CP}^1 .

Lemma 3.8. Fix integers d and x such that $d \ge 2$ and $0 \le x \le 1 + \frac{d(d-1)}{2}$. Let $S \subset \mathbb{CP}^2$ be a general finite subset of \mathbb{CP}^2 with |S| = x. Then $h^1(\mathcal{I}_S(d)) = 0$ and S is the scheme-theoretic base locus of $|\mathcal{I}_S(d)|$.

Proof. By the semicontinuity theorem for cohomology it is sufficient to find a finite set $A \subset \mathbb{CP}^2$ such that |A| = x, $h^1(\mathcal{I}_A(d)) = 0$ and A is the scheme-theoretic base locus of $|\mathcal{I}_A(d)|$. Fix d-1general lines R_1, \ldots, R_{d-1} , a point $p \in \mathbb{CP}^2 \setminus (R_1 \cup \cdots \cup R_{d-1})$, and sets $B_i \subset R_i$, $i = 1, \ldots, d-1$ with $|B_i| = i$ for all i and $B_i \cap R_j = \emptyset$ for all $j \neq i$. Set $B_0 := \{p\}$ and $B := \bigcup_{i=0}^{d-1} B_i$. We have $|B| = 1 + \sum_{i=1}^{d-1} i = 1 + (d-1)d/2$. Let $R_0 \subset \mathbb{CP}^2$ be a general line through p.

Claim 1: We have $h^1(\mathcal{I}_B(d)) = 0$.

Claim 2: B is the scheme-theoretic base locus of $|\mathcal{I}_B(d)|$.

To prove Claim 1 and Claim 2 we use induction on d. For both of them, the case d = 1 is trivial. *Proof of Claim 1:* For the inductive proof use the residual exact sequence

(12)
$$0 \to \mathcal{I}_{B \setminus B_{d-1}}(d-1) \to \mathcal{I}_B(d) \to \mathcal{I}_{B_{d-1},R_{d-1}}(d) \to 0.$$

Proof of Claim 2: The scheme-theoretic base locus \mathcal{B} of $|\mathcal{I}_B(d)|$ is contained in $R_0 \cup \cdots \cup R_{d-1}$. Remark 3.7 gives $\mathcal{B} \cap R_{d-1} = B_{d-1}$. By the inductive assumption $B \setminus B_{d-1}$ is the base locus \mathcal{B}' of $|\mathcal{I}_{B\setminus B_{d-1}}(d-1)|$ and $h^1(\mathcal{I}_{B\setminus B_{d-1}}(d-1)) = 0$. Take $q \in R_0 \cup \cdots \cup R_{d-1}$ such that $q \notin B$. If $q \in R_{d-1}$ we have $h^1(R_{d-1}, \mathcal{I}_{q \cup B_{d-1}, R_{d-1}}(d-1)) = 0$ [17, Corollary 3.13 pag 150]. Since $h^1(\mathcal{I}_{B \setminus B_{d-1}}(d-1)) = 0$ 0, the residual exact sequence of $B \cup \{q\}$ with respect to R_{d-1} gives $h^1(\mathcal{I}_{B \cup \{q\}}(d)) = 0$, i.e. $q \notin \mathcal{B}$. Now assume $q \in R_i$ for some $i \leq d-2$. The inductive assumption for Claim 2 gives $q \notin \mathcal{B}'$ and so $h^1(\mathcal{I}_{(B \setminus B_{d-1}) \cup \{q\}}(d-1)) = 0$. The residual exact sequence of $B \cup \{q\}$ with respect to R_{d-1} gives $h^1(\mathcal{I}_{B\cup\{q\}}(d)) = 0$. Thus $q \notin \mathcal{B}$. Thus B is the set-theoretic base locus of $|\mathcal{I}_B(d)|$. Since $\mathcal{B} \subset R_0 \cup \cdots \cup R_{d-1}$ and B is contained in the smooth locus of $R_0 \cup \cdots \cup R_{d-1}$, to prove Claim 2 it is sufficient to prove that for each $q \in B$, say $q \in R_i$, $h^1(\mathcal{I}_{(B \setminus \{q\}) \cup E}(d)) = 0$, where E is the degree 2 zero-dimensional subscheme of R_i with q as its reduction. First assume i = d - 1. We have $h^1(R_{d-1}, \mathcal{I}_{(B_{d-1} \setminus \{q\}) \cup E, R_{d-1}}(d)) = 0$, because $deg((B_{d-1} \setminus \{q\}) \cup E) = d$ [17, Corollary 3.13 pag 150]. Since $h^1(\mathcal{I}_{B \setminus B_{d-1}}(d-1)) = 0$, a residual exact sequence with respect to R_{d-1} like (12) with $(B \setminus \{q\}) \cup E$ instead of B gives $h^1(\mathcal{I}_{(B \setminus \{q\}) \cup E}(d)) = 0$. Now assume $i \leq d-2$. Since $B \setminus B_{d-1} = \mathcal{B}'$ as schemes, we have $h^1(\mathcal{I}_{((B \setminus B_{d-1}) \setminus \{q\}) \cup E}(d-1)) = 0$. Use the residual exact sequence of $(B \setminus \{q\}) \cup E$ with respect to R_{d-1}

If x = 1 + (d-1)d/2 we take A := B. If x < 1 + (d-1)d/2 Remark 3.7 and the proofs of Claims 1 and 2 show that we may take as A any subset of B with cardinality x.

Proof of Theorem 1.4 case (2): Let $E \subset \mathbb{CP}^3$ be a general union of $\nu_n(d)$ twistor lines. Let E' be a general union of E and $\nu(d) - \nu_n(d)$ twistor lines. By case (1) we have $|\mathcal{I}_{E'}(d)| \neq \emptyset$ and every element of $|\mathcal{I}_{E'}(d)|$ is integral. Thus $|\mathcal{I}_E(d)| \neq \emptyset$ and a general $Y \in |\mathcal{I}_E(d)|$ is irreducible. By Corollary 3.5 we have $h^1(\mathcal{I}_E(t)) = 0$ for $t \geq d-1$ and in particular dim $|\mathcal{I}_E(d)| = \binom{d+3}{3} - (d+1)\nu_n(d) - 1$. Hence the thesis follows from the following two claims.

Claim 1: A general $Y \in |\mathcal{I}_E(d)|$ is such that $E \cap Sing(Y) = \emptyset$.

Claim 2: A general $Y \in |\mathcal{I}_E(d)|$ is such that $\operatorname{Sing}(Y)$ is finite.

Proof of Claim 1: Since dim E = 1, to prove that a general $Y \in |\mathcal{I}_E(d)|$ is smooth at all points of E it is sufficient to prove that $h^0(\mathcal{I}_{2q\cup E}(d)) = h^0(\mathcal{I}_E(d)) - 2$ for all $q \in E$, i.e. we are going to prove that the space of surfaces singular at some point q of E has dimension strictly less than $|\mathcal{I}_E(d)|$ (recall Remark 2.1). Since $q \in E$, the scheme $2q \cap E$ has degree 2 and we have $h^0(\mathcal{I}_{2q\cup E}(d)) \leq h^0(\mathcal{I}_E(d)) - 2$. Hence to prove that the last inequality is an equality it is sufficient to prove that $h^1(\mathcal{I}_{2q\cup E}(d)) = 0$. Call L the connected component of E containing q. Set $A := E \setminus L$.

Let $H \subset \mathbb{CP}^3$ be a general plane containing L and set $S := A \cap H$. Since any two lines of H meet, S is the union of $\nu_n(d) - 1$ distinct points. As $\operatorname{Res}_H(2q) = \{q\}$ and $q \in L$, we have $\operatorname{Res}_H(2q \cup E) = \{q\} \cup A$. Moreover, since $L \cap A = \emptyset$, we have $H \cap (2q \cup E) = (2q, H) \cup L \cup S$. We have a residual exact sequence

(13)
$$0 \to \mathcal{I}_{\{q\}\cup A}(d-1) \to \mathcal{I}_{2q\cup E}(d) \to \mathcal{I}_{(2q,H)\cup L\cup S,H}(d) \to 0.$$

Since $h^1(\mathcal{I}_E(d-1)) = 0$, we have $h^1(\mathcal{I}_A(d-1)) = 0$ and L imposes d independent conditions to $|\mathcal{I}_A(d-1)|$. Since $L \cong \mathbb{CP}^1$ and $\mathcal{O}_L(d-1)$ is a degree d-1 line bundle, we get that any union of d points of L imposes d independent conditions to $|\mathcal{I}_A(d-1)|$. Hence dim $|\mathcal{I}_{\{q\}\cup A}(d-1)| = \dim |\mathcal{I}_A(d-1)| - 1$. Since $h^1(\mathcal{I}_A(d-1)) = 0$, we get $h^1(\mathcal{I}_{\{q\}\cup A}(d-1)) = 0$. By the long cohomology exact sequence of (13), to conclude the proof that a general $Y \in |\mathcal{I}_E(d)|$ is smooth at all points of E it is sufficient to prove that $h^1(H, \mathcal{I}_{(2q,H)\cup L\cup S,H}(d)) = 0$. We have $L \cap S = \emptyset$ and

 $L \cap (2q, H) = (2q, L)$ and so $\operatorname{Res}_{L,H}((2q, H) \cup S) = \{q\} \cup S$. Hence we have

$$h^{1}(H, \mathcal{I}_{(2q,H)\cup L\cup S,H}(d)) = h^{1}(H, \mathcal{I}_{\{q\}\cup S}(d-1)).$$

Since $q \in L$ and $d \geq 2$, we have

$$h^{1}(H, \mathcal{I}_{\{q\}\cup S}(d-1)) \leq h^{1}(H, \mathcal{I}_{L\cup S}(d-1)) = h^{1}(H, \mathcal{I}_{S,H}(d-2)).$$

The plane H depends only from L, not from the choice of q. Thus for a general E, we may see S as the intersection of H with $\nu_n(d) - 1 = \nu(d-1) - 1$ general twistor lines. Thus S has the same cohomological properties of a general subset of H with cardinality $\nu(d-1) - 1$. Since $\nu(d-1) - 1 \leq {d \choose 2}$, we have $h^1(H, \mathcal{I}_{S,H}(d-2)) = 0$ and so $h^1(\mathcal{I}_{2q\cup E}(d)) = 0$ and a general Y is smooth at E.

Proof of Claim 2: By Bertini's theorem [13, page 137], [14, II.8.18] and the generality of Y the set $\operatorname{Sing}(Y)$ is contained in the base locus \mathcal{B} of $|\mathcal{I}_E(d)|$. Let $B_1, \ldots, B_s, s \geq 0$, the positive dimensional components of \mathcal{B} which are not one of the lines of E. Fix a general plane $H \subset \mathbb{CP}^3$ and set $S := E \cap H$. Since E is a general union of $\nu_n(d)$ twistor lines and (for this part of the proof) we may first choose H independent from E, we may see S as a general subset of H with cardinality $\nu_n(d)$. By Lemma 3.8 for each $p \in H \setminus S$ we have $h^1(H, \mathcal{I}_{\{p\}\cup S, H}(d)) = 0$. Since we saw that $h^1(\mathcal{I}_E(d-1)) = 0$, the residual exact sequence of H for the sheaf $\mathcal{I}_{\{p\}\cup E}(d)$

$$0 \to \mathcal{I}_E(d-1) \to \mathcal{I}_{\{p\}\cup E}(d) \to \mathcal{I}_{\{p\}\cup A,H}(d) \to 0,$$

gives $h^1(\mathcal{I}_{E \cup \{p\}}(d)) = 0$ and so p is not in the base locus of $|\mathcal{I}_E(d)|$. Thus either s = 0 (and so $\operatorname{Sing}(Y)$ is finite) or each B_i is an irreducible curve meeting E. Since Y is smooth at each point of E, each B_i contains at most finitely many points of $\operatorname{Sing}(Y)$. Thus $\operatorname{Sing}(Y)$ is finite.

To prove the third part of Theorem 1.4, we need the following lemma.

Lemma 3.9. Let $T \subset \mathbb{CP}^3$ be any smooth quadric. Fix integers $b \ge a > 0$ and x with $0 \le x \le (a+1)(b-1)$. Let $S \subset T$ be a general union of x points. Then $h^1(T, \mathcal{I}_{S,T}(a, b)) = 0$ and S is the scheme-theoretic base locus of $|\mathcal{I}_{S,T}(a, b)|$.

Proof. Since S is general and $|S| = x = (a+1)(b-1) \leq (a+1)(b+1) = h^0(T, \mathcal{O}_T(a, b))$, we have $h^1(T, \mathcal{I}_{S,T}(a, b)) = 0$. It is then sufficient to find a finite set $A \subset \mathbb{CP}^2$ such that |A| = x, $h^1(\mathcal{I}_{A,T}(d)) = 0$ and A is the scheme-theoretic base locus of $|\mathcal{I}_{A,T}(d)|$. Take a+1 distinct elements R_1, \ldots, R_{a+1} of $|\mathcal{O}_T(1,0)|$ and any $B_i \subset R_i$ with $|B_i| = b-1$. Set $B := \bigcup_{i=1}^{a+1} B_i$. The divisor R_{a+1} of T gives the residual exact sequence on T:

(14)
$$0 \to \mathcal{I}_{B \setminus B_{a+1}}(a-1,b) \to \mathcal{I}_{B,T}(a,b) \to \mathcal{I}_{B_1,R_{a+1}}(a,b) \to 0$$

Note that $\mathcal{O}_{R_1}(a, b)$ is the degree *b* line bundle on $R_1 \cong \mathbb{CP}^1$. Following the proof of Lemma 3.8 we get Claims 1 and 2, i.e. we prove that we may take A = B when x = (a + 1)(b - 1). If x < (a + 1)(b - 1) the same inductive proof using (14) shows that we may take as A any subset of B with cardinality x.

In the last step of the proof of Theorem 1.4, we prove that $\nu_s(d)$ general twistor lines are contained in a smooth degree d surface of \mathbb{CP}^3 .

Proof of Theorem 1.4 case (3): Thanks to the results contained in [15] and the Density Lemma 3.2 we have $h^1(\mathcal{I}_E(d)) = 0$ for a general union of $\nu_s(d)$ lines. By Bertini's theorem it is sufficient to prove that $|\mathcal{I}_E(d)|$ has no base points outside E and that a general $Y \in |\mathcal{I}_E(d)|$ is smooth at all points of E. The latter condition is satisfied thanks to the Proof of Theorem 1.4 case (2), because $\nu_s(d) \leq \nu_n(d)$. We saw that it is sufficient to check separately these two Zariski open properties.

Since $\nu_s(d) \leq d$ for all $d \leq 7$, by Theorem 1.3 we could assume $d \geq 8$. However, we want to prove by induction on d the following statement:

Statement \star_d , $d \ge 0$: For any general union $F \subset \mathbb{CP}^3$ of $\nu_s(d)$ twistor lines the linear system $|\mathcal{I}_F(d)|$ has no base point outside F.

When $\nu_s(d) \leq d$ (and in particular if $d \leq 7$) we proved \star_d in the proof of Theorem 1.3 (the part studying dim Σ_q when $q \in \mathbb{CP}^3 \setminus E$).

Thus we may assume $d \ge 8$ and assume \star_{d-2} is true, i.e. assume that $|\mathcal{I}_G(d-2)|$ has no base points in $\mathbb{CP}^3 \setminus G$ for a general union G of $\nu_s(d-2)$ twistor lines. Let $T \subset \mathbb{CP}^3$ be a smooth jinvariant quadric in the sense of [21] (also called *real*), i.e. assume that one of the two rulings of T, say the ruling $|\mathcal{O}_T(1,0)|$, is formed by twistor lines. Let $M \subset T$ be any union of $x := \nu_s(d) - \nu_s(d-2)$ distinct elements of $|\mathcal{O}_T(1,0)|$ and take a general union $F \subset \mathbb{CP}^3$ of $\nu_s(d-2)$ twistor lines. By the inductive assumption $|\mathcal{I}_F(d-2)|$ has no base point outside F. Since any two different twistor lines are disjoint, we have $M \cap F := \emptyset$. Since we fixed F after fixing T, we may assume that each line of F is transversal to T. Thus $|F \cap T| = 2\nu_s(d-2)$. Note that $F \cap T \subset T \setminus M$. Set $E := F \cup M$. Since E is not a general union of $\nu_s(d)$ lines, we need to prove the following claim.

Claim: For the just defined set of lines E it holds $h^1(\mathcal{I}_E(d)) = 0$.

Proof of Claim: First of all, any two points of T not contained in a line of T are the intersection of T with a line of \mathbb{CP}^3 . Thus the union D of $\nu_s(d-2)$ general lines of \mathbb{CP}^3 has the property that $D \cap T$ is a general union of $2\nu_s(d-2)$ lines. Notice that $D \cap T \subset T \setminus M$ and that $2\nu_s(d-2) \leq (d+1)(d+1-x)$. Since $D \cap T$ is general, we get $h^1(T, \mathcal{I}_{D\cap T,T}(d-x,d)) = 0$. The residual exact sequence of M in T gives $h^1(T, \mathcal{I}_{(D\cap T)\cup M,T}(d,d)) = 0$. By the Density Lemma the same is true for a general union of $\nu_s(d-2)$ twistor lines, i.e. $h^1(T, \mathcal{I}_{(F\cap T)\cup M,T}(d,d)) = 0$. Since $h^1(\mathcal{I}_F(d-2)) = 0 = h^1(T, \mathcal{I}_{(F\cap T)\cup M,T}(d,d))$, to prove that $h^1(\mathcal{I}_E(d)) = 0$ it is sufficient to use the residual exact sequence

$$0 \to \mathcal{I}_F(d-2) \to \mathcal{I}_E(d) \to \mathcal{I}_{(F \cap T) \cup M, T}(d, d) \to 0.$$

Having proved the claim, we now check that $|\mathcal{I}_E(d)|$ has no base point outside E. Since $h^1(\mathcal{I}_E(d)) = 0$, it would be sufficient to prove that $\nu_s(d)$ twistor lines are contained in a smooth degree d surface. Since $|\mathcal{I}_F(d-2)|$ has no base point outside F, $|\mathcal{I}_E(d)|$ has no base point outside $F \cup T$. Thus it is sufficient to prove that $h^1(\mathcal{I}_{\{p\}\cup E}(d)) = 0$ for each $p \in T \setminus ((F \cap T) \cup M)$. Fix $p \in T \setminus ((F \cap T) \cup M)$. The residual exact sequence of T gives

(15)
$$0 \to \mathcal{I}_F(d-2) \to \mathcal{I}_{\{p\}\cup E}(d) \to \mathcal{I}_{(F\cap T)\cup M\cup\{p\},T}(d,d) \to 0.$$

Since $h^1(\mathcal{I}_F(d-2)) = 0$, the long cohomology exact sequence of (15) shows that it is sufficient to prove the vanishing of $h^1(T, \mathcal{I}_{(F\cap T)\cup M\cup \{p\},T}(d,d))$. Since $F\cap M = \emptyset$ and $p \notin M$, we have

$$h^{1}(T, \mathcal{I}_{(F\cap T)\cup M\cup \{p\}, T}(d, d)) = h^{1}(T, \mathcal{I}_{(F\cap T)\cup \{p\}, T}(d - x, d))$$

Note that $2\nu_s(d-2) \leq (d-x+1)(d-1)$. By Lemma 3.9 we have $h^1(T, \mathcal{I}_{(D\cap T)\cup\{p\},T}(d-x,d)) = 0$ for a general union of $\nu_s(d-2)$ lines of \mathbb{CP}^3 . By the Density Lemma 3.2 this is true for a general union of $\nu_s(d-2)$ twistor lines, i.e. $h^1(T, \mathcal{I}_{(F\cap T)\cup\{p\},T}(d-x,d)) = 0$. Therefore we have proved \star_d and hence the Theorem.

Having proved Theorem 1.4, we pass to Theorem 1.5.

Proof of Theorem 1.5: Fix an integer $k \leq \nu_j(d)$. Let $E \subset \mathbb{CP}^3$ be a general union of k twistor lines. The case k = 0 is trivial, because a general surface of degree ≥ 5 contains no line. Thus we may assume k > 0 and in particular $d \geq 10$. Let $E \subset \mathbb{CP}^3$ be a general union of k twistor lines. Since $k \leq \nu_s(d-6) \leq \nu(d-8)$, we have $h^1(\mathcal{I}_E(t)) = 0$ for all $t \geq d-8$. By Theorem 1.4 case (3) for each $t \geq d-8$ a general $W \in |\mathcal{I}_E(t)|$ is smooth. Thus we only need to prove that a general $Y \in |\mathcal{I}_E(d)|$ contains no line L with $L \cap E = \emptyset$. For any line $L \subset \mathbb{CP}^3$ set $A(L) := |\mathcal{I}_{E \cup L}(d)|$ and $a(L) := h^0(\mathcal{I}_E(d)) - h^0(\mathcal{I}_{E \cup L}(d))$. A(L) is a linear subspace of the projective space $|\mathcal{I}_E(d)|$ with codimension a(L) and it parametrizes all degree d surfaces containing $E \cup L$. By the semicontinuity theorem for cohomology [14, III.12.8] for any integer $i \geq 0$ the set $\omega(i)$ of all lines $L \subset \mathbb{CP}^3$ such that $L \cap E = \emptyset$ and a(L) = i is a locally closed subset of Gr(2, 4). We call τ_i its dimension, i.e. the maximal dimension of one of the irreducible components of $\omega(i)$. To prove that a general $Y \in |\mathcal{I}_E(d)|$ contains no line L with $L \cap E = \emptyset$, it is sufficient to prove that $\tau_i < i$ (i.e. $\omega(0) = \emptyset$, $\omega(1)$ is finite or empty, and so on). Since dim Gr(2, 4) = 4, it is sufficient to prove that $\tau_i < i$ for $0 \leq i \leq 4$. Note that $\nu_j(d) < \nu(d)$ and in particular $(d+1)(k+1) \leq {d+3 \choose 3}$. By Corollary 3.5 we have $h^1(\mathcal{I}_{E \cup L}(d)) = 0$ for a general $L \in Gr(2, 4)$, i.e. a(L) = d + 1 for a general $L \in Gr(2, 4)$. Thus $\tau_i < 3$ for all $i \leq d$. Thus it is sufficient to prove that $\tau_i < i$ for i = 0, 1, 2, 3.

Fix 4 distinct quadric surfaces Q_1 , Q_2 , Q_3 and Q_4 such that $Q_1 \cap Q_2 \cap Q_3 \cap Q_4 = \emptyset$. We fix these quadrics before fixing E. Let \mathcal{E} be the set of all unions F of k distinct twistor lines with $h^1(\mathcal{I}_F(t)) = 0$ for $t \ge d - 6$ and intersecting transversally each quadric Q_j , j = 1, 2, 3, 4. By the Density Lemma the set of all $F \cap Q_j$, $F \in \mathcal{E}$, is Zariski dense in the 2-dimensional variety $Q_j[k]$ parametrizing all subsets of Q_j with cardinality 2k. Note that $2k \leq 2\nu_j(d) \leq (d-5)^2 - 2(d-5)$. Again, by the Density Lemma for a general $E \in \mathcal{E}$ we have $h^1(Q_j, \mathcal{I}_{E \cap Q_j, Q_j}(a, b)) = 0$ for all $a \geq d-6, b \geq d-6, j = 1, 2, 3, 4$, and each linear system $|\mathcal{I}_{E \cap Q_j}(a, b)|$ has no base point. Fix a line $L \subset \mathbb{CP}^3$ such that $L \cap E = \emptyset$.

First assume that L is contained in one of the quadrics Q_1, Q_2, Q_3, Q_4 , say in Q_j as an element of $|\mathcal{O}_{Q_i}(0,1)|$. Since no line in E is contained in Q_i , we have the following residual exact sequence

(16)
$$0 \to \mathcal{I}_E(d-2) \to \mathcal{I}_{E \cup L}(d) \to \mathcal{I}_{E \cap Q_j, Q_j}(d) \to 0$$

Since $E \in \mathcal{E}$ we have $h^1(\mathcal{I}_E(d-2)) = 0$ and $h^1(Q_j, \mathcal{I}_{E \cap Q_j, Q_j}(d, d-1)) = 0$. Since we have $h^{1}(Q_{j}, \mathcal{I}_{E \cap Q_{j}, Q_{j}}(d)) = h^{1}(Q_{j}, \mathcal{I}_{E \cap Q_{j}, Q_{j}}(d, d-1)), (16) \text{ gives } L \in \omega(d+1).$

Now assume $L \nsubseteq Q_j$ for any j. In this case we want to prove that $h^0(\mathcal{I}_{E \cup L}(d)) \leq h^0(\mathcal{I}_E(d)) - 4$, i.e. $L \notin \omega(i)$ for $i \leq 3$. Fix $p_j \in L \cap Q_j$. To prove that $L \notin \omega(i)$ for $i \leq 3$ (and so that $\tau_i < i$ for $i \leq 3$ it is sufficient to prove that $h^0(\mathcal{I}_{E \cup \{p_1, p_2, p_3, p_4\}}(d)) = h^0(\mathcal{I}_E(d)) - 4$, i.e. that $h^1(\mathcal{I}_{E\cup\{p_1,p_2,p_3,p_4\}}(d)) = 0$. We prove that this is the case if p_j is not contained in Q_i when $i \neq j$. With this assumption we have the 4 residual exact sequences

(17)
$$0 \to \mathcal{I}_E(d-8) \to \mathcal{I}_{E \cup \{p_1\}}(d-6) \to \mathcal{I}_{(E \cap Q_1) \cup \{p_1\}, Q_1}(d-6) \to 0$$

(18)
$$0 \to \mathcal{I}_{E \cup \{p_1\}}(d-6) \to \mathcal{I}_{E \cup \{p_1, p_2\}}(d-4) \to \mathcal{I}_{(E \cap Q_2) \cup \{p_2\}, Q_3}(d-4) \to 0$$

(19)
$$0 \to \mathcal{I}_{E \cup \{p_1, p_2\}}(d-4) \to \mathcal{I}_{E \cup \{p_1, p_2, p_3\}}(d-2) \to \mathcal{I}_{(E \cap Q_3) \cup \{p_3\}, Q_3}(d-2) \to 0$$

(20)
$$0 \to \mathcal{I}_{E \cup \{p_1, p_2, p_3\}}(d-2) \to \mathcal{I}_{E \cup \{p_1, p_2, p_3, p_4\}}(d) \to \mathcal{I}_{(E \cap Q_4) \cup \{p_4\}, Q_4}(d) \to 0$$

We now use repeatedly Lemma 3.9 and its consequences.

- Since $h^1(\mathcal{I}_E(d-8)) = 0$ and $h^1(Q_1, \mathcal{I}_{(E\cap Q_1)\cup \{p_1\}, Q_1}(d-6)) = 0$, (17) gives $h^1(\mathcal{I}_{E\cup \{p_1\}}(d-6)) = 0$ (6)) = 0.
- Since we have $h^1(Q_2, \mathcal{I}_{(E \cap Q_2) \cup \{p_2\}, Q_3}(d-4)) = 0$, (18) gives $h^1(\mathcal{I}_{E \cup \{p_1, p_2\}}(d-4)) = 0$.
- Since $h^1(Q_3, \mathcal{I}_{(E \cap Q_3) \cup \{p_3\}, Q_3}(d-2)) = 0$, (19) gives $h^1(\mathcal{I}_{E \cup \{p_1, p_2, p_3\}}(d-2)) = 0$.
- Since we have $h^1(Q_4, \mathcal{I}_{(E \cap Q_4) \cup \{p_4\}, Q_4}(d)) = 0$, (20) gives $h^1(\mathcal{I}_{E \cup \{p_1, p_2, p_3, p_4\}}(d)) = 0$, concluding the proof in this case.

Assume now that $p_i \in Q_j$ for some $i \neq j$. First of all, to do the construction without any modification we only need $p_i \notin Q_j$ for all i < j (we do not care if Q_3 contains p_4 , if Q_2 contains p_3 or p_2 or if Q_1 contains p_2 or p_3 or p_4). Instead of fixing 4 smooth quadrics, we fix 10 smooth quadrics Q[h], $1 \le h \le 10$, such that any 4 of these quadrics have no common point. We set $Q_1 := Q[1]$ and fix $p_1 \in L \cap Q_1$. Since any 4 of the quadrics Q[h] have empty intersection, there is $Q[h_1]$ such that $p_1 \notin Q[h_1]$. We set $Q_2 := Q[h_1]$ and take $p_2 \in L \cap Q_2$. Since any 4 of the quadrics Q[h] have empty intersection, there is $Q[h_2]$ with $p_1 \notin Q[h_2]$ and $p_2 \notin Q[h_2]$. Set $Q_3 := Q[h_2]$ and fix $p_3 \in Q_3 \cap L$. Since any 4 of the quadrics Q[h] have empty intersection, there is $Q[h_3]$ with $\{p_1, p_2, p_3\} \cap Q[h_3] = \emptyset$. Set $Q_4 := Q[h_3]$ and take any $p_4 \in L \cap Q_4$. Using again the same argument as before for these new four quadrics, we conclude the proof. \square

We now pass to the last theorem. To precise what stated in the introduction we set the following notation. Let $z \in \mathbb{C}^4$. For any $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$, set

$$z^{\alpha} = z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$$

For any $\alpha \in \mathbb{N}^4$ we write $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. We now extend the map j to the $\binom{d+3}{3}$ dimensional vector space $H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$. Consider the map $j: H^0(\mathcal{O}_{\mathbb{CP}^3}(d)) \to H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$ defined in the following way

$$H^{0}(\mathcal{O}_{\mathbb{CP}^{3}}(d)) \ni f = \sum_{|\alpha|=d} c_{\alpha} z^{\alpha} \mapsto j(f) = \sum_{|\alpha|=d} \hat{c}_{\alpha} z^{\alpha},$$

where $\hat{c}_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = (-1)^{\alpha_0+\alpha_2} \overline{c}_{\alpha_1,\alpha_0,\alpha_3,\alpha_2}$. Note that j is \mathbb{R} -linear and that $j^2(f) = (-1)^d(f)$. For any $z \in \mathbb{C}^4$ we have $f(j(z)) = (-1)^d(f)$. $(-1)^d \overline{j(f)(z)}$. We say that $f = \sum_{\alpha} c_{\alpha} z^{\alpha}$ is j-invariant if and only if there is a constant $a \in \mathbb{C} \setminus \{0\}$ such that j(f) = af i.e.: if j fix the line $\mathbb{C}f$.

By the Density Lemma, the set of all elements of $H^0(\mathcal{O}_{\mathbb{CP}^3}(d))$ associated to a fixed $a \in \mathbb{C} \setminus \{0\}$ is a real vector space of dimension $\binom{d+3}{3}$ over \mathbb{R} . Hence j induces a bijection between the $\binom{d+3}{3} - 1$ complex projective space $|\mathcal{O}_{\mathbb{CP}^3}(d)|$ and itself.

Definition 3.10. We say that a surface $Y \in |\mathcal{O}_{\mathbb{CP}^3}(d)|$ is *j*-invariant if $j(z) \in Y$ for every $z \in Y$, i.e.: if j(Y) = Y.

In some paper (e.g. [21]), a *j*-invariant surface Y is called *real* with respect to *j*.

We pass now to the proof of Theorem 1.6. More precisely, for a general union $E \subset \mathbb{CP}^3$ of k twistor line we will prove the existence of a smooth $Y \in |\mathcal{I}_E(d)|$ such that j(Y) = Y and Y contains no line $L \subset \mathbb{CP}^3$ with $L \cap E = \emptyset$ and in particular it contains no other twistor line.

Proof of Theorem 1.6. By Corollary 3.5 we have $h^1(\mathcal{I}_E(d)) = 0$ and so $h^0(\mathcal{I}_E(d)) = \binom{d+3}{3} - k(d+1)$. The integer $\binom{d+3}{3} = (d+3)(d+2)(d+1)/6$ is even if and only if either d is odd or $d \equiv 2 \pmod{4}$. The integer d+1 is odd if and only if d is even, thus $\binom{d+3}{3} - k(d+1)$ is odd, i.e. dim $|\mathcal{I}_E(d)|$ is even, if and only if either $d \equiv 2 \pmod{4}$ and k is odd or $d \equiv 0 \pmod{4}$ and k is even, i.e. in the cases needed to prove Theorem 1.6.

Fix any union $E \subset \mathbb{CP}^3$ of finitely many twistor fibers. For each $t \in \mathbb{Z}$ the projective space $|\mathcal{I}_E(t)|$ is *j*-invariant.

Claim: If dim $|\mathcal{I}_E(t)|$ is even, then the set of all *j*-invariant element of $|\mathcal{I}_E(t)|$ is Zariski dense in $|\mathcal{I}_E(t)|$.

Proof of Claim: First assume $h^0(\mathcal{I}_E(t)) = 1$, i.e. assume $|\mathcal{I}_E(t)| = \{Y\}$ for some Y. Since j(E) = E, w have $\{j(Y)\} = \{Y\}$ and hence j(Y) = Y. Now assume $h^0(\mathcal{I}_E(t)) > 1$ and $h^0(\mathcal{I}_E(t))$ odd, say $h^0(\mathcal{I}_E(t)) = 2m + 1$ for some positive integer m. Fix a general set $S \subset \mathbb{CP}^3$ such that |S| = m. Since S is general, we have $h^0(\mathcal{I}_{E \cup S \cup j(S)}(t)) = 1$. Since $j(E \cup S \cup j(S)) = E \cup S \cup j(S)$, the proof just given shows that the only element $A \in |\mathcal{I}_{E \cup S \cup j(S)}(t)|$ satisfies j(Y) = Y. By our assumptions on d and k there is $Y \in |\mathcal{I}_E(d)|$ such that j(Y) = Y. Thus j induces an anti-holomorphic involution on the projective space $|\mathcal{I}_E(d)|$ with at least one fixed point. Thus the set Γ of all $Y \in |\mathcal{I}_E(d)|$ with j(Y) = Y is Zariski dense in $|\mathcal{I}_E(d)|$.

Hence (by the proof of Theorem 1.5) there is a smooth $Y \in \Gamma$ containing no line L with $L \cap E = \emptyset$.

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Altavilla Amedeo: Dipartimento Di Matematica, Università di Roma "Tor Vergata", Via Della Ricerca Scientifica 1, 00133, Roma, Italy

E-mail address: altavilla@mat.uniroma2.it

Edoardo Ballico: Dipartimento Di Matematica, Università di Trento, Via Sommarive 14, 38123, Povo, Trento, Italy

E-mail address: edoardo.ballico@unitn.it