Construction of a surface integral under local Malliavin assumption and integration by parts formulae

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In this paper, we consider convex sets $K_r = \{g \geq r\}$ in an infinite dimensional Hilbert space, where g is suitably related to a reference Gaussian measure μ in H. We first show how to define a surface measure on the level sets $\{g = r\}$ that is related to μ. This allows to introduce an integration-by-parts formula in H. This formula can be applied in several important constructions, as for instance the case where μ is the law of a (Gaussian) stochastic process and H is the space of its trajectories.

Keywords: Gaussian measures, surface measures in infinite dimensional spaces, integration-by-parts formulae, Neumann problem

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Contents

1. Introduction

Let H be a separable Hilbert space (with inner product denoted by $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$) endowed with a non degenerate centered Gaussian measure $\mu = N_Q$; we are given a continuous mapping $g : H \to \mathbb{R}$ and an open subset I of $\mathbb{R}$ such that $I \subset g(H)$.

Our aim is to construct surface measures defined on level sets $\{g = r\}$ of the mapping g under local Malliavin conditions on I, see Hypothesis 1 below, and provide several integration-by-parts formulae. Our construction extends previous results in the literature, and in particular we shall rely on the procedure recently introduced in [DaLuTu14]: under Hypothesis 1 below, for each $r \in I$ there exists a Borel measure σ_r in H, concentrated on the level surface $\{g = r\}$ of g such that for any $\varphi \in UC_b(H)$ we have

$$F_\varphi(r) := \int_{\{g=r\}} \varphi \, d\sigma_r = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon\le g\le r+\epsilon\}} \varphi \, d\mu, \quad r \in I. \tag{1.1}$$

We shall call σ_r the *surface measure* related to μ on $\{g = r\}$.

When σ_r exists, the measure $(\varphi\mu) \circ g^{-1}$ is absolutely continuous with respect to the Lebesgue measure λ restricted on I and possesses a continuous density which we denote by $\rho_\varphi(r)$, $r \in I$. Moreover, for any $r \in I$ there exists a version of $\mathbb{E}^\mu[\varphi|g = r]$ such that

$$\int_{\{g=r\}} \varphi \, d\sigma_r = \mathbb{E}^\mu[\varphi|g = r] \, \rho_1(r), \quad \forall\, r \in I. \tag{1.2}$$

As by-product, we prove the following integration by parts formula

$$\int_{\{g=r\}} \langle Mg, z\rangle \, d\sigma_r = -\int_{\{g\le r\}} (\langle M\varphi, z\rangle - \varphi\, \langle Q^{-1/2}x, z\rangle) \, d\mu, \tag{1.3}$$

under the assumptions $\varphi \in C_b^1(H)$, $g \in \mathbb{D}^{1,2}(H,\mu)$, $z \in H$, $\langle Mg, z\rangle \in UC_b(H) \cup \mathbb{D}^{1,2}(H,\mu)$ and $r \in I$, where M is the Malliavin derivative and $\mathbb{D}^{1,2}(H,\mu)$ its domain (see below for further details).

Our basic assumption is the following

Hypothesis 1. *There exist two random variables $u : H \to H$ and $\gamma : H \to \mathbb{R}$ (both depending on I) such that*

$$\langle Mg(x), u(x)\rangle = \gamma(x), \quad \forall\, x \in g^{-1}(I) \tag{1.4}$$

and

$$\frac{u}{\gamma} \in D(M_p^*) \quad \forall\, p \ge 1. \tag{1.5}$$

Notice that if $I = \mathbb{R}$, $u = Mg$ and $\gamma = |Mg|^2$, Hypothesis 1 reduces to the classical assumption from Airault–Malliavin [AiMa88], namely,

$$\frac{Mg}{|Mg|^2} \in D(M^*). \tag{1.6}$$

Several papers have been devoted to the construction of surface integrals under assumption (1.6) (requiring possibly some additional regularity on g), see for instance [AiMa88], [Bo98], [DaLuTu14] and the references therein.

It is well known, however, that (1.6) requires strong regularity for the level surfaces $\{g = r\}$ and it is not fulfilled for the function

$$g(x) = \inf_{t\in[0,1]} x(t), \quad x \in L^2(0,1),$$

which arises in studying reflection problems on the set of positive functions, see [NuPa92], [Za01].

The *local Malliavin condition* provided by Hypothesis 1 is not new, since it has already been introduced by D. Nualart, see [Nu06] (with other additional regularity assumptions both on u and on γ) but with a different purpose, namely for proving the existence of a density of some Gaussian random variables with respect to the Lebesgue measure. As we shall see, this is only a first step in constructing a surface measure.

In Section 3 we provide sufficient conditions on g ensuring Hypothesis 1. They require that g has the special form

$$g(x) = \inf_{t\in[0,1]} X(t)(x), \quad x \in H, \tag{1.7}$$

where $X(t)$, $t \in [0,1]$, is the solution of a stochastic differential equation on $\mathbb{R}$ with smooth coefficients. We shall need some results about the unique existence of a minimum of $g(x)$ μ–a.s. (Proposition 3.1) and a formula for the Malliavin derivative of g (Proposition 3.2) which are probably known. We presents the proofs, however, for the reader's convenience.

We are able to check Hypothesis 1 only in few particular situations: more precisely, when: i) X a real Brownian motion, ii) X is a distorted Brownian motion, iii) X is a geometric Brownian motion. We shall explain why we are not able so far to handle more general cases, see Remark 3.7.

In Section 4 we concentrate on integration by parts formulae in the set $L^2_+(0,1)$ of all nonnegative functions. In this case we provide more precise results than (1.3). The first results in this case where obtained by L. Zambotti, see [Za01]. See also [Ot09].

More precisely we consider a Gaussian process X in $[0,1]$ and its law $\mu = N_Q$ concentrated on $E = C([0,1])$ and set

$$g(x) := \min_{t\in[0,1]} X(t)(x), \quad x \in E$$

assuming that the law of g is absolutely continuous with respect to the Lebesgue measure on $(-\infty, 0]$ with a density ρ and that for μ–almost all $x \in E$, $g(x)$ attains the minimum at a unique point τ_x, so that for each $z \in E$ we can write $Dg(x)\cdot z := z(\tau_x)$, see Hypothesis 2 below. Under these conditions we prove the integration by parts formula, equation (4.7) below

$$\begin{aligned}\mathbb{E}\big[z(\tau_x)\varphi\big|g{=}r\big]\rho(r) \\ = -\int_{\{g\geq r\}} [D\varphi\cdot z - \langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\varphi]\,\mathrm{d}\mu,\ \forall\, r<0.\end{aligned}$$

We also study the limit case $r = 0$ and apply the obtained results when X is respectively: (i) a Brownian motion, (ii) a distorted Brownian motion, (iii) a Brownian Bridge. In case (i) and (iii) we recover by a different method the celebrate integration by parts formulae from [Za01] and [BoZa04].

Finally, we shall give some concluding remarks to some open problems, in particular to the Neumann problem on $\{g \geq r\}$ for the Kolmogorov operator

$$\mathscr{L}\varphi = \frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi] - \frac{1}{2}\,\langle Q^{-1}x, D\varphi\rangle.$$

We end this section with some notations. For a differentiable function $\varphi\colon H \mapsto \mathbb{R}$ we denote by D the gradient operator in H. By $C_b(H)$ (resp. $UC_b(H)$) we mean the space of all real continuous (resp. uniformly continuous) and bounded mappings $\varphi\colon H \to \mathbb{R}$ endowed with the sup norm $\|\cdot\|_\infty$.

Moreover, $C_b^1(H)$ is the subspace of $C_b(H)$ of all continuously differentiable functions, with bounded derivative.

It is well known that the operator $M := Q^{1/2}D$, defined in $C_b^1(H)$, is closable in $L^p(H,\mu)$ for all $p \in [1,+\infty)$; we denote by M_p its closure, by $\mathbb{D}^{1,p}(H,\mu)$ the domain of M_p and by M_p^* the adjoint of M_p. We shall call M_p the *Malliavin derivative* and M_p^* the *Skorokhod integral*. When no confusion may arise we shall omit the sub-index p.

2. Constructing a surface measure

2.1. Differentiability of F_φ.

We assume here Hypothesis 1 and set

$$F_\varphi(r) := \int_{\{g\le r\}} \varphi \, \mathrm{d}\mu, \quad \forall\, r \in I,\ \forall\, \varphi \in L^1(H,\mu). \tag{2.1}$$

We start by proving that when $\varphi \in \mathbb{D}^{1,p}(H,\mu)$ then $F_\varphi(\cdot)$ is continuously differentiable. This result is slightly different than a similar one from [Nu06]; we present a proof, however, both for reader's convenience and because the assumptions from [Nu06] are somewhat stronger.

Lemma 2.1. *Assume Hypotheses 1 and take $\varphi \in \mathbb{D}^{1,p}(H,\mu)$ for some $p \ge 1$. Then F_φ is continuously differentiable at any $r \in I$ and we have*

$$F'_\varphi(r) = \int_{\{g\ge r\}} M^*\left(\tfrac{u}{\gamma}\,\varphi\right) \mathrm{d}\mu, \tag{2.2}$$

Proof. Fix $\epsilon_0 < \mathrm{dist}(r, I^c)$ [(1)]; for every $\epsilon < \epsilon_0$ we write

$$\frac{1}{\epsilon}(F_\varphi(r+\epsilon) - F_\varphi(r)) = \frac{1}{\epsilon}\int_H \mathbb{1}_{\{r<g\le r+\epsilon\}}\,\varphi\, d\mu = \int_H h'_\epsilon(g)\,\varphi\,\mathrm{d}\mu,$$

where

$$h_\epsilon(z) = \frac{1}{\epsilon}\int_{-\infty}^{z} \mathbb{1}_{[r,r+\epsilon]}(s)\,\mathrm{d}s.$$

Since h_ϵ is Lipschitz continuous, we can apply the chain rule (formally, but the result can be obtained by approximating h_ϵ with functions in $C^1(\mathbb{R})$)

$$\varphi\, M(h_\epsilon(g)) = \varphi\, h'_\epsilon(g) Mg. \tag{2.3}$$

Then, multiplying both sides of (2.3) scalarly by u it follows that

$$\varphi(x)\,\langle((M h_\epsilon(g))(x), u(x)\rangle = \varphi(x)\, h'_\epsilon(g(x))\langle(Mg)(x), u(x)\rangle, \quad \forall\, x \in H. \tag{2.4}$$

For $x \in g^{-1}(I)$, the right-hand side of previous equation is equal to

$$\varphi(x)\, h'_\epsilon(g(x))\,\gamma(x) \tag{2.5}$$

thanks to Hypothesis 1, while it is equal to 0 for every $x \notin g^{-1}(I)$, by the definition of h'_ϵ; however, since clearly also (2.5) vanishes for $x \notin g^{-1}(I)$, we can use it on the whole H to get

$$\varphi(x)\, h'_\epsilon(g(x)) = \varphi(x)\,\langle(M(h_\epsilon(g))(x), \tfrac{u(x)}{\gamma(x)}\rangle, \quad \forall x \in H,\ \gamma(x) \ne 0. \tag{2.6}$$

Integrating with respect to μ over H, yields

$$\int_H \varphi(x)\, h'_\epsilon(g(x))\,\mu(dx) = \int_H \varphi(x)\,\langle M h_\epsilon(g(x)), \tfrac{u(x)}{\gamma(x)}\rangle\,\mu(dx). \tag{2.7}$$

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(1) By I^c we denote the complement of I.

Notice that $\varphi \in \mathbb{D}^{1,p}(H)$ by assumption, while $\frac{u}{\gamma} \in D(M_q^*)$ for any $q \geq 1$ by Hypothesis 1, therefore, Hölder's inequality implies that $\frac{u}{\gamma}\,\varphi \in D(M_q^*)$ for any $q > p$. So, by duality and using Fubini's theorem, we have

$$\begin{aligned}
\frac{1}{\epsilon}(F_\varphi(r+\epsilon) - F_\varphi(r)) &= \int_H \varphi(x)\, h_\epsilon'(g(x))\, \mu(dx) \\
&= \int_H \varphi(x)\, \langle M h_\epsilon(g(x)), \tfrac{u(x)}{\gamma(x)}\rangle\, \mu(dx) \\
&= \int_H M^* \left(\varphi(x)\, \tfrac{u(x)}{\gamma(x)}\right)\, h_\epsilon(g(x))\, \mu(dx) \\
&= \int_H \frac{1}{\epsilon} \int_{\mathbb{R}} M^* \left(\varphi(x)\, \tfrac{u(x)}{\gamma(x)}\right)\, \mathbb{1}_{[r,r+\epsilon]}(s) \mathbb{1}_{\{g(x)\geq s\}}\, ds\, \mu(dx) \\
&= \frac{1}{\epsilon} \int_r^{r+\epsilon} \int_H M^* \left(\varphi(x)\, \tfrac{u(x)}{\gamma(x)}\right)\, \mathbb{1}_{\{g(x)\geq s\}}\, \mu(dx)\, \mathrm{d}s. \qquad (2.8)
\end{aligned}$$

If we could apply the integral mean theorem to the right-hand side of (2.8), then letting $\epsilon \to 0$ we obtain (2.2). This requires a little work.
Let $\varphi = 1$ and notice that (2.8) implies the continuity of the mapping

$$s \mapsto F_1(s), \qquad s \in [r, r+\epsilon];$$

however, by definition (2.1), $F_1(s) = \mu\{g \leq s\}$, which implies the continuity of the integrand function in the right-hand side of (2.8) and, therefore, the proof is concluded.
By a similar argument, by using again Hölder's inequality, we can further prove continuity and even hölderianity of $F_\varphi'(\cdot)$. □

Let us introduce the following notation. For $\varphi \in \mathbb{D}^{1,p}(H,\mu)$, $p \geq 1$, we set

$$\rho_\varphi(r) = F_\varphi'(r) = \int_{\{g \geq r\}} M^* \left(\tfrac{u}{\gamma}\,\varphi\right)\, \mathrm{d}\mu, \qquad r \in I.$$

Remark 2.2. By Lemma 2.1 it follows that for all $\varphi \in \mathbb{D}^{1,p}(H,\mu)$ the measure $(\varphi\mu) \circ g^{-1}$ is absolutely continuous with respect to the Lebesgue measure in I having a continuous density ρ_φ.

Notice that, in particular,

$$\rho_1(r) = \int_{\{g \geq r\}} M^* \left(\tfrac{u}{\gamma}\right)\, \mathrm{d}\mu, \qquad r \in I.$$

By (2.2) it follows that

Corollary 2.3. *Let $\varphi \in \mathbb{D}^{1,p}(H,\mu)$, $r \in I$. Then there exists a constant $K > 0$ and $q > p$ such that*

$$|\rho_\varphi(r)| \leq K \|\varphi\|_{\mathbb{D}^{1,q}(H,\mu)}. \qquad (2.9)$$

The following lemma will be useful later.

Lemma 2.4. *Assume, besides Hypothesis 1, that $\varphi \in \mathbb{D}^{1,p}(H,\mu)$. Then there is a continuous version of the function*

$$\mathbb{E}[\varphi | g = r]\, \rho_1(r), \quad r \in I$$

such that

$$\rho_\varphi(r) = \mathbb{E}[\varphi | g = r]\, \rho_1(r), \quad \forall\, r \in I. \qquad (2.10)$$

If in addition $\varphi \in UC_b(H)$ we have

$$|\rho_\varphi(r)| \le \|\varphi\|_\infty \, \rho_1(r), \qquad \forall r \in I, \ \forall \varphi \in \mathbb{D}^{1,2}(H,\mu). \tag{2.11}$$

Proof. Write for $r \in I$

$$\begin{aligned}\rho_\varphi(r) &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_H \mathbb{1}_{\{r-\epsilon \le g \le r+\epsilon\}} \, \varphi \, \mathrm{d}\mu \\ &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_H \mathbb{1}_{\{r-\epsilon \le g \le r+\epsilon\}} \, \mathbb{E}[\varphi|g] \, \mathrm{d}\mu \\ &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{r-\epsilon}^{r+\epsilon} \mathbb{E}[\varphi|g=s] \, \rho_1(s) ds = \mathbb{E}[\varphi|g=r] \, \rho_1(r), \quad r\text{–a.s. in } I.\end{aligned} \tag{2.12}$$

Since $\rho_\varphi(\cdot)$ is continuous, the first statement of the Lemma follow from Lemma 2.1.
To show (2.11) write

$$|\mathbb{E}[\varphi|g] \le \mathbb{E}[|\varphi||g] \le \|\varphi\|_\infty.$$

Therefore the conclusion follows. □

2.2. **The surface measure.** Now we are going to prove the main result of this section, namely that the positive functional $r \to \rho_\varphi(r)$ is in fact the integral of φ with respect to a given (surface) measure σ_r. The key point is to extend the functional to all $\varphi \in UC_b(H)$ as the following proposition shows.

Proposition 2.5. *For any $\varphi \in UC_b(H)$, F_φ is continuously differentiable on I.*

Proof. If $\varphi \in \mathbb{D}^{1,p}(H,\mu)$ the result follows from Lemma 2.1. Now let $\varphi \in UC_b(H)$ and let (φ_n) be a sequence in $UC^1_b(H)$ convergent to φ in $UC_b(H)$. Since $\varphi_n \in D^{1,p}(H,\mu)$ by (2.11) we have

$$|\rho_{\varphi_n}(r) - \rho_{\varphi_m}(r)| = |\rho_{\varphi_n - \varphi_m}(r)| \le \|\varphi_n - \varphi_m\|_\infty \, \rho_1(r), \quad \forall \, r \in I,$$

hence $\{\rho_{\varphi_n}\}$ is a Cauchy sequence in $UC_b(H)$ and the conclusion follows.

□

Remark 2.6. Assume that $M^*(\frac{u}{\gamma}) \in UC_b(H) \cup \mathbb{D}^{1,2}(H,\mu)$. Then F_φ is twice continuously differentiable on I.

Now we we can prove

Theorem 2.7. *Assume Hypothesis 1. Then for every $r \in I$ there exists a unique Borel measure σ_r on H such that*

$$F'_\varphi(r) = \rho_\varphi(r) = \int_H \varphi(x) \, \sigma_r(dx), \quad \forall \, \varphi \in UC_b(H). \tag{2.13}$$

Moreover, $\sigma_r(H) = \rho_1(r)$ and if g is continuous the support of σ_r is included in $\{g = r\}$.

Proof. The proof is similar to that of [DaLuTu14, Theorem 3.5], so it will be only sketched. Let us fix $r \in I$. By Proposition 2.5 the functional

$$UC_b(H) \to \mathbb{R}, \quad \varphi \to \rho_r(\varphi),$$

is well defined and clearly positive. To show that it is a measure we follow a classical method. First we construct a suitable increasing sequence (K_n) of compact sets of H converging to H. Then we introduce the restrictions $\rho^n_\varphi(r)$ of $\rho_\varphi(r)$ to K_n for all $\varphi \ge 0$ setting

$$\rho^n_\varphi(r) = \inf \Big\{ \rho_\psi(r) : \ \psi \in UC_b(H), \ \psi = \varphi \text{ on } K_n, \ \ \psi \ge 0 \text{ on } H \Big\},$$

while if φ takes both positive and negative values, $\rho^n_\varphi(r)$ is defined by

$$\rho^n_\varphi(r) = (\rho^n_\varphi(r))^+ - (\rho^n_\varphi(r))^-,$$

where φ^+ and φ^- denote the positive and the negative part of φ. Then $\rho_\varphi(r)$ is a positive linear functional in $C(K_n)$ as easily checked. Now for each $r \in I$ $\varphi \to \rho^n_\varphi(r)$ are measures in view of the Riesz representation theorem. Finally, it is not difficult to show that $\rho^n_1(r) \uparrow \rho_1(r)$, which implies that $\rho_\varphi(r)$ is a measure as well. □

In the applications it is important to know whether a Borel function $\varphi : H \to \mathbb{R}$ (not necessarily belonging to $UC_b(H)$) has a *trace* on the surface $\{g = r\}$ for some $r \in I$. When the Malliavin condition (1.6) is fulfilled this problem was investigated in [CeLu14], see also [DaLuTu14]. Given $r \in I$, we shall say that φ possesses a trace $T\varphi$ on $\{g = r\}$ if there exists a sequence $(\varphi_n) \subset UC_b(H)$ such that

$$\varphi_n \to T\varphi, \quad \text{on } L^1(H, \mathscr{B}(H), \sigma_r).$$

Proposition 2.8. *Let $\varphi \in \mathbb{D}^{1,2}(H,\mu)$ and let $(\varphi_n) \subset C^1_b(H)$ be a sequence convergent to φ in $\mathbb{D}^{1,2}(H,\mu)$. Then (φ_n) is Cauchy in $L^1(H,\sigma_r)$, so that φ possesses a trace $T\varphi$ on $\{g = r\}$.*

Proof. We first notice that by passing if necessary to an approximating sequence in $UC^1_b(H)$, and using Lemma 2.4 it follows that

$$F'_\varphi(r) = \int_{\{g=r\}} \varphi \, \mathrm{d}\sigma_r = \mathbb{E}[\varphi | g = r] \, \rho_1(r), \quad \forall \, \varphi \in UC_b(H), \ r \in I. \tag{2.14}$$

Let now $\varphi \in \mathbb{D}^{1,2}(H,\mu)$ and let $(\varphi_n) \subset C^1_b(H)$ be a sequence convergent to φ in $\mathbb{D}^{1,2}(H,\mu)$. We claim that (φ_n) is Cauchy in $L^1(H,\sigma_r)$. In fact

$$\int_H |\varphi_n - \varphi_m| \, \mathrm{d}\sigma_r = \rho_{|\varphi_n - \varphi_m|}(r) \le K(r) \, \|\varphi_n - \varphi_m\|_{\mathbb{D}^{1,2}(H,\mu)},$$

thanks to Corollary 2.3. □

3. Fulfilling Hypothesis 1

3.1. **The maximum of a stochastic flow.** Let us start with some general results concerning a stochastic differential equation on $\mathbb{R}$

$$\mathrm{d}X = b(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}B(t), \quad X(0) = \xi \in \mathbb{R}, \tag{3.1}$$

where b and σ are of class C^2 and Lipschitz continuous and $B(\cdot)$ is a Brownian motion on $(H, \mathscr{B}(H), \mu)$. We denote by $X(t,\xi) = X(t)$ the strong solution of (3.1).

We are going to consider the function on H

$$g(x) = \sup_{t \in [0,1]} X(t)(x), \quad x \in H. \tag{3.2}$$

Since the trajectories of $X(t)$ are μ–a.s. continuous the supremum in (3.2) is indeed a maximum μ–a.s.

Proposition 3.1. *Let X be the solution to* (3.1) *and assume that the joint probability distributions of any order admit a density with respect to the Lebesgue measure. Then $X(t)(x)$, $t \in [0,1]$, attains the maximum at a unique point τ_x of $[0,1]$ for μ–almost $x \in H$.*

Proof. We proceed by steps. First, we consider two time points $0 \le s < t$: we have that

$$\mu(X(s) = X(t)) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{x=y\}} f_{(X(t),X(s))}(x,y) \,\mathrm{d}x \,\mathrm{d}y = 0,$$

where $f_{(X(t),X(s))}$ is the joint density of $(X(t),\, X(s))$.
Next, we extend the analysis to three times. Consider $s_1 < s_2 < t$ and the corresponding random variables $g_{(s)} = \max\{X(s_i), i = 1,2\}$ and X_t, we have

$$\begin{aligned}\mu(g_{(s)} \le \xi, X(t) \le \eta) =& \mu(X(s_1) \le \xi, X(s_2) \le \xi, X(t) \le \eta) \\ =& \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} f(x_1, x_2, y) \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}y,\end{aligned}$$

that has a density given by

$$\int_{-\infty}^{\xi} f(x_1, \xi, \eta) \,\mathrm{d}x_1 + \int_{-\infty}^{\xi} f(\xi, x_2, \eta) \,\mathrm{d}x_2.$$

Considering now $s_1 < s_2 < t_1 < t_2$ and the corresponding random variables $g_{(s)} = \max\{X(s_i), i = 1,2\}$ and $g_{(t)} = \max\{X(t_i), i = 1,2\}$, we have

$$\begin{aligned}\mu(g_{(s)} \le \xi, g_{(t)} \le \eta) &= (X(s_1) \le \xi, X(s_2) \le \xi, X(t_1) \le \eta), X(t_2) \le \eta) \\ &= \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \int_{-\infty}^{\eta} f(x_1, x_2, y_1, y_2) \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}y_1 \,\mathrm{d}y_2\end{aligned}$$

that has a density given by

$$\begin{aligned}&\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} f(x_1, \xi, y_1, \eta) \,\mathrm{d}x_1 \,\mathrm{d}y_1 + \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} f(x_1, \xi, \eta, y_2) \,\mathrm{d}x_1 \,\mathrm{d}y_2 \\ &\qquad + \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} f(\xi, x_2, y_1, \eta) \,\mathrm{d}x_2 \,\mathrm{d}y_1 + \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} f(\xi, x_2, \eta, y_2) \,\mathrm{d}x_2 \,\mathrm{d}y_2.\end{aligned}$$

Let us consider a system of points

$$s_1 < s_2 < \dots < s_m < t_1 < t_2 < \dots < t_n$$

and define $g_{(s)} = \max\{X(s_i), i = 1, 2, \dots, m\}$ and $g_{(t)} = \max\{X(t_i), i = 1, 2, \dots, n\}$: we claim that

$$\mu(g_{(s)} = g_{(t)}) = 0. \tag{3.3}$$

In fact, the joint probability distribution of $(g_{(s)}, g_{(t)})$ (using the cumulative distribution function) given by

$$\begin{aligned}&\mu(g_{(s)} \le \xi, g_{(t)} \le \eta) = \\ &\mu(X(s_1) \le \xi, X(s_2) \le \xi, \dots, X(s_m) \le \xi, X(t_1) \le \eta, X(t_2) \le \eta, \dots, X(t_n) \le \eta) \\ &= \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \cdots \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \int_{-\infty}^{\eta} \cdots \int_{-\infty}^{\eta} f(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y},\end{aligned}$$

admits a density with respect to the Lebesgue measure given by

$$\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} \Big[\sum_{i,j} f(\mathbf{x}_i, \mathbf{y}_j) \Big] \, dx \, dy$$

where

$$\mathbf{x}_i = (x_1, x_2, \dots, x_{i-1}, x, x_{i+1}, \dots, x_m)$$

and analogously

$$\mathbf{y}_j = (y_1, y_2, \dots, y_{j-1}, y, y_{j+1}, \dots, y_n),$$

from which we get (3.3).
Now, given two disjoint intervals $I = [a_1, b_1]$ and $J = [a_2, b_2]$ (with $b_1 < a_2$) in $[0, T]$, we get, by the continuity of paths,

$$\mu(\max_{s \in I} X(s) = \max_{t \in J} X(t)) = \mu(\sup_{s \in \mathbb{Q} \cap I} X(s) = \sup_{t \in \mathbb{Q} \cap J} X(t))$$

where $\mathbb{Q}$ is the set of rational numbers. On the other hand let $\mathbb{Q}_n$ be an increasing sequence of finite number of rationals such that $\bigcup \mathbb{Q}_n = \mathbb{Q}$; we have

$$\mu\left(\sup_{s \in \mathbb{Q} \cap I} X(s) = \sup_{t \in \mathbb{Q} \cap J} X(t)\right) \le \mu\left(\sup_{s \in \mathbb{Q}_n \cap I} X(s) = \sup_{t \in \mathbb{Q}_n \cap J} X(t)\right) = 0.$$

Therefore, the proposition is proved. □

Let us compute the Malliavin derivative of g.

Proposition 3.2. *Function g defined by* (3.2) *belongs to* $\mathbb{D}^{1,2}(H, \mu)$ *and it results*

$$Mg(x) = MX(t)\Big|_{t=\tau_x} \qquad \mu\text{–a.s. in } H. \tag{3.4}$$

Proof. The first part follows by the general criterium stated in [Nu06, Proposition 2.1.10], which holds for general continuous processes X.
According to such result, in order to establish (3.4) we fix a countable and dense set $\{t_n,\ n \ge 1\}$ in $[0, 1]$; notice that we can approximate g with a sequence of discrete random variables $g_n := \max\{X(t_1), \dots, X(t_n)\}$. Let $\phi_n(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$; then ϕ_n is a Lipschitz continuous function with

$$M\phi_n(X(t_1), \dots, X(t_n)) = MX(\tau_n),$$

where $\tau_n \in \{t_1, t_2, \dots, t_n\}$ is the time when $X(\tau_n) = \max\{X(t_1), \dots, X(t_n)\}$. Passing to the limit as $n \to \infty$, due to the continuity of the trajectories, the claim follows. □

3.2. **Applications.** We start with the maximum of the Brownian motion. Set

$$S(t) = \max_{s \in [0,t]} B(s), \quad \forall\, t \in [0, 1]. \tag{3.5}$$

By Proposition 3.1 for almost every $x \in H$ and any $t \in [0, 1]$, $S(\cdot)(x)$ attains the maximum on $[0, t]$ at a unique point denoted τ_x^t. Moreover, by Proposition 3.2 for any $t \in [0, 1]$, $S(t) \in \mathbb{D}^{1,2}(H, \mu)$ and the Malliavin derivative of $S(t)$ is given by

$$MS(t)(x) = \mathbb{1}_{[0, \tau_x^t]}. \tag{3.6}$$

Let us fix $a > 0$ and set $I = (a, +\infty)$. Our aim is to show the following result.

Proposition 3.3. *The function*

$$g(x) = S(1)(x), \quad x \in H,$$

fulfills the local Malliavin condition on I with

$$u_t(x) = \psi(S(t)(x)), \quad \gamma(x) = \int_0^1 \psi(S(t)(x))\, \mathrm{d}t, \tag{3.7}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is C^∞ and such that

$$\psi(r) = \begin{cases} 1 & \text{if } r \in [0, a/2] \\ \\ 0 & \text{if } r \geq a. \end{cases} \tag{3.8}$$

Proof. We have to show that

$$\int_0^1 M_t g(x)\, \psi(S(t)x)\, \mathrm{d}t = \gamma(x), \quad \forall\, x \in g^{-1}(I) \tag{3.9}$$

and

$$\frac{u}{\gamma} \in D(M^*). \tag{3.10}$$

We proceed in several steps.

Step 1. Identity (3.9) is fulfilled.

Let x be fixed such that $g(x) > a$ and write, taking into account (3.6)

$$\int_0^1 M_t g(x)\, \psi(S(t)x)\, \mathrm{d}t = \int_0^{\tau_x^1} \psi(S(t)x)\, \mathrm{d}t.$$

Function $t \to S(t)x$ is increasing, so there is $t_x < \tau_x^1$ such that $S(t_x)x = a$. Therefore (since $\psi(r) = 0$ for $r \geq a$)

$$\int_0^1 M_t g(x)\, \psi(S(t)x)\, \mathrm{d}t = \int_0^{t_x} \psi(S(t)x)\, \mathrm{d}t = \int_0^1 \psi(S(t)x)\, \mathrm{d}t = \gamma(x).$$

Step 2. $\gamma \in \mathbb{D}^{1,p}(H,\mu)$ for all $p \geq 1$.
By the chain rule and (3.6) we have that $\gamma \in \mathbb{D}^{1,p}(H,\mu)$ and

$$M\gamma(x) = \int_0^1 \psi'(S(t)x)\, \mathbb{1}_{[0,\tau_x^t]}\, \mathrm{d}t.$$

Step 3. $u \in D(M_p^*)$ for all $p \geq 1$.
Actually, u is an adapted process, $u \leq 1$, hence u is Itô integrable and, *a fortiori*, Skorohod integrable.

Step 4. $\frac{1}{\gamma} \in L^p(H,\mu)$ for all $p \geq 1$.
First notice that (recall that $S(\cdot)x$ is not decreasing)

$$\gamma(x) = \int_0^1 \psi(S(t)x)\, \mathrm{d}t \geq (S(\cdot)x)^{-1}(\tfrac{a}{2})$$
$$= (S(.)x)^{-1}(r) = \inf\{s \geq 0 : S(s)x \geq r\}.$$

Therefore

$$\frac{1}{\gamma(x)} \leq \frac{1}{(S(\cdot)x)^{-1}(\frac{a}{2})} =: Z(x).$$

So, it is enough to show that $Z \in L^p(H,\mu)$, equivalently that

$$\int_H Z^p\, \mathrm{d}\mu = p \int_0^\infty \mu(Z > \epsilon)\, \epsilon^{p-1}\, \mathrm{d}\epsilon < \infty. \tag{3.11}$$

Notice now that for any $q>1$ we have

$$\{Z>\epsilon\}=\left\{\frac{a}{2}<S(\tfrac{1}{\epsilon})x\right\}\le\left(\frac{2}{a}\right)^q\int_H\sup_{s\in[0,1/\epsilon]}B(s)^q\mathrm{d}\mu\le c_q\left(\tfrac{1}{\epsilon}\right)^{q/2}.$$

So the conclusion follows from the arbitrariness of q.

Step 5. $\frac{u}{\gamma}\in D(M^*)$.
Let us first notice that $\gamma^{-1}\in D^p(H,\mu)$ since by the chain rule

$$M_p(\gamma^{-1})=-\gamma^{-2}M_p\gamma.$$

Therefore $\frac{u}{\gamma}\in D(M^*)$ and by Step 3 we have

$$M_p^*(\tfrac{u}{\gamma})=\gamma^{-1}\,M_p^*u-\langle M_p(\gamma^{-1}),u\rangle.$$

The proof is complete. □

Remark 3.4. The proof above was inspired by the paper from [FlNu95] about the supremum of the Brownian sheet, but it is more elementary. In particular, it does not require fractional Sobolev spaces and the Garsia, Rodemich and Rumsey result as in the quoted paper.

Now we consider a *distorted Brownian motion*

$$B_{b,\sigma}(t):=b\,t+\sigma B(t),\quad\forall\,t\in[0,1],\tag{3.12}$$

where $b\in\mathbb{R}$ and $\sigma>0$ are given. Set

$$S_{b,\sigma}(t):=\sup_{s\in[0,t]}(bs+\sigma B(s)),\quad\forall\,t\in[0,1].\tag{3.13}$$

Again, by Proposition 3.1 $S_{b,\sigma}(t)$ attains the maximum at a unique point τ_x^t of $[0,t]$ μ–a.s. Moreover, by (3.4) we have

$$MS_{b,\sigma}(t)(x)=\sigma\mathbb{1}_{[0,\tau_x^t]}.\tag{3.14}$$

Set $I=(a,+\infty)$ where $a>0$ is fixed. By proceeding as in the proof of Proposition 3.3 we show

Proposition 3.5. *The function*

$$g(x)=S_{b,\sigma}(1)(x),\quad x\in H,$$

fulfills Hypothesis 1 on I *with*

$$u_t(x)=\psi(B_{b,\sigma}(t)x),\quad\gamma(x)=\sigma\int_0^1\psi(B_{b,\sigma}(t)x)\,\mathrm{d}t,\tag{3.15}$$

where $\psi:[0,+\infty)\to[0,+\infty)$ *is* C^∞ *and such that* (3.8) *holds.*

We finally consider the *Geometric Brownian motion*

$$X(t):=e^{(b-\frac{1}{2}\sigma^2)t+\sigma B(t)},\quad\forall\,t\in[0,1],$$

where $b\in\mathbb{R}$ and $\sigma>0$. $X(t)$, $t\in[0,1]$, is the strong solution of the following SDE:

$$\mathrm{d}X=bX\,\mathrm{d}t+\sigma X\,\mathrm{d}B(t),\quad X(0)=1.$$

Set

$$S_X(t)=\max_{s\in[0,t]}X(s),\quad\forall\,t\in[0,1].$$

Arguing as before we see that μ–a.s. $X(s)$, $s\in[0,t]$, attains the maximum at a unique point τ_x^t of $[0,t]$ and that

$$MS_X(t)(x)=\sigma\,S_X(t)\mathbb{1}_{[0,\tau_x^t]},\quad \forall\, t\in[0,1].$$

Now let us fix $a>0$ and set $I=(a,+\infty)$. By proceeding as in the proof of Proposition 3.3 we show

Proposition 3.6. *The function*

$$g(x)=S_X(1)(x),\quad x\in H,$$

fulfills Hypothesis 1 with

$$u_t(x)=\psi(S_X(t)(x)),\quad \gamma(x)=\sigma\,S_X(1)(x)\int_0^1\psi(S_X(t)(x))\,\mathrm{d}t, \tag{3.16}$$

where $\psi:[0,+\infty)\to[0,+\infty)$ *is* C^∞ *and such that* (3.8) *is fulfilled.*

Remark 3.7. Let us consider the *Brownian Bridge*,

$$B_0(t)=B(t)-tB(1),\quad t\in[0,1].$$

Also $B_0(t)$, $t\in[0,1]$, attains the maximum at a unique point τ_x^t of $[0,t]$, μ–a.s.. Set

$$S_0(t)=\max_{s\in[0,t]}(X(t)),\quad \forall\, t\in[0,1], \tag{3.17}$$

so that

$$MS_0(t)(x)=B(\tau_x)-\tau_x, \tag{3.18}$$

In this case we are not able to show that the function

$$g(x)=\max_{t\in[0,1]}S_0(t)(x),$$

fulfills Hypothesis 1. One realizes in fact that, due to the additional term $-\tau_x$ in equation (3.18), the proof of Proposition 3.3 does not work in this case.

A similar difficulty arises with the *Ornstein–Uhlenbeck* process

$$X(t)=\int_0^t e^{-(t-s)a}\,\mathrm{d}B(s),\quad t\in[0,1],$$

where $a>0$ is fixed. The Malliavin derivative of $X(t)$ is given by

$$M_\tau X(t)x=e^{-a(t-\tau)}\mathbb{1}_{[0,t]}(\tau). \tag{3.19}$$

Setting

$$S(1)=\max_{t\in[0,1]}(X(t)), \tag{3.20}$$

we have by (3.4)

$$MS(1)(x)=e^{-a(1-\tau_x)}\mathbb{1}_{[0,\tau_x]}, \tag{3.21}$$

where τ_x is the the time when $X(t)(x)$ reaches the maximum value as $t\in[0,1]$.
As before we cannot repeat the proof of Proposition 3.3 due to the exponential term in equation (3.21).

4. Integration by parts formulae in $L^2_+(0,1)$

4.1. **Setting of the problem.** We are given a Gaussian process $X(t)$, $t \in [0,1]$, in $H = L^2(0,1)$. Its law is a Gaussian measure which we denote by $\mu = N_Q$. We assume that μ is non degenerate.

The following integration by parts formula is well known

$$\int_H \langle D\varphi(x), z\rangle \,\mu(\mathrm{d}x) = \int_H \varphi(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\mu(\mathrm{d}x), \tag{4.1}$$

for all $\varphi \in C^1_b(H)$ and all $z \in Q^{1/2}(H)$. We shall also assume that μ is concentrated in $E := C([0,1])$ and that the Cameron–Martin space $Q^{1/2}(H)$ is included in E. Then we can find easily an integration by parts formula on E. For this we need the following elementary lemma, see e.g., [CeDa14, Lemma 2.1].

Lemma 4.1. *For any $\varphi \in C^1_b(E)$ there exists a sequence $(\varphi_n) \in C^1_b(H)$ such that*

(i) $\lim_{n\to\infty} \varphi_n(x) \to \varphi(x), \quad \forall\, x \in E.$
(ii) $\lim_{n\to\infty} \langle D\varphi_n(x), z\rangle_H = D\varphi(x)\cdot z$, *for all* $x, z \in E$.

Now we can prove

Proposition 4.2. *For all $\varphi \in C^1_b(E)$ and any $z \in Q^{1/2}(H)$ the following integration by parts formula holds*

$$\int_E D\varphi(x)\cdot z\,\mu(\mathrm{d}x) = \int_E \varphi(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\mu(\mathrm{d}x). \tag{4.2}$$

Proof. Let $\varphi_n \in C^1_b(H)$ be a sequence as in Lemma 4.1. Then by (4.1) we have

$$\int_H \langle D\varphi_n(x), z\rangle_H \,\mu(\mathrm{d}x) = \int_H \varphi_n(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\mu(\mathrm{d}x). \tag{4.3}$$

The conclusion follows letting $n \to \infty$. □

Corollary 4.3. *For all $\varphi, \psi \in C^1_b(E)$ and any $z \in Q^{1/2}(H)$ the following integration by parts formula holds*

$$\int_E D\varphi\cdot z\,\psi\,\mathrm{d}\mu = -\int_E D\psi\cdot z\,\varphi\,\mathrm{d}\mu + \int_E \varphi\,\psi\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\mathrm{d}\mu. \tag{4.4}$$

Remark 4.4. By (4.2) it follows, by standard arguments, that the gradient operator D is closable in $L^p(E,\mu)$ for any $p \ge 1$, we shall still denote by D its closure. As a consequence identity (4.2) also holds for φ belonging to the domain of the closure of D.

The main object of this section is the following function in E

$$g(x) := \min_{t\in[0,1]} X(t)(x), \quad x \in E. \tag{4.5}$$

We shall assume that

Hypothesis 2.
(i) The law of g is absolutely continuous with respect to the Lebesgue measure λ on $(-\infty, 0]$; we shall denote ρ the corresponding density.

(ii) For μ–almost all $x \in E$, trajectories of $X(t)$ attain their minimum $g(x)$ at a unique point $\tau_x \in [0,1]$. Moreover, there exists the directional derivative of $g(x)$ in all directions $z \in E$ and its result

$$Dg(x)\cdot z := z(\tau_x). \tag{4.6}$$

Proposition 4.5. *Assume Hypothesis 2. Then for all $\varphi \in C_b^1(E)$ and all $z \in Q^{1/2}(H)$ the following identity holds for every $r < 0$:*

$$\mathbb{E}\big[z(\tau)\varphi\big|g = r\big]\,\rho(r) = -\int_{\{g\ge r\}} \Big(D\varphi\cdot z - \langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\varphi\Big)\,\mathrm{d}\mu. \tag{4.7}$$

Proof. We fix $r < 0$ and $\epsilon > 0$ such that $r + \epsilon < 0$. Then we apply (4.4) setting $\psi = \theta_\epsilon(g)$, where θ_ϵ is given by

$$\theta_\epsilon(\xi) = \begin{cases} 0 & \text{if} \quad \xi < r-\epsilon \\ \frac{\xi-r+\epsilon}{2\epsilon} & \text{if} \quad \xi \in [r-\epsilon, r+\epsilon] \\ 1 & \text{if} \quad \xi > r+\epsilon. \end{cases}$$

By the chain rule we have

$$D(\theta_\epsilon(g)) = \theta_\epsilon'(g)Dg = \frac{1}{2\epsilon}\,\mathbb{1}_{\{r-\epsilon\le g\le r+\epsilon\}}\,Dg,$$

and by (4.6) we deduce

$$\frac{1}{2\epsilon}\int_E \mathbb{1}_{\{r-\epsilon\le g\le r+\epsilon\}} z(\tau)\,\varphi\,\mathrm{d}\mu = -\int_{\{g\ge r-\epsilon\}} \theta_\epsilon(g)\Big(D\varphi\cdot z - \langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\varphi\Big)\,\mathrm{d}\mu. \tag{4.8}$$

On the one hand,

$$\frac{1}{2\epsilon}\int_E \mathbb{1}_{\{r-\epsilon\le g\le r+\epsilon\}} z(\tau)\,\varphi\,\mathrm{d}\mu = \frac{1}{2\epsilon}\int_E \mathbb{1}_{\{r-\epsilon\le g\le r+\epsilon\}} \mathbb{E}[z(\tau)\varphi|g]\,\mathrm{d}\mu.$$

Since the law of g is given by the measure $\rho(r)\,\mathrm{d}r$ we have

$$\frac{1}{2\epsilon}\int_{\{r-\epsilon\le g\le r+\epsilon\}} z(\tau)\,\varphi\,\mathrm{d}\mu = \frac{1}{2\epsilon}\int_{r-\epsilon}^{r+\epsilon} \mathbb{E}\big[z(\tau)\,\varphi\big|g{=}\xi\big]\,\rho(\xi)\,\mathrm{d}\xi. \tag{4.9}$$

Now letting $\epsilon \to 0$, we have that for almost every $r < 0$

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\{r-\epsilon\le g\le r+\epsilon\}} z(\tau)\,\varphi\,\mathrm{d}\mu = \mathbb{E}\big[z(\tau)\,\varphi\big|g{=}r\big]\,\rho(r).$$

We next consider the right hand side of (4.8), and we prove that it converges to [(2)]

$$-\int_{\{g\ge r\}} \Big(D\varphi\cdot z - \langle Q^{-1/2}x, Q^{-1/2}z\rangle\,\varphi\Big)\,d\mu$$

[(2)] if $\psi(x) = D\varphi(x)\cdot z - \varphi(x)\langle Q^{-1/2}x, Q^{-1/2}z\rangle$ then $\mathbb{E}|\psi|^2 \le \|\varphi\|^2_{C_b^1}\left(|z|^2 + |Q^{-1/2}z|^2\right)$, and since the right hand side of (4.8) is dominated by ψ and $\theta_\epsilon(g(x)) \to \mathbb{1}_{[r,+\infty)}(g(x))$, we can apply the dominated convergence theorem and since this is a continuous function in r [3], formula (4.7) follows for every $r < 0$. $\square$

Remark 4.6. Assume, besides the assumptions of Proposition 4.5 that $\varphi\langle Mg, z\rangle \in \mathbb{D}^{1,2}(H,\mu)$ or $\varphi\langle Mg, z\rangle \in UC_b(H)$. Then if $r<0$, identity (4.7) can be written as

$$\int_{\{g=r\}} \langle Mg, z\rangle\, \varphi\, \mathrm{d}\sigma_r = -\int_{\{g\geq r\}} \langle M\varphi, z\rangle\, \mathrm{d}\mu + \int_{\{g\geq r\}} W_z\, \varphi\, \mathrm{d}\mu, \tag{4.10}$$

where σ_r is the surface measure introduced before.

The following result will be useful later to pass to the limit for $r \to 0$.

Proposition 4.7. *Assume, besides Hypothesis 2, that the joint law of the random vector (g,τ) can be written as $\pi(\mathrm{d}y,\mathrm{d}s) = \pi(y,s)\,\mathrm{d}y\mathrm{d}s$ on $\mathbb{R}\times[0,1]$, with the map $y \to \pi(y,s)$ continuous. Then for all $\varphi \in C^1_b(E)$ and all $z \in Q^{1/2}(H)$ the following identity holds $\forall\, r<0$*

$$\begin{aligned}\int_0^1 \mathbb{E}\big[\varphi|g{=}r, \tau{=}s\big]\, z(s)\, \pi(r,s)\, \mathrm{d}s \\ = -\int_{\{g\geq r\}} [D\varphi\cdot z - \langle Q^{-1/2}x, Q^{-1/2}z\rangle\, \varphi]\, \mathrm{d}\mu, \quad \forall\, r<0.\end{aligned} \tag{4.11}$$

Proof. Let us consider the left hand side of (4.8): by conditioning with respect to $\{g=y, \tau=s\}$ we get

$$\frac{1}{2\epsilon}\int_{\{r-\epsilon\leq g\leq r+\epsilon\}} z(\tau)\,\varphi\,\mathrm{d}\mu = \frac{1}{2\epsilon}\int_{r-\epsilon}^{r+\epsilon}\int_0^1 z(s)\,\mathbb{E}\big[\varphi|g{=}y,\tau{=}s\big]\,\pi(\mathrm{d}y,\mathrm{d}s)$$

Passing to the limit for $\epsilon \to 0$ and taking into account that $\pi(\mathrm{d}y,\mathrm{d}s) = \pi(y,s)\,\mathrm{d}y\,\mathrm{d}s$, we get the thesis. $\square$

4.2. **Examples.**

4.2.1. *Brownian motion.* Let $X(t) = B(t)$, $t\in[0,1]$. Then the law μ of X is the Wiener measure, it is is concentrated on $\{x \in C([0,1]) :\, x(0)=0,\ x'(1)=0\} \subset E$. Moreover, the law of g (defined by (4.5)) is given by, see e.g. [BoSa02, 1.2.4 page 154],

$$(\mu\circ g^{-1})(\mathrm{d}r) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}r^2}\, \mathbb{1}_{(-\infty,0]}\,\mathrm{d}r \tag{4.12}$$

As well known, for almost all $x \in E$ $B(\cdot)x$ has a unique minimum point at τ_x so that Hypothesis 2 is fulfilled see e.g. [EnSt93] .

Consequently, applying the integration by parts formula (4.7), we obtain that for all $\varphi \in C^1_b(H)$, $z \in Q^{1/2}(H)$ and all $r<0$ we have

$$\begin{aligned}\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}r^2}\, \mathbb{E}\left[z(\tau)\,\varphi|g{=}r\right] \\ = -\int_{\{g\geq r\}} (D\varphi(x){\cdot}z - \varphi(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle)\,\mu(\mathrm{d}x).\end{aligned} \tag{4.13}$$

---

[3] with the notation of formula (2.1), we shall prove that $F_\psi(r)$ is a continuous function; notice that $F_\psi(r+\epsilon) - F_\psi(r) = \int \theta'_\epsilon(g(x))\psi(x)\,\mu(dx)$ and, again, we can apply the dominated convergence theorem by noticing that $\mathbb{P}(g(x)\in[r,r+\epsilon)) = \int_r^{r+\epsilon}\rho(t)\,\mathrm{d}t \to 0$ due to the absolute continuity of the law of g

We want now to extend identity (4.13) up to $r = 0$. To this purpose we recall two facts. The first one is a result from [DuIgMi77, page 118, Thm 2.1]; namely defining the probability measure ν_r, $r < 0$, as

$$\int_H \varphi \, \mathrm{d}\nu_r = \tfrac{1}{\mu(\{g \geq r\})} \int_{\{g \geq r\}} \varphi \, \mathrm{d}\mu, \quad \varphi \in C_b(H) \tag{4.14}$$

then ν_r converges weakly to ν, where ν is the law of the Brownian meander.
The second one is the expression for the joint density $\pi(y, s)$ of $g = B(\tau)$ and τ, which is given by, see [BoSa02, 1.14.4 page 172]

$$\pi(y, s) = \tfrac{|y|}{\sqrt{\pi^2 \, s^3 \, (1-s)}} \, e^{-\frac{y^2}{2s}}, \quad y \leq 0, \, s \in [0, 1]. \tag{4.15}$$

Now we can prove

Proposition 4.8. *For all $\varphi \in C_b^1(E)$ and all $z \in Q^{-1/2}(H)$ we have*

$$\begin{aligned} \tfrac{1}{\sqrt{2\pi}} \int_0^1 \mathbb{E}\left[\varphi | g = 0, \tau = s\right] \tfrac{z(s)}{\sqrt{s^3 \, (1-s)}} \, \mathrm{d}s \\ = - \int_{\{g=0\}} [D\varphi(x) \cdot z - \varphi(x) \langle Q^{-1/2} x, Q^{-1/2} z \rangle] \, \nu(\mathrm{d}x), \end{aligned} \tag{4.16}$$

where ν is the law of the Brownian meander.

Proof. By (4.11) we have, taking into account (4.15),

$$\begin{aligned} - \int_{\{g \geq r\}} [D\varphi \cdot z - \langle Q^{-1/2} x, Q^{-1/2} z \rangle \, \varphi] \, \mathrm{d}\mu \\ = \int_0^1 \mathbb{E}\left[\varphi | g = r, \tau = s\right] \tfrac{|r| \, z(s)}{\sqrt{\pi^2 \, s^3 \, (1-s)}} \, e^{-\frac{r^2}{2s}} \, \mathrm{d}s. \end{aligned} \tag{4.17}$$

On the other hand, by (4.12) we have

$$\mu(g \geq r) = \tfrac{2}{\sqrt{2\pi}} \int_r^0 e^{-\frac{1}{2}s^2} \, \mathrm{d}s$$

it follows that

$$\lim_{r \to 0} \tfrac{\mu(g \geq r)}{|r|} = \frac{2}{\sqrt{2\pi}}.$$

Therefore, dividing both sides of (4.17) by $\mu(g \geq r)$ and letting r tend to zero we find identity (4.16). $\square$

Remark 4.9. Identity (4.16) was proved by a different method by S. Bonaccorsi and L. Zambotti, [BoZa04].

4.2.2. *Distorted Brownian motion.* Now we consider a *distorted Brownian motion*

$$X(t) := b \, t + \sigma B(t), \quad \forall \, t \in [0, 1], \tag{4.18}$$

where $b > 0$ and $\sigma > 0$ are given. Set as usual

$$g := \inf_{s \in [0,1]} (b \, s + \sigma B(s)). \tag{4.19}$$

Then the law μ of X is the Wiener measure. Moreover, the law of g (defined by (4.19)) has a density with respect to the Lebesgue measure given by, see e.g. [BoSa02, 1.2.4 page 251] [(4)]

$$\rho_{b,\sigma}(r) = \tfrac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\tfrac{(r-b)^2}{2\sigma^2}\right) + \tfrac{b}{\sigma^2} e^{2\frac{b}{\sigma^2}r}\,\mathrm{Erfc}\left(-\tfrac{r+b}{\sigma\sqrt{2}}\right) \mathbb{1}_{(-\infty,0]}(r). \tag{4.20}$$

By a direct computation, we have

$$\mu(g \geq r) \approx C_{b,\sigma} r + o(r), \qquad r \to 0,$$

where $C_{b,\sigma}$ can be explicitly computed. [(5)]

For almost all $x \in E$, $X(\cdot)x$ has a unique minimum point at τ_x so that Hypothesis 2 is fulfilled (see e.g. Proposition 3.1).

Consequently, applying the integration by parts formula (4.7), we obtain that for all $\varphi \in C_b^1(H)$, $z \in Q^{1/2}(H)$ and all $r < 0$ we have

$$\begin{aligned}\rho_{b,\sigma}(r)\,\mathbb{E}\left[z(\tau)\,\varphi|g=r\right]& \\ &= -\int_{\{g\geq r\}} (D\varphi(x)\cdot z - \varphi(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle)\,\mu(\mathrm{d}x).\end{aligned} \tag{4.21}$$

Now, to apply Proposition 4.7 we need the expression of the joint density of g and τ which is given by, see e.g. [BoSa02, 1.13.4 page 268]

$$\pi(r,s) = \tfrac{|r|}{\sqrt{\pi\sigma^2}\,s^{3/2}} e^{-\frac{(|r|+b\,s)^2}{2\sigma^2 s}} \left(\tfrac{e^{-\frac{b^2}{2\sigma^2}(1-s)}}{\sqrt{\pi\sigma^2(1-s)}} + \tfrac{b}{\sqrt{2\sigma^2}}\mathrm{Erfc}\left(-\tfrac{b\sqrt{1-s}}{\sqrt{2\sigma^2}}\right)\right) \tag{4.22}$$

Now we can prove

Proposition 4.10. *For all $\varphi \in C_b^1(E)$ and all $z \in Q^{1/2}(H)$ we have*

$$\begin{aligned}\int_0^1 \mathbb{E}\left[\varphi|g{=}0,\tau{=}s\right]\tilde{\pi}(s)\,\mathrm{d}s& \\ &= -\tfrac{1}{\int_{\{g=0\}} e^{-b\,x(1)}\,\nu(\mathrm{d}x)} \int_{\{g=0\}} e^{-b\,x(1)}[D\varphi(x)\cdot z - \varphi(x)\langle Q^{-1/2}x, Q^{-1/2}z\rangle]\,\nu(\mathrm{d}x),\end{aligned} \tag{4.23}$$

where $\tilde{\pi}$ is a given function, that is defined in (4.25) *below, and ν is the law of the Brownian meander.*

Proof. First we notice that for all $\varphi \in C_b^1(E)$ and all $z \in Q^{-1/2}(H)$ we have

$$\begin{aligned}\int_0^1 \mathbb{E}\left[\varphi|g{=}r,\tau{=}s\right] z(s)\,\pi(r,s)\,\mathrm{d}s& \\ &= -\int_{\{g\geq r\}} [D\varphi(x)\cdot z - \varphi(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle]\,\mu(\mathrm{d}x).\end{aligned} \tag{4.24}$$

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(4)

$$\begin{aligned}\mathbb{P}_0(\inf_{0\leq t\leq 1} \mu s + \sigma W_s \leq y) &= \mathbb{P}_0(\inf_{0\leq t\leq \sigma^2} \frac{\mu}{\sigma^2}t + W_t \leq y) \\ &= \frac{1}{2}\,\mathrm{Erfc}\left(-\frac{y-\mu}{\sigma\sqrt{2}}\right) + \frac{1}{2}e^{2\mu y/\sigma^2}\,\mathrm{Erfc}\left(-\frac{y+\mu}{\sigma\sqrt{2}}\right)\end{aligned}$$

(5) $C_{b,\sigma} = \dfrac{b}{\sigma^2}\left(\mathrm{Erfc}\left(\tfrac{b}{\sigma\sqrt{2}}\right) - 2\right) - \tfrac{\sqrt{2}}{\sqrt{\pi\sigma^2}} e^{-b^2/2\sigma^2}$

Now we have to divide both sides of (4.24) by $\mu(g \ge r)$ and pass to the limit for $r \uparrow 0$.
By using the explicit formulas available for $\pi(r,s)$ and $\mu(g \ge r)$, which show that these are (asymptotically) linear functions for r around 0, we have

$$\begin{aligned}\lim_{r\uparrow 0} \frac{\pi(r,s)}{\mu(g\ge r)} &= \tfrac{1}{C_{b,\sigma}\sqrt{\pi\sigma^2}\, s^{3/2}} \left(\tfrac{e^{-\frac{b^2}{2\sigma^2}}}{\sqrt{\pi\sigma^2(1-s)}} + \tfrac{b}{\sqrt{2\sigma^2}} e^{-\frac{b^2}{2\sigma^2}s} \operatorname{Erfc}\left(-\tfrac{b\sqrt{1-s}}{\sqrt{2\sigma^2}}\right)\right) \\ &= \tilde\pi(s). \end{aligned} \tag{4.25}$$

or the right hand side we will use Girsanov's theorem. Let us consider the process $Y(t) = \frac{1}{\sigma}X(t) = \frac{b}{\sigma}t + B(t)$, and the probability measure

$$\gamma(\mathrm{d}x) = \rho(x)\mu(\mathrm{d}x) = e^{\frac{b}{\sigma}B(1) - \frac12\frac{b^2}{\sigma^2}}\,\mu(\mathrm{d}x).$$

Then Y is a Brownian motion on $(H, \mathscr{B}(H), \gamma)$. On the other hand

$$\mu(\mathrm{d}x) = \rho^{-1}(x)\,\gamma(\mathrm{d}x) = e^{-\frac{b}{\sigma}Y(1) - \frac12\frac{b^2}{\sigma^2}}\,\gamma(\mathrm{d}x).$$

We let $\tilde g = \min_{s\in[0,1]} Y(s) = \frac{1}{\sigma}g$; since Y is a Brownian motion under γ, we can reason as in previous section to prove that there exists the limit

$$\lim_{r\to 0} \tfrac{1}{\gamma(\tilde g \ge r/\sigma)} \int_{\{\tilde g\ge r/\sigma\}} \varphi\,\mathrm{d}\gamma = \int_{\{g=0\}} \varphi\,\mathrm{d}\nu, \quad \forall\, \varphi \in C_b(H),$$

where ν, as in subsection 4.2.1, is the law of the Brownian meander.
Then it follows, by taking the limit after a change of measure, that

$$\begin{aligned}\tfrac{1}{\mu(g\ge r)} \int_{\{g\ge r\}} \varphi\,\mathrm{d}\mu = \frac{\int_{\{\tilde g\ge r/\sigma\}} \varphi\,\rho^{-1}\,d\gamma}{\int_{\{\tilde g\ge r/\sigma\}} \rho^{-1}\,d\gamma} &= \frac{\frac{1}{\gamma(\tilde g\ge r/\sigma)}\int_{\{\tilde g\ge r/\sigma\}} \varphi\,\rho^{-1}\,\mathrm{d}\gamma}{\frac{1}{\gamma(\tilde g\ge r/\sigma)}\int_{\{\tilde g\ge r/\sigma\}} \rho^{-1}\,\mathrm{d}\gamma} \\ &\longrightarrow \frac{\int_{\{g=0\}} \varphi\,\rho^{-1}\,\mathrm{d}\nu}{\int_{\{g=0\}} \rho^{-1}\,\mathrm{d}\nu}\end{aligned}$$

that we can write in the form

$$\tfrac{1}{\mu(g\ge r)} \int_{\{g\ge r\}} \varphi\,\mathrm{d}\mu \longrightarrow \frac{\int_{\{g=0\}} \varphi\, e^{-b\,x(1)}\,\nu(\mathrm{d}x)}{\int_{\{g=0\}} e^{-b\,x(1)}\,\nu(\mathrm{d}x)}.$$

So, the conclusion follows. □

4.2.3. *Brownian bridge.* Let us consider now the Brownian bridge,

$$X(t) = B_0(t) = B(t) - tB(1), \quad t\in[0,1],$$

whose law μ is concentrated on the Banach space $\{x \in C([0,1] : x(0) = x(1) = 0\} \subset E$. The law of g can be obtained easily by conditioning with respect to $B(1)$, see [BoSa02, 1.2.8 page 154]

$$(\mu\circ g^{-1})(\mathrm{d}r) = 4\,|r|\,e^{-2r^2}\,\mathrm{d}r\,\mathbb{1}_{(-\infty,0]}, \qquad r<0. \tag{4.26}$$

Moreover, also in this case Hypothesis 2 is fulfilled. Therefore, by formula (4.7) we deduce that for any $z \in H$ and any $r < 0$

$$\begin{aligned} &4\,|r|\,e^{-2r^2}\,\mathbb{E}\left[z(\tau)\,\varphi | g = r\right] \\ &\qquad = -\int_{\{g\ge r\}} [D\varphi(x)\cdot z - \varphi(x)\,\langle Q^{-1/2}x, Q^{-1/2}z\rangle]\,\mu(dx). \end{aligned} \tag{4.27}$$

Now for letting $r \to 0$ we proceed as before. First we notice that defining the probability measure μ_r, $r<0$, as

$$\int_H \varphi \, d\nu_r = \tfrac{1}{\mu(g\geq r)} \int_{\{g\geq r\}} \varphi \, d\mu, \quad \varphi \in C_b(H) \tag{4.28}$$

then by [DuIgMi77, page126, Thm 5.1] ν_r converges weakly to ν, where ν is the law of the 3–D Bessel bridge.

Moreover, we can deduce the joint density $\psi_0(y,s)$ of $g = B_0(\tau)$ and τ by conditioning a previous formula with respect to $B(1)$, that is

$$\mu\left(\tau \in ds, \inf_{0\leq t\leq 1} B_0(t) \in dr\right) = \mu\left(\tau \in ds, \inf_{0\leq t\leq 1} B(t) \in dr \mid B(1)=0\right),$$

using the following expression in [BoSa02, 1.14.8 page 173]

$$\mu(\tau \in ds, \inf_{0\leq t\leq 1} B(t) \in dr, B(1)\in dz)$$
$$= \tfrac{|r|(z+|r|)}{\pi\sqrt{s^3(1-s)^3}} \exp\left(-\tfrac{r^2}{2s} - \tfrac{(z+|r|)^2}{2(1-s)}\right) \, dr \, dz,$$

with $r < 0 \wedge z$. Finally, we obtain

$$\psi_0(y,s) dy \, ds = \mu(\tau \in ds, \inf_{0\leq t\leq 1} B_0(t) \in dy) \tag{4.29}$$
$$= \sqrt{\tfrac{2}{\pi}} \, \tfrac{y^2}{\sqrt{s^3(1-s)^3}} e^{-\frac{y^2}{2s(1-s)}} \, ds \, dy.$$

Now we can prove

Proposition 4.11. *For all $\varphi \in C_b^1(E)$ and all $z \in Q^{-1/2}(H)$ we have*

$$\tfrac{1}{\sqrt{2\pi}} \int_0^1 \mathbb{E}\left[\varphi | g{=}r, \tau{=}s\right] \tfrac{z(s)}{\sqrt{s^3(1-s)^3}} \, ds \tag{4.30}$$
$$= -\int_{\{g=0\}} [D\varphi(x)\cdot z - \varphi(x)\, \langle Q^{-1/2}x, Q^{-1/2}z\rangle] \, \nu(dx).$$

Proof. By (4.11), taking into account (4.29), we have

$$\int_0^1 \mathbb{E}\left[\varphi | g{=}r, \tau = s\right] \sqrt{\tfrac{2}{\pi}} \, \tfrac{r^2\ (z(s))}{\sqrt{s^3(1-s)^3}} e^{-\frac{r^2}{2s(1-s)}} \, ds$$
$$= -\int_{\{g\geq r\}} [D\varphi(x)\cdot z - \varphi(x)\, \langle Q^{-1/2}x, Q^{-1/2}z\rangle] \, \mu(dx).$$

Since

$$\mu(g\geq r) = 4\int_r^0 |s| \, e^{-2s^2} \, ds,$$

we have

$$\lim_{r\to 0} \frac{\mu(g\geq r)}{r^2} = 2,$$

and the conclusion follows. □

Remark 4.12. Identity (4.30) was proved by a different method by L. Zambotti, [Za01]. We also quote [Ot09] for a similar formula but assuming $r<0$.

4.2.4. *Ornstein–Uhlenbeck.* Let

$$X(t)=\int_0^t e^{-(t-s)a}\,\mathrm{d}B(s),\quad t\in[0,1],$$

where $a>0$. Then the law of g is given by, see [BoSa02, 1.2.4 Pag. 522]

$$(\mu\circ g^{-1})(dr)=\frac{2}{\sqrt{\pi}}\frac{\sqrt{a}}{\sqrt{e^{2a}-1}}\;e^{-\frac{a\,r^2}{e^{2a}-1}}\;dr\;\mathbb{1}_{(-\infty,0]}(r),\qquad \forall r<0 \tag{4.31}$$

By formula (4.7) we deduce that for any $z\in H$ and any $r<0$

$$\begin{aligned}&\frac{2}{\sqrt{\pi}}\frac{\sqrt{a}}{\sqrt{e^{2a}-1}}\;e^{-\frac{a\,r^2}{e^{2a}-1}}\;\mathbb{E}\left[z(\tau)\,\varphi\mid g=r\right]\\ &=-\int_{\{g\ge r\}}[D\varphi(x)\cdot z-\varphi(x)\,\langle Q^{-1/2}x,Q^{-1/2}z\rangle]\,\mu(dx).\end{aligned} \tag{4.32}$$

We emphasize once more that this formula holds for $r<0$. As opposite to previous sections, however, at this point we do not know the existence of the joint density $\pi(r,s)$ of g and τ, therefore we are not able to apply Proposition 4.7 and pass to the limit as $r\to 0$.

4.3. **Some remarks about the Neumann problem.** Let $X(t)$, $t\in[0,1]$, be the Gaussian process considered in Section 4.1 with law $\mu=N_Q$ assuming that Hypothesis 2 is fulfilled. We denote by (e_h) and (λ_h) eigen–sequences such that

$$Qe_h=\lambda_h e_h,\quad h\in\mathbb{N}.$$

Let us consider the following stochastic equation in the infinite dimensional Hilbert space H

$$\begin{cases}\mathrm{d}Z=-\frac{1}{2}\,Q^{-1}Z\,\mathrm{d}t+\mathrm{d}W(t),\\[2mm] Z(0)=x,\end{cases} \tag{4.33}$$

where W is a cylindrical white noise in H. It is well known that equation (4.33) has a unique mild solution $Z(t,x)$ (which is an Ornstein–Uhlenbeck process) and that $\mu=N_Q$ is invariant for the corresponding transition semigroup

$$P_t\varphi(x)=\mathbb{E}[\varphi(Z(t,x))],\quad \varphi\in C_b(H). \tag{4.34}$$

We denote by $\mathscr{L}$ the Kolmogorov operator

$$\mathscr{L}\varphi=\frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi]-\frac{1}{2}\,\langle Q^{-1}x,D\varphi\rangle,\quad \varphi\in\mathscr{E}_A(H), \tag{4.35}$$

where $\mathscr{E}_A(H)$ is the space of all exponential functions, see e.g. [Da04]. We recall the identity

$$\int_H \mathscr{L}\varphi\;\psi\,\mathrm{d}\mu=-\frac{1}{2}\int_H\langle D\varphi,D\psi\rangle\,\mathrm{d}\mu,\quad \forall\,\varphi,\psi\in\mathscr{E}_A(H). \tag{4.36}$$

Let g be the function defined by (4.5) and set

$$K_r=\{x\in H:\,g(x)\ge r\}.$$

We are interested in the Neumann problem in K_r. For this we fix $r<0$ and introduce the symmetric Dirichlet form

$$a_r(\varphi,\psi)=\frac{1}{2}\int_{K_r}\langle D\varphi,D\psi\rangle\,\mathrm{d}\mu. \tag{4.37}$$

If a is closable (this had to be checked in the different situations), then thanks to the Lax–Milgram Lemma, for any $\lambda>0$ and $f\in L^2(H,\mu)$ there exists $\varphi\in W^{1,2}(H,\mu)$ such that

$$\lambda\int_{K_r}\varphi\,\psi\,\mathrm{d}\mu+a(\varphi,\psi)=\int_{K_r}f\,\psi\,\mathrm{d}\mu,\quad\forall\,\psi\in\mathscr{E}_A(H). \tag{4.38}$$

φ is called the *weak* solution to the *Neumann* problem

$$\lambda\,\varphi-\mathscr{L}\varphi=f\quad\text{in }\;K_r. \tag{4.39}$$

To see whether φ fulfills a Neumann type condition on the boundary of K_r it is important to extend the integration formula (4.36) to K_r. This is provided by the following result.

Proposition 4.13. *Under the assumptions of Proposition 4.7, let φ, $\psi\in\mathscr{E}_A(H)$, $r<0$ and assume that the series*

$$\langle Dg,D\varphi\rangle:=\sum_{h=1}^{\infty}e_h(\tau)\,D_h\varphi \tag{4.40}$$

is convergent in $L^1(H,\nu)$ (τ is the unique point where the trajectories of $X(t)$ attain their minimum, see Hypothesis 2). Then the following identity holds

$$\int_{\{g\ge r\}}\mathscr{L}\varphi\,\psi\,\mathrm{d}\mu=-\frac{1}{2}\int_{\{g\ge r\}}\langle D\varphi,D\psi\rangle\,\mathrm{d}\mu-\frac{1}{2}\,\mathbb{E}\left[\langle Dg,D\varphi\rangle\,\psi|g=r\right]\rho(r). \tag{4.41}$$

Proof. Let us start from identity (4.8),

$$\mathbb{E}\big[z(\tau)\varphi\big|g=r\big]\rho(r)=-\int_{\{g\ge r\}}[D\varphi\cdot z-\langle Q^{-1/2}x,Q^{-1/2}z\rangle\,\varphi]\,\mathrm{d}\mu,\quad\forall\,r<0. \tag{4.42}$$

Setting $z=e_h$, $x_h=\langle x,e_h\rangle$, yields

$$\mathbb{E}\big[e_h(\tau_x)\,\varphi\big|g=r\big]\rho(r)=-\int_{\{g\ge r\}}(D_h\varphi-\tfrac{x_h}{\lambda_h}\,\varphi)\,\mathrm{d}\mu. \tag{4.43}$$

Replacing φ by $D_h\varphi\,\psi$ and consequently $D_h\varphi$ by $D_h^2\varphi\,\psi+D_h\varphi\,D_h\psi$, yields

$$\mathbb{E}\big[e_h(\tau)\,D_h\varphi\,\psi\big|g=r\big]\rho(r)=-\int_{\{g\ge r\}}(D_h^2\varphi\,\psi-\tfrac{x_h}{\lambda_h}\,D_h\varphi\,\psi+D_h\varphi\,D_h\psi)\,\mathrm{d}\mu. \tag{4.44}$$

Summing up on h the conclusion follows. □

Remark 4.14. Let φ be the weak solution of the Neumann problem. If φ is sufficiently regular then, by comparing (4.41) with (4.36) we deduce

$$\mathbb{E}\left[\langle Dg,D\varphi\rangle\,\psi|g=r\right]=0,\quad\forall\,\psi\in L^2(H,\mu). \tag{4.45}$$

Identity (4.45) can be interpreted as a generalized Neumann condition.
Recall in fact that when g is a very regular function as for instance $g=|x|^2$ then (4.45) reduces to

$$\langle D\varphi,Dg\rangle=0\quad\text{if }|x|^2=r, \tag{4.46}$$

see [BaDaTu09], [BaDaTu11] and [DaLu15]. In the present situation, where g is the infimum of a suitable process, we cannot hope that a boundary condition as (4.46) is fulfilled but another condition should be guessed with the help of (4.45).

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