EFFICIENT CONSTRUCTION OF 2-CHAINS WITH A PRESCRIBED BOUNDARY

ANA ALONSO RODRÍGUEZ*, ENRICO BERTOLAZZI †, RICCARDO GHILONI*, AND RUBEN SPECOGNA‡

Abstract. Let Ω be a bounded domain of $\mathbb{R}^3$ whose closure $\overline{\Omega}$ is polyhedral, and let $\mathcal{T}$ be a triangulation of $\overline{\Omega}$. We devise a fast algorithm for the computation of homological Seifert surfaces of any 1-boundary of $\mathcal{T}$; namely, 2-chains of $\mathcal{T}$ whose boundary is γ. Assuming that the boundary of Ω is sufficiently regular, we provide an explicit formula for a homological Seifert surface of any 1-boundary γ of $\mathcal{T}$. It is based on the existence of special spanning trees of the complete dual graph, and on the computation of certain linking numbers associated with those spanning trees. If the triangulation $\mathcal{T}$ is fine, the explicit formula is too expensive to be used directly. To overcome this difficulty, we adopt an easy and very fast elimination procedure, that sometimes fails. In such a case a new unknown can be computed using the explicit formula and the elimination algorithm restarts. The numerical experiments we performed illustrate the efficiency of the resulting algorithm even when the homology of Ω is not trivial and the triangulation $\mathcal{T}$ of $\overline{\Omega}$ consists of millions of tetrahedra.

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1. Introduction. A basic concept of knot theory is the one of Seifert surface. A Seifert surface of a smooth knot of $\mathbb{R}^3$ is an orientable compact smooth surface of $\mathbb{R}^3$ having the knot as its boundary (see [29]). If the Seifert surface has minimum area, then it is called minimal surface of the knot. The notion of Seifert surface has a natural counterpart in simplicial homology theory. Let Ω be a bounded domain of $\mathbb{R}^3$ whose closure $\overline{\Omega}$ in $\mathbb{R}^3$ is polyhedral, and let $\mathcal{T}$ be a triangulation of $\overline{\Omega}$. A 1-cycle γ of $\mathcal{T}$ is a formal linear combination (with integer coefficients) of oriented edges of $\mathcal{T}$ with zero boundary. The 1-cycle γ is said to be a 1-boundary of $\mathcal{T}$ if it is equal to the boundary of a 2-chain of $\mathcal{T}$; namely, equal to the boundary of a formal linear combination S of oriented faces of $\mathcal{T}$. If such a S exists, we call it *homological Seifert surface of γ in $\mathcal{T}$*.

Given an orientation of the edges and of the faces of the triangulation $\mathcal{T}$ of $\overline{\Omega}$, the problem of constructing a homological Seifert surface of γ in $\mathcal{T}$ can be formulated as a linear system with as many unknowns as faces and as many equations as edges of $\mathcal{T}$. The matrix A of this linear system is the incidence matrix between faces and edges of $\mathcal{T}$. This matrix is very sparse because it has just three nonzero entries per column and the number of nonzero entries on each row is equal to the number of faces incident on the edge corresponding to the row. We are looking for an integer solution of this sparse rectangular linear system. These kinds of problems are usually solved using the Smith normal form, a computationally demanding algorithm even in the case of sparse matrices (see e.g. [26] and [21, 17]). In this way, the natural linear algebra formulation of the mentioned problem leads to a high complexity algorithm.

The aim of this paper is to devise a fast and robust algorithm to compute a homological Seifert surface S of any given 1-boundary γ of $\mathcal{T}$. Here we are not interested in questions concerning the regularity or the minimality of S. Even if the 1-boundary γ of $\mathcal{T}$ is a polygonal knot (without

*Dipartimento di Matematica, Università di Trento, Italy
†Dipartimento di Ingegneria Industriale, Università di Trento, Italy
‡Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica, Università di Udine, Italy

self-intersections), in general, our algorithm gives a homological Seifert surface of γ in $\mathcal{T}$, which is neither a genuine polyhedral Seifert surface (it may have self-intersections) nor a polyhedral minimal surface in $\mathcal{T}$. In fact, our motivations for studying homological Seifert surfaces are completely different, as we explain below. However, we think that, in future investigations, our approach could be taken as a new starting point to obtain polyhedral Seifert and minimal surfaces.

The identification of homological Seifert surfaces is a fundamental task in very different fields. Let us recall two remarkable examples.

They appear in Stokes' theorem: given a sufficiently regular vector field $\mathbf{Z}$ defined in $\overline{\Omega}$ and a 1-boundary γ of $\mathcal{T}$, we have that $\oint_\gamma \mathbf{Z}\cdot ds = \int_S \operatorname{curl} \mathbf{Z}\cdot\nu$, where S is any homological Seifert surface of γ in $\mathcal{T}$. As a consequence, homological Seifert surfaces are a powerful tool in computational electromagnetism, in particular for the construction of vector fields with assigned discrete curl. This is an initialization step required in many algorithms, because, thanks to Ampère's law, we know that the curl of the magnetic field equals the current density (see, e.g., [8, 19, 5]).

Homological Seifert surfaces are also a key point in the construction of bases of the relative homology group $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$. Let $\{\sigma'_m\}_{m=1}^g$ be 1-boundaries of $\mathcal{T}$ contained in $\partial\Omega$ whose homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of the first homology group of $\mathbb{R}^3 \setminus \Omega$ (for a construction of such a basis see, e.g., [20] and [4]). If S_m is a homological Seifert surface of σ'_m in $\mathcal{T}$ for each $m \in \{1, \ldots, g\}$, then the Poincaré-Lefschetz and the Alexander duality theorems ensure that the relative homology classes $[S_m]$ of the S_m's form a basis of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$. We refer the reader to [10, 19] for the possible applications of the surfaces S_m's.

There is extensive literature concerning the construction of minimal surfaces, which is a more difficult problem (see [13, 14, 15]). In [31], Sullivan formulated this problem as a linear programming problem. This idea was developed by Dey, Hirani and Krishnamoorthy in [12] and by Dunfield and Hirani in [18] (see also [27, 28]). However, in the two possible applications of homological Seifert surfaces described above regarding Stokes' theorem and the bases of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$, the computation of genuine Seifert surfaces or of minimal surfaces does not offer any advantage. On the contrary, the inevitable increase of the computational cost needed to obtain such types of surfaces may prevent their effective use in the mentioned applications.

Even if the question of computing homological Seifert surfaces is very natural and significant, there are relatively few works on efficient and general algorithms to compute such surfaces. To the best knowledge of the authors, the first papers on this subject are those of Allili and Kaczynski [3, 22] (see also [24]). There the motivation for studying this problem is the computation of the homomorphism induced in homology by a continuous map, which is an important tool in the theory of dynamical systems (see [2]). In [3], the authors consider a d-dimensional cycle γ of a rectangular domain Ω of $\mathbb{R}^n$, equipped with a cubical subdivision. Taking advantage of the product structure of such a domain and of such a subdivision, they obtain an efficient algorithm to compute a $(d+1)$-chain S in $\overline{\Omega}$ with boundary equal to γ. The algorithm proposed by Kaczynski in [22] works in polyhedral domains of $\mathbb{R}^n$ with trivial homology. It is based on a reduction strategy introduced in [23], and hence the coefficients used belong to a field, and not to $\mathbb{Z}$. The computational complexity of this algorithm is expected to be at most cubic in the number of edges of the triangulation of $\overline{\Omega}$. Potentially, this complexity is linear in some particular cases (for example, when γ is a trivial polygonal knot in a triangulated cube $\overline{\Omega}$ of $\mathbb{R}^3$). However, numerical experiments are not reported.

The novelty of the algorithm that we propose and analyze in the sequel is the use of a combinatorial technique, instead of an algebraic one, to greatly improve computational time. It works in general polyhedral domains of $\mathbb{R}^3$ (with $\partial\Omega$ satisfying a mild regularity condition) and it uses integer coefficients. The theoretical worst-case complexity is cubic. However, making use of breadth-first spanning trees, the complexity turns out to be linear in all the numerical experiments we performed (see Section 5). It avoids any reduction strategies that are, in fact, quite time consuming (see [16] for some numerical experiments comparing different state-of-the-art reduction strategies). Another feature of our algorithm is that it can be immediately vectorized: if many homological Seifert surfaces are required on the same triangulation $\mathcal{T}$ of $\overline{\Omega}$, then all the surfaces can be generated at once. Finally, the algorithm can also be used to detect if a 1-cycle of $\mathcal{T}$ is a 1-boundary of $\mathcal{T}$ or not (see Remark 14).

Let Ω be a bounded domain of $\mathbb{R}^3$ whose closure $\overline{\Omega}$ in $\mathbb{R}^3$ is polyhedral, let $\mathcal{T}$ be a triangulation of $\overline{\Omega}$ and let γ be a 1-boundary of $\mathcal{T}$. A first difficulty to devise a general and efficient algorithm to compute a homological Seifert surface S of γ in $\mathcal{T}$ is that this problem does not have a unique solution. Indeed, the kernel of the edge-face incidence matrix A of $\mathcal{T}$ is never trivial. If $\mathfrak{t}$ is the number of tetrahedra of $\mathcal{T}$ and $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ are the connected components of $\partial\Omega$, then $\ker(A)$ is a free abelian group of rank $\mathfrak{t}+p$; namely, $\ker(A)$ is isomorphic to $\mathbb{Z}^{\mathfrak{t}+p}$. One of its bases is given by the boundaries of tetrahedra of $\mathcal{T}$ and by the 2-chains $\gamma_1, \ldots, \gamma_p$ associated with the triangulations of $\Gamma_1, \ldots, \Gamma_p$ induced by $\mathcal{T}$. This follows easily from the fact that the third homology group of $\overline{\Omega}$ is null and the 2-chains $\gamma_1, \ldots, \gamma_p$ represent a basis of the second homology group of $\overline{\Omega}$ (see Remark 9 below).

A natural strategy to obtain a unique solution S is to add $\mathfrak{t}+p$ equations, by setting equal to zero the unknowns corresponding to suitable faces $f_1, \ldots, f_{\mathfrak{t}+p}$ of $\mathcal{T}$. From the geometric point of view, this is equivalent to imposing that the homological Seifert surface S of γ does not contain the faces $f_1, \ldots, f_{\mathfrak{t}+p}$. Now the problem is to understand how to choose such faces. Our idea to make this choice is to use a suitable spanning tree of the dual complex of $\mathcal{T}$. More precisely, we introduce the *complete dual graph of* $\mathcal{T}$ denoted by $\mathcal{A}'$. Let F be the set of faces of $\mathcal{T}$, F_∂ the set of faces of $\mathcal{T}$ contained in $\partial\Omega$ and E_∂ the set of edges of $\mathcal{T}$ contained in $\partial\Omega$. The dual edge ϵ'_f of a face $f \in F$ and the dual edge ϵ'_ℓ of an edge $\ell \in E_\partial$ are defined in the following way. If $f \in F_\partial$, then it is contained in a unique tetrahedron t and $\epsilon'_f := \{B(f), B(t)\}$, where $B(f)$ is the barycenter of f and $B(t)$ the barycenter of t. If f is an internal face of $\mathcal{T}$ (namely $f \in F \setminus F_\partial$), then it is the common face of exactly two tetrahedra t_1 and t_2, and $\epsilon'_f := \{B(t_1), B(t_2)\}$. Similarly, if $\ell \in E_\partial$, then it is the common edge of exactly two faces f_1, f_2 in F_∂, and $\epsilon'_\ell := \{B(f_1), B(f_2)\}$. The vertices of $\mathcal{A}'$ are the barycenters of tetrahedra of $\mathcal{T}$ and the barycenters of faces in F_∂, and the edges of $\mathcal{A}'$ are the dual edges $\{\epsilon'_f\}_{f\in F}$ and $\{\epsilon'_\ell\}_{\ell\in E_\partial}$. Let $\mathcal{B}'$ be a spanning tree of $\mathcal{A}'$. Denote by $N_{\mathcal{B}'}$ the number of faces of $\mathcal{T}$ whose dual edge belongs to $\mathcal{B}'$; namely, the number of edges of $\mathcal{B}'$ not contained in $\partial\Omega$. It is not difficult to see that, for all spanning trees $\mathcal{B}'$ of $\mathcal{A}'$, $N_{\mathcal{B}'} \geq \mathfrak{t}+p$. The equality holds true if and only if, for each $i \in \{0, 1, \ldots, p\}$, the graph induced by $\mathcal{B}'$ on Γ_i is a spanning tree of the graph induced by $\mathcal{A}'$ on Γ_i (see Remark 9). If the spanning tree $\mathcal{B}'$ of $\mathcal{A}'$ has the latter property of induced graphs, then we call it *Seifert dual spanning tree of* $\mathcal{T}$ (see Definition 8).

Our main result, Theorem 10, shows that if $\mathcal{B}'$ is a Seifert dual spanning tree, then, for every 1-boundary γ of $\mathcal{T}$, there exists a unique homological Seifert surface S of γ in $\mathcal{T}$, which does not contain faces of $\mathcal{T}$ whose dual edges belong to $\mathcal{B}'$. Furthermore, if f is a face of $\mathcal{T}$ whose dual edge ϵ'_f does not belong to $\mathcal{B}'$, then f appears in S with a coefficient equal to the linking number between γ (suitably retracted inside $\overline{\Omega}$) and the unique 1-cycle $\sigma_{\mathcal{B}'}(\epsilon'_f)$ of $\mathcal{A}'$ with all the edges except ϵ'_f contained in $\mathcal{B}'$. As a byproduct, in Theorem 12, we solve the related problem concerning the existence and the construction of internal homological Seifert surfaces of γ; namely, homological Seifert surfaces of γ formed only by internal faces of $\mathcal{T}$.

The construction of Seifert dual spanning trees of $\mathcal{T}$ is quite easy and the computation of the linking number between two simplicial 1-cycles of $\mathbb{R}^3$ can be performed in a very accurate and efficient way (see [6]). However, for a fine triangulation $\mathcal{T}$, the number of faces whose dual edge does not belong to a given Seifert dual spanning tree of $\mathcal{T}$ is very large: it is equal to $\mathfrak{e} - \mathfrak{v} + 1 - g \geq \frac{1}{2}\mathfrak{v} + 1 - g$, where $\mathfrak{e}$ is the number of edges of $\mathcal{T}$, $\mathfrak{v}$ is the number of vertices of $\mathcal{T}$, and g is the first Betti number of $\overline{\Omega}$ (see Section 4). Thus, the use of the explicit formula in terms of linking number turns out to be too expensive. To overcome this difficulty, we adopt an elimination procedure, similar to the one proposed by Webb and Forghani in [32] for the solution of three-dimensional magnetostatic problems. When this procedure fails, one can compute a new unknown by using the explicit formula and then restart the elimination algorithm. Most often, the elimination procedure itself computes directly the homological Seifert surfaces. In the numerical experiments we performed, the elimination procedure fails only in one example where the computational domain is the complement of a thickened trefoil knot in a cube and the boundary γ embraces two branches of the knot. In this example it is enough to use the explicit formula once to restart the elimination algorithm (see Section 5.3).

We remark that what developed in this paper for simplicial complexes extends to general polyhedral cell complexes; namely, finite regular CW complexes.

The remainder of the paper is organized as follows. In Section 2, we specify the topological requirements on the domain Ω, recall some classical homological notions and constructions, and introduce some new geometric concepts, such as *corner edge, coil*, and *plug*. Section 3 is devoted to the presentation and the proof of our main theoretical result (Theorem 10) and of some of its consequences (Theorem 12 and Corollary 13). In Section 4, we describe the above mentioned elimination algorithm to improve the implementation of our main theorem. Finally, in Section 5, we perform several numerical experiments with the algorithm.

2. Preliminary homological notions. Throughout the remainder of this paper, Ω will denote a bounded polyhedral domain of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is locally flat; that is, for every point $x \in \partial\Omega$, there exists an open neighborhood U_x of x in $\mathbb{R}^3$ and a homeomorphism $\phi_x : U_x \longrightarrow \mathbb{R}^3$ such that $\phi_x(U_x \cap \partial\Omega) = P$, where P is the coordinate plane $\{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0\}$ (see [9, 7]). This kind of domains includes all Lipschitz polyhedral domains, but also domains like the crossed bricks (see, e.g., Fig. 3.1 in [25]).

2.1. Cycles, boundaries, and homological Seifert surfaces. We recall some notions of homology theory. Most of them are classical and well-known (see, e.g., [26]), but we recall them in order to fix the notation for the introduction of new concepts such as *corner-free* 1*-chain* and *internal homological Seifert surface.*

The basic concept is that of a chain. A 0-chain of $\mathbb{R}^3$ is a finite formal linear combination $\sum_{i=1}^n p_i \mathbf{v}_i$ of points $\mathbf{v}_i \in \mathbb{R}^3$ with integer coefficients p_i. We denote by $C_0(\mathbb{R}^3, \mathbb{Z})$ the abelian group of 0-chains of $\mathbb{R}^3$.

Given two different points $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$, we denote by $[\mathbf{a}, \mathbf{b}]$ the oriented segment of $\mathbb{R}^3$ from $\mathbf{a}$ to $\mathbf{b}$. The segment of $\mathbb{R}^3$ with vertices $\mathbf{a}$, $\mathbf{b}$ is called the support of $[\mathbf{a}, \mathbf{b}]$ and it is denoted by $|[\mathbf{a}, \mathbf{b}]|$. The unit tangent vector $\boldsymbol{\tau}([\mathbf{a}, \mathbf{b}])$ of the oriented segment $[\mathbf{a}, \mathbf{b}]$ is given by $\boldsymbol{\tau}([\mathbf{a}, \mathbf{b}]) := \frac{\mathbf{b}-\mathbf{a}}{|\mathbf{b}-\mathbf{a}|}$. A (piecewise linear) 1-chain of $\mathbb{R}^3$ is a finite formal linear combination $\sum_{i=1}^m a_i e_i$ of oriented segments $e_i = [\mathbf{a}_i, \mathbf{b}_i]$ of $\mathbb{R}^3$ with integer coefficients a_i. We identify $[\mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{b}]$ and we denote by $C_1(\mathbb{R}^3, \mathbb{Z})$ the abelian group of 1-chains in $\mathbb{R}^3$.

Analogously, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three different non-collinear points in $\mathbb{R}^3$, we denote by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ the oriented triangle of $\mathbb{R}^3$. The triangle of $\mathbb{R}^3$ with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is called the support of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ and it is denoted by $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. The unit normal vector $\boldsymbol{\nu}([\mathbf{a}, \mathbf{b}, \mathbf{c}])$ of the oriented triangle $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is obtained by the right hand rule: $\boldsymbol{\nu}([\mathbf{a}, \mathbf{b}, \mathbf{c}]) := \frac{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})}{|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|}$. A (piecewise linear) 2-chain of $\mathbb{R}^3$ is a finite formal linear combination $\sum_{i=1}^{p} b_i f_i$ of oriented triangles $f_i = [\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i]$ of $\mathbb{R}^3$ with integer coefficients b_i. If $\rho : \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \longrightarrow \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a permutation, we identify $[\rho(\mathbf{a}), \rho(\mathbf{b}), \rho(\mathbf{c})] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$ if $\boldsymbol{\nu}([\rho(\mathbf{a}), \rho(\mathbf{b}), \rho(\mathbf{c})]) = \boldsymbol{\nu}([\mathbf{a}, \mathbf{b}, \mathbf{c}])$, and $[\rho(\mathbf{a}), \rho(\mathbf{b}), \rho(\mathbf{c})] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ if $\boldsymbol{\nu}([\rho(\mathbf{a}), \rho(\mathbf{b}), \rho(\mathbf{c})]) = -\boldsymbol{\nu}([\mathbf{a}, \mathbf{b}, \mathbf{c}])$. We denote by $C_2(\mathbb{R}^3, \mathbb{Z})$ the abelian group of 2-chains in $\mathbb{R}^3$.

Finally, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are four different non-coplanar points in $\mathbb{R}^3$, we denote by $[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ the oriented tetrahedron of $\mathbb{R}^3$. The tetrahedron of $\mathbb{R}^3$ with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is called the support of the oriented tetrahedron $[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ and it is denoted by $|[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]|$. A (piecewise linear) 3-chain of $\mathbb{R}^3$ is a finite formal linear combination $\sum_{i=1}^{q} d_i t_i$ of oriented tetrahedra $t_i = [\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i]$ of $\mathbb{R}^3$ with integer coefficients d_i. If $\rho : \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\} \longrightarrow \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is a permutation, we identify $[\rho(\mathbf{a}), \rho(\mathbf{b}), \rho(\mathbf{c}), \rho(\mathbf{d})] = [\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ if ρ is an even permutation and $[\rho(\mathbf{a}), \rho(\mathbf{b}), \rho(\mathbf{c}), \rho(\mathbf{d})] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ if ρ is an odd permutation. We denote by $C_3(\mathbb{R}^3, \mathbb{Z})$ the abelian group of 3-chains in $\mathbb{R}^3$.

We remark that, if all the coefficients in one of the preceding finite formal linear combinations are equal to zero, then we obtain the null element of the corresponding abelian group.

Let $k \in \{0, 1, 2, 3\}$ and let $c = \sum_{i=1}^{r} c_i z_i$ be a k-chain of $\mathbb{R}^3$, where the c_i's are integers and the z_i's are points, oriented segments, oriented triangles, or oriented tetrahedra of $\mathbb{R}^3$ if $k = 0, 1, 2$ or 3, respectively. Denote by I_c the set of indices $i \in \{1, \ldots, r\}$ such that $c_i \neq 0$. The support $|c|$ of c is the subset of $\mathbb{R}^3$ defined as the union $\bigcup_{i \in I_c} |z_i|$. In particular $|c| = \emptyset$ if $c = 0$. Moreover $|z_i| = \{z_i\}$ (and hence $|c| = \{z_i \in \mathbb{R}^3 \,|\, c_i \neq 0\}$) if $k = 0$.

For every $k \in \{1, 2, 3\}$, let us define the boundary operator $\partial_k : C_k(\mathbb{R}^3; \mathbb{Z}) \longrightarrow C_{k-1}(\mathbb{R}^3; \mathbb{Z})$. For every oriented segment $e = [\mathbf{a}, \mathbf{b}]$, for every oriented triangle $f = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, and for every oriented tetrahedron $t = [\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ of $\mathbb{R}^3$, we set $\partial_1 e := \mathbf{b} - \mathbf{a}$, $\partial_2 f := [\mathbf{b}, \mathbf{c}] - [\mathbf{a}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}]$ and $\partial_3 t := [\mathbf{b}, \mathbf{c}, \mathbf{d}] - [\mathbf{a}, \mathbf{c}, \mathbf{d}] + [\mathbf{a}, \mathbf{b}, \mathbf{d}] - [\mathbf{a}, \mathbf{b}, \mathbf{c}]$. Now we extend these definitions to all the k-chains of $\mathbb{R}^3$ by linearity. The reader observes that $\partial_1(\partial_2 f) = (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{b}) - (\mathbf{c} - \mathbf{a}) = 0$. In this way, by linearity, we have that $\partial_1 \circ \partial_2 = 0$ on the whole $C_2(\mathbb{R}^3; \mathbb{Z})$. Analogously, we have that $\partial_2 \circ \partial_3 = 0$ on the whole $C_3(\mathbb{R}^3; \mathbb{Z})$.

A 1-chain γ of $\mathbb{R}^3$ is called *1-cycle of* $\mathbb{R}^3$ if $\partial_1 \gamma = 0$. The 1-chain γ is said to be a *1-boundary of* $\mathbb{R}^3$ if there exists a 2-chain S of $\mathbb{R}^3$ such that $\partial_2 S = \gamma$. In this situation, we say that S is a *homological Seifert surface of* γ *in* $\mathbb{R}^3$. Since $\partial_1 \circ \partial_2 = 0$, every 1-boundary of $\mathbb{R}^3$ is also a 1-cycle of $\mathbb{R}^3$. Actually, $\mathbb{R}^3$ is contractible (namely, it can be continuously deformed to a point) and hence the converse is true as well: every 1-cycle of $\mathbb{R}^3$ is also a 1-boundary of $\mathbb{R}^3$. In other words, a 1-chain of $\mathbb{R}^3$ has a homological Seifert surface in $\mathbb{R}^3$ if and only if it is a 1-cycle of $\mathbb{R}^3$.

Let Y be a subset of $\mathbb{R}^3$ and let η be a 1-cycle of $\mathbb{R}^3$ with $|\eta| \subset Y$. We say that η *bounds in* Y if η admits a homological Seifert surface S in $\mathbb{R}^3$ with $|S| \subset Y$. Given another 1-cycle η' of $\mathbb{R}^3$ with $|\eta'| \subset Y$, we say that η *and* η' *are homologous in* Y if $\eta - \eta'$ bounds in Y.

Let Ω be a fixed bounded polyhedral domain of $\mathbb{R}^3$ with locally flat boundary and let $\mathcal{T} = (V, E, F, K)$ be a finite triangulation of $\overline{\Omega}$, where V is the set of vertices, E the set of edges, F

the set of faces and K the set of tetrahedra of $\mathcal{T}$.

Let us fix an orientation (namely, an ordering of vertices) of each edge, face, and tetrahedron of $\mathcal{T}$. This can be done as follows. Choose a total ordering $(\mathbf{v}_1, \ldots, \mathbf{v}_{\mathfrak{v}})$ of the elements of V. If $e = \{\mathbf{v}_i, \mathbf{v}_j\} \in E$ is an edge of $\mathcal{T}$ with vertices $\mathbf{v}_i, \mathbf{v}_j$ with $1 \leq i < j \leq \mathfrak{v}$, then e determines the oriented segment $[\mathbf{v}_i, \mathbf{v}_j]$ of $\mathbb{R}^3$. Analogously, the face $f = \{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k\} \in F$ of $\mathcal{T}$ with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ with $1 \leq i < j < k \leq \mathfrak{v}$ and the tetrahedron $t = \{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l\} \in K$ of $\mathcal{T}$ with $1 \leq i < j < k < l \leq \mathfrak{v}$ determine the oriented triangle $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$ of $\mathbb{R}^3$ and the oriented tetrahedron $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l]$ of $\mathbb{R}^3$, respectively. In what follows, we denote again by e, f, and t, an *oriented edge* of $\mathcal{T}$, an *oriented face* of $\mathcal{T}$, and an *oriented tetrahedron* of $\mathcal{T}$, respectively. We indicate by $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{K}$ the sets of oriented edges, oriented faces and oriented tetrahedra of $\mathcal{T}$, respectively.

A k-chain of $\mathcal{T}$ is a formal linear combination of vertices in V, oriented edges in $\mathcal{E}$, oriented faces in $\mathcal{F}$, and oriented tetrahedra in $\mathcal{K}$ for $k = 0, 1, 2$, and 3, respectively. We denote by $C_k(\mathcal{T}; \mathbb{Z})$ the abelian subgroup of $C_k(\mathbb{R}^3; \mathbb{Z})$ consisting of all k-chains of $\mathcal{T}$. Observe that the boundary operators ∂_k preserve the chains of $\mathcal{T}$; namely, $\partial_k(C_k(\mathcal{T}; \mathbb{Z})) \subset C_{k-1}(\mathcal{T}; \mathbb{Z})$ if $k \in \{1, 2, 3\}$.

A 1-chain γ of $\mathcal{T}$ is called 1-*cycle of* $\mathcal{T}$ if $\partial_1 \gamma = 0$, and it is called 1-*boundary of* $\mathcal{T}$ if there exists a 2-chain S of $\mathcal{T}$ such that $\partial_2 S = \gamma$.

Let $\mathcal{T}_\partial = (V_\partial, E_\partial, F_\partial)$ be the triangulation of $\partial\Omega$ induced by $\mathcal{T}$. Denote by $\mathcal{E}_\partial$ and $\mathcal{F}_\partial$ the sets of oriented edges and of oriented faces of $\mathcal{T}$ determined by the edges in E_∂ and the faces in F_∂, respectively. We have:

$$\mathcal{E}_\partial = \big\{ e \in \mathcal{E} \,\big|\, |e| \subset \partial\Omega \big\} \ \text{ and } \ \mathcal{F}_\partial = \big\{ f \in \mathcal{F} \,\big|\, |f| \subset \partial\Omega \big\}.$$

A 1-chain of $\mathcal{T}_\partial$ is a formal linear combination of oriented edges in $\mathcal{E}_\partial$ and a 2-chain of $\mathcal{T}_\partial$ a formal linear combination of oriented faces in $\mathcal{F}_\partial$. We denote by $C_k(\mathcal{T}_\partial; \mathbb{Z})$ the abelian subgroup of $C_k(\mathcal{T}; \mathbb{Z})$ consisting of k-chains of $\mathcal{T}_\partial$ for $k = 1, 2$. The notions of 1-cycle and of 1-boundary of $\mathcal{T}_\partial$ can be defined in the natural way: a 1-chain γ of $\mathcal{T}_\partial$ is a 1-cycle of $\mathcal{T}_\partial$ if $\partial_1 \gamma = 0$, and it is a 1-boundary of $\mathcal{T}_\partial$ if there exists a 2-chain S of $\mathcal{T}_\partial$ such that $\partial_2 S = \gamma$.

Let us introduce the notions of corner edge, of corner face, and of corner tetrahedron of $\mathcal{T}$. Let $e = \{\mathbf{v}, \mathbf{w}\}$ be an edge of $\mathcal{T}$. We say that e is a *corner edge of* $\mathcal{T}$ if $e \in E_\partial$ and there exist two distinct vertices $\mathbf{z}^*$ and $\mathbf{z}^{**}$ in $V_\partial \setminus \{\mathbf{v}, \mathbf{w}\}$ such that the 3-sets $f^* = \{\mathbf{v}, \mathbf{w}, \mathbf{z}^*\}$ and $f^{**} = \{\mathbf{v}, \mathbf{w}, \mathbf{z}^{**}\}$ are faces of $\mathcal{T}$ in F_∂, and the 4-set $t^* = \{\mathbf{v}, \mathbf{w}, \mathbf{z}^*, \mathbf{z}^{**}\}$ is a tetrahedron in $\mathcal{T}$. If e has this property, then we call f^* and f^{**} *corner faces of* $\mathcal{T}$ *associated with* e, and t^* *corner tetrahedron of* $\mathcal{T}$ *associated with* e, see Figure 2.1. A corner face of $\mathcal{T}$ associated with some corner edge of $\mathcal{T}$ is called a *corner face of* $\mathcal{T}$. Similarly, a corner tetrahedron of $\mathcal{T}$ associated with some corner edge of $\mathcal{T}$ is called a *corner tetrahedron of* $\mathcal{T}$.

We denote by $E_\partial^\angle$, $F_\partial^\angle$, and $K_\partial^\angle$ the sets of corner edges, of corner faces, and of corner tetrahedra of $\mathcal{T}$, respectively. Moreover, we indicate by $\mathcal{E}_\partial^\angle$ the sets of oriented edges in $\mathcal{E}_\partial$ determined by the corner edges of $\mathcal{T}$. Given a 1-chain $\gamma = \sum_{e \in \mathcal{E}} a_e e$ of $\mathcal{T}$, we say that γ is *corner-free* if it does not contain any corner oriented edge; namely, if $a_e = 0$ for every $e \in \mathcal{E}_\partial^\angle$. Moreover, we call γ *internal* if it does not contain any boundary oriented edge; namely, if $a_e = 0$ for every $e \in \mathcal{E}_\partial$. Evidently, if γ is internal, then it is also corner-free. Similarly, given a 2-chain $S = \sum_{f \in \mathcal{F}} b_f f$ of $\mathcal{T}$, we say that S is *internal* if it does not contain any boundary oriented face; namely, if $b_f = 0$ for every $f \in \mathcal{F}_\partial$. The reader observes that, if $\mathcal{T}$ is the first barycentric subdivision of some triangulation of $\overline{\Omega}$, then $E_\partial^\angle = \emptyset$ and hence every 1-chain of $\mathcal{T}$ is corner-free. On the other hand, there are examples in which $E_\partial^\angle \neq \emptyset$: if $\overline{\Omega}$ is a tetrahedron of $\mathbb{R}^3$ equipped with its natural triangulation $\mathcal{T}$, then $E_\partial^\angle = E_\partial \neq \emptyset$.

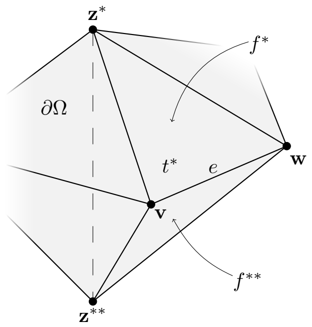

FIG. 2.1. *The corner edge e and the corner faces f^* and f^{**}.*

We conclude this subsection by introducing the notions of homological Seifert surface and of internal homological Seifert surface.

DEFINITION 1. *Given a 1-boundary γ of $\mathcal{T}$, we say that a 2-chain S of $\mathcal{T}$ is a* homological Seifert surface of γ in $\mathcal{T}$ *if $\partial_2 S = \gamma$. If, in addition, S is internal, then we call S* internal homological Seifert surface of γ in $\mathcal{T}$.

2.2. Complete dual graph, coils, and plugs. We begin by describing parts of the closed block dual barycentric complex of $\mathcal{T}$ (see [26, Section 64] for the general definition).

Denote by $B : V \cup E \cup F \cup K \longrightarrow \mathbb{R}^3$ the barycenter map: if $\mathbf{v} \in V$, $\ell = \{\mathbf{v}, \mathbf{w}\} \in E$, $g = \{\mathbf{v}, \mathbf{w}, \mathbf{y}\} \in F$, and $t = \{\mathbf{v}, \mathbf{w}, \mathbf{y}, \mathbf{z}\} \in K$, then we have $B(\mathbf{v}) = \mathbf{v}$, $B(\ell) = (\mathbf{v} + \mathbf{w})/2$, $B(g) = (\mathbf{v} + \mathbf{w} + \mathbf{y})/3$, and $B(t) = (\mathbf{v} + \mathbf{w} + \mathbf{y} + \mathbf{z})/4$. Extend B to the oriented edges in $\mathcal{E}$ and to the oriented faces in $\mathcal{F}$ in the natural way: if $e = [\mathbf{v}, \mathbf{w}] \in \mathcal{E}$ and $f = [\mathbf{v}, \mathbf{w}, \mathbf{y}] \in \mathcal{F}$, then we set $B(e) := (\mathbf{v} + \mathbf{w})/2$ and $B(f) := (\mathbf{v} + \mathbf{w} + \mathbf{y})/3$.

Let us recall the definitions of dual vertices, of dual edges, and of dual faces of $\mathcal{T}$. We equip the dual edges and the dual faces with the natural orientation induced by the right hand rule.

- For every tetrahedron $t \in K$, the dual vertex $D(t)$ of $\mathcal{T}$ associated with t is defined as the barycenter of t: $D(t) := B(t)$.
  We denote by V' the set $\{D(t) \in \mathbb{R}^3 \,|\, t \in K\}$ of all dual vertices of $\mathcal{T}$.
- For every oriented face $f = [\mathbf{v}, \mathbf{w}, \mathbf{y}] \in \mathcal{F}$, the oriented dual edge $D(f)$ of $\mathcal{T}$ associated with f is the element of $C_1(\mathbb{R}^3; \mathbb{Z})$ defined as follows: if $K(f)$ denotes the set $\big\{t \in K \,\big|\, \{\mathbf{v}, \mathbf{w}, \mathbf{y}\} \subset t\big\}$; namely, the set of tetrahedra of $\mathcal{T}$ incident on f, we set
  $$D(f) := \sum_{t \in K(f)} \operatorname{sign}\big(\boldsymbol{\nu}(f) \cdot \boldsymbol{\tau}([B(f), B(t)])\big)\, [B(f), B(t)],$$
  where $\operatorname{sign} : \mathbb{R} \setminus \{0\} \longrightarrow \{-1, 1\}$ denotes the function given by $\operatorname{sign}(s) := -1$ if $s < 0$ and $\operatorname{sign}(s) := 1$ otherwise.
  $D(f)$ can be described as follows. If the (oriented) face f is internal, then f is the common face of two tetrahedra t_1 and t_2 of $\mathcal{T}$, and the support of $D(f)$ is the union of the segment joining $B(f)$ with $B(t_1)$ and of the segment joining $B(f)$ and $B(t_2)$, see Figure 2.2 (on the left). If f is a boundary face, then f is face of just one tetrahedron t, and the support of $D(f)$ is the segment joining $B(f)$ with $B(t)$, see Figure 2.2 (on the right). In both cases, $D(f)$ is endowed with the orientation induced by f via the right hand rule.

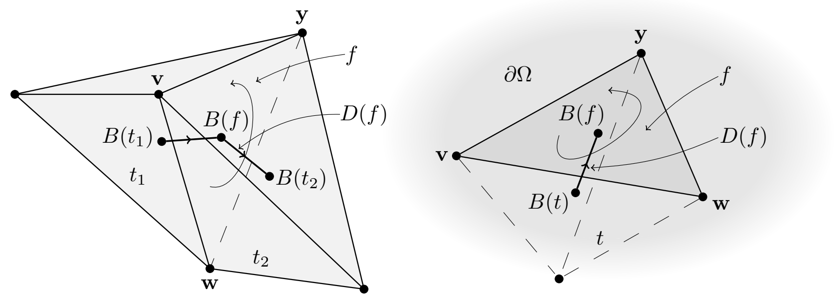

FIG. 2.2. *The dual edge $D(f)$ in the case of an internal face (on the left) and in the case of a boundary face (on the right).*

We denote by $\mathcal{E}'$ the set $\{D(f) \in C_1(\mathbb{R}^3;\mathbb{Z}) \,|\, f \in \mathcal{F}\}$ of all oriented dual edges of $\mathcal{T}$. Moreover, we call a (non-oriented) dual edge of $\mathcal{T}$ a 2-subset $\{v', w'\}$ of $\mathbb{R}^3$ such that $\{v', w'\} = |\partial_1 e'|$ for some $e' \in \mathcal{E}'$. We indicate by E' the set of all (non-oriented) dual edges of $\mathcal{T}$.

- For every oriented edge $e = [\mathbf{v}, \mathbf{w}] \in \mathcal{E}$, the oriented dual face $D(e)$ of $\mathcal{T}$ associated with e is the element of $C_2(\mathbb{R}^3;\mathbb{Z})$ defined as follows: if $F(e)$ denotes the set $\big\{f \in F \,\big|\, \{\mathbf{v}, \mathbf{w}\} \subset f\big\}$, namely, the set of oriented faces of $\mathcal{T}$ incident on e, then we set

$$D(e) := \sum_{f \in F(e)} \sum_{t \in K(f)} \mathrm{sign}\big(\boldsymbol{\tau}(e) \cdot \boldsymbol{\nu}([B(e), B(f), B(t)])\big)\, [B(e), B(f), B(t)],$$

see Figure 2.3. The reader observes that the support of $D(e)$ is the union of triangles of $\mathbb{R}^3$ with vertices $B(e)$, $B(f)$, and $B(t)$, where f varies in $F(e)$ and t in $K(f)$. Such triangles are oriented by e via the right hand rule.

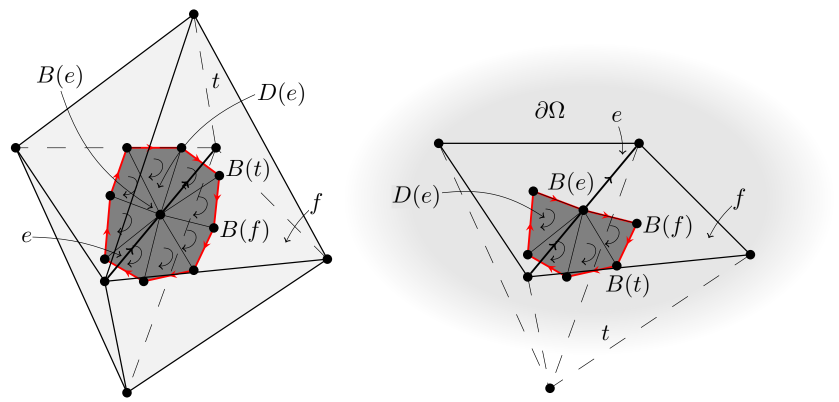

FIG. 2.3. *The dual face $D(e)$ in the case of an internal edge together with its boundary $\partial_2 D(e)$ in red (on the left) and in the case of a boundary edge (on the right).*

We denote by $\mathcal{F}'$ the set $\{D(e) \in C_2(\mathbb{R}^3;\mathbb{Z}) \,|\, e \in \mathcal{E}\}$ of all oriented dual faces of $\mathcal{T}$.

The preceding three definitions determine the bijection $D : K \cup \mathcal{F} \cup \mathcal{E} \longrightarrow V' \cup \mathcal{E}' \cup \mathcal{F}'$ such that $D(K) = V'$, $D(\mathcal{F}) = \mathcal{E}'$, and $D(\mathcal{E}) = \mathcal{F}'$.

We need also to describe part of the closed block dual barycentric complex of the triangulation $\mathcal{T}_\partial$ of $\partial\Omega$ induced by $\mathcal{T}$. Recall that V_∂, $\mathcal{E}_\partial$, and $\mathcal{F}_\partial$ denote the sets of vertices, of oriented edges, and of oriented faces of $\mathcal{T}_\partial$, respectively.

Let us define the dual vertices and the oriented dual edges of $\mathcal{T}_\partial$.

- For every oriented face $f \in \mathcal{F}_\partial$, the dual vertex $D_\partial(f)$ of $\mathcal{T}_\partial$ associated with f is defined as the barycenter of f: $D_\partial(f) := B(f)$.
  We denote by V'_∂ the set $\{D_\partial(f) \in \mathbb{R}^3 \,|\, f \in \mathcal{F}_\partial\}$ of all dual vertices of $\mathcal{T}_\partial$.
- For every oriented edge $e \in \mathcal{E}_\partial$, the oriented dual edge $D_\partial(e)$ of $\mathcal{T}_\partial$ associated with e is the element of $C_1(\mathbb{R}^3;\mathbb{Z})$ defined as follows. Let f_1 and f_2 be the oriented faces in $\mathcal{F}_\partial$ incident on e, and let $\mathbf{n}(f_1)$ and $\mathbf{n}(f_2)$ be the outward unit normals of $\partial\Omega$ at $B(f_1)$ and at $B(f_2)$, respectively. Then we set

$$D_\partial(e) := \sum_{i=1}^{2} \operatorname{sign}\big(\boldsymbol{\tau}(e) \cdot (\mathbf{n}(f_i) \times \boldsymbol{\tau}([B(e), B(f_i)]))\big)[B(e), B(f_i)].$$

  $D_\partial(e)$ can be described as follows. By interchanging f_1 with f_2 if necessary, we can suppose that f_1 is on the left of e and f_2 on the right of e with respect to the orientation of $\partial\Omega$ induced by its outward unit vector field. Then we have:

$$D_\partial(e) = [B(f_1), B(e)] + [B(e), B(f_2)],$$

  see Figure 2.4.

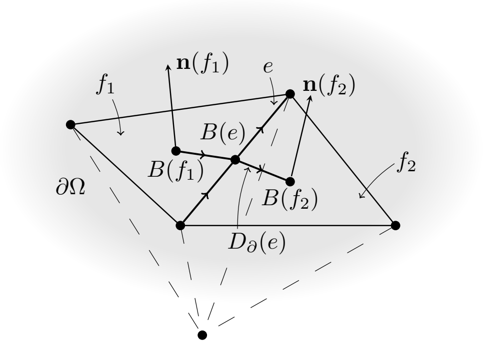

FIG. 2.4. *The boundary dual edge $D_\partial(e)$.*

We denote by $\mathcal{E}'_\partial$ the set $\{D_\partial(e) \in C_1(\mathbb{R}^3;\mathbb{Z}) \,|\, e \in \mathcal{E}_\partial\}$; namely, the set of all oriented dual edges of $\mathcal{T}_\partial$. Moreover, we call a (non-oriented) dual edge of $\mathcal{T}_\partial$ a 2-subset $\{\mathbf{v}', \mathbf{w}'\}$ of V'_∂ such that $\{\mathbf{v}', \mathbf{w}'\} = |\partial_1 e'|$ for some $e' \in \mathcal{E}'_\partial$. We indicate by E'_∂ the set of all (non-oriented) dual edges of $\mathcal{T}_\partial$.

Let us give four definitions, which will prove to be useful later.

DEFINITION 2. *We call* $\mathcal{A}' := (V' \cup V'_\partial, E' \cup E'_\partial)$ complete dual graph of $\mathcal{T}$. *A* 1-chain *of $\mathcal{A}'$ is a formal linear combination of oriented dual edges in $\mathcal{E}' \cup \mathcal{E}'_\partial$ with integer coefficients. A* 1*-chain γ of $\mathcal{A}'$ is called* 1-cycle of $\mathcal{A}'$ *if $\partial_1\gamma = 0$. We denote by $C_1(\mathcal{A}';\mathbb{Z})$ the abelian subgroup of $C_1(\mathbb{R}^3;\mathbb{Z})$ consisting of all* 1*-chains of $\mathcal{A}'$, and by $Z_1(\mathcal{A}';\mathbb{Z})$ the abelian subgroup of $Z_1(\mathbb{R}^3;\mathbb{Z})$ consisting of all* 1*-cycles of $\mathcal{A}'$.*

Definition 3. *For every* $e \in \mathcal{E}$*, we define the* coil of e (in $\mathcal{T}$)*, denoted by* $\mathrm{Coil}(e)$*, as the* 1*-cycle of* $\mathcal{A}'$ *given by* $\mathrm{Coil}(e) := \partial_2 D(e)$.

In Figure 2.3 above, in red, we show the coil of an internal edge on the left and of a boundary edge on the right. The reader observes that, for every $e \in \mathcal{E}_\partial$, $\mathrm{Coil}(e) - D_\partial(e)$ is a 1-chain of $\mathcal{A}'$, whose expression as a formal linear combination contains only oriented edges in $\mathcal{E}'$; namely, $\mathrm{Coil}(e) - D_\partial(e) = \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a_{e'} e'$ for some (unique) integer $a_{e'}$ such that $a_{e'} = 0$ for every $e' \in \mathcal{E}'_\partial$.

Let us introduce the notion of plug of $\mathcal{T}$.

Definition 4. *Given a dual edge* $\mathrm{e}' \in E'$*, we say that* e' *is a* plug of $\mathcal{T}$ *if there exists a face* $f \in F_\partial$ *such that* $\mathrm{e}' = \{B(f), B(t)\}$*, where* t *is the unique tetrahedron in* $\mathcal{T}$ *containing* f*. Such a plug* e' *is said to be* induced by f*. The plug* e' *is called* corner plug of $\mathcal{T}$ *if it is induced by a corner face* $f \in F_\partial^\angle$*, see Figure 2.5 (on the right). On the contrary, if the face inducing* e' *belongs to* $F_\partial \setminus F_\partial^\angle$*, then* e' *is called* regular plug of $\mathcal{T}$*, see Figure 2.5 (on the left).*

Let $\mathrm{J}_\mathcal{T}$ be the set of all plugs of $\mathcal{T}$, and let $\mathrm{J}_\mathcal{T}^\angle$ and $\mathrm{J}_\mathcal{T}^r$ be the subsets of $\mathrm{J}_\mathcal{T}$ consisting of corner plugs and of regular plugs of $\mathcal{T}$, respectively.

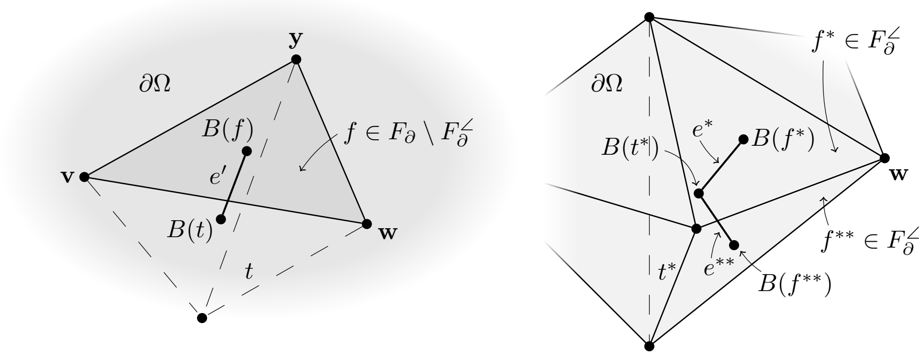

Fig. 2.5. *A regular plug* e' *(on the left) and two corner plugs* e^* *and* e^{**} *(on the right).*

Definition 5. *Given a subset* J *of* $\mathrm{J}_\mathcal{T}$*, we say that* J *is a* plug-set of $\mathcal{T}$ *if, for every* $\mathrm{e}', \mathrm{e}'' \in J$ *with* $\mathrm{e}' \neq \mathrm{e}''$*,* e' *and* e'' *do not have any vertex in common; namely,* $\mathrm{e}' \cap \mathrm{e}'' = \emptyset$*. Moreover, we say that such a plug-set* J *is* maximal *if there is no plug-set of* $\mathcal{T}$ *that strictly contains* J*.*

Remark 6. *Notice that a regular plug does not intersect any other plug so if* $E_\partial^\angle = \emptyset$ *(or, equivalently, if* $K_\partial^\angle = \emptyset$*), then all the plugs of* $\mathcal{T}$ *are regular and hence the set* $\mathrm{J}_\mathcal{T}$ *itself is the unique maximal plug-set of* $\mathcal{T}$*. Suppose* $E_\partial^\angle \neq \emptyset$*. In this case, a subset* J *of* $\mathrm{J}_\mathcal{T}$ *is a maximal plug-set of* $\mathcal{T}$ *if and only if it can be constructed as follows. For every* $t \in K_\partial^\angle$*, choose one of the corner faces of* $\mathcal{T}$ *contained in* t *and denote it by* $f_t^\angle$*. Define* $F^\angle := \{f_t^\angle \in F_\partial^\angle \,|\, t \in K_\partial^\angle\}$ *and indicate by* J' *the set of corner plugs of* $\mathcal{T}$ *induced by the corner faces in* $F^\angle$*. Then* $J = \mathrm{J}_\mathcal{T}^r \cup J'$*.*

In Figure 2.6 we consider a mesh where there are no regular plugs, $\mathrm{J}_\mathcal{T}^r = \emptyset$*. On the left we show the set* $\mathrm{J}_\mathcal{T}$ *with 12 elements. On the right we show a maximal plug-set; it has 4 elements that are one plug for each one of the corner tetrahedra of the mesh.*

2.3. Linking number and retractions. We begin by recalling the notion of linking number. See, e.g., Rolfsen [29, pp. 132–136], Seifert and Threlfall [30, Sects. 70, 73, 77].

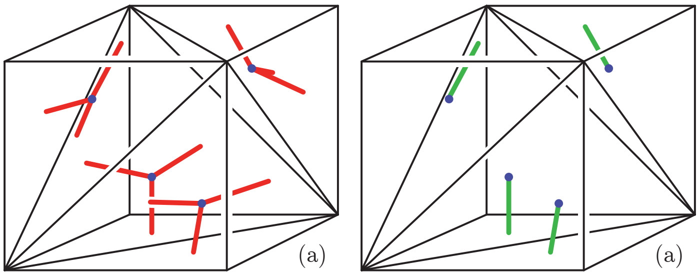

FIG. 2.6. *The whole set of plugs (on the left) and a maximal plug set (on the right).*

Consider two 1-cycles γ and η of $\mathbb{R}^3$ with disjoint supports; namely, $|\gamma| \cap |\eta| = \emptyset$. A possible geometric way to define the linking number $\ell_{\mathcal{K}}(\gamma, \eta)$ between γ and η is as follows. Choose a homological Seifert surface $S_\eta = \sum_{q=1}^{k} b_q f_q$ of η in $\mathbb{R}^3$. It is well-known (and easy to see) that there exists a 1-cycle $\widehat{\gamma} = \sum_{p=1}^{h} \widehat{a}_p \widehat{e}_p$ homologous to γ in $\mathbb{R}^3 \setminus |\eta|$ (and "arbitrarily close to γ" if necessary), which is transverse to S_η in the following sense: for every $p \in \{1, \ldots, h\}$ and for every $q \in \{1, \ldots, k\}$, the intersection $|\widehat{e}_p| \cap |f_q|$ is either empty or consists of a single point, which does not belong to $|\partial_1 \widehat{e}_p| \cup |\partial_2 f_q|$.

For every $p \in \{1, \ldots, h\}$ and for every $q \in \{1, \ldots, k\}$, define $L_{pq} := 0$ if $|\widehat{e}_p| \cap |f_q| = \emptyset$ and $L_{pq} := \mathrm{sign}(\boldsymbol{\tau}(\widehat{e}_p) \cdot \boldsymbol{\nu}(f_q))$ otherwise. The linking number $\ell_{\mathcal{K}}(\gamma, \eta)$ between γ and η is the integer defined as follows:

$$\ell_{\mathcal{K}}(\gamma, \eta) := \sum_{p=1}^{h} \sum_{q=1}^{k} \widehat{a}_p b_q L_{pq}. \tag{2.1}$$

This definition is well-posed: it depends only on γ and η, not on the choice of S_η and of $\widehat{\gamma}$. The reader observes that the preceding construction fully justifies the usual heuristic description of the linking number between γ and η as the number of times that γ winds around η.

The linking number has some remarkable properties. It is "symmetric", $\ell_{\mathcal{K}}(\gamma, \eta) = \ell_{\mathcal{K}}(\eta, \gamma)$, and "bilinear", $\ell_{\mathcal{K}}(a\gamma, \eta) = a\, \ell_{\mathcal{K}}(\gamma, \eta)$ for every $a \in \mathbb{Z}$ and, if $\gamma^* \in Z_1(\mathbb{R}^3; \mathbb{Z})$ with $|\gamma^*| \cap |\eta| = \emptyset$, $\ell_{\mathcal{K}}(\gamma + \gamma^*, \eta) = \ell_{\mathcal{K}}(\gamma, \eta) + \ell_{\mathcal{K}}(\gamma^*, \eta)$.

The linking number is a homological invariant in the following sense: if a 1-cycle γ^* of $\mathbb{R}^3$ is homologous to γ in $\mathbb{R}^3 \setminus |\eta|$, then

$$\ell_{\mathcal{K}}(\gamma, \eta) = \ell_{\mathcal{K}}(\gamma^*, \eta). \tag{2.2}$$

In particular, we have:

$$\ell_{\mathcal{K}}(\gamma, \eta) = 0 \text{ if } \gamma \text{ bounds in } \mathbb{R}^3 \setminus |\eta|. \tag{2.3}$$

The linking number can be computed via an integral formula. Write γ and η explicitly: $\gamma = \sum_{i=1}^{n} a_i e_i$ and $\eta = \sum_{j=1}^{m} c_j g_j$ for some integers a_i, c_j and for some oriented segment $e_i = [\mathbf{a}_i, \mathbf{b}_i]$ and $g_j = [\mathbf{c}_j, \mathbf{d}_j]$ of $\mathbb{R}^3$. The following Gauss formula holds:

$$\ell_{\mathcal{K}}(\gamma, \eta) = \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{m} a_i c_j \left(\int_0^1 \int_0^1 \frac{e_i(r) - g_j(s)}{|e_i(r) - g_j(s)|^3} \times \vec{e}_i \right) \cdot \vec{g}_j \, dr\, ds, \tag{2.4}$$

where $\vec{e}_i := \mathbf{b}_i - \mathbf{a}_i$, $\vec{g}_j := \mathbf{d}_j - \mathbf{c}_j$ and $e_i(r) := \mathbf{a}_i + r\vec{e}_i$, $g_j(s) := \mathbf{c}_j + s\vec{g}_j$ for $r, s \in [0,1]$. The computational cost is of the order of the product of the number of edges in the support of the two 1-cycles, namely, $n\, m$.

Retractions. Now we define two "retraction operators" $R_+ : Z_1(\mathcal{T};\mathbb{Z}) \longrightarrow Z_1(\mathbb{R}^3;\mathbb{Z})$ and $R_- : Z_1(\mathcal{A}';\mathbb{Z}) \longrightarrow Z_1(\mathbb{R}^3;\mathbb{Z})$, and we prove a useful invariance property of certain linking numbers with respect to the application of such "retractions".

Let us define R_+. For every oriented edge $e = [\mathbf{v}, \mathbf{w}]$ in $\mathcal{E}_\partial$, choose a tetrahedron $t_e \in K$ incident on e (namely, $\{\mathbf{v}, \mathbf{w}\} \subset t_e$), denote by $\mathbf{d}_e$ the barycenter of the triangle of $\mathbb{R}^3$ of vertices $\mathbf{v}$, $\mathbf{w}$, $B(t_e)$, and define the 1-chain $r_+(e)$ of $\mathbb{R}^3$ and the oriented triangle S_e of $\mathbb{R}^3$ by setting

$$r_+(e) := [\mathbf{v}, \mathbf{d}_e] + [\mathbf{d}_e, \mathbf{w}] \quad \text{and} \quad S_e := [\mathbf{v}, \mathbf{d}_e, \mathbf{w}].$$

The reader observes that $\partial_2 S_e = r_+(e) - e$, see Figure 2.7.

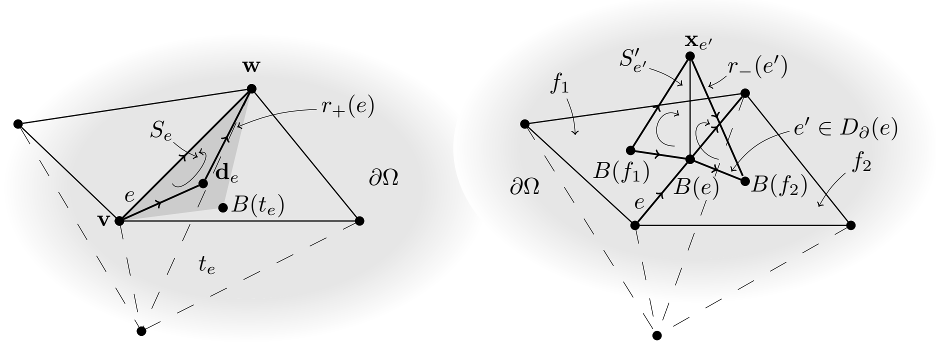

FIG. 2.7. *On the left is the* 1*-chain* $r_+(e)$ *and the oriented triangle* S_e*. On the right is the* 1*-chain* $r_-(e')$ *and the* 2*-chain* $S'_{e'}$*.*

Given $\xi = \sum_{e\in\mathcal{E}} \alpha_e e \in Z_1(\mathcal{T};\mathbb{Z})$, we define:

$$R_+(\xi) := \sum_{e\in\mathcal{E}\setminus\mathcal{E}_\partial} \alpha_e e + \sum_{e\in\mathcal{E}_\partial} \alpha_e r_+(e).$$

Evidently, $R_+(\xi)$ belongs to $Z_1(\mathbb{R}^3;\mathbb{Z})$ and $R_+(\xi) - \xi$ is a 1-boundary of $\mathbb{R}^3$:

$$R_+(\xi) - \xi = \partial_2 \left(\textstyle\sum_{e\in\mathcal{E}_\partial} \alpha_e S_e \right). \tag{2.5}$$

Now we introduce R_-. First, we recall that since $\partial\Omega$ is assumed to be locally flat, we know that it has a collar in $\mathbb{R}^3 \setminus \Omega$ (see [9]); namely, there exists an open neighborhood U of $\partial\Omega$ in $\mathbb{R}^3 \setminus \Omega$ and a homeomorphism $\psi : \partial\Omega \times [0,1) \longrightarrow U$, called collar of $\partial\Omega$ in $\mathbb{R}^3 \setminus \Omega$, such that $\psi(x,0) = x$ for every $x \in \partial\Omega$.

Let $e' \in \mathcal{E}'_\partial$. By definition of $\mathcal{E}'_\partial$, there exist unique $e \in \mathcal{E}_\partial$ and $f_1, f_2 \in \mathcal{F}_\partial$ such that $e' = D_\partial(e) = [B(f_1), B(e)] + [B(e), B(f_2)]$. Thanks to the existence of a collar of $\partial\Omega$ in $\mathbb{R}^3 \setminus \Omega$, one can choose a point $\mathbf{x}_{e'} \in \mathbb{R}^3 \setminus \overline{\Omega}$ arbitrarily close to $B(e)$ with the following property: if $S'_{e'}$ is the 2-chain of $\mathbb{R}^3$ defined by setting

$$S'_{e'} := [B(f_1), \mathbf{x}_{e'}, B(e)] + [B(e), \mathbf{x}_{e'}, B(f_2)], \tag{2.6}$$

then $\overline{\Omega} \cap |S'_{e'}| = |e'|$. Denote by $r_-(e')$ the 1-chain $[B(f_1), \mathbf{x}_{e'}] + [\mathbf{x}_{e'}, B(f_2)]$ of $\mathbb{R}^3$, see Figure 2.7. Observe that $\partial_2 S'_{e'} = r_-(e') - e'$.

For every $\xi' = \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} \alpha'_{e'} e' \in Z_1(\mathcal{A}'; \mathbb{Z})$, we define:

$$R_-(\xi') := \sum_{e' \in \mathcal{E}'} \alpha'_{e'} e' + \sum_{e' \in \mathcal{E}'_\partial} \alpha'_{e'} r_-(e'). \tag{2.7}$$

We remark that $R_-(\xi')$ is a 1-cycle of $\mathbb{R}^3$ and $R_-(\xi') - \xi'$ is a 1-boundary of $\mathbb{R}^3$:

$$R_-(\xi') - \xi' = \partial_2 \left(\textstyle\sum_{e' \in \mathcal{E}'_\partial} \alpha'_{e'} S'_{e'} \right). \tag{2.8}$$

The following result holds true.

Lemma 2.1. *For every $\xi \in Z_1(\mathcal{T}; \mathbb{Z})$ and for every $\xi' \in Z_1(\mathcal{A}'; \mathbb{Z})$, we have: $\ell_{\mathcal{K}}(R_+(\xi), \xi') = \ell_{\mathcal{K}}(\xi, R_-(\xi'))$.*

Proof. First, observe that $|R_+(\xi)| \cap |\xi'| = \emptyset$, $|\xi| \cap |R_-(\xi')| = \emptyset$ and hence the linking numbers $\ell_{\mathcal{K}}(R_+(\xi), \xi')$ and $\ell_{\mathcal{K}}(\xi, R_-(\xi'))$ are defined. Moreover, we have:

$$|R_+(\xi)| \cap \bigcup_{e' \in \mathcal{E}'_\partial} |S'_{e'}| = \emptyset \tag{2.9}$$

and

$$|R_-(\xi')| \cap \bigcup_{e \in \mathcal{E}_\partial} |S_e| = \emptyset. \tag{2.10}$$

By combining points (2.8) and (2.10), we obtain that ξ' and $R_-(\xi')$ are homologous in $\mathbb{R}^3 \setminus |R_+(\xi)|$. Thanks to (2.2), we infer that $\ell_{\mathcal{K}}(R_+(\xi), \xi') = \ell_{\mathcal{K}}(R_+(\xi), R_-(\xi'))$. Similarly, points (2.5), (2.9), and (2.2) ensure that $\ell_{\mathcal{K}}(\xi, R_-(\xi')) = \ell_{\mathcal{K}}(R_+(\xi), R_-(\xi'))$. It follows that $\ell_{\mathcal{K}}(R_+(\xi), \xi') = \ell_{\mathcal{K}}(\xi, R_-(\xi'))$, as desired. □

Remark 7. *We have introduced the retraction R_- in order to simplify the proof of some results. However, it will be never used in the construction of the homological Seifert surfaces presented below.*

We will provide an explicit formula for a homological Seifert surface where, roughly speaking, the coefficients of the faces in the surface are the linking number between a 1-chain ξ of $\mathcal{T}$ and a 1-chain ξ' of $\mathcal{A}'$. If $e \in \mathcal{E}_\partial$ belongs to ξ and $e' = D_\partial(e)$ belongs to ξ', then $|\xi|$ and $|\xi'|$ intersect at $B(e)$ and it is necessary to replace e or e' with its retraction. The previous Lemma shows that it is equivalent to pull e inside the domain (see Figure 2.7 on the left) and to compute $\ell_{\mathcal{K}}(R_+(\xi), \xi')$ or to push e' outside the domain (see Figure 2.7 on the right) and to compute $\ell_{\mathcal{K}}(\xi, R_-(\xi'))$.

3. The main results.

3.1. The statements. Consider the complete dual graph $\mathcal{A}' = (V' \cup V'_\partial, E' \cup E'_\partial)$ of $\mathcal{T}$. Choose a spanning tree $\mathcal{B}' = (V' \cup V'_\partial, E'_{\mathsf{S}})$ of $\mathcal{A}'$ and denote by $\mathcal{E}'_{\mathsf{S}}$ the set of oriented dual edges in $\mathcal{E}' \cup \mathcal{E}'_\partial$ corresponding to E'_{S}; namely, we set $\mathcal{E}'_{\mathsf{S}} := \{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial \,|\, |\partial_1 e'| \in E'_{\mathsf{S}}\}$. We call $\mathcal{E}'_{\mathsf{S}}$ *set of oriented dual edges of $\mathcal{B}'$*.

Fix a dual vertex $\mathbf{a}' \in V' \cup V'_\partial$, which we consider a root of $\mathcal{B}'$. Let us give the rigorous definition of "(unique) 1-chain $C'_{\mathbf{v}'}$ of $\mathcal{B}'$ from the root $\mathbf{a}'$ to another vertex $\mathbf{v}'$". Consider a dual vertex $\mathbf{v}'$ in $V' \cup V'_\partial$. First, suppose $\mathbf{v}' \neq \mathbf{a}'$. Since $\mathcal{B}'$ is a tree, there exist, and are unique, a positive integer m and an ordered sequence $(\mathbf{w}'_0, \mathbf{w}'_1, \ldots, \mathbf{w}'_m)$ of vertices in $V' \cup V'_\partial$ such that $\mathbf{w}'_0 = \mathbf{a}'$, $\mathbf{w}'_m = \mathbf{v}'$, $\mathbf{w}'_i \neq \mathbf{w}'_j$ for every $i, j \in \{0, 1, \ldots, m\}$ with $i \neq j$ and $\{\mathbf{w}'_{k-1}, \mathbf{w}'_k\} \in E'_\mathsf{S}$ for every $k \in \{1, \ldots, m\}$. In this way, for every $k \in \{1, \ldots, m\}$, there exist, and are unique, $e'_k \in \mathcal{E}'_\mathsf{S}$ and $\delta_k \in \{-1, 1\}$ such that $\partial_1(\delta_k e'_k) = \mathbf{w}'_k - \mathbf{w}'_{k-1}$. We can now define $C'_{\mathbf{v}'} \in C_1(\mathcal{A}'; \mathbb{Z})$ as follows:

$$C'_{\mathbf{v}'} := \sum_{k=1}^{m} \delta_k e'_k. \tag{3.1}$$

Evidently, we have $\partial_1(C'_{\mathbf{v}'}) = \mathbf{v}' - \mathbf{a}'$. If $\mathbf{v}' = \mathbf{a}'$, then we define $C'_{\mathbf{v}'}$ as the zero 1-chain in $C_1(\mathcal{A}'; \mathbb{Z})$.

For every oriented dual edge $e' \in \mathcal{E}' \cup \mathcal{E}'_\partial$ with $\partial_1 e' = \mathbf{v}' - \mathbf{w}'$, we define the 1-cycle $\sigma_{\mathcal{B}'}(e')$ of $\mathcal{A}'$ by setting $\sigma_{\mathcal{B}'}(e') := C'_{\mathbf{w}'} + e' - C'_{\mathbf{v}'}$.

The reader observes that $\sigma_{\mathcal{B}'}(e')$ depends only on $\mathcal{B}'$ and on e', and not on the chosen root $\mathbf{a}'$ of $\mathcal{B}'$. Moreover, if $e' \in \mathcal{E}'_\mathsf{S}$, then $\sigma_{\mathcal{B}'}(e') = 0$.

Denote by $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ the connected components of $\partial\Omega$. For every $i \in \{0, 1, \ldots, p\}$, we define $V'_{\partial,i}$ as the set of vertices in V'_∂ belonging to Γ_i, and $E'_{\partial,i}$ as the set of dual edges $\{\mathbf{v}', \mathbf{w}'\}$ in E'_∂ such that $\{\mathbf{v}', \mathbf{w}'\} \subset \Gamma_i$. Indicate by $\mathcal{A}'_i$ the graph $(V'_{\partial,i}, E'_{\partial,i})$. It is the graph induced by $\mathcal{A}'$ on Γ_i.

Definition 8. *Let $\mathcal{B}' = (V' \cup V'_\partial, E'_\mathsf{S})$ be a spanning tree of $\mathcal{A}'$. We say that $\mathcal{B}'$ is a* Seifert dual (barycentric) spanning tree of $\mathcal{T}$ *if it restricts to a spanning tree on each connected component Γ_i of $\partial\Omega$; more precisely, if*

$$(V'_{\partial,i}, E'_\mathsf{S} \cap E'_{\partial,i}) \text{ is a spanning tree of } \mathcal{A}'_i \text{ for every } i \in \{0, 1, \ldots, p\}. \tag{3.2}$$

This kind of dual spanning tree is also used in computational electromagnetism; see, for instance, [1].

Remark 9. *We pointed out in the introduction that, given a spanning tree $\mathcal{B}'$ of $\mathcal{A}'$, the number $N_{\mathcal{B}'}$ of oriented faces of $\mathcal{T}$ whose dual edge belongs to $\mathcal{B}'$ is $\geq \mathfrak{t}+p$, where $\mathfrak{t}$ is the number of tetrahedra of $\mathcal{T}$. Moreover, the equality holds if and only if $\mathcal{B}'$ is a Seifert dual spanning tree of $\mathcal{T}$. The following simple argument of graph theory explains why. Let $i \in \{0, 1, \ldots, p\}$. Indicate by $\mathfrak{v}'_i$ the number of vertices of $\mathcal{A}'_i$ or, equivalently, the number of faces of F_∂ contained in Γ_i. Evidently, the number of vertices of $\mathcal{A}'$ is $\mathfrak{t} + \sum_{i=0}^{p} \mathfrak{v}'_i$. Denote by $\mathcal{B}'_i$ the graph induced by $\mathcal{B}'$ on Γ_i and by k_i the number of connected components of $\mathcal{B}'_i$. Bearing in mind that $\mathcal{B}'$ is a spanning tree of $\mathcal{A}'$, we infer at once that $\mathcal{B}'_i$ is a subgraph of $\mathcal{A}'_i$ with the same vertices as $\mathcal{A}'_i$, whose connected components are trees. In particular, $\mathcal{B}'_i$ is a spanning tree of $\mathcal{A}'_i$ if and only if $k_i = 1$. Since in a finite tree the number of edges is equal to the number of vertices minus 1, we have that the number of edges of $\mathcal{B}'$ is $(\mathfrak{t} + \sum_{i=0}^{p} \mathfrak{v}'_i) - 1$ and the number of edges of $\mathcal{B}'_i$ is $\mathfrak{v}'_i - k_i$. It follows that*

$$N_{\mathcal{B}'} = (\mathfrak{t} + \textstyle\sum_{i=0}^{p} \mathfrak{v}'_i) - 1 - \sum_{i=0}^{p}(\mathfrak{v}'_i - k_i) = \mathfrak{t} - 1 + \sum_{i=0}^{p} k_i \geq \mathfrak{t} + p$$

and $N_{\mathcal{B}'} = \mathfrak{t} + p$ if and only if each k_i is equal to 1 or, equivalently, if and only if the graph $\mathcal{B}'_i$ is a spanning tree of $\mathcal{A}'_i$ for each $i \in \{0, 1, \ldots, p\}$; namely, if $\mathcal{B}'$ is a Seifert dual spanning tree of $\mathcal{T}$.

The reader observes that a Seifert dual spanning tree of $\mathcal{T}$ always exists and it is easy to construct. Indeed, it suffices to choose a spanning tree $\mathcal{B}'_i$ of each $\mathcal{A}'_i$ and to extend the union of the $\mathcal{B}'_i$'s to a spanning tree of the whole $\mathcal{A}'$.

Our main result reads as follows:

THEOREM 10. *Let $\mathcal{B}' = (V' \cup V'_\partial, E'_\mathsf{S})$ be a Seifert dual spanning tree of $\mathcal{T}$ and let $\mathcal{E}'_\mathsf{S}$ be its set of oriented dual edges. Then, for every 1-boundary γ of $\mathcal{T}$, there exists, and is unique, a homological Seifert surface $S = \sum_{f \in \mathcal{F}} b_f f$ of γ in $\mathcal{T}$ such that $b_f = 0$ for every $f \in \mathcal{F}$ with $D(f) \in \mathcal{E}'_\mathsf{S}$. Moreover, we have:*

$$b_f = \ell_K\big(R_+(\gamma), \sigma_{\mathcal{B}'}(D(f))\big). \tag{3.3}$$

for every $f \in \mathcal{F}$.

We consider also the problem of the existence and of the construction of internal homological Seifert surfaces. To this end, we need a definition, in which we will employ the notion of maximal plug-set of $\mathcal{T}$ introduced in Definition 5.

DEFINITION 11. *Given a spanning tree $\mathcal{B}' = (V' \cup V'_\partial, E'_\mathsf{S})$ of $\mathcal{A}'$, we say that $\mathcal{B}'$ is a* strongly-Seifert dual (barycentric) spanning tree of $\mathcal{T}$ *if it satisfies* (3.2) *and the set E'_S of its edges contains a maximal plug-set of $\mathcal{T}$.*

Once again, strongly-Seifert dual spanning trees of $\mathcal{T}$ always exist, and are easy to construct. Let $i \in \{0, 1, \dots, p\}$. Choose a spanning tree $\mathcal{B}'_i = (V'_{\partial,i}, E'_{\mathsf{S},i})$ of each $\mathcal{A}'_i$. Denote by $\mathrm{J}^r_{\mathcal{T},i}$ the set of regular plugs of $\mathcal{T}$ induced by the faces $f \in F_\partial \setminus F^\angle_\partial$ with $f \subset \Gamma_i$. Let $K_{\partial,i}$ be the set of tetrahedra $t \in K$ such that t contains at least one face in Γ_i and let $K^\angle_{\partial,i} := K_{\partial,i} \cap K^\angle_\partial$. For every $t \in K^\angle_{\partial,i}$, choose one of the corner faces of $\mathcal{T}$ contained in t and denote it by $f^\angle_{t,i}$. Let J'_i be the set of corner plugs of $\mathcal{T}$ induced by the chosen corner faces $\{f^\angle_{t,i}\}_{t \in K^\angle_{\partial,i}}$, let $J''_i := \mathrm{J}^r_{\mathcal{T},i} \cup J'_i$ and let V''_i be the set of dual vertices of $\mathcal{T}$ of the form $B(t)$ with $t \in K_{\partial,i}$; namely, $V''_i = \{B(t) \in V' \,|\, t \in K_{\partial,i}\}$. By construction, the graph $\mathcal{B}''_i := (V'_{\partial,i} \cup V''_i, E'_{\mathsf{S},i} \cup J''_i)$ is a tree containing $\mathcal{B}'_i$. Moreover, it is immediate to verify that, for every $i, j \in \{0, 1, \dots, p\}$ with $i \neq j$, $\mathcal{B}''_i$ and $\mathcal{B}''_j$ have neither vertices nor edges in common. In particular, the set $\bigcup_{i=0}^p J''_i$ is a maximal plug-set of $\mathcal{T}$. Now one can extend the union of the $\mathcal{B}''_i$'s to a spanning tree of $\mathcal{A}'$, which turns out to be a strongly-Seifert dual spanning tree of $\mathcal{T}$.

The reader observes that the maximal plug-set of $\mathcal{T}$ contained in the set of edges of a given strongly-Seifert dual spanning tree of $\mathcal{T}$, which exists by definition, is unique.

As a consequence of Theorem 10, we have the following result, which settles the above-mentioned problem of the existence and of the construction of internal homological Seifert surfaces.

THEOREM 12. *The following assertions hold.*

(i) *A 1-boundary of $\mathcal{T}$ has an internal homological Seifert surface in $\mathcal{T}$ if and only if it is corner-free.*

(ii) *Let $\mathcal{B}' = (V' \cup V'_\partial, E'_\mathsf{S})$ be a strongly-Seifert dual spanning tree of $\mathcal{T}$ and let $\mathcal{E}'_\mathsf{S}$ be its set of oriented dual edges. Then, for every corner-free 1-boundary γ of $\mathcal{T}$, there exists, and is unique, an internal homological Seifert surface $S = \sum_{f \in \mathcal{F}} b_f f$ of γ in $\mathcal{T}$ such that $b_f = 0$ for every $f \in \mathcal{F}$ with $D(f) \in \mathcal{E}'_\mathsf{S}$. Moreover, each coefficient b_f satisfies formula* (3.3).

In particular, we have:

COROLLARY 13. *The following assertions hold.*

(i) *Every internal* 1*-boundary of* $\mathcal{T}$ *has an internal homological Seifert surface in* $\mathcal{T}$.
(ii) *If* $\mathcal{T}$ *is the first barycentric subdivision of some triangulation of* $\overline{\Omega}$*, then every* 1*-boundary of* $\mathcal{T}$ *has an internal homological Seifert surface in* $\mathcal{T}$.

3.2. The proofs. We begin by proving Theorem 10. First, we need three preliminary lemmas.

Let $\mathcal{B}' = (V' \cup V'_\partial, E'_\mathsf{S})$ be a Seifert dual spanning tree of $\mathcal{T}$ and let $\mathcal{E}'_\mathsf{S}$ be its set of oriented dual edges. We define $\mathcal{G} := \{f \in \mathcal{F} \,|\, D(f) \notin \mathcal{E}'_\mathsf{S}\}$ and, for every $f \in \mathcal{F}$, we simplify the notation by writing $\sigma(f)$ in place of $\sigma_{\mathcal{B}'}(D(f))$.

Lemma 3.1. *For every* $f, g \in \mathcal{G}$*, we have:*

$$\ell_\mathcal{K}(\partial_2 f, R_-(\sigma(g))) = \begin{cases} 1 & \text{if } f = g \\ 0 & \text{if } f \neq g \end{cases}.$$

Proof. Let $f, g \in \mathcal{G}$ and let $\mathbf{v}', \mathbf{w}' \in V' \cup V'_\partial$ such that $\partial_1 D(g) = \mathbf{v}' - \mathbf{w}'$. By definition of $\sigma(g)$, there exist, and are unique, an integer $\ell \geq 2$, a $(\ell+1)$-tuple of pairwise disjoint vertices $(p'_0, p'_1, \ldots, p'_\ell)$ of $V' \cup V'_\partial$ and, for every $i \in \{1, \ldots, \ell\}$, $\delta_i \in \{-1, 1\}$, and $e'_i \in \mathcal{E}'_\mathsf{S}$ such that $p'_0 = \mathbf{v}'$, $p'_\ell = \mathbf{w}'$, $\partial_1(\delta_i e'_i) = p'_i - p'_{i-1}$ for every $i \in \{1, \ldots, \ell\}$ and $\sigma(g) = D(g) + \sum_{i=1}^{\ell} \delta_i e'_i$.

There are only two cases in which the intersection $|f| \cap |R_-(\sigma(g))|$ is non-empty, and hence the linking number $\ell_\mathcal{K}(\partial_2 f, R_-(\sigma(g)))$ may be different from zero.

Case 1: Assume $f = g$. In this case, we have that $|f| \cap |R_-(\sigma(g))| = \{B(f)\}$. We must prove that $\ell_\mathcal{K}(\partial_2 f, R_-(\sigma(g))) = 1$. Suppose that $f \notin \mathcal{F}_\partial$. Observe that the intersection between f and $R_-(\sigma(g))$ is not transverse, because $D(g) = [\mathbf{w}', B(f)] + [B(f), \mathbf{v}']$. Let a'_1 be a point of the segment $|[\mathbf{w}', B(f)]|$ different from $B(f)$, let b'_1 be a point of the segment $|[B(f), \mathbf{v}']|$ different from $B(f)$ and let $\widehat{\gamma}_1$ be the 1-cycle of $\mathbb{R}^3$ defined by setting

$$\widehat{\gamma}_1 := [\mathbf{w}', a'_1] + [a'_1, b'_1] + [b'_1, \mathbf{v}'] + \sum_{i=1}^{\ell} \delta_i r_-(e'_i),$$

see Figure 3.1 on the left. If a'_1 and b'_1 are chosen sufficiently close to $B(f)$, we have that $\widehat{\gamma}_1$ is homologous to $R_-(\sigma(g))$ in $\mathbb{R}^3 \setminus |\partial_2 f|$, it intersects f transversely in one point belonging to $|[a'_1, b'_1]| \setminus \{a'_1, b'_1\}$ and $\mathrm{sign}(\boldsymbol{\tau}([a'_1, b'_1]) \cdot \boldsymbol{\nu}(f)) = 1$. By the definition of linking number, we infer that $\ell_\mathcal{K}(\partial_2 f, R_-(\sigma(g))) = 1$.

Suppose now that $f \in \mathcal{F}_\partial$. Changing the orientation of f if necessary, we may also suppose that $\mathbf{v}' = B(f)$. It follows that p'_1 is the barycenter of an oriented face f_1 in $\mathcal{E}_\partial$ having an (oriented) edge e in common with f and hence $\delta_1 r_-(e'_1) = [\mathbf{v}', \mathbf{x}_{e'_1}] + [\mathbf{x}_{e'_1}, p'_1]$ for some point $\mathbf{x}_{e'_1} \in \mathbb{R}^3 \setminus \overline{\Omega}$ close to $B(e)$ (see Subsection 2.3 for the definition of r_-). Let us proceed as above. Choose a point $a'_2 \in |[\mathbf{w}', \mathbf{v}']| \setminus \{\mathbf{v}'\}$ close to $\mathbf{v}'$ and a point $b'_2 \in |[\mathbf{v}', \mathbf{x}_{e'_1}]| \setminus \{\mathbf{v}'\}$ close to $\mathbf{v}'$. Then the 1-cycle $\widehat{\gamma}_2$ of $\mathbb{R}^3$ defined by setting

$$\widehat{\gamma}_2 := [\mathbf{w}', a'_2] + [a'_2, b'_2] + [b'_2, \mathbf{x}_{e'_1}] + [\mathbf{x}_{e'_1}, p'_1] + \sum_{i=2}^{\ell} \delta_i r_-(e'_i),$$

see Figure 3.1 on the right, is homologous to $R_-(\sigma(g))$ in $\mathbb{R}^3 \setminus |\partial_2 f|$, it intersects f transversely in one point belonging to $|[a'_2, b'_2]| \setminus \{a'_2, b'_2\}$ and $\mathrm{sign}(\boldsymbol{\tau}([a'_2, b'_2]) \cdot \boldsymbol{\nu}(f)) = 1$. It follows that $\ell_\mathcal{K}(\partial_2 f, R_-(\sigma(g))) = 1$, as desired.

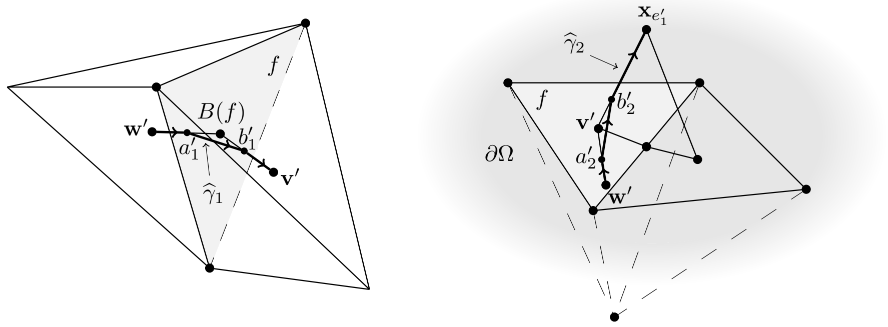

FIG. 3.1. *The 1-cycles $\widehat{\gamma}_1$ (on the left) and $\widehat{\gamma}_2$ (on the right).*

Case 2. Assume that $f \neq g$, $f \in \mathcal{F}_\partial$ and there exists $h \in \{1, \dots, \ell - 1\}$ such that $p'_h = B(f)$ and both e'_h and e'_{h+1} belong to $\mathcal{E}'_\partial$. We know that $\delta_h r_-(e'_h) = [p'_{h-1}, \mathbf{x}_{e'_h}] + [\mathbf{x}_{e'_h}, p'_h]$ and $\delta_{h+1} r_-(e'_{h+1}) = [p'_h, \mathbf{x}_{e'_{h+1}}] + [\mathbf{x}_{e'_{h+1}}, p'_{h+1}]$ for some $\mathbf{x}_{e'_h}, \mathbf{x}_{e'_{h+1}} \in \mathbb{R}^3 \setminus \overline{\Omega}$. In particular, we have:

$$R_-(\sigma(g)) = c + [p'_{h-1}, \mathbf{x}_{e'_h}] + [\mathbf{x}_{e'_h}, p'_h] + [p'_h, \mathbf{x}_{e'_{h+1}}] + [\mathbf{x}_{e'_{h+1}}, p'_{h+1}],$$

where $c := D(g) + \sum_{i \in \{1,\dots,\ell\} \setminus \{h,h+1\}} \delta_i r_-(e'_i)$. Let $a'_3 \in |[\mathbf{x}_{e'_h}, p'_h]| \setminus \{p'_h\}$, let $b'_3 \in |[p'_h, \mathbf{x}_{e'_{h+1}}]| \setminus \{p'_h\}$ and let $\widehat{\gamma}_3$ be the 1-cycle of $\mathbb{R}^3$ defined by setting

$$\widehat{\gamma}_3 := c + [p'_{h-1}, \mathbf{x}_{e'_h}] + [\mathbf{x}_{e'_h}, a'_3] + [a'_3, b'_3] + [b'_3, \mathbf{x}_{e'_{h+1}}] + [\mathbf{x}_{e'_{h+1}}, p'_{h+1}],$$

see Figure 3.2. If a'_3 and b'_3 are chosen sufficiently close to p'_h, then $\widehat{\gamma}_3$ is homologous to $R_-(\sigma(g))$ in $\mathbb{R}^3 \setminus |\partial_2 f|$ and it does not intersect $|f|$. It follows that $\ell_{\mathcal{K}}(\partial_2 f, R_-(\sigma(g))) = 0$.

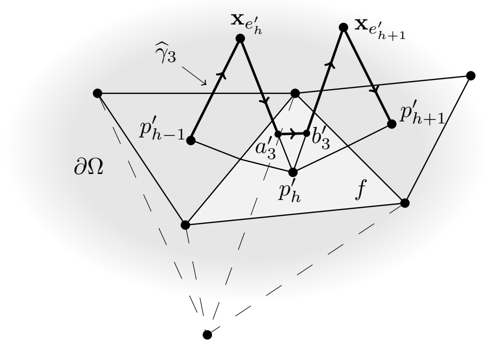

FIG. 3.2. *The 1-cycle $\widehat{\gamma}_3$.*

This completes the proof. □

LEMMA 3.2. *Let $\xi = \sum_{e \in \mathcal{E}} \alpha_e e$ be a 1-cycle of $\mathcal{T}$. Then, for every $e^* \in \mathcal{E}$, we have:*

$$\ell_{\mathcal{K}}\big(\xi, R_-(\mathrm{Coil}(e^*))\big) = \alpha_{e^*}. \tag{3.4}$$

In particular, $\xi = 0$ if and only if $\ell_{\mathcal{K}}\big(\xi, R_-(\mathrm{Coil}(e^*))\big) = 0$ for every $e^* \in \mathcal{E}$.

Proof. Fix $e^* \in \mathcal{E}$, a spanning tree (V, L) of the graph (V, E) such that $|\partial_1 e^*| \notin L$ and a vertex $\mathbf{a} \in V$, which is a root of (V, L). Denote by $\mathcal{L}$ the set of oriented edges in $\mathcal{E}$ determined by the corresponding edges in L; namely, $\mathcal{L} := \{e \in \mathcal{E} \,\big|\, |\partial_1 e| \in L\}$. For every $\mathbf{v} \in V$, denote by $C_{\mathbf{v}}$ the (unique) 1-chain of $\mathcal{T}$ such that $|C_{\mathbf{v}}| \subset \bigcup_{e \in \mathcal{L}} |e|$ and $\partial_1 C_{\mathbf{v}} = \mathbf{v} - \mathbf{a}$. Given $e = [\mathbf{a}_e, \mathbf{b}_e] \in \mathcal{E}$, we denote by σ_e the 1-cycle of $\mathcal{T}$ given by $\sigma_e := C_{\mathbf{a}_e} + e - C_{\mathbf{b}_e}$.

By hypothesis, ξ is a 1-cycle of $\mathcal{T}$ and hence $0 = \partial_1 \xi = \sum_{e \in \mathcal{E}} \alpha_e (\mathbf{b}_e - \mathbf{a}_e)$ in $C_0(\mathcal{T}; \mathbb{Z})$. Since $C_{\mathbf{a}_e}$ and $C_{\mathbf{b}_e}$ depend only on $\mathbf{a}_e$ and $\mathbf{b}_e$, respectively, it follows that $0 = \sum_{e \in \mathcal{E}} \alpha_e (C_{\mathbf{b}_e} - C_{\mathbf{a}_e})$ in $C_1(\mathcal{T}; \mathbb{Z})$ as well. In this way, we obtain that

$$\sum_{e \in \mathcal{E}} \alpha_e \sigma_e = \sum_{e \in \mathcal{E}} \alpha_e (C_{\mathbf{a}_e} + e - C_{\mathbf{b}_e}) = \xi - \sum_{e \in \mathcal{E}} \alpha_e (C_{\mathbf{b}_e} - C_{\mathbf{a}_e}) = \xi.$$

Then $\ell_{\mathcal{K}}\big(\xi, R_-(\mathrm{Coil}(e^*))\big) = \sum_{e \in \mathcal{E}} \alpha_e \, \ell_{\mathcal{K}}\big(\sigma_e, R_-(\mathrm{Coil}(e^*))\big)$. Thanks to the latter equality, it suffices to show that

$$\ell_{\mathcal{K}}\big(\sigma_e, R_-(\mathrm{Coil}(e^*))\big) = \begin{cases} 1 & \text{if } e = e^* \\ 0 & \text{if } e \neq e^* \end{cases}.$$

To do this, we use an argument similar to the one employed in the proof of the preceding lemma. However, contrarily to such a proof, we omit the details concerning the construction of "small deformations of σ_e" to obtain transversality. If $e \in \mathcal{L}$, then $e \neq e^*$ (because $e^* \notin \mathcal{L}$), $\sigma_e = 0$ and hence $\ell_{\mathcal{K}}\big(\sigma_e, R_-(\mathrm{Coil}(e^*))\big) = 0$. If $e \notin \mathcal{L} \cup \{e^*\}$, then $|\sigma_e| \cap |D(e^*)| = \emptyset$, so $\ell_{\mathcal{K}}\big(\sigma_e, R_-(\mathrm{Coil}(e^*))\big) = 0$. Suppose $e = e^* \in \mathcal{E} \setminus \mathcal{E}_\partial$. In this case, we have that $R_-(\mathrm{Coil}(e)) = \mathrm{Coil}(e) = \partial_2 D(e)$ and $|\sigma_e| \cap |D(e)| = \{B(e)\}$. By Equation (2.1), it follows immediately that $\ell_{\mathcal{K}}\big(\sigma_e, R_-(\mathrm{Coil}(e))\big) = \pm 1$. The sign of such a linking number is positive, because the triangles forming $D(e)$ were oriented by e via the right hand rule. Finally, consider the case in which $e = e^* \in \mathcal{E}_\partial$. By construction (see Definition 3 and points (2.6) and (2.7)), we have that $R_-(\mathrm{Coil}(e)) = \partial_2\big(D(e) + S'_{D_\partial(e)}\big)$ and $|\sigma_e| \cap \big|D(e) + S'_{D_\partial(e)}\big| = \{B(e)\}$. Once again, we infer that $\ell_{\mathcal{K}}\big(\sigma_e, R_-(\mathrm{Coil}(e))\big) = 1$. □

LEMMA 3.3. *Let γ be a 1-boundary of $\mathcal{T}$. Then, for every $e' \in \mathcal{E}'_\partial$, we have:*

$$\ell_{\mathcal{K}}\big(\gamma, R_-(\sigma_{\mathcal{B}'}(e'))\big) = 0\,.$$

Proof. If $e' \in \mathcal{E}'_{\mathsf{S}}$, then $\sigma_{\mathcal{B}'}(e') = 0$ and the result is trivial. Choose $e' \in \mathcal{E}'_\partial \setminus \mathcal{E}'_{\mathsf{S}}$ and indicate by i the unique index in $\{0, 1, \dots, p\}$ such that $|\partial_1 e'| \in E'_{\partial,i}$ or, equivalently, $|e'| \subset \Gamma_i$. Since $\mathcal{B}'_{\partial,i} := (V'_{\partial,i}, E'_{\mathsf{S}} \cap E'_{\partial,i})$ is a spanning tree of $\mathcal{A}'_i$, there exists a unique vertex $\mathbf{b}'_i$ in $V'_{\partial,i}$ such that $|C'_{\mathbf{b}'_i}| \subset \bigcup_{e' \in \mathcal{E}'} |e'|$; namely, in the expression of $C'_{\mathbf{b}'_i}$, the oriented dual edges in $\mathcal{E}'_\partial$ appear with null coefficients (see (3.1) for the definition of $C'_{\mathbf{b}'_i}$). Let $\mathcal{E}'_{\partial,i}$ be the set of oriented dual edges in $\mathcal{E}'_\partial$ corresponding to the edges in $E'_{\partial,i}$; namely, $\mathcal{E}'_{\partial,i} := \{e' \in \mathcal{E}'_\partial \,\big|\, |\partial_1 e'| \in E'_{\partial,i}\}$. For every $\mathbf{v}' \in V'_{\partial,i}$, denote by $c'_{i,\mathbf{v}'}$ the unique 1-chain of $\mathcal{B}'_{\partial,i}$ from $\mathbf{b}'_i$ to $\mathbf{v}'$. Let $e' \in \mathcal{E}'_{\partial,i}$ with $\partial_1 e' = \mathbf{v}' - \mathbf{w}'$. Observe that $C'_{\mathbf{v}'} = C'_{\mathbf{b}'_i} + c'_{i,\mathbf{v}'}$, $C'_{\mathbf{w}'} = C'_{\mathbf{b}'_i} + c'_{i,\mathbf{w}'}$ and hence

$$\sigma_{\mathcal{B}'}(e') = c'_{i,\mathbf{w}'} + e' - c'_{i,\mathbf{v}'}.$$

It follows that $|\sigma_{\mathcal{B}'}(e')| \subset \Gamma_i$ and hence $|R_-(\sigma_{\mathcal{B}'}(e'))| \subset (\mathbb{R}^3 \setminus \overline{\Omega}) \cup V'_{\partial,i}$. Since $\partial\Omega$ has a collar in $\mathbb{R}^3 \setminus \Omega$, it is easy to find a 1-cycle η of $\mathbb{R}^3$ such that $|\eta| \subset \mathbb{R}^3 \setminus \overline{\Omega}$ and η is homologous to $R_-(\sigma_{\mathcal{B}'}(e'))$ in $(\mathbb{R}^3 \setminus \overline{\Omega}) \cup V'_{\partial,i} \subset \mathbb{R}^3 \setminus |\gamma|$. Thanks to (2.2), we infer that $\ell_{\mathcal{K}}\big(\gamma, R_-(\sigma_{\mathcal{B}'}(e'))\big) = \ell_{\mathcal{K}}(\gamma, \eta)$. On the other hand, by hypothesis, γ bounds in $\overline{\Omega}$. Since $\overline{\Omega} \subset \mathbb{R}^3 \setminus |\eta|$, γ bounds in $\mathbb{R}^3 \setminus |\eta|$ as well. Equality (2.3) ensures that $\ell_{\mathcal{K}}(\gamma, \eta) = 0$, as desired. □

We are now in position to prove our results.

Proof of Theorem 10. We start by proving the uniqueness of the solution. Suppose that $S = \sum_{f \in \mathcal{F}} b_f f$ is a homological Seifert surface of γ in $\mathcal{T}$ such that $b_f = 0$ for every f with $D(f) \in \mathcal{E}'_{\mathsf{S}}$; namely, for every $f \in \mathcal{F} \setminus \mathcal{G}$. We must show that $b_f = \ell_{\mathcal{K}}\big(R_+(\gamma), \sigma(f)\big)$ for every $f \in \mathcal{G}$. The reader observes that, if $f \in \mathcal{F} \setminus \mathcal{G}$, then $\sigma(f) = 0$ and hence $\ell_{\mathcal{K}}\big(R_+(\gamma), \sigma(f)\big)$ is automatically equal to $0 = b_f$. Choose $f^* \in \mathcal{G}$. By Lemma 2.1, we infer that $\ell_{\mathcal{K}}\big(R_+(\gamma), \sigma(f^*)\big) = \ell_{\mathcal{K}}\big(\gamma, R_-(\sigma(f^*))\big) = \ell_{\mathcal{K}}\Big(\sum_{f \in \mathcal{G}} b_f \partial_2 f, R_-(\sigma(f^*))\Big) = \sum_{f \in \mathcal{G}} b_f \, \ell_{\mathcal{K}}\big(\partial_2 f, R_-(\sigma(f^*))\big)$.

Now Lemma 3.1 implies that $\sum_{f \in \mathcal{G}} b_f \, \ell_{\mathcal{K}}\big(\partial_2 f, R_-(\sigma(f^*))\big) = b_{f^*}$. In this way, we have that $\ell_{\mathcal{K}}\big(R_+(\gamma), \sigma(f^*)\big) = b_{f^*}$ for every $f^* \in \mathcal{G}$, as desired.

It remains to prove that, if $b_f := \ell_{\mathcal{K}}\big(R_+(\gamma), \sigma(f)\big)$ for every $f \in \mathcal{G}$, then the boundary of the 2-chain $S := \sum_{f \in \mathcal{G}} b_f f$ of $\mathcal{T}$ is equal to γ. This is equivalent to showing that the 1-cycle $\eta := \gamma - \partial_2 S = \gamma - \sum_{f \in \mathcal{G}} b_f \, \partial_2 f$ of $\mathcal{T}$ is equal to the zero 1-chain of $\mathcal{T}$. Thanks to Lemma 3.2, this is in turn equivalent to showing that $\ell_{\mathcal{K}}\big(\eta, R_-(\mathrm{Coil}(e))\big) = 0$ for every $e \in \mathcal{E}$.

Fix $e \in \mathcal{E}$ and write Coil(e) explicitly as follows: $\mathrm{Coil}(e) = \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a'_{e'} e'$ for some (unique) integer $a'_{e'}$. For every $e' \in \mathcal{E}' \cup \mathcal{E}'_\partial$, denote by $\mathbf{v}'(e')$ and $\mathbf{w}'(e')$ the dual vertices in V' such that $\partial_1 e' = \mathbf{v}'(e') - \mathbf{w}'(e')$. Since Coil$(e)$ is a 1-cycle of $\mathcal{A}'$ (a 1-boundary of $\mathcal{A}'$ indeed), we have that $0 = \partial_1 \mathrm{Coil}(e) = \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a'_{e'}(\mathbf{v}'(e') - \mathbf{w}'(e'))$. It follows that $\sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a'_{e'}(C'_{\mathbf{v}'(e')} - C'_{\mathbf{w}'(e')}) = 0$ as well, and hence

$$\mathrm{Coil}(e) = \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a'_{e'} e' - \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a'_{e'}(C'_{\mathbf{v}'(e')} - C'_{\mathbf{w}'(e')}) = \sum_{e' \in \mathcal{E}' \cup \mathcal{E}'_\partial} a'_{e'} \sigma_{\mathcal{B}'}(e'). \tag{3.5}$$

In this way, in order to complete the proof, it suffices to prove that $\ell_{\mathcal{K}}\big(\eta, R_-(\sigma_{\mathcal{B}'}(e')\big) = 0$ for every $e' \in \mathcal{E}' \cup \mathcal{E}'_\partial$.

We distinguish three cases: $e' \in \mathcal{E}'_{\mathsf{S}}$, $e' \in \mathcal{E}' \setminus \mathcal{E}'_{\mathsf{S}}$, and $e' \in \mathcal{E}'_\partial \setminus \mathcal{E}'_{\mathsf{S}}$.

If $e' \in \mathcal{E}'_{\mathsf{S}}$, then $\sigma_{\mathcal{B}'}(e') = 0$ and hence $\ell_{\mathcal{K}}\big(\eta, R_-(\sigma_{\mathcal{B}'}(e'))\big) = 0$.

If $e' \in \mathcal{E}' \setminus \mathcal{E}'_{\mathsf{S}}$, then $e' = D(f^*)$ for some (unique) $f^* \in \mathcal{G}$. Bearing in mind Lemma 3.1, we obtain:

$$\begin{aligned} \ell_{\mathcal{K}}\big(\eta, R_-(\sigma_{\mathcal{B}'}(e'))\big) &= \ell_{\mathcal{K}}\big(\eta, R_-(\sigma(f^*))\big) \\ &= \ell_{\mathcal{K}}\big(\gamma, R_-(\sigma(f^*))\big) - \textstyle\sum_{f \in \mathcal{G}} b_f \, \ell_{\mathcal{K}}\big(\partial_2 f, R_-(\sigma(f^*))\big) \\ &= b_{f^*} - b_{f^*} = 0. \end{aligned}$$

Finally, if $e' \in \mathcal{E}'_\partial \setminus \mathcal{E}'_{\mathsf{S}}$, then Lemma 3.3 ensures that $\ell_{\mathcal{K}}\big(\eta, R_-(\sigma_{\mathcal{B}'}(e'))\big) = 0$, because η is a 1-boundary of $\mathcal{T}$. □

We conclude with the proofs of Theorem 12 and of its Corollary 13.

Proof of Theorem 12. Let γ be a 1-boundary of $\mathcal{T}$. It is evident that the boundary of any internal 2-chain of $\mathcal{T}$ cannot contain oriented edges determined by corner edges of $\mathcal{T}$. Hence if γ admits an internal homological Seifert surface in $\mathcal{T}$, then it must be corner-free.

Suppose γ is corner-free. Let $\mathcal{B}' = (V' \cup V'_\partial, E'_{\mathsf{S}})$ and $\mathcal{E}'_{\mathsf{S}}$ be as in the statement of point (ii), and let J be the maximal plug-set of $\mathcal{T}$ contained in E'_{S}. Write J as in Remark 6: $J = \mathrm{J}^r_{\mathcal{T}} \cup J'$, where J' is the set of corner plugs of $\mathcal{T}$ belonging to J. Denote by $F^{\angle}$ the set of corner faces of $\mathcal{T}$ inducing the corner plugs in J'.

By Theorem 10, there exists, and is unique, a homological Seifert surface $S = \sum_{f \in \mathcal{F}} b_f f$ of γ in $\mathcal{T}$ such that $b_f = 0$ for every $f \in \mathcal{F}$ with $D(f) \in \mathcal{E}'_{\mathsf{S}}$. Moreover, each b_f satisfies formula (3.3).

We must prove that S is internal; namely, $b_f = 0$ for every $f \in \mathcal{F}_\partial$. Since $J \subset E'_{\mathsf{S}}$, it suffices to show the following: if g is an oriented face in $\mathcal{F}_\partial$ such that the corresponding (non-oriented) face belongs to $F^{\angle}_\partial \setminus F^{\angle}$, then $b_g = 0$. Let g be such an oriented face in $\mathcal{F}_\partial$. Then there exist vertices $\mathbf{v}, \mathbf{w}, \mathbf{z}^*, \mathbf{z}^{**} \in V_\partial \cap \Gamma_i$ for some (unique) $i \in \{0, 1, \dots, p\}$ such that the tetrahedron $\{\mathbf{v}, \mathbf{w}, \mathbf{z}^*, \mathbf{z}^{**}\}$ of $\mathcal{T}$ is a corner tetrahedron, its face $\{\mathbf{v}, \mathbf{w}, \mathbf{z}^*\}$ belongs to $F^{\angle}$ and the oriented face in $\mathcal{F}$ corresponding to $\{\mathbf{v}, \mathbf{w}, \mathbf{z}^{**}\}$ is equal to g. Indicate by f the oriented face in $\mathcal{F}$ corresponding to $\{\mathbf{v}, \mathbf{w}, \mathbf{z}^*\}$, by e the oriented edge in $\mathcal{E}_\partial$ corresponding to $\{\mathbf{v}, \mathbf{w}\}$, by e' the oriented dual edge $D_\partial(e)$ in $\mathcal{E}'_\partial$ and by $\mathbf{v}', \mathbf{w}'$ the vertices in V'_∂ such that $\partial_1(e') = \mathbf{v}' - \mathbf{w}'$. Observe that there exist, and are unique, $s_1, s_2 \in \{-1, 1\}$ such that

$$\mathrm{Coil}(e) = e' + s_1 D(f) + s_2 D(g). \tag{3.6}$$

For instance, in Figure 3.3 we have $f = [\mathbf{v}, \mathbf{w}, \mathbf{z}^*]$ and $g = [\mathbf{w}, \mathbf{v}, \mathbf{z}^{**}]$ so $s_1 = 1$ and $s_2 = -1$.

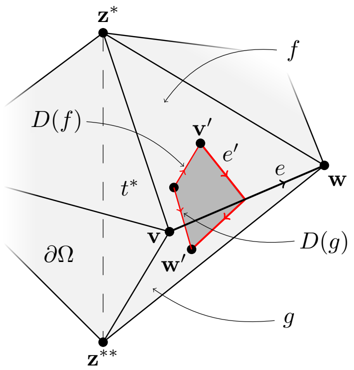

FIG. 3.3. *The coil of the corner edge e in terms of e', $D(f)$ and $D(g)$.*

In particular, since $\partial_1(\mathrm{Coil}(e)) = 0$, we have:

$$\mathbf{v}' - \mathbf{w}' = \partial_1(-s_1 D(f) - s_2 D(g)). \tag{3.7}$$

By hypothesis, $\mathcal{B}'_{\partial,i} := (V'_{\partial,i}, E'_{\mathsf{S}} \cap E'_{\partial,i})$ is a spanning tree of $\mathcal{A}'_i$. In this way, there exists a unique 1-chain C in $\mathcal{B}'_{\partial,i}$ such that $\partial_1(C) = \mathbf{w}' - \mathbf{v}'$. It follows that $\sigma_{\mathcal{B}'}(e') = e' + C$. Moreover, by combining (3.7) with the fact that $D(f) \in \mathcal{E}'_{\mathsf{S}}$, we infer at once that

$$\sigma(g) = -s_2(-s_1 D(f) - s_2 D(g) + C) = D(g) + s_1 s_2 D(f) - s_2 C.$$

On the other hand, by Equation (3.6), we have also that $-s_1 D(f) - s_2 D(g) = e' - \mathrm{Coil}(e)$ and hence

$$\sigma(g) = -s_2(e' - \mathrm{Coil}(e) + C) = -s_2\big(\sigma_{\mathcal{B}'}(e') - \mathrm{Coil}(e)\big) = -s_2 \sigma_{\mathcal{B}'}(e') + s_2 \mathrm{Coil}(e). \tag{3.8}$$

By Lemma 3.3, we know that $\ell_{\mathcal{K}}(\gamma, R_-(\sigma_{\mathcal{B}'}(e'))) = 0$. Moreover, since γ is corner-free and $e \in \mathcal{E}^{\angle}_\partial$, Lemma 3.2 ensures that $\ell_{\mathcal{K}}(\gamma, R_-(\mathrm{Coil}(e))) = 0$. In this way, bearing in mind (3.8) and Lemma 2.1, we have:

$$\begin{aligned} b_g &= \ell_{\mathcal{K}}(R_+(\gamma), \sigma(g)) = -s_2\, \ell_{\mathcal{K}}(R_+(\gamma), \sigma_{\mathcal{B}'}(e')) + s_2\, \ell_{\mathcal{K}}(R_+(\gamma), \mathrm{Coil}(e)) = \\ &= -s_2\, \ell_{\mathcal{K}}(\gamma, R_-(\sigma_{\mathcal{B}'}(e'))) + s_2\, \ell_{\mathcal{K}}(\gamma, R_-(\mathrm{Coil}(e))) = 0, \end{aligned}$$

as desired. This completes the proof. □

Proof of Corollary 13. (i) An internal 1-boundary of $\mathcal{T}$ is corner-free and hence it has an internal homological Seifert surface in $\mathcal{T}$ by Theorem 12.

(ii) As above, this point follows immediately from Theorem 12. Indeed, if $\mathcal{T}$ is the first barycentric subdivision of some triangulation of $\overline{\Omega}$, then $K_\partial^\angle = \emptyset$ and hence every 1-boundary of $\mathcal{T}$ is corner-free. □

4. An elimination algorithm. Let $\gamma = \sum_{e \in \mathcal{E}} a_e e$ be a given 1-boundary of $\mathcal{T}$. A 2-chain $S = \sum_{f \in \mathcal{F}} b_f f$ of $\mathcal{T}$ is a homological Seifert surface of γ in $\mathcal{T}$ if its coefficients $\{b_f\}_{f \in \mathcal{F}}$ satisfy the following equation in $C_1(\mathcal{T};\mathbb{Z})$:

$$\sum_{f \in \mathcal{F}} b_f \partial_2 f = \sum_{e \in \mathcal{E}} a_e e. \tag{4.1}$$

Let us write this equation more explicitly as a linear system with as many equations as edges and as many unknowns as faces of $\mathcal{T}$. Given $e \in \mathcal{E}$, let $\mathcal{F}(e)$ be the set $\{f \in \mathcal{F} \,\big|\, |e| \subset |f|\}$ of oriented faces in $\mathcal{F}$ incident on e and let $\mathrm{o}_e : \mathcal{F}(e) \longrightarrow \{-1,1\}$ be the function sending $f \in \mathcal{F}(e)$ into the coefficient of e in the expression of $\partial_2 f$ as a formal linear combination of oriented edges in $\mathcal{E}$. Equation (4.1) is equivalent to the linear system

$$\sum_{f \in \mathcal{F}(e)} \mathrm{o}_e(f) b_f = a_e \quad \text{if } e \in \mathcal{E},$$

where the unknowns $\{b_f\}_{f \in \mathcal{F}}$ are integers. Theorem 10 ensures that, if $\mathcal{B}' = (V' \cup V'_\partial, E'_\mathsf{S})$ is a Seifert dual spanning tree of $\mathcal{T}$ and $\mathcal{E}'_\mathsf{S}$ is its set of oriented dual edges, then the linear system

$$\sum_{f \in \mathcal{F}(e)} \mathrm{o}_e(f) b_f = a_e \quad \text{if } e \in \mathcal{E} \tag{4.2}$$

$$b_f = 0 \quad \text{if } D(f) \in \mathcal{E}'_\mathsf{S} \tag{4.3}$$

has a unique solution given by the formula:

$$b_f = \ell_\mathcal{K}\big(R_+(\gamma), \sigma_{\mathcal{B}'}(D(f))\big) \tag{4.4}$$

for every $f \in \mathcal{G}$, where $\mathcal{G} = \{f \in \mathcal{F} \,|\, D(f) \notin \mathcal{E}'_\mathsf{S}\}$.

As we have just recalled in the introduction, the linking number can be computed accurately. However, the use of formula (4.4) is too expensive if $\mathcal{T}$ is fine. In fact, if $\mathfrak{v}$ is the number of vertices of $\mathcal{T}$, g is the first Betti number of $\overline{\Omega}$ and $\sharp\mathcal{G}$ is the cardinality of $\mathcal{G}$, then $\sharp\mathcal{G}$ is greater than or equal to $\frac{1}{2}\mathfrak{v} + 1 - g$, which is usually huge if $\mathcal{T}$ is fine. Let us explain the latter assertion. Let $\mathfrak{e}$, $\mathfrak{f}$, and $\mathfrak{t}$ be the numbers of edges, of faces, and of tetrahedra of $\mathcal{T}$, respectively. Let us prove that $\sharp\mathcal{G} = \mathfrak{e} - \mathfrak{v} + 1 - g \geq \frac{1}{2}\mathfrak{v} + 1 - g$. We know that $\sharp\mathcal{G} = \mathfrak{f} - (\mathfrak{t} + p)$ (see Remark 9). The Euler characteristic $\chi(\mathcal{T}) = \mathfrak{v} - \mathfrak{e} + \mathfrak{f} - \mathfrak{t}$ of $\mathcal{T}$ is equal to the sum $\sum_{j=0}^3 (-1)^j r_j$, where r_j is the rank of the j^{th} homology group $H_j(\mathcal{T};\mathbb{Z})$ of $\mathcal{T}$. Since $r_0 = 1$, $r_1 = g$, $r_2 = p$, and $r_3 = 0$, we infer that $\mathfrak{v} - \mathfrak{e} + \mathfrak{f} - \mathfrak{t} = 1 - g + p$ and hence $\sharp\mathcal{G} = \mathfrak{e} - \mathfrak{v} + 1 - g$. Recall that, in a finite graph, the sum of degrees of its vertices equals two times the number of its edges. Apply this result to the graph $\mathcal{A} = (V, E)$. Since each vertex v in V belongs to at least one tetrahedron of $\mathcal{T}$, the degree of v, as a vertex of $\mathcal{A}$, is ≥ 3. It follows that $\mathfrak{e} \geq \frac{3}{2}\mathfrak{v}$ and hence $\sharp\mathcal{G} \geq \frac{1}{2}\mathfrak{v} + 1 - g$.

We present below a simple elimination algorithm that simplifies drastically the construction of homological Seifert surfaces given by Theorem 10. Let us denote by $\mathcal{R}$ the set of oriented faces

f in $\mathcal{F}$ for which the corresponding coefficient b_f is already known. Initially, thanks to (4.3), we have that $\mathcal{R} = \mathcal{F} \setminus \mathcal{G}$. If there exist edges e such that exactly one oriented face $f^* \in \mathcal{F}(e)$ does not belong to $\mathcal{R}$; namely, if there exist equations of linear system (4.2) with just one remaining unknown, then we compute the coefficients b_{f^*} via such equations and update $\mathcal{R}$. If there are no such edges and $\mathcal{R} \neq \mathcal{F}$, then we pick an oriented face $f \in \mathcal{F} \setminus \mathcal{R}$, compute b_f using explicit formula (4.4) and update $\mathcal{R}$. More precisely, the algorithm reads as follows:

ALGORITHM 1.

1. $\mathcal{R} := \mathcal{F} \setminus \mathcal{G}$, $\mathcal{D} := \mathcal{E}$.
2. *while* $\mathcal{R} \neq \mathcal{F}$
   - (a) $n_{\mathcal{R}} := card(\mathcal{R})$
   - (b) *for every* $e \in \mathcal{D}$
     - i. *if every oriented face of* $\mathcal{F}(e)$ *belongs to* $\mathcal{R}$
       - A. $\mathcal{D} = \mathcal{D} \setminus \{e\}$
     - ii. *if exactly one oriented face* $f^* \in \mathcal{F}(e)$ *does not belong to* $\mathcal{R}$
       - A. *compute* b_f *via* (4.2)
       - B. $\mathcal{R} = \mathcal{R} \cup \{f\}$
       - C. $\mathcal{D} = \mathcal{D} \setminus \{e\}$
   - (c) *if* $card(\mathcal{R}) = n_{\mathcal{R}}$
     - i. *pick* $f \notin \mathcal{R}$ *and compute* $b_f = \ell_{\mathcal{K}}(R_+(\gamma), \sigma_{\mathcal{B}'}(D(f)))$
     - ii. $\mathcal{R} = \mathcal{R} \cup \{f\}$

It is always possible to choose a Seifert dual spanning tree $\mathcal{B}'$ of $\mathcal{T}$ in such a way that, for some $e \in \mathcal{E}$, exactly one oriented face $f^* \in \mathcal{F}(e)$ does not belong to $\mathcal{E}'_{\mathsf{S}}$. In fact, in many of the numerical experiments we have considered, including knotted 1-boundaries and homologically non-trivial computational domains, when we use breadth-first spanning trees (BFS) [11], the elimination algorithm determines the homological Seifert surface directly, without computing any linking number. The only experiment in which the elimination procedure fails is the one with a computational domain that is a cube with a cavity following a trefoil knot and a 1-boundary γ embracing the cavity. However, to compute the homological Seifert surface, it is enough, in this case, to use once the explicit formula $b_f = \ell_{\mathcal{K}}(R_+(\gamma), \sigma_{\mathcal{B}'}(D(f)))$.

Concerning the complexity of this algorithm, if the elimination procedure does not fail, it is linear in the number of faces of the mesh. However the computational cost of a linking number is, in the worst case, of the order of the product of the number of faces times the number of edges and the (non-realistic) worst scenario corresponds to a spanning tree $\mathcal{B}'$ that requires the explicit computation of the coefficient b_f for each f with $D(f) \notin \mathcal{E}'_{\mathsf{S}}$. In this case, the computational cost is of the order of the square of the number of faces times the number of edges.

REMARK 14. *The reader observes that Algorithm 1 works also if* γ *is an arbitrarily* 1*-cycle of* $\mathcal{T}$*. In this way, the* 1*-cycle* γ *of* $\mathcal{T}$ *is a* 1*-boundary of* $\mathcal{T}$ *if and only if the* 2*-chain* S *of* $\mathcal{T}$ *computed by the algorithm applied to* γ *has* γ *itself as its boundary; namely, if and only if* $\partial_2 S = \gamma$*.*

5. Numerical results. In this section we illustrate the performance of the method analyzed in the previous sections. We consider three sets of test problems. In the first set we focus on the simplest situation: the computational domain is a cube and the considered 1-boundary is a trivial polygonal knot. Then, we present two more complicated benchmark problems where the computational domain is still a cube but the 1-boundary is a non-trivial knot or a link. In the last set of tests the computational domain is homologically non-trivial: first a torus with a concentric toroidal cavity and second a cube with a knotted cavity. In the first case the 1-boundary γ that we consider has two connected components that are circumferences, one on the external boundary of the computational domain and the other one on the boundary of the cavity. In the second case γ is a trivial polygonal knot embracing two branches of the cavity.

As we anticipated at the end of the previous section, in all these examples but the last one, the elimination procedure in Algorithm 1, 2.(b).ii.A, provides the homological Seifert surface. Only in the last example it is necessary to use once the explicit formula $b_f = \ell_{\mathcal{K}}(R_+(\gamma), \sigma_{\mathcal{B}'}(D(f)))$ in Algorithm 1, 2.(c).i.

The algorithm has been implemented in Fortran 90 compiled with Intel Visual Fortran. All the numerical computations have been performed by a Intel Core i7-3720QM, with a processor at 2.60 GHz in a laptop with 16 GB of RAM.

We adopt the following strategy for the construction of a Seifert dual spanning tree $\mathcal{B}'$ of $\mathcal{T}$ containing a maximal plug-set J; namely, a strongly-Seifert spanning tree (see Definition 11):

1. Using a breadth-first search (BFS) [11] build a spanning tree on each graph $\mathcal{A}'_i$ induced by $\mathcal{A}'$ on the connected component Γ_i of $\partial\Omega$. (We remark that this step is usually not required in practice as indicated in Remark 15.)
2. Build a maximal plug-set J. That is, for each tetrahedron with at least one face in F_∂, add exactly one plug induced by one of its faces in F_∂.
3. Form a tree in (V', E') with the BFS strategy, by using all tetrahedra with at least one face in F_∂ as root.
4. If $\partial\Omega$ has more than one connected component, the preceding steps return a forest. To obtain a spanning tree of $\mathcal{A}'$, one may run the Kruskal algorithm [11] starting from the forest already constructed.

5.1. Trivial polygonal knot in a cubic computational domain. We start with a toy problem obtained by triangulating a cube using just 48 tetrahedra, see Figure 5.1 (a). A possible maximal plug-set for the toy problem is represented in Figure 5.1 (b). In the same picture, the dotted dual edges represent the plugs induced by corner faces whose plugs do not belong to the maximal plug-set J. The tree extended to the interior of the domain by running the BFS algorithm is represented in Figure 5.1 (c). We consider first (Example 1) the 1-boundary γ represented in Figure 5.1 (a). By running Algorithm 1, one obtains the 2-chain S, whose support is represented in Figure 5.1 (d). Then we consider (Example 2) the 1-boundary γ represented in Figure 5.2 (a) obtaining the 2-chain illustrated in 5.2 (b). For Example 2 we repeat the computation using a finer mesh with 479,435 tetrahedra obtaining now the homological Seifert surface in Figure 5.2 (c). It is worth noticing that for both Example 1 or Example 2 the surface obtained is non-self-intersecting.

Table 5.1 contains the information about the number of geometric elements of the triangulation $\mathcal{T}$ and of the edges belonging to the support of the 1-boundary γ. It also shows the number of faces contained in the support of the homological Seifert surface obtained by the elimination procedure of Algorithm 1 with a strongly-Seifert dual spanning tree, together with the time (in milliseconds) required to obtain them.

5.2. Link and knot in a cubic computational domain. Now we present results for more complicated benchmark problems. First we take γ as the non-trivial knot 8_{21} inside a cube, see Figure 5.3 (a) (see also [29, p. 394]). Figure 5.3 (b) represents a zoom on γ, while Figure 5.3

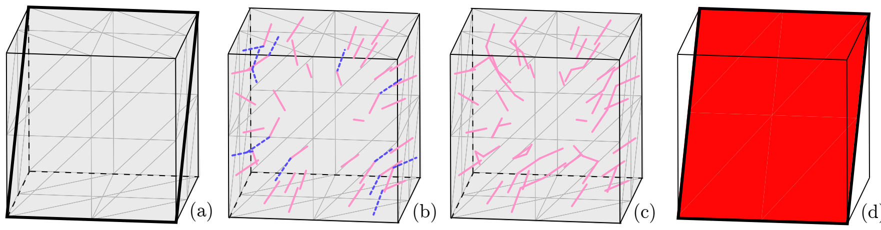

FIG. 5.1. *Example 1: (a) A toy problem is obtained by triangulating a cube using 48 tetrahedra. Thicker edges represent the support of the* 1*-boundary* γ*, whereas thin edges represent the edges of the triangulation of the cube contained in its boundary. (b) Continuous dual edges represent a maximal plug-set* J*, whereas the dotted dual edges are the plugs induced by corner faces that do not belong to* J*. (c) The tree is completed in the interior of the triangulation by a BFS strategy (the tree in* $\mathcal{A}'_0$ *is not shown). (d) The support of the homological Seifert surface obtained with Algorithm 1, which appears to be minimal in this case.*

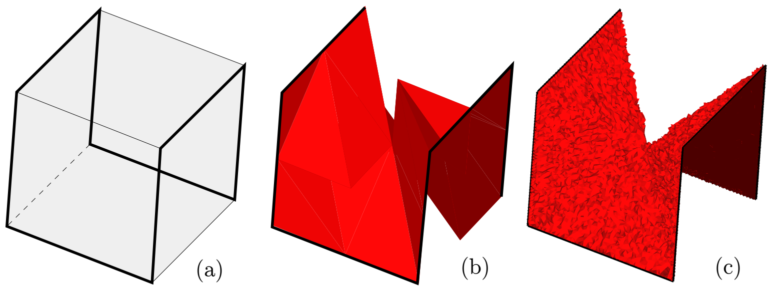

FIG. 5.2. *Example 2. (a) The* 1*-boundary* γ *is represented by the thicker edges. (b) The support of the* 2*-chain obtained for the coarse mesh. (c) The support of the* 2*-chain obtained for the fine mesh.*

(c) illustrates the support of the 2-chain obtained by the elimination procedure of Algorithm 1 with a strongly-Seifert dual spanning tree. Then we consider γ as the Hopf link inside a cube, see Figure 5.3 (d); in this case the support of γ has two connected components. Figure 5.3 (e) represents a zoom on γ. Figure 5.3 (f) shows the support of the obtained homological Seifert surface.

As in the previous set of tests, Table 5.2 contains the information about the number of geometric elements of the triangulation $\mathcal{T}$ and of the edges belonging to the support of the 1-boundary γ. It also shows the number of faces contained in the support of the computed homological Seifert surface, together with the time (in milliseconds) required to obtain them. It is worth noticing that, in a mesh with more that 800,000 tetrahedra, Algorithm 1 computes the homological Seifert surface of a Hopf link in under half a second. In these two examples the computed homological Seifert surface is self-intersecting.

5.3. Non-trivial computational domain. The last set of test problems concerns computational domains that are topologically non-trivial. First we consider a toric shell (a solid torus with a concentric toroidal cavity) and a 1-boundary γ given by two disjoint circumferences, one in the exterior boundary of the computational domain and the other one in the boundary of the cavity, see Figure 5.4 (a). The support of the homological Seifert surface computed using Algorithm 1 with a strongly-Seifert dual spanning tree is illustrated in Figure 5.4 (b). It is worth noticing that it is non-self-intersecting.

Name	Tetrah.	Faces	Edges	Vertices	$\|\gamma\|$	$\|S\|$	Time
Example 1	48	120	98	27	8	8	< 1
Example 2	48	120	98	27	12	28	< 1
Example 2	479,435	973,963	583,183	88,656	341	15,023	233

TABLE 5.1

Cubic computational domain with a 1-boundary that is a trivial polygonal knot. Column $|\gamma|$ reports the number of edges in the 1-boundary γ and column $|S|$ the number of faces in the computed homological Seifert surface. The computational time in the last column is expressed in milliseconds.

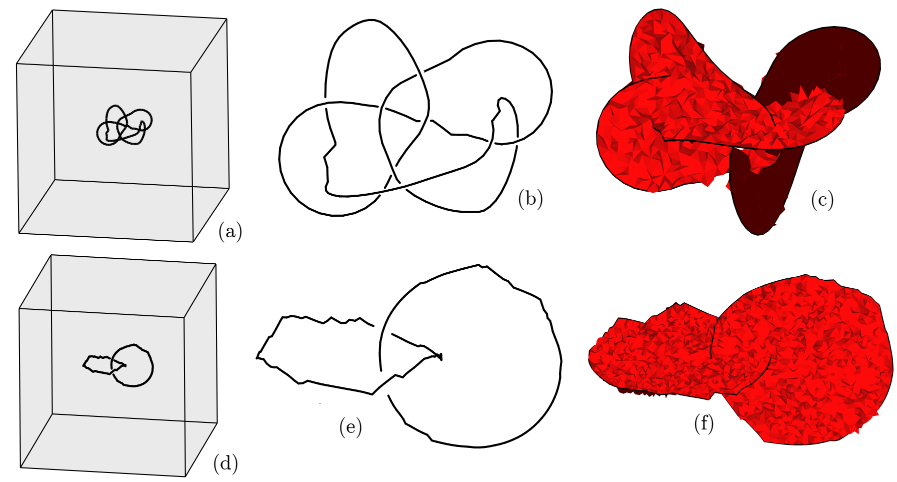

FIG. 5.3. *A non-trivial knot in a cube (top): (a) the support of the 1-boundary γ is a 8_{21} knot placed inside a box outlined in the picture, (b) a zoom on γ, (c) the support of the homological Seifert surface. The Hopf link in a cube (bottom): (d) the 1-boundary γ is a Hopf link placed inside a cube, (e) a zoom on γ, (f) the support of the homological Seifert surface.*

Second, we consider a cube with a knotted cavity, a thickened trefoil knot, and a 1-boundary γ that is a trivial polygonal knot embracing two branches of the cavity, see Figure 5.5 (a). Figure 5.5 (b) represents a zoom on γ. For this test case the support of the homological Seifert surface computed is illustrated in Figures 5.5 (c) and 5.5 (d). Also in this case, it is non-self-intersecting. Topologically it is a torus without a disk. It is in fact a genuine Seifert surface of the polygonal knot γ in $\overline{\Omega}$ of minimal genus; namely, genus 1. (It is like a "swollen" version of the surface in Figure 5.7 (f) in order to obtain an internal homological Seifert surface).

The geometrical information and the computational time for these two examples are summarized in Table 5.3. Again these two experiments illustrate the effectiveness of Algorithm 1: in the first one the algorithm computes the homological Seifert surface in a mesh with 1.8 millions tetrahedra under a second; in the second one, even if the elimination algorithm fails and it is necessary to use once the explicit formula $b_f = \ell_{\mathcal{K}}(R_+(\gamma), \sigma_{\mathcal{B}'}(D(f)))$, Algorithm 1 is extremely fast. In fact, it takes less than 10 milliseconds in a mesh with 8,000 tetrahedra and 0.3 seconds in a mesh with half a million of tetrahedra.

Concerning the computational cost of Algorithm 1, in Figure 5.6 we plot computational time versus the number of tetrahedra of the meshes with more than 10,000 tetrahedra to illustrate

Name	Tetrah.	Faces	Edges	Vertices	$\lvert\gamma\rvert$	$\lvert S\rvert$	Time
8_{21} knot	87,221	175,317	102,212	14,117	170	2663	37
Hopf link	800,020	1,600,537	937,631	137,115	235	4841	407

TABLE 5.2

A non-trivial knot and the Hopf link in a cube. Column $|\gamma|$ reports the number of edges in the 1-boundary γ and column $|S|$ the number of faces in the computed homological Seifert surface. The computational time in the last column is expressed in milliseconds.

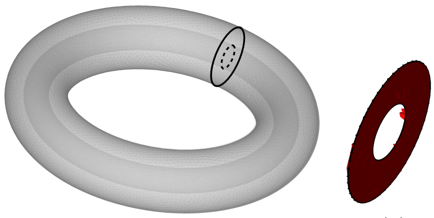

FIG. 5.4. *Toric shell: (a) The support of the 1-boundary γ is a pair of disjoint circumferences, outlined in the picture, placed on the boundary of the toric shell (namely, the difference between two coaxial solid tori); (b) the support of the computed homological Seifert surface.*

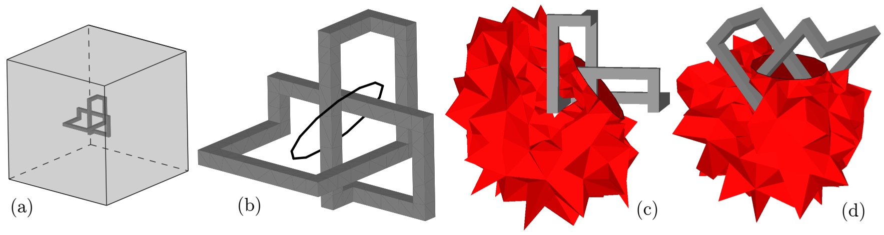

FIG. 5.5. *Knotted cavity: (a) The support of the 1-boundary γ is a trivial polygonal knot embracing two branches of the cavity, (b) a zoom on γ, (c), and (d) two different views of the support of the computed homological Seifert surface.*

the linear trend (for meshes with less than 10,000 tetrahedra the computational time is under 10 milliseconds). We also plot the regression line for the four examples with more than 10,000 tetrahedra where the elimination procedure success. For the considered examples, we can clearly see the linear behavior when the elimination procedure succeeds and the minor influence in the computational time of the computation of a linking number in the one example where it is required.

5.4. Using a Seifert dual spanning tree without a maximal plug-set. To conclude this section on numerical experiments we show, in Figure 5.7, the homological Seifert surfaces computed with Algorithm 1 using a Seifert dual spanning tree that does not contain a maximal plug-set. The meshes and the 1-boundaries considered are the same as in the previous examples, the difference is just in the choice of the spanning tree.

To construct this non-strongly-Seifert dual spanning tree we proceed (heuristically) in this way:

1. Build a BFS spanning tree on each graph $\mathcal{A}'_i$ induced by $\mathcal{A}'$ on the connected component

Name	Tetrah.	Faces	Edges	Vert.	$\lvert\gamma\rvert$	$\lvert S\rvert$	Time
Toric shell	1,851,494	3,871,379	2,419,350	399,465	176	1662	961
Knotted cavity	8,267	16,913	10,187	1,551	13	225	< 10
Knotted cavity	529,664	1,065,104	626,566	91,127	52	10,407	309

TABLE 5.3

The number of geometric elements of the triangulation and of the edges belonging to the support of the 1-boundary γ.

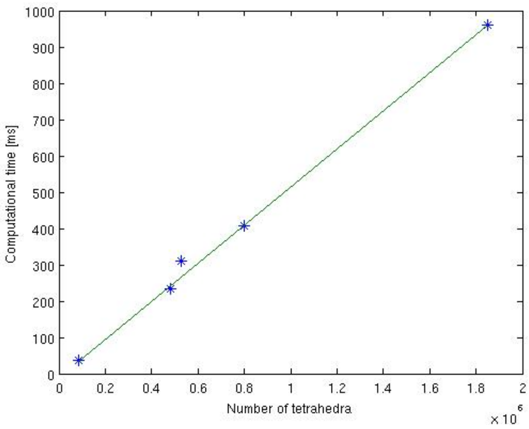

FIG. 5.6. *Computational complexity: computational time versus the number of tetrahedra of the mesh and the regression line for the examples where the elimination procedure succeeds.*

Γ_i of $\partial\Omega$. We remark that this step is usually not required in practice as indicated in Remark 15.

2. Build an "internal" spanning tree of the graph (V', E').
3. For each Γ_i, add exactly one plug induced by a face in Γ_i.

It is worth noticing that, as proved in Theorem 12, when using a strongly-Seifert dual spanning tree, Algorithm 1 computes an internal homological Seifert surface but this is no longer true when considering a Seifert dual spanning tree that does not contain a maximal plug-set. In Figure 5.7 we show the homological Seifert surface computed using a Seifert dual spanning tree of this kind and the surfaces computed in Example 1, Example 2 and the two examples in non-trivial computational domains are not internal surfaces, (see Figure 5.7 (a), (b), (e), and (f) respectively). It can also be noticed that, in general, the strongly-Seifert dual spanning tree provides homological Seifert surfaces with reduced support with respect to the Seifert dual spanning tree without a maximal plug-set. The unique exception is the last example; compare Figure 5.5 (c) or (d) and Figure 5.7 (f).

Notice that when many homological Seifert surfaces are required on the same triangulation, Algorithm 1 can be vectorized in such a way that all surfaces are generated at once.

REMARK 15. *It is worth noticing that in all but one of the examples considered (the exception is the example concerning a cubic domain with a knotted cavity), Algorithm 1 is able to construct the homological Seifert surface without the computation of any linking number. Therefore, there is no need to compute a spanning tree of each graph $\mathcal{A}'_i$ and even to consider the dual graph*

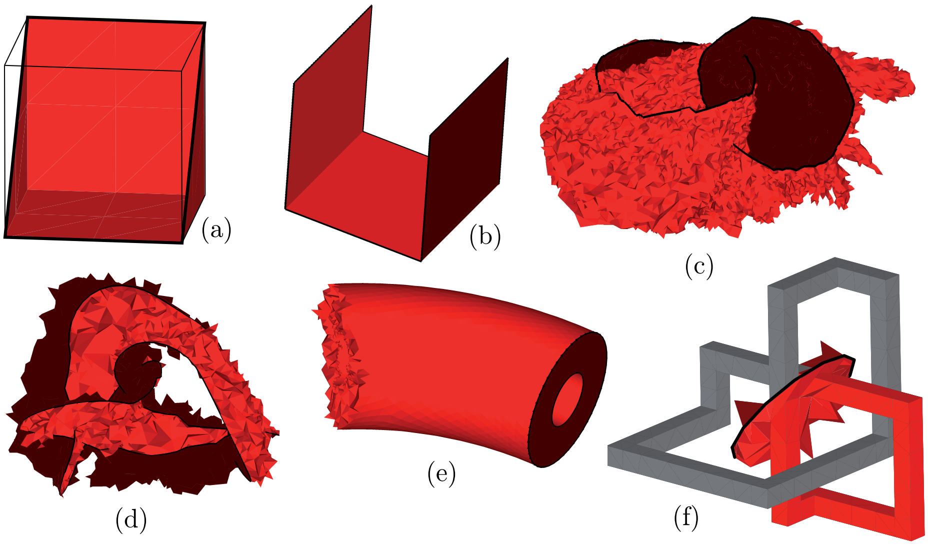

FIG. 5.7. *Homological Seifert surfaces computed using a Seifert dual spanning tree without a maximal plug-set. (a) Example 1. (b) Example 2. (c) A non-trivial knot in a cube. (d) The Hopf link in a cube. (e) A 1-boundary γ with two connected components in a toric shell. (f) A trivial polygonal knot embracing two branches of a knotted cavity.*

$(V'_\partial, E'_\partial)$ on the boundary of Ω. In fact, in the elimination step 2.(b), only $\mathcal{E}'_S \cap \mathcal{E}'$ is used. The complete knowledge of $\mathcal{E}'_S$; namely, the construction of $\mathcal{B}'_i$ for every $i \in \{0, 1, 2, \ldots, p\}$, is required just in the direct computation step.

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