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PATTERN POSETS: ENUMERATIVE, ALGEBRAIC AND ALGORITHMIC ISSUES

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Notation

$\lvert A \rvert$	the number of elements in a set A
$[n]$	the set $\{1, 2, 3, ..., n\}$ when $n \geq 1$ and the empty set $\emptyset$ otherwise
$f^{\longleftarrow}(C)$	the preimage of the set C under the function f
C_n	the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$
$C(z)$	the generating function $\frac{1-\sqrt{1-4z}}{2z}$ for the Catalan numbers
φ_n	the n^{th} Fibonacci number
$[z^n]f(z)$	the n^{th} coefficient of the formal power series $f(z)$
$\mathcal{A}_n$	the set of all elements with size n in a combinatorial class $\mathcal{A}$.
$\mathcal{A}(z)$	the generating function of $\mathcal{A}$
$\mathcal{P}(S)$	the class of all elements in a pattern poset $\mathcal{P}$ avoiding each pattern in S
$\mathcal{P}_n(S)$	the set of all elements in $\mathcal{P}(S)$ with size n
$\mathcal{P}(S, z)$	the generating function of $\mathcal{P}(S)$
$\mathcal{T}_\vartheta$	the generating tree of the ECO operator ϑ
$\mathcal{S}$	the class of all permutations
$\mathcal{D}$	the class of all Dyck paths
$\mathcal{M}$	the class of all matchings
$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, 3, ...\}$
$\mathbb{N}^*$	the set of non-zero natural numbers $\{1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integer numbers $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}_{<0}$	the set of negative integers $\{k \in \mathbb{Z} : \ k < 0\}$
$R^\times$	the multiplicative group of the ring R
$R[X]$	the ring of polynomials in the indeterminate X with coefficients in the ring R
$R(X)$	the field of rational functions in the indeterminate X with coefficients in the ring R
$R[[X]]$	the ring of formal power series in the indeterminate X with coeffiecients in the ring R
$R[X, X^{-1}]$	the ring of Laurent polynomials in the indeterminate X with coefficients in the ring R
$R((X))$	the ring of formal Laurent series in the indeterminate X with coeffiecients in the ring R

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Chapter 1

Introduction

The study of patterns in permutations is currently one of the most active trends of research in combinatorics. The notion of pattern in a permutation historically emerged from the problem of sorting permutations with certain devices, however the richness of this notion became especially evident from its plentiful appearances in several very different disciplines, such as algebra, geometry, analysis, theoretical computer science, biology, and many others. In the last deacades, similar notions of pattern have been considered on discrete structures other than permutations, such as integer sequences, lattice paths, graphs, matchings and set partitions. The formal definition of pattern poset attempts to encompass this research trend into a more general framework. Although pattern posets have received an increasing attention in the last few years, this vast landscape is still largely unexplored and many fundamental issues remain completely open. This thesis aims to shed some light on these posets by collecting some enumerative, algebraic and algorithmic results about a few classes of pattern posets that appear to be promising for future developements.

1.0.1 Enumerative combinatorics

Enumerative combinatorics is usually defined as the discipline that deals with the problem of counting the elements in a finite set. This is of course a somewhat coarse definition, as the notion of counting is not entirely transparent itself. In fact, a finite set A is usually assigned by means of an intensional definition, i.e. A is assigned as the subset of all the elements in a broader finite set B satisfying a given characteristic property P, therefore, given an explicit list of the elements of B, to count the elements in A one can simply scan the list of the elements of B, check which elements satisfy the property P to produce an explicit list of the elements of A, attach the label 1 to the first element of A you find, the label 2 to the second one and so on: the last label one attaches is the number of elements in A. This is of course a brute force algorithm to count the elements in a finite set (and it is sometimes useful for computer experiments), however it is easily seen to quickly become computationally intractable, depending on the paramenters characterizing the finite set, hence we need to clarify what we expect to be a good answer for this kind of problems. A more precise discussion could get phylosophical, depending on the criteria one can choose to consider a counting algorithm as computationally efficient (see for instance [W]). Here we will content ourselves with informally listing certain classes of answers that are usually accepted as solutions for a counting problem. Usually, one is given a sequence of finite sets $\{A_n\}_{n\in\mathbb{N}}$ and is asked to count the number $f(n)$ of elements in A_n for every $n \in \mathbb{N}$.

1. A generally accepted answer for this counting problem is a formula for $f(n)$ involving only "nice" functions of n. Of course it remains a matter of discussion which functions should be regarded as "nice" and, as a formula for $f(n)$ become more complicated, our willingness to accept it as a satisfying answer decreases. For instance, for every $n \in \mathbb{N}$, let A_n be the set of all subsets of $[n]$ with two elements, then a formula for $f(n)$ is given by the binomial coefficient $\binom{n}{2} = \frac{n(n-1)}{2}$. Unfortunately, only in rare cases will such an explicit formula exist, therefore one has to look for a slightly more implicit answer.

2. In this sense, a possible answer can also be a recurrence relation allowing to compute $f(n)$ from previously computed terms $f(k)$ for $k < n$. For instance, when A_n is the set of all subsets of $[n]$ with two elements, the recurrence relation $f(n) = f(n-1) + n - 1$ holds for $n \geq 2$.

3. Another useful and combinatorially meaningful method to compute $f(n)$ is to provide a bijection with a set already counted. For instance the number of edges in the complete graph K_n on n vertices is clearly in bijection with the set of all subsets of $[n]$ with two elements, therefore they are counted by the same sequence.

4. Finally, a more implicit, but compact, answer can be an analytic expression or a functional equation for the generating function of the sequence $\{f(n)\}_{n\in\mathbb{N}}$. The generating function of the sequence $\{f(n)\}_{n\in\mathbb{N}}$ is the formal power series

$$f(z) = \sum_{n\geq 0} f(n) z^n$$

which is meant to be a purely algebraic object, living in the ring of formal power series $\mathbb{C}[[z]]$ and encoding the seuquence $\{f(n)\}_{n\in\mathbb{N}}$ regardless of convergence issues, or to put in in Wilf's worlds it is "a clothesline on which we hang up a sequence of numbers for display". For instance, when A_n is the set of all subsets of $[n]$ with two elements, one can write $f(z)$ in the compact form

$$f(z) = \frac{z^2}{(1-z)^3}.$$

Of course, the more efficient the counting method, the more cleverly it will rely on the properties of the elements in the finite set, therefore enumerative combinatorics is much more than just counting, but it has to do with understanding the structure of the objects we are counting.

1.0.2 Pattern posets

In this section we give a general introduction to pattern posets, which are the main topic of this thesis. We can trace back the root of the study of pattern posets to the 1960's book "The art of computer programming" [K] by Donald Knuth, where the stack sorting problem for permutations was introduced, characterizing the structure of stack-sortable permutations in terms of pattern avoidance. For a more detailed introduction to permutation patterns in particular see [B1] and [Ki1]. Roughly speaking, a stack is a data structure to store items, consisting of a linear sequence accessed at one end, called the top. Items can be added and removed from the top end of the stack by push and pop operations. Given $n \in \mathbb{N}$, we will write a permutation π of the set $[n]$ in the so-called one-line notation, i.e. regarding it as word $\pi_1\pi_2...\pi_n$ with distinct letters in the alphabet $[n]$ where π maps i to π_i for every $i \in [n]$. Sometimes it will also be useful to represent π as the set of all points (i, π_i) in a cartesian plane, where $i \in [n]$. The integer n will be called the length of π and denoted by $|\pi|$. The class of all permutations will be denoted by $\mathcal{S}$ and the set of all permutations of length n will be denoted by $\mathcal{S}_n$. A stack can be used to rearrange a permutation π of length n as follows: the elements of π are pushed onto an initially empty stack and an output permutation is obtained by popping elements from the stack; the output permutation obviously depends on how the push and pop operations are interleaved. One wishes the output permutation to be $12...n$, so that one has sorted the input. A necessary but not sufficient condition for this to be possible is that the stack values should always be increasing (red from the top end). Forcing this condition leads to a greedy algorithm that only pops a stack symbol if pushing the next input symbol would have violated the increasing property of the stack. We let $s(\pi)$ denote the output resulting from this greedy algorithm. More formally, given any word π with distinct letters in the alphabet $\mathbb{N}^*$, write $\pi = \lambda n \rho$, where n is the greatest letter of π, then define recursively $s(\pi) = s(\lambda)s(\rho)n$. Thus we say that a permutation π of length n is *stack-sortable* when $s(\pi) = 12...n$. In general a permutation is not stack-sortable, for instance $s(231) = s(2)s(1)3 = 213$. Knuth proved that a permutation is not stack-sortable if and only if it contains a (non necessarily consecutive) subword whose items are in the same relative order as the items of the permutation 231, hence characterizing stack-sortable permutations as those permutations avoiding such subwords. For instance 53142 is not stack-sortable because it contains the subword 342 whose items are in the same relative order as the items of 231, indeed $s(53142) = 13245$. Using this characterization, Knuth also proved that stack-sortable permutations of length n are counted by the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

This characterization led to the more general notion of permutation pattern. Let us first fix some terminology: given a poset $\mathcal{P}$ and two words x and y in the alphabet $\mathcal{P}$, we say that x and y are *order isomorphic* when they have the same length and $x_i \leq x_j$ if and only if $y_i \leq y_j$ for every $i, j \in [|x|]$. Let now σ and τ be non-empty permutations. We say that σ is a *pattern* of τ, and write $\sigma \leq \tau$, when one can find some $i = (i_1, ..., i_{|\sigma|}) \in [|\tau|]^{|\sigma|}$ such that $i_1 < ... < i_{|\sigma|}$ and the words $\tau_{i_1}...\tau_{i_{|\sigma|}}$ and σ are order-isomorphic, we say that τ *avoids* σ otherwise. Any $i \in [|\tau|]^{|\sigma|}$ satisfying this property is called an *occurrence* of σ in

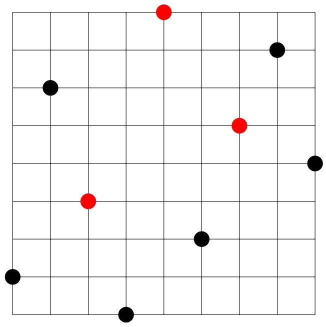


Figure 1.1: (3,5,7) is an occurrence of 132 in the permutation 274193685.

τ. It is easily seen that this relation turns the set of all permutations into a poset, which we call the *permutation pattern poset*. Let now $n \in \mathbb{N}$. We will denote by $\mathcal{S}(\sigma)$ the class of all permutations avoiding σ and by $\mathcal{S}_n(\pi)$ the set of all the elements in $\mathcal{S}(\sigma)$ with length n. Thus Knuth proved that $\mathcal{S}(231)$ is exactly the class of stack-sortable permutations and proved that $|\mathcal{S}_n(231)| = C_n$. In fact, one can prove that $|\mathcal{S}_n(\pi)| = C_n$ for every permutation π of length 3. When $|\mathcal{S}_n(\sigma)| = |\mathcal{S}_n(\tau)|$ for every $n \in \mathbb{N}$, we say that σ and τ are *Wilf-equivalent*. Thus all permutations of length 3 are Wilf-equivalent. Many Wilf-equivalences stem from the trivial fact that $|\mathcal{S}_n(\pi)| = |\mathcal{S}_n(\pi^{rev})| = |\mathcal{S}_n(\pi^c)| = |\mathcal{S}_n(\pi^{-1})|$ for every $n \in \mathbb{N}$, where $\pi^{rev} = \pi_{|\pi|}...\pi_2\pi_1$ and $\pi^c = (|\pi| - \pi_1 + 1)...(|\pi| - \pi_{|\pi|} + 1)$, however, there are also numerous examples of nontrivial Wilf-equivalences (such as that between 123 and 231). In general, enumeration of $\mathcal{S}_n(\pi)$ turns out to be a very difficult problem for patterns π of length greater than or equal to 4. It is known that the set of patterns of length 4 splits into 3 Wilf-equivalence classes represented by the patterns 1234, 1342 and 1324 respectively. Explicit formulas to enumerate the classes $\mathcal{S}(1234)$ and $\mathcal{S}(1342)$ were found by Gessel [G] in 1990 and Bona [B2] in 1997 respectively, explicitly

$$
\begin{aligned}
|\mathcal{S}_n(1234)| =& \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1} \\
|\mathcal{S}_n(1342)| =& (-1)^{n-1}\frac{7n^2-3n-2}{2} + 3\sum_{k=0}^{n}(-1)^{n-k}2^{k+1}\frac{(2k-4)!}{k!(k-2)!}\binom{n-k+2}{2}
\end{aligned}
$$

for every $n \geq 1$. Instead, the class $\mathcal{S}(1324)$ still resists any attempt of enumeration and it provides one of the most challenging problems in enumerative combinatorics. Finding an explicit formula to enumerate $\mathcal{S}_n(\pi)$ is an open problem for most of the patterns π longer than 4, however, a long-standing conjecture of Stanley and Wilf (proved in 2003 [MT] by Adam Marcus and Gabor Tardos) connected all patterns by claiming that no matter what π is, the number of permutations of length n avoiding π is very small compared to $n!$ and most of the permutations will actually contain the pattern π. More precisely, the Marcus-Tardos theorem states that, for every permutation pattern π, the sequence $|\mathcal{S}_n(\pi)|$ grows subexponentially with respect to n, i.e. one can find some constant $c(\pi)$

such that $|\mathcal{S}_n(\pi)| \leq c(\pi)^n$ for every $n \geq 1$. Actually, an (apparently) stronger statement holds: the limit $\lim_{n\to+\infty} \sqrt[n]{|\mathcal{S}_n(\pi)|}$ exists and it is finite. This limit is called the *growth rate* of π and it is denoted by $L(\pi)$. For instance, the exact enumerative results mentioned above imply that $L(\pi) = 4$ for every $\pi \in \mathcal{S}_3$, moreover $L(1234) = 9$ and $L(1342) = 8$, whereas only upper and lower bounds are known for $L(1324)$.

Since then, many variations of the notion of pattern have been investigated, either by imposing restrictions on the relative position of the items in an occurrence (as in the case of vincular patterns, bivincular patterns or mesh patterns) or by introducing similar partial order relations on classes of discrete structures other than permutations, such as integer sequences, lattice paths, graphs, matchings and set partitions. The latter research trend constitutes the main topic addressed in this thesis and in the following discussion we aim at encompassing this trend into a quite general framework provided by Smith in [Sm], namely a general definition of pattern poset. Informally speaking, given a specific class of combinatorial objects, a pattern can be thought of as an occurrence of a small object inside a larger one; the word "inside" means that the pattern is suitably embedded into the larger object. This idea can be formalized as follows.

Let $\mathscr{A}$ be a non-empty alphabet and let $\mathscr{A}^*$ denote the language generated by $\mathscr{A}$, i.e. $\mathscr{A}^* = \bigcup_{n\in\mathbb{N}} \mathscr{A}^n$ (where by convention $\mathscr{A}^0 = \{\varepsilon\}$ and ε is an element, called the *empty word*, such that $\varepsilon \notin \mathscr{A}$). Given a word σ in the alphabet $\mathscr{A}$, the number of letters of σ is called the *length* of σ and it will be denoted by $|\sigma|$. Let now $\sim$ be an equivalence relation on $\mathscr{A}^*$, let σ and τ be two words in the alphabet $\mathscr{A}$ and suppose $i \in [|\tau|]^{|\sigma|}$. We say that i is a $\sim$ *−occurrence* of σ in τ when $i_1 < i_2 < ... < i_{|\sigma|}$ and $\tau_{i_1}\tau_{i_2}...\tau_{i_{|\sigma|}} \sim \sigma$. We say that σ is a $\sim$ *−pattern* of τ, and we write $\sigma \leq_\sim \tau$, when either $\sigma = \varepsilon$ or one can find a $\sim$ −occurrence of σ in τ, we say that τ *avoids* σ as a $\sim$ *−pattern* otherwise. Let now $\mathcal{P}$ be a language in the alphabet $\mathscr{A}$, i.e. a subset of $\mathscr{A}^*$. It is routine to check that $\leq_\sim$ is a partial order relation on $\mathcal{P}$, turning it into a poset, which we call the $(\sim, \mathcal{P})-$*pattern poset*. Let now S be a subset of $\mathscr{A}^*$ and $n \in \mathbb{N}$. The set of all words in $\mathcal{P}$ avoiding each pattern in S will be denoted by $\mathcal{P}(S)$ and the set of all $\sigma \in \mathcal{P}(S)$ such that $|\sigma| = n$ will be denoted by $\mathcal{P}_n(S)$. In particular, if $S = \{\pi_1, ..., \pi_m\}$ we will also denote $\mathcal{P}(S) = \mathcal{P}(\pi_1, ..., \pi_m)$. We will usually denote by $\mathcal{P}(S, z)$ the generating function

$$\mathcal{P}(S, z) = \sum_{\sigma\in\mathcal{P}(S)} z^{|\sigma|} = \sum_{n\geq 0} |\mathcal{P}_n(S)| z^n.$$

We will say that two subsets S and T of $\mathscr{A}^*$ are *Wilf-equivalent* on $\mathcal{P}$ when $\mathcal{P}(S, z) = \mathcal{P}(T, z)$. In analogy with the permutation pattern case, for a given pattern poset $\mathcal{P}$ and a given subset S of $\mathscr{A}^*$, counting the elements of $\mathcal{P}_n(S)$, for every $n \in \mathbb{N}$, is one of the most interesting issues in current research on pattern posets and Chapter 2 will be devoted to expoloring the various declinations of this problem in the pattern posets of permutations, lattice paths and matchings.

Example 1.0.1. *The following will serve as reference examples of pattern posets throughout this thesis.*

(i) Let $\mathscr{A}$ be an alphabet and consider the equivalence relation $\sim$ given by equality. Then $\leq_\sim$ is called the subword order *on $\mathscr{A}^*$. When $\mathscr{A} = \mathbb{Z}$, some subclasses $\mathcal{P}$ of*

$\mathscr{A}^*$ give rise to particularly interesting $(\sim,\mathcal{P})-$pattern posets, whose elements can be represented as lattice paths on the Cartesian plane subject to suitable constraints, prominent examples being the Dyck pattern poset (see [BBFGPW]), the Motzkin pattern poset and the Schröder pattern poset.

(ii) *Suppose now $\mathscr{A}$ is a poset (e.g. $\mathscr{A} = \mathbb{N}$ with the natural total order) and $\sim$ is the order isomorphism equivalence relation on $\mathscr{A}^*$. Then the $\leq_\sim$ is called the* pattern order *on $\mathscr{A}^*$. When $\mathscr{A} = \mathbb{N}$ (endowed with the usual order), some subclasses $\mathcal{P}$ of $\mathscr{A}^*$ give rise to prominent examples of $(\sim,\mathcal{P})$-pattern posets, some of which are listed below.*

(i) *The class of all the permutations, which is the clasical* permutation pattern poset*.*

(ii) *The class of all inversion sequences, where an* inversion sequence *is a word w in the alphabet $\mathbb{N}$ such that $w_i < i$ for every $i \in [|w|]$.*

(iv) *The class of all Catalan words, where a* Catalan word *is a a word w in the alphabet $\mathbb{N}$ such that $w_1 = 0$ and $w_i < w_{i-1}$ for every $i \in [|w|] - \{1\}$.*

(iii) *The class of all set partitions, as every partition σ of $[n]$ can be identified with a word w with length n in the alphabet $\mathbb{N}$ such that every element of $[k]$ occurs as a letter in w for some $k \in [n]$ and the leftmost occurrences of the repeated letters in w appear in increasing order from left to right, where, for every $i, j \in [n]$, one has $w_i = w_j$ if and only if i and j are contained in the same block of σ (e.g. the partition $\{\{1,3\},\{2,5,6\},\{4\},\{7,8\}\}$ can be represented by the sequence 12132244). If $w(\sigma)$ denotes the word corresponding to a set partition σ, then it is easily seen that $w(\sigma) \leq_\sim w(\tau)$ if and only if σ can be obtained from τ by deleting some elements in the blocks of τ and consistently relabelling the remaining elements. In Section 2.3 we will be particularly concerned with* matchings*, that are set partitions into blocks with exactly two elements.*

It is worth noting that the items in an occurrence of a classical pattern in a combinatorial object are not required to be necessarily consecutive, forcing this further condition we obtain the consecutive pattern poset on $\mathcal{P}$. With the notations as above, we say that i is a *consecutive* $\sim$ *−occurrence* of σ in τ when i is a $\sim$ −occurrence of σ in τ (in the classical sense) and either $|\sigma| = 1$ or $i_{j+1} = i_j + 1$ for every $j \in [|\sigma| - 1]$. In the same fashion, we say that σ is a *consecutive* $\sim$ *−pattern* of τ when one can find a consecutive $\sim$ −occurrence of σ in τ. The resulting poset on $\mathcal{P}$ will be called the *consecutive* $(\sim,\mathcal{P})-$*pattern poset.* Usually, the consecutive pattern poset reveals a much simpler structure than the classical pattern poset. For instance, the Möbius function of the consecutive permutation pattern poset is completely understood, whereas it is largely unknown in the classical case.

Consecutive and classical patterns are special (actually, extremal) cases of the more general notion of vincular patterns. Vincular patterns were introduced by Babson and Steingrimsson (under the name of generalized patterns), and constitute a vast intermediate continent between the two lands of consecutive patterns and classical patterns.

Investigation of this intermediate notion could hopefully shed some light on differences and analogies between the extremal cases of consecutive and classical patterns. An occurrence of a vincular pattern is basically an occurrence of that pattern in which entries are subject to given adjacency conditions. As described in [BF], adjacency conditions on the entries of a pattern can be encoded into an infinite lower triangular $\{0,1\}-$matrix, leading to the following formal definitions.

A *dashed word* in the alphabet $\mathscr{A}$ is a word in the alphabet $\mathscr{A}$ in which some dashes are possibly inserted between any two consecutive letters. For instance, 5-13-42 is a dashed permutation (of length 5). The type of a dashed word π such that $|\pi| \geq 2$ in the alphabet $\mathscr{A}$ is the $\{0,1\}-$vector $r = (r_1, ..., r_{|\pi|-1}) \in \{0,1\}^{|\pi|-1}$ such that $r_i = 0$ whenever there is no dash between π_i and π_{i+1} and $r_i = 1$ whenever there is a dash between π_i and π_{i+1}, for every $i \in [|\pi| - 1]$. For example, the above dashed permutation 5-13-42 has type $(1,0,1,0)$. Given a dashed word π in the alphabet $\mathscr{A}$, the *reduced word* of π is the word in the alphabet $\mathscr{A}$ obtained by removing the dashes from π, conversely, given a word σ in the alphabet $\mathscr{A}$ such that $|\sigma| \geq 2$ and a vector $r \in \{0,1\}^{|\sigma|-1}$, the $r -$ *dashed word* of σ is the dashed word in the alphabet $\mathscr{A}$ obtained by inserting a dash between σ_i and σ_{i+1} whenever $r_i = 1$ for every $i \in [|\sigma|-1]$. Let π be a dashed word in the alphabet $\mathscr{A}$. With the same notations as before, given some $i = (i_1, ..., i_{|\pi|}) \in [|\tau|]^{|\pi|}$, we say that i is an *occurrence* of π in τ when it is a (classical) $\sim -$occurrence of the reduced word of π and $i_{j+1} = i_j + 1$ whenever $j \in [|\pi| - 1]$ and π_j and π_{j+1} are not separated by a dash. As before, given a set S of dashed words in the alphabet $\mathscr{A}$, the set of all words in $\mathcal{P}$ avoiding each pattern in S will be denoted by $\mathcal{P}(S)$ and the set of all $\sigma \in \mathcal{P}(S)$ such that $|\sigma| = n$ will be denoted by $\mathcal{P}_n(S)$. Let now A be an infinite lower triangular $\{0,1\}-$matrix. For every $k \in \mathbb{N}$ we denote by r_k the k^{th} row of A. Again, with the same notations as before, given $i = (i_1, ..., i_{|\sigma|}) \in [|\tau|]^{|\sigma|}$, we say that i is a $(\sim, A) -$ *occurrence* of σ in τ when either $|\sigma| = 1$ and i is a $\sim -$occurrence of σ in τ or i is a $\sim -$occurrence of the $r_{|\sigma|-1}-$dashed word of σ in τ. We say that σ is a $(\sim, A) -$ *vincular pattern* of τ, and we write $\sigma \in_{(\sim,A)} \tau$, when one can find a $(\sim, A)-$occurrence of σ in τ, we say that τ *avoids* σ as a $(\sim, A) -$ *vincular pattern* otherwise. Note that for the infinite lower triangular $\{0,1\}-$matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

one recovers the usual notions of classical pattern and consecutive pattern containment respectively. However, it is easy to see that in general $\in_{(\sim,A)}$ is not a transitive relation. Write $\sigma \preceq_{(\sim,A)} \tau$ when $\sigma \in_{(\sim,A)} \tau$ and $|\sigma| \in \{|\tau| - 1, |\tau|\}$. Then $\preceq_{(\sim,A)}$ is a covering relation on $\mathcal{P}$, the transitive closure of $\preceq_{(\sim,A)}$ is a partial order relation on $\mathcal{P}$ denoted by $\leq_{(\sim,A)}$ and the related poset is called the $(\sim, A) -$ *vincular pattern poset* on $\mathcal{P}$.

1.0.3 Symbolic method

A general framework to formalize many enumerative issues concerning a wide range of combinatorial structures is provided by the notion of combinatorial class, which easily allows to translate combinatorial descriptions of a structure into equations for the corresponding generating function. A broader overview can be found in [FS]. This approach will later be useful to settle some enumerative problems related to lattice paths and matchings. Let $\mathcal{A}$ be a set, $|\cdot| : \mathcal{A} \to \mathbb{N}$ be a map, $n \in \mathbb{N}$ and denote by $\mathcal{A}_n$ the set of all $A \in \mathcal{A}$ such that $|A| = n$. Then the pair $(\mathcal{A}, |\cdot|)$ is a *combinatorial class* when $\mathcal{A}_n$ is a finite set for every $n \in \mathbb{N}$, in which case $|\cdot|$ is called the *size map* of $\mathcal{A}$ and $|A|$ is called the *size* of A for every $A \in \mathcal{A}$. In general we will identify a combinatorial class $(\mathcal{A}, |\cdot|)$ with its underlying set $\mathcal{A}$ and we will denote its size map simply by $|\cdot|$ or by $|\cdot|_{\mathcal{A}}$ to stress that the size map is relative to $\mathcal{A}$ when some risk of confusion arises. Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes and $f : \mathcal{A} \to \mathcal{B}$ be a map. Then f is a *morphism* of combinatorial classes when $|f(A)|_{\mathcal{B}} = |A|_{\mathcal{A}}$ for every $A \in \mathcal{A}$. Moreover, f is an *isomorphism* of combinatorial classes if it is a morphism and it is bijective. The *ordinary generating function* of $\mathcal{A}$ is the ordinary genrating function of $\{|\mathcal{A}_n|\}_{n\in\mathbb{N}}$ and it is denoted by $\mathcal{A}(z)$, thus

$$\mathcal{A}(z) = \sum_{n\geq 0} |\mathcal{A}_n| z^n = \sum_{A\in\mathcal{A}} z^{|A|}.$$

Note that, if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then $|\mathcal{A}_n| = |\mathcal{B}_n|$, indeed, given an isomorphism $f : \mathcal{A} \to \mathcal{B}$, by definition $|f(A)|_{\mathcal{B}} = |A|_{\mathcal{A}} = n$ and $|f^{-1}(B)|_{\mathcal{A}} = |B|_{\mathcal{B}} = n$ for every $A \in \mathcal{A}_n$ and $B \in \mathcal{B}_n$, so that $f(\mathcal{A}_n) = \mathcal{B}_n$ and $f|_{\mathcal{A}_n}$ is a one to one correspondence between $\mathcal{A}_n$ and $\mathcal{B}_n$, in particular it follows that $\mathcal{A}(z) = \mathcal{B}(z)$. For convenience, let us introduce a combinatorial class, denoted by $\mathcal{E}$, whith a unique element ε such that $|\varepsilon|_{\mathcal{E}} = 0$. Similarly it will be convenient to introduce a combinatorial class, denoted by $\mathcal{Z}$, with a unique element $\bullet$ such that $|\bullet|_{\mathcal{Z}} = 1$. Note that $\mathcal{E}(z) = 1$ and $\mathcal{Z}(z) = z$. Let us now introduce some combinatorial classes construction admitting a direct translation as operations over generating functions, namely: the sum, the product and the sequence. This dictionary often allows us to translate a combinatorial description of a class into a functional equation for the corresponding generating function, which can hopefully be solved by means of algebraic or analytic methods to extract enumerative results.

(i) (*Sum*) Denote by $\mathcal{A}+\mathcal{B}$ the disjoint union of $\mathcal{A}$ and $\mathcal{B}$ and denote $|(C, \mathcal{C})|_{\mathcal{A}+\mathcal{B}} = |C|_{\mathcal{C}}$ for every $\mathcal{C} \in \{\mathcal{A}, \mathcal{B}\}$ and $C \in \mathcal{C}$. Then $(\mathcal{A} + \mathcal{B}, |\cdot|_{\mathcal{A}+\mathcal{B}})$ is a combinatorial class, called the *sum* of $\mathcal{A}$ and $\mathcal{B}$. Note that $(\mathcal{A}+\mathcal{B})_n$ is the disjoint union of $\mathcal{A}_n$ and $\mathcal{B}_n$, therefore $|(\mathcal{A}+\mathcal{B})_n| = |\mathcal{A}_n| + |\mathcal{B}_n|$, proving

$$(\mathcal{A} + \mathcal{B})(z) = \mathcal{A}(z) + \mathcal{B}(z).$$

More in general, suppose $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is a sequence of combinatorial classes. Then denote by $\sum_{n\geq 0} \mathcal{A}_n$ the disjoint union of $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ and set $|(C, n)|_{\sum_{n\geq 0} \mathcal{A}_n} = |C|_{\mathcal{A}_n}$ for every $n \in \mathbb{N}$ and $C \in \mathcal{A}_n$. Then it is clear that $(\sum_{k\geq 0} \mathcal{A}_k)_n$ is the disjoint union of $\{(\mathcal{A}_k)_n\}_{k\in\mathbb{N}}$, therefore $\sum_{n\geq 0} \mathcal{A}_n$ is a combinatorial class if and only if for every

$n \in \mathbb{N}$ the set of all $k \in \mathbb{N}$ such that $(\mathcal{A}_k)_n \neq \emptyset$ is finite. In this case it follows by the very definition that

$$\left(\sum_{k\geq 0} \mathcal{A}_k\right)(z) = \sum_{k\geq 0} \mathcal{A}_k(z)$$

where the convergence of the right hand side in the $(z)-$adic topology of $\mathbb{C}[[z]]$ is guaranteed by the assumptions on $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ (see [FS] for further details).

(ii) (*Product*) Denote by $\mathcal{A}\times\mathcal{B}$ the cartesian product of $\mathcal{A}$ and $\mathcal{B}$ and denote $|(A,B)|_{\mathcal{A}\times\mathcal{B}} = |A| + |B|$ for every $(A,B) \in \mathcal{A}\times\mathcal{B}$. Then $(\mathcal{A}\times\mathcal{B}, |\cdot|_{\mathcal{A}\times\mathcal{B}})$ is a combinatorial class, called the *product* of $\mathcal{A}$ and $\mathcal{B}$. Note that clearly

$$(\mathcal{A}\times\mathcal{B})_n = \bigcup_{\substack{(h,k)\in\mathbb{N}^2 \\ h+k=n}} \mathcal{A}_h \times \mathcal{B}_k$$

therefore $|(\mathcal{A}\times\mathcal{B})_n| = \sum_{k=0}^{n} |\mathcal{A}_k||\mathcal{B}_{n-k}|$, proving that

$$(\mathcal{A}\times\mathcal{B})(z) = \mathcal{A}(z)\mathcal{B}(z).$$

(iii) (*Sequence*) Denote $\mathrm{Seq}(\mathcal{A}) = \sum_{n\geq 0} \mathcal{A}^n$. In case $|A| \neq 0$ for every $A \in \mathcal{A}$, the class $\mathrm{Seq}(\mathcal{A})$ is a combinatorial class, because, if $n \in \mathbb{N}$, then $(\mathcal{A}^k)_n = \emptyset$ whenever $k \in \mathbb{N}$ and $k \geq n+1$. In particular, it follows that

$$\mathrm{Seq}(\mathcal{A})(z) = \sum_{n\geq 0} (\mathcal{A}(z))^n = \frac{1}{1-\mathcal{A}(z)}.$$

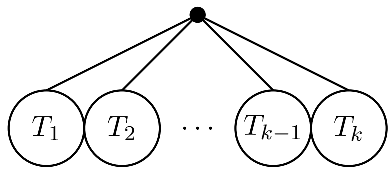
Finally, we state a classical and powerful result to extract the coefficients of a formal power series satisfying a suitable functional equation (see [FS] for a proof). This result proves to be often useful to settle a wide variety of enumerative problems.

Theorem 1.0.1 (Lagrange's inversion theorem). *Let $\phi(w) \in \mathbb{C}[[w]]^\times$ and $f(z) \in z\mathbb{C}[[z]]$. Then $f(z) = z\phi(f(z))$ if and only if $[z^0]f(z) = 0$ and*

$$[z^n]f(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n$$

for every $n \geq 1$.

As an application of the theory developed in this section, we work out a closed formula to enumerate the class of $k-$ary trees, which will turn useful later in Section 2.3.4. Recall that, for $k \in \mathbb{N}^*$, a *k-ary tree* is an ordered rooted tree such that every node has at most k children. Let $\mathcal{T}_k$ denote the combinatorial class of $k-$ary trees. Note that every $k-$ary tree is either empty or it can be decomposed as in the following figure:



where $\bullet$ is the root and $T_1, ..., T_k \in \mathcal{T}_k$. Therefore the combinatorial classes $\mathcal{T}_k$ and $\{\emptyset\} + \{\bullet\} \times (\mathcal{T}_k)^k$ are isomorphic and the isomorphism translates into the functional equation $\mathcal{T}_k(z) = 1 + z\mathcal{T}_k(z)^k$ for the generating function $\mathcal{T}_k(z)$ of the class $\mathcal{T}_k$. This equation can be classically solved by Lagrange's inversion as follows

$$[z^n]\mathcal{T}_k(z) = [z^n](\mathcal{T}_k(z) - 1) = \frac{1}{n}[w^{n-1}](1+w)^{kn} =$$

$$\frac{1}{n}[w^{n-1}]\sum_{i=0}^{kn}\binom{kn}{i}w^i = \frac{1}{n}\binom{kn}{n-1} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

In particular, when $k = 3$, we thus get

$$[z^n]\mathcal{T}_3(z) = \frac{1}{2n+1}\binom{3n}{n}$$

for every $n \in \mathbb{N}^*$.

1.0.4 ECO method

The ECO method (Enumerating Combinatorial Objects method) was introduced by Barcucci, Del Lungo, Pergola and Pinzani [BDLPP] and it is quite a natural approach to generation and enumeration of combinatorial classes of objects according to their size. The main idea of the ECO method consists of looking for a way to grow objects from smaller to larger ones by making some local expansions, where each object should be obtained from a unique father so that this construction gives rise to a tree that allows us to recursively generate all the objects in the class. If the shape of this tree can be described with a simple rule there is hope for exact enumeration results, translating this description into equations for the generating function of the class.

Let $\mathcal{A}$ be a combinatorial class and denote as usual by $\mathcal{A}_n$ the set of all $x \in \mathcal{A}$ such that $|x|_{\mathcal{A}} = n$ for every $n \in \mathbb{N}$. Let $\vartheta : \mathcal{A} \longrightarrow 2^{\mathcal{A}}$ be a map. We say that ϑ is an *ECO operator* on $\mathcal{A}$ when for every $n \in \mathbb{N}$ the following conditions hold:

(i) $\vartheta(\mathcal{A}_n) \subseteq 2^{\mathcal{A}_{n+1}}$.

(ii) for every $y \in \mathcal{A}_{n+1}$ one can find some $x \in \mathcal{A}_n$ such that $y \in \vartheta(x)$.

(iii) $\vartheta(x_1) \cap \vartheta(x_2) = \emptyset$ for every $x_1, x_2 \in \mathcal{A}_n$ such that $x_1 \neq x_2$.

In particular $\{\vartheta(x) : x \in \mathcal{A}_n\}$ is a partition of $\mathcal{A}_{n+1}$. In other words, the map ϑ generates all the objects of the class $\mathcal{A}$ in such a way that each object in $\mathcal{A}_{n+1}$ is obtained from a unique object in $\mathcal{A}_n$ for every $n \in \mathbb{N}$. Actually, we can also characterize ECO operators as follows. We say that a map $\rho : \mathcal{A} \setminus \mathcal{A}_0 \longrightarrow \mathcal{A}$ is a *reduction operator* on $\mathcal{A}$ when $\rho(\mathcal{A}_{n+1}) \subseteq \mathcal{A}_n$ for every $n \in \mathbb{N}$. The next proposition states that defining an ECO operator is essentially equivalent to defining a reduction operator.

Proposition 1.0.1. *Let $\mathcal{A}$ be a combinatorial class and $\vartheta : \mathcal{A} \longrightarrow 2^{\mathcal{A}}$ be a map. Then ϑ is an ECO operator on $\mathcal{A}$ if and only if $\vartheta = \rho^{\longleftarrow}$ for some reduction operator ρ on $\mathcal{A}$.*

Proof. Suppose ϑ is an ECO operator and $y \in \mathcal{A} \setminus \mathcal{A}_0$, then $y \in \mathcal{A}_{n+1}$ for some $n \in \mathcal{N}$, hence $y \in \vartheta(x)$ for a unique $x \in \mathcal{A}_n$, because ϑ is an ECO operator, and we set $\rho(y) = x$. Let now $\rho : \mathcal{A} \longrightarrow \mathcal{A}$ denote the map assigning the object $\rho(y)$ to each $y \in \mathcal{A}$. Then by definition $\rho(\mathcal{A}_{n+1}) \subseteq \mathcal{A}_n$ for every $n \in \mathbb{N}$, furthermore, for every $x \in \mathcal{A}$, again by definition $y \in \vartheta(x)$ if and only if $x = \rho(y)$, equivalently $y \in \rho^{\longleftarrow}(x)$, hence $\vartheta(x) = \rho^{\longleftarrow}(x)$ and $\vartheta = \rho^{\longleftarrow}$. Conversely, suppose $\vartheta = \rho^{\longleftarrow}$ for some reduction operator ρ on $\mathcal{A}$. Take $n \in \mathbb{N}$, $x \in \mathcal{A}_n$ and $y \in \vartheta(x) = \rho^{\longleftarrow}(x)$, then $\rho(y) = x$, hence $y \in \mathcal{A}_{n+1}$ because $\rho(\mathcal{A}_{n+1}) \subseteq \mathcal{A}_n$. Pick now $y \in \mathcal{A}_{n+1}$ and let $x = \rho(y) \in \mathcal{A}_n$, then by assumption $y \in \rho^{\longleftarrow}(x) = \vartheta(x)$. Finally, suppose $x_1, x_2 \in \mathcal{A}_n$ and $y \in \vartheta(x_1) \cap \vartheta(x_2) = \rho^{\longleftarrow}(x_1) \cap \rho^{\longleftarrow}(x_2)$, then $x_1 = \rho(y) = x_2$. This proves that ϑ is an ECO operator on $\mathcal{A}$. □

Let ϑ be an ECO operator on $\mathcal{A}$. Say that $\mathcal{A}$ is *rooted* when $\mathcal{A}$ contains a unique object with minimum size. Suppose $\mathcal{A}$ is rooted. In this case, one can represent ϑ by means of a tree, called the *generating tree* of ϑ and denoted by $\mathcal{T}_\vartheta$, which is a rooted tree having the objects of $\mathcal{A}$ as nodes, the object in $\mathcal{A}$ with minimum size as root and such that $\vartheta(x)$ is the set of children of x for every $x \in \mathcal{A}$. This representation of ϑ can be useful for enumeration purpose when $\mathcal{T}_\vartheta$ displays enough regularity to be described by a so-called succession rule. Let S be a set, let a be an element of S, let e be a sequence of maps in S^S and let $p : S \longrightarrow \mathbb{N}$ be a map. The triple (a, e, p) will be called the *succession rule* with *axiom* a, *production rule* e and *production parameter* p and it will be denoted by the symbol

$$\begin{cases} (a) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))...(e_{p(k)}(k)) \end{cases}$$

Let now $\mathcal{T}$ be a rooted tree with set of nodes T and root R, let $\Omega = (a, e, p)$ be a succession rule and let $\ell : T \longrightarrow S$ be a map. We say that ℓ is a $\Omega-$*labelling* of $\mathcal{T}$ when $\ell(R) = a$ and, if $v \in T$, then v has $p(\ell(v))$ children $v_1, ..., v_{p(\ell(v))}$ and $\ell(v_i) = e_i(\ell(v))$ for every $i \in [p(\ell(v))]$. We say that ϑ and $\mathcal{A}$ are *described* by Ω when one can find some $\Omega-$labelling of $\mathcal{T}_\vartheta$. Suppose now $S \subseteq \mathbb{N}$ and ℓ is an $\Omega-$labelling of $\mathcal{T}_\vartheta$. Denote by $\mathcal{A}(z, u)$ the generating function

$$\mathcal{A}(z, u) = \sum_{\sigma \in \mathcal{A}} z^{|\sigma|} u^{\ell(\sigma)} = \sum_{n,k \geq 0} |\mathcal{A}_{n,k}| z^n u^k$$

where $\mathcal{A}_{n,k}$ denotes the set of all $\sigma \in \mathcal{A}_n$ such that $\ell(\sigma) = k$ for every $(n, k) \in \mathbb{N}^2$. We can translate the $\Omega-$description of $\mathcal{A}$ into a functional equation for $\mathcal{A}(z, u)$ as follows. Denote by L_Ω the unique $\mathbb{Z}[[z]]-$linear map $L_\Omega : \mathbb{Z}[[z, u]] \longrightarrow \mathbb{Z}[[z, u]]$ such that

$$L_\Omega(u^k) = \begin{cases} u^a & k = 0 \\ u^{e_1(k)} + ... + u^{e_{p(k)}(k)} & k \geq 1 \end{cases}$$

Then it is easily seen that

$$\mathcal{A}(z,u) = \mathcal{A}(0,u) + zL_{\Omega}(\mathcal{A}(z,u)).$$

In the luckiest cases, these kind of equations can be solved using kernel type methods or other standard tools.

In Section 2 we will use the ECO method to generate all the permutations avoiding the vincular pattern $1-32-4$.

Chapter 2

Pattern avoidance

2.1 Permutation pattern avoidance

In this section we continue the study of the vincular permutation pattern poset. We refer to Section 1.0.2 for a general introduction to vincular pattern posets.

In Section 2.1.1 we characterize the infinite lower triangular $\{0,1\}-$matrices A such that the binary relation $\in_A$ defined in Section 1.0.2 actually coincide with the partial order relation $\leq_A$, clarifying a theoretical issue already posed in [BF] about the vincular permutation pattern poset.

In Section 2.1.3 we construct a single label generating tree for permutations avoiding the vincular pattern $1-32-4$, thus providing a direct explanation for a recursive formula to enumerate these permutations, which was originally obtained by Callan in [C] only as a consequence of a very intricated bijection with a certain class of ordered rooted trees.

2.1.1 Vincular permutation pattern posets

Recall from Section 1.0.2 that, given an infinite lower triangular $\{0,1\}-$ matrices A and two non-empty permutations σ and τ, we write $\sigma \in_A \tau$ when one can find some $i = (i_1, ..., i_{|\sigma|}) \in [|\tau|]^{|\sigma|}$ such that $i_1 < ... < i_{|\sigma|}$, $\tau_{i_1} ... \tau_{i_{|\sigma|}}$ and σ are order isomorphic and $i_{j+1} = i_j + 1$ whenever $j \in [|\sigma| - 1]$ and $A(|\sigma| - 1, j) = 0$. Now we write $\sigma \preceq_A \tau$ when $\pi \in_A \sigma$ and $|\pi| \in \{|\sigma| - 1, |\sigma|\}$. The transitive closure of $\preceq_A$ is a partial order relation on $\mathcal{S}$ denoted by $\leq_A$. The first thing we observe is that $\in_A$ and $\leq_A$ do not coincide in general, as it is immediately pointed out by the following counterexample. Take as A the infinite lower triangular $\{0,1\}-$matrices A all whose entries are 0, except for its third row, which is $(0,1,0)$, and take $\sigma = 1234$ and $\tau = 342156$, then clearly $\sigma \in_A \tau$ (the only occurrence of σ as an $A-$vincular pattern in τ being 3456), however $\sigma \not\leq_A \tau$, for otherwise $\sigma \prec_A \sigma_1 \prec_A \tau$ for some permutation σ_1, hence $\sigma_1 = 23145$ because 3456 is the only occurrence of σ as an $A-$vincular pattern in τ, on the other hand $23145 \notin_A \tau$, because the only occurrences of 23145 in τ as a classical pattern are 34156 and 34256, which are not occurrences as an $A-$vincular pattern by the assumptions on A, and this yields a contradiction. As a consequence, in [BF], the authors asked for a characterization of the lower triangular $\{0,1\}-$ matrices A for which $\in_A$ and $\leq_A$ coincide. We actually succeded in finding such a characterization and this result is the content of the next theorem. Before showing its statement, let us introduce a few necessary notations. First, recall from Section 1.0.2 that

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Now, for every $(h,k) \in \mathbb{N}^2$ denote by A_{hk} the infinite binary matrix A defined for every $(i,j) \in (\mathbb{N}^*)^2$ by setting $A(i,j) = 0$ if and only if either $i < j$ or $j > h$ and $i \geq j + k$, i.e. the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & 1 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & \vdots & \ddots & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & \ddots & \vdots & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & \ddots & 1 & \vdots & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & \ddots & 1 & 1 & \vdots & 1 & 0 & 0 & \dots \\ 1 & 1 & \ddots & 1 & 0 & 1 & \vdots & 1 & 0 & \dots \\ 1 & 1 & \ddots & 1 & 0 & 0 & 1 & \vdots & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

with i^{th} row

$$(\underbrace{1, ..., 1}_{i})$$

when $i \in \{1, 2, ..., h+k\}$ and

$$(\underbrace{1, ..., 1}_{h}, \underbrace{0, ..., 0}_{i-(h+k)}, \underbrace{1, ..., 1}_{k})$$

when $i \in \{(h+k)+1, (h+k)+2, ...\}$. For a matter of details, denote also by A^* the infinite matrix defined setting $A^*(1,1) = 1 - A(1,1)$ and $A^*(i,j) = A(i,j)$ when $(i,j) \in (\mathbb{N}^*)^2 \setminus \{(1,1)\}$. Note that clearly $\in_{A^*}$ and $\in_A$ coincide, therefore $\in_A$ and $\leq_A$ coincide if and only if $\in_{A^*}$ and $\leq_{A^*}$ coincide. Now we are in a position to state and prove the main result of this section.

Theorem 2.1.1. *Let A be an infinite lower triangular binary matrix. Then $\in_A$ and $\leq_A$ coincide if and only if either $A \in \{A_i, A_i^*\}$ for some $i \in \{1, 0\}$ or $A \in \{A_{hk}, A_{hk}^*\}$ for some $(h,k) \in \mathbb{N}^2$.*

Proof. Suppose first $A = A_{hk}$ for some $(h,k) \in \mathbb{N}^2$ and let us prove that $\in_A$ and $\leq_A$ coincide. Let $\sigma, \tau \in \mathcal{S}$ and assume σ has type

$$(\underbrace{1,...,1}_{h}, \underbrace{0,...,0}_{r}, \underbrace{1,...,1}_{k})$$

for some $r \in \mathbb{N}^*$, while τ has type

$$(\underbrace{1,...,1}_{h}, \underbrace{0,...,0}_{s}, \underbrace{1,...,1}_{k})$$

for some $s \in \{r+2, r+3, ...\}$. Suppose first $\sigma \leq_A \tau$, so that by induction $\sigma \in_A \pi \prec_A \tau$. Now let i be an occurrence of σ as an $A-$vincular pattern in π and j be an occurrence of π as an $A-$vincular pattern in τ, then $i_{h+1}, ..., i_{h+r+1}$ are consecutive by assumption on the type of σ, therefore $\{i_{h+1}, ..., i_{h+r+1}\} \subseteq \{h+1, ..., h+s\}$, which forces $j \circ i = (j_{i_1}, j_{i_2}, ..., j_{i_{|\sigma|}})$ to be an occurrence of σ in τ as an $A-$vincular pattern because by assumption π has type

$$(\underbrace{1,...,1}_{h}, \underbrace{0,...,0}_{s-1}, \underbrace{1,...,1}_{k})$$

which proves $\sigma \in_A \tau$. Conversely, assume $\sigma \in_A \tau$, so that by induction it is enough to show that $\sigma \prec_A \pi \in_A \tau$ for some $\pi \in \mathcal{S}$. Let i be an occurrence of σ in τ as an $A-$vincular pattern, now choose any $j \in \{1, ..., |\tau|\} \smallsetminus \{i_1, ..., i_{|\sigma|}\}$ such that reordering $i_1, ..., i_{|\sigma|}, j$ in increasing order $p_1 < p_2 < ... < p_{|\sigma|+1}$ the elements $p_{h+1}, ..., p_{h+r+2}$ are consecutive, then denote by π the unique $\pi \in \mathcal{S}_{|\sigma|+1}$ such that $\pi \sim \tau_{p_1}...\tau_{p_{|\sigma|+1}}$, now certainly $\sigma \prec_A \pi$, while also $\pi \in_A \tau$ because π has type

$$(\underbrace{1,...,1}_{h}, \underbrace{0,...,0}_{r+1}, \underbrace{1,...,1}_{k}).$$

Conversely, suppose now that $\in_A$ and $\leq_A$ coincide and let us prove that either $A = A_i$ for some $i \in \{1,0\}$ or $A = A_{hk}$ for some $(h,k) \in \mathbb{N}^2$. Let us first show that one cannot find some $k \in \{3,4,...\}$ and some $i \in \{1, ..., k-1\}$ such that $A(k-1,i) = 1$ and $A(k,i) = A(k,i+1) = 0$, i.e. a compact minor of A of the form

$$\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

for some $x \in \{0,1\}$. Let us argue by contradiction and assume such a minor exists. Suppose first $i \neq k-1$ and let $\sigma = 123...k$ and $\tau = 123...i(k+2)(k+1)(i+1)...k$. Then it is clear that $\sigma \in_A \tau$, thanks to the assumptions on the minor, where $123...k$ is indeed the only occurrence of σ as an $A-$vincular pattern in τ because $i \neq k-1$. However, let us show that $\sigma \not\leq_A \tau$. Indeeed, if $\sigma \leq_A \tau$, then one can find some chain

$\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ in $\mathcal{S}$, which forces $\sigma_1 = 123...i(k+1)(i+1)...k$ because $123...k$ is the only occurrence of σ as a pattern in τ, but then the only occurrences of σ_1 as a pattern in τ are $123...i(k+1)(i+1)...k$ and $123...i(k+2)(i+1)...k$, which are not occurrences as an $A-$vincular pattern because of the assumptions on the minor, contradicting $\sigma_1 \prec_A \tau$. Now let us suppose $i = k-1$ and let $\sigma = k...321$ and $\tau = k...32(k+1)(k+2)1$. Then clearly $\sigma \in_A \tau$ thanks to the assumptions on the minor, where $k...321$ is the only occurrence of σ as an ($A-$vincular) pattern in τ because $k \neq 2$. However, let us show that $\sigma \not\leq_A \tau$. Indeed, if $\sigma \leq_A \tau$, then one can find some chain $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ in $\mathcal{S}$, which forces $\sigma_1 = k...32(k+1)1$ because $k...321$ is the only occurrence of σ as a pattern in τ, but then the only occurrences of σ_1 as a pattern in τ are $k...32(k+1)1$ and $k...32(k+2)1$, which are not occurrences as an $A-$vincular pattern because of the assumptions on the minor, again contradicting $\sigma_1 \prec_A \tau$. Now let us prove that one cannot find any $k \in \{2, 3, ...\}$ and any $i \in \{1, ..., k-1\}$ such that $A(k-1, i) = 0$ and $A(k, i) = 1$, i.e. any compact minor of A of the form

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us argue by contradiction and assume such a minor exists. Suppose first $i \neq 1$ and let $\sigma = k...321$ and $\tau = k...(k-i+1)(k+2)(k-i)...321(k+1)$. Then clearly $\sigma \leq_A \tau$ because $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ where $\sigma_1 = k...321(k+1)$. However $\sigma \notin_A \tau$, indeed the only occurrence of σ as a pattern in τ is $k...321$ because $i \neq 1$, but this is not an occurrence as an $A-$vincular pattern. Suppose now $i = 1$ and $k \in \{4, 5, ...\}$, then let $\sigma = 1324...k$ and $\tau = 1(k+2)324...k(k+1)$. Now, clearly $\sigma \leq_A \tau$, because $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ where $\sigma_1 = 1324...k(k+1)$. On the other hand $\sigma \notin_A \tau$, indeed every occurrence of σ as a pattern in τ has necessarily the form 132... and therefore it is not an occurrence as an $A-$vincular pattern, according to the assumptions on the minor. For the remaining case $i = 1$ and $k = 3$ one can pick $\sigma = 213$ and $\tau = 25134$, then again $\sigma \leq_A \tau$ because $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ where $\sigma_1 = 2134$, on the other hand $\sigma \notin_A \tau$, because the only occurrences of σ as a pattern in τ are 213 and 214, which are not occurrences as an $A-$vincular pattern. Now let us show that one cannot find any $k \in \{2, 3, ...\}$ and any $i \in \{1, ..., k-1\}$ such that $A(k-1, i-1) = A(k, i-1) = 0$ and $A(k-1, i) = A(k, i) = 1$, i.e. any compact minor of A of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let us argue by contradiction and assume such a minor exists. Let $\sigma = 123...k$ and $\tau = (k+1)123...(i-1)(k+2)i...k$. Then clearly $\sigma \leq_A \tau$ because $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ where $\sigma_1 = (k+1)123...k$. However $\sigma \notin_A \tau$, indeed the only occurrence of σ as a pattern in τ is $123...k$ because $i \neq k$, but this is not an occurrence as an $A-$vincular pattern, according to the assumptions on the minor. Finally, let us show that one cannot find any $k \in \{3, 4, ...\}$ such that $A(k-1, k-1) = 0$ and $A(k, k) = 1$, i.e. any compact minor of A on its diagonal with the form

$$\begin{pmatrix} 0 & 0 \\ x & 1 \end{pmatrix}.$$

Let us a argue by contradiction and assume such a minor exists. Suppose first $k \neq 4$ and let $\sigma = k...4231$ and $\tau = (k+1)k...423(k+2)1$. Then clearly $\sigma \leq_A \tau$ because $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ where $\sigma_1 = (k+1)k...4231$. On the other hand $\sigma \not\leq_A \tau$, because the only occurrences of σ in τ have the form ...231, which are not occurrences as an $A-$vincular pattern thanks to the assumptions on the minor. In case $k = 3$, pick $\sigma = 312$ and $\tau = 43152$, then clearly $\sigma \leq_A \tau$ because $\sigma = \sigma_0 \prec_A \sigma_1 \prec_A \sigma_2 = \tau$ where $\sigma_1 = 4312$, however $\sigma \not\leq_A \tau$, because the only occurrences of σ in τ are 312 and 412, which are not occurrences as an $A-$vincular pattern thanks to the assumptions on the minor. Now the claim follows by induction on the columns of A. □

2.1.2 Previous work

We refer to [Ste] for a more detailed survey on vincular permutation pattern avoidance. The first systematic study of vincular patterns of length 3 was done by Claesson in [Cl] and permutations avoiding π have been enumerated for every vincular pattern π of length 3. It is worth mentioning that the fact that, as it turns out, $\mathcal{S}(1-23)$ is counted by the Bell numbers shows that the analogous of the Stanley-Wilf conjecture does not hold for some vincular patterns. Moreover, the same fact shows that the conjecture of Noonan and Zeilberger stated in [NZ] is also false for vincular patterns, namely, the number of permutations avoiding a vincular pattern is not necessarily polynomially recursive.

As for vincular patterns of length 4, it is stated in [Ste] that there are 48 symmetry classes of vincular patterns of length 4, and computer experiments show that there are at least 24 Wilf-equivalent classes (although their exact number seems to be unknown). For vincular non-classical patterns enumerative results are known for seven Wilf classes (out of at least 24) which are as follows:

- Elizalde and Noy [EN] gave the exponential generating functions for the number of occurrences of a consecutive pattern of length 4 for three out of the seven Wilf-equivalence classes for consecutive patterns, namely the classes with representatives 1234, 1243 and 1342.

- Kitaev [Ki2] and Elizalde [E] decomposed the class $\mathcal{S}(\sigma - k)$ in a suitable boxed product, where σ is any consecutive pattern and $k = |\sigma| + 1$. This decomposition allows to provide an expression for the exponential generating function of $\mathcal{S}(\sigma - k)$ in terms of the exponential generating function of $\mathcal{S}(\sigma)$. In particular, if σ is any consecutive pattern of length 3, this, together with the results of Elizalde and Noy [EN], yields an explicit formula for the exponential generaing function $\mathcal{S}(\sigma - 4)$, where σ is any consecutive pattern of length 3. Since there are precisely two Wilf-equivalence classes of consecutive patterns of length 3, with representatives 123 and 132, the result of Kitaev yields explicit formulas for the exponential generating functions of $\mathcal{S}(\pi)$ where π is any vincular pattern Wilf-equivalent to $123 - 4$ or

132 − 4. Explicitly, these formulas are

$$\exp\left(\frac{\sqrt{3}}{2}\int_0^x \frac{e^{\frac{t}{2}}}{\cos\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}\right)}\right) \quad \text{and} \quad \exp\left(\int_0^x \frac{dt}{1-\int_0^t e^{-\frac{u^2}{2}}du}\right)$$

for a vincular pattern Wilf-equivalent to 123 − 4 or 132 − 4, respectively.

- Callan gave a recursion for $a_n = |\mathcal{S}_n(31-4-2)|$, which goes as follows. Set $a_0 = c_1 = 1$ and
  1. $a_n = \sum_{i=0}^{n-1} a_i c_{n-i}$ for $n \geq 1$.
  2. $c_n = \sum_{i=0}^{n-1} i a_{(n-1),i}$ for $n \geq 2$.
  3. $a_{n,k} = \begin{cases} \sum_{i=0}^{k} a_i \sum_{j=k-i}^{n-1-i} a_{(n-1-i),j} & 1 \leq k \leq n-1 \\ a_{n-1} & k = n \end{cases}$
- Finally, Callan also showed in [C] that $|\mathcal{S}_n(1-32-4)| = \sum_{k=1}^{n} u(n,k)$ where, for every $1 \leq k \leq n$, the triangle $u(n,k)$ satisfies the recurrence relation

$$u(n,k) = u(n-1,k-1) + k\sum_{j=k}^{n-1} u(n-1,j) \tag{2.1}$$

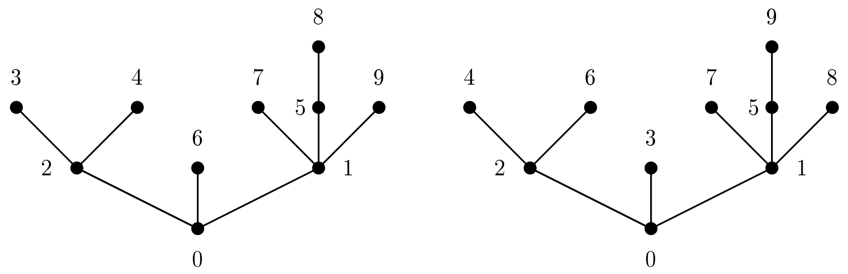
  with initial conditions $u(0,0) = 1$ and $u(n,0) = 0$ for every $n \geq 1$.

These seem to be the only explicit enumerative results concerning vincular patterns of length 4. An alternative, and we believe more explicative, proof of the recursive formula in Equation 2.1 will be the main issue of the next section.

2.1.3 Permutations avoiding the pattern 1-32-4

The vincular pattern 1 − 32 − 4 is a dashed version of the classical pattern 1324, which attracted great attention among combinatorialists, as enumeration of permutations avoiding this classical pattern has proven to be one of the hardest open problems in permutation pattern combinatorics. Hopefully, our insight into the class of permutations avoiding the classical pattern could benefit from a closer study of permutations avoiding one of its vincular counterparts. However, these two patterns seem also to display quite a different behaviour, for instance it is not difficult to see that 1 − 32 − 4 is actually Wilf-equivalent to 1 − 23 − 4 (see [E]), whereas this does not hold for their classical counterparts.

Permutations avoiding 1 − 32 − 4 are counted by sequence A113227 in [S], whose first ten terms are 1, 1, 2, 6, 23, 105, 549, 3207, 20577, 143239. As mentioned in Section 2.1.2, an efficient bivariate recursive formula to count permutations avoiding 1 − 32 − 4 was first discovered by Callan in [C]. This formula actually relies on a very intricated bijection (involving several contrived discrete structures defined ad hoc in order to break this transformation into somewhat simpler steps) between permutations of length n avoiding $1-32-4$ and increasing ordered rooted trees on $n+1$ nodes with increasing leaves for every $n \geq 1$. An ordered rooted tree on $n+1$ nodes $\{0,1,2,...,n\}$ is *increasing* when every node is smaller than each of its children. If in addition its leaves are increasing from left to right we say that it has *increasing leaves*. For instance, the figure below shows two increasing ordered rooted trees, the first has increasing leaves while the second does not.



Let $\mathcal{I}$ denote the combinatorial class of such trees. If, for every $1 \leq k \leq n$, we denote by $u(n)$ the number of trees in $\mathcal{I}_n$ and by $u(n,k)$ the number of trees in $\mathcal{I}_n$ such that the root has k children, so that $u(n) = \sum_{k=1}^{n} u(n,k)$, then it is easily proved in [C] that the triangle $u(n,k)$ satisfies the recurrence relation

$$u(n,k) = u(n-1,k-1) + k\sum_{j=k}^{n-1} u(n-1,j) \tag{2.2}$$

when $1 \leq k \leq n$, with initial conditions $u(0,0) = 1$ and $u(n,0) = 0$ for every $n \geq 1$. Thanks to the bijection established by Callan, this recursive formula allows also to count permutations of length n avoiding $1-32-4$. Although the recursive formula given in Equation (2.2) provides an efficient way to enumerate these permutations, we believe it provides only little insight into their structure, as it is not transparent at all from the bijection constructed by Callan how to read this recursive formula directly from a description of these permutations. Actually, it is not even clear which statistic on these permutations should correspond to the number of children of the root. Some unsuccesful attempts to read the recursive formula given in Equation (2.2) directly from a description of permutations avoiding $1-32-4$ have been made. As far as we know, the best result in this direction has been achieved by Duchi, Guerrini and Rinaldi, who constructed a two labels generating tree for this class of permutations as a consequence of a certain insertion algorithm called INSERTPOINT (see [DGR]). However, in the same paper, they suggest that the recursive formula given in Equation (2.2) appears to be difficult to understand directly on $\mathcal{S}(1-32-4)$. Additionally, sequence A113227 also happens to count quite a wide variety of combinatorial objects, among which we find increasing ordered rooted trees with increasing leaves, valley marked Dyck paths and inversion sequences avoiding the pattern 101 (see [C] and [CMSW]). In all these cases, it is instead relatively easy to read the recurrence relation given in Equation (2.2) from the same structural description of these objects and it is actually not hard to construct quite a straightforward bijection between them.

In this section we construct a single label generating tree for permutations avoiding the pattern $1-32-4$. We believe that this construction finally break the annoying asimmetry between the aforementioned combinatorial objects and permutations avoiding $1-32-4$, by providing a better insight into the structure of these permutations and, compared to Callan's bijection, a clearer explanation of why the recursive formula given in Equation (2.2) actually counts them. As a remarkable byproduct of this construction, we also obtain an explicit algorithm to generate all permutations avoiding $1-32-4$.

Although not explicitly stated in [C], the class $\mathcal{I}$ can be described by a succession rule as follows. Consider the map $e : \mathcal{I} \longrightarrow \mathbb{N}$ attaching to every $T \in \mathcal{I}$ the number $e(T)$ of the children of its root. One can easily construct an ECO operator η on the class $\mathcal{I}$ such that e is a $\Lambda-$labelling of $\mathcal{T}_\eta$ where Λ is the succession rule

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)(2)^2(3)^3...(k)^k(k+1) \end{cases}$$

The ECO operator ϑ is implicitly described by Callan in [C] when he proves that the class $\mathcal{I}$ is enumerated by the recursive formula in Equation (2.2), but we omit further details. We will show that the class $\mathcal{S}(1-32-4)$ can be described by pretty the same rule, which proves that the combinatorial classes $\mathcal{S}(1-32-4)$ and $\mathcal{I}$ are isomorphic.

The outline of the proof is as follows. First, we attach to every permutation π a label $\ell(\pi)$ defined as the number of right-to-left maxima of π on the right of 1 (e.g. $\ell(84617523) = 3$ as the right-to-left maxima of 84617523 on the right of 1 are exactly $7, 5$ and 3). Next, we define an ECO operator ϑ on the combinatorial class $\mathcal{S}(1-32-4)$ and we show that the map $\ell : \mathcal{S}(1-32-4) \longrightarrow \mathbb{N}$ attaching to every $\pi \in \mathcal{S}(1-32-4)$ the label $\ell(\pi)$ is actually an $\Omega-$labelling of $\mathcal{T}_\vartheta$ where Ω is the succession rule

$$\begin{cases} (0) \\ (k) \rightsquigarrow (0)(1)^2(2)^3...(k)^{k+1}(k+1) \end{cases} \tag{2.3}$$

This is the same as the succession rule

$$\begin{cases} (1) \\ (h) \rightsquigarrow (1)(2)^2(3)^3...(h)^h(h+1) \end{cases}$$

up to the change of label $h = k+1$. Observe also that it actually takes not much effort to deduce Equation (2.2) directly from the succession rule Ω, without any reference to increasing ordered rooted trees with increasing leaves.

Actually, it is easier to construct our ECO operator ϑ moving backwards, i.e. by first defining a reduction operator ρ on $\mathcal{S}(1-32-4)$ and then setting $\vartheta(\pi) = \rho^{\longleftarrow}(\pi)$ for every $\pi \in \mathcal{S}(1-32-4)$. Suppose π is a permutation of length $n \geq 1$ avoiding $1-32-4$ such that $\ell(\pi) = k$. As already noted in [E], any permutation of this kind can be written in the form

$$\pi = m_1\ell_{11}...\ell_{1k_1}m_2\ell_{21}...\ell_{2k_2}...m_h\ell_{h1}...\ell_{hk_h} \tag{2.4}$$

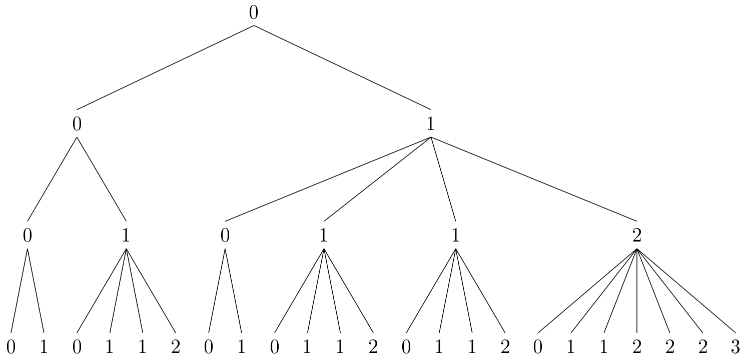


Figure 2.1: The first 3 levels of the labeled tree defined by the succession rule 2.3.

where $m_1, ..., m_h$ are the left to right minima of π (and of course $m_h = 1$) for some $h \geq 1$, while, for every $1 \leq i \leq h$, the letters $\ell_{i1}, ..., \ell_{ik_i}$ (where possibly $k_i = 0$, with obvious meaning) denote non-empty increasing sequences such that $\max(\ell_{ij}) > \max(\ell_{i(j+1)})$ for every $j \in [k_i - 1]$ when $k_i \geq 2$. In particular $k_h = k$ by definition of $\ell(\pi)$. Actually, a permutation that can be written in the form displayed in (2.4) avoids $1-32-4$ if and only if $\max(\ell_{(i+1)1}) < \max(\ell_{i(k_i-1)})$ whenever $h \geq 2$, $i \in [h-1]$ and $k_i \geq 2$. For instance, the permutation $\pi = (8, 9, 14, 12, 5, 2, 4, 10, 11, 1, 3, 13, 6, 7)$ avoids $1-32-4$ and decomposes as follows

$$\pi = (\underset{m_1}{\color{red}{8}}, \underset{\ell_{11}}{\boxed{9,14}}, \underset{\ell_{12}}{\boxed{12}}, \underset{m_2}{\color{red}{5}}, \underset{m_3}{\color{red}{2}}, \underset{\ell_{31}}{\boxed{4,10,11}}, \underset{m_4}{\color{red}{1}}, \underset{\ell_{41}}{\boxed{3,13}}, \underset{\ell_{42}}{\boxed{6,7}})$$

where the left to right minima of π are marked in red. Note that in this case $h = 4$, while $k_1 = 2$, $k_2 = 0$, $k_3 = 1$ and $k_4 = k = 2$. There is quite a natural way to reduce π to another permutation $\rho(\pi)$ of length $n-1$ avoiding the pattern $1-32-4$, namely by deleting 1, performing some further almost forced operations to restore the avoidance of the pattern $1-32-4$ and taking the standard reduction of the sequence obtained in this way (i.e. subtracting 1 to each item of the sequence). Indeed, we can distinguish two cases:

(i) Suppose 2 occurs to the left of 1 in π, i.e. $h \geq 2$ and $m_{h-1} = 2$. In this case we say that π has *type* $(2,1)$ and we can construct $\rho(\pi)$ as follows. We delete 1 from π and restore the structure displayed in (2.4) by sorting the list $\ell_{(h-1)1}, ..., \ell_{(h-1)k_{h-1}}$, $\ell_{h1}, ..., \ell_{hk_h}$ of increasing sequences to the right of 2 in such a way that their maximum elements are in decreasing order from left to right. Finally, we define $\rho(\pi)$ as the standard reduction of the integer sequence obtained in this way. It is clear by construction that $\rho(\pi)$ avoids $1-32-4$.

(ii) Suppose 2 occurs to the right of 1 in π, i.e. either $h = 1$ or $h \geq 2$ and $m_{h-1} \geq 3$, so that $\ell_{hi} = 2\ell'_{hi}$ for some $i \in [k]$ and a possibly empty increasing sequence ℓ'_{hi}. In this case we say that π has *type* $(1,2)$ and we can construct $\rho(\pi)$ as follows. We delete 1 and restore the structure displayed in (2.4) by moving 2 to the position previously occupied by 1 (in other words we swap 1 and 2 and then delete 1), thus obtaining the integer sequence $m_1\ell_{11}...\ell_{1k_1}m_2\ell_{21}...\ell_{2k_2}...2\ell_{h1}...\ell'_{hi}...\ell_{hk_h}$. Finally, we define $\rho(\pi)$ as the standard reduction of this sequence. Again, it is clear by construction that $\rho(\pi)$ avoids $1-32-4$.

Example 2.1.1. *Let us illustrate the previous construction with two examples.*

(i) Take the permutation $\pi = (8,9,14,12,5,2,4,10,11,1,3,13,6,7)$ *and let us compute* $\rho(\pi)$*. Note that* π *has type* $(2,1)$*, therefore we first delete* 1 *to obtain the sequence* $(8,9,14,12,5,2,4,10,11,3,13,6,7)$*, then we sort the increasing sequences* $(4,10,11)$*,* $(3,13)$ *and* $(6,7)$ *on the right of* 2 *in such a way that their maximum elements* $11,13$ *and* 7 *are in decreasing order. Therefore the correct order is given by* $(3,13)(4,10,11)(6,7)$*, which yields the sequence* $(8,9,14,12,5,2,3,13,4,10,11,6,7)$*. Taking the standard reduction of this sequence returns the permutation* $\rho(\pi) = (7,8,13,11,4,1,2,12,3,9,10,5,6)$*.*

(ii) Take the permutation $\pi = (8,9,14,12,5,3,4,10,11,1,6,13,2,7)$ *and let us compute* $\rho(\pi)$*. Note that* π *has type* $(1,2)$*, hence we first delete* 1 *to obtain the sequence* $(8,9,14,12,5,3,4,10,11,6,13,2,7)$*, then we move* 2 *to the position previously occupied by* 1*, so to obtain the sequence* $(8,9,14,12,5,3,4,10,11,2,6,13,7)$*. Taking the standard reduction of this sequence returns the permutation* $\rho(\pi) = (7,8,13,11,4,2,3,9,10,1,5,12,6)$*.*

This construction induces a reduction operator ρ on $\mathcal{S}(1-32-4)$ and thus, as mentioned before, an ECO operator ϑ on $\mathcal{S}(1-32-4)$. Now we want to show that ϑ can be described by the succession rule Ω given by (2.3). For this purpose, we will explicitly describe all the elements of $\vartheta(\pi)$ and compute their labels by reversing the previous construction. More specifically, we will expand π by appending 0 at the end of π, then moving some of the increasing sequences $\ell_{h1},...,\ell_{hk_h}$ to the right of 0 and finally normalizing the sequence thus obtained (i.e. adding 1 to each item of the sequence). In fact, the range of possibilities to perform this operation is quite constrained because we have to preserve the avoidance of the pattern $1-32-4$. First append a 0 at the end of π.

(i) Of course, no occurrence of $1-32-4$ will appear if we move the whole sequence $\ell_{h1}...\ell_{hk_h}$ to the right of 0. In this way we get the sequence $m_1\ell_{11}...\ell_{1k_1}...m_h0\ell_{h1}...\ell_{hk_h}$, whose normalization is a permutation which we denote by $\pi^{(k)}$. Note that $\ell(\pi^{(k)}) = k$.

(ii) Suppose instead that $k \geq 1$ and we want to move only $i \in \{0,...,k-1\}$ increasing sequences among $\ell_{h1},...,\ell_{hk_h}$ to the right of 0. Then it is easy to see that there is a unique way to perform this operation so to prevent an occurrence of $1-32-4$ to appear in the resulting expansion of π, which is the following way. Choose some $j \in [i+1]$ and move the $(i+1)^{th}$ suffix $\ell_{h(k_h-i)},...,\ell_{hk_h}$ of the list $\ell_{h1},...,\ell_{hk_h}$, except for its j^{th} increasing sequence $\ell_{h(k_h-i+j-1)}$, to the right of 0. In other words, move the sequence $\ell_{h(k_h-i)}...\hat{\ell}_{h(k_h-i+j-1)}...\ell_{hk_h}$ (where the hat over an item means that it must be omitted) to the right of 0, thus obtaining the sequence

$$m_1\ell_{11}...\ell_{1k_1}...m_h\ell_{h1}...\ell_{h(k_h-i-1)}\ell_{h(k_h-i+j-1)}0\ell_{h(k_h-i)}...\hat{\ell}_{h(k_h-i+j-1)}...\ell_{hk_h}.$$

Finally normalize this sequence to a permutation, which we denote by $\pi^{(i,j)}$. Note that $\ell(\pi^{(i,j)}) = i$. Hence, this operation produces $i+1$ children with label i, for every $0 \leq i \leq k-1$, from a node with label k.

Note that all permutations defined in (i) and (ii) will have type $(2,1)$, therefore these permutations cannot exhaust the whole class $\mathcal{S}(1-32-4)$ and we need to construct other expansions of π to include also permutations of type $(1,2)$. To this purpose we also move $m_h = 1$ to the right of 0 and perform some further transformations.

(iii) Of course, no occurrence of $1-32-4$ will appear if we move the whole sequence $1\ell_{h1}...\ell_{hk_h}$ to the right of 0. In this way, we get the sequence $m_1\ell_{11}...\ell_{1k_1}...01\ell_{h1}...\ell_{hk_h}$, whose normalization is a permutation which we denote by $\pi^{[1]}$. More generally, it is clear that no occurrence of $1-32-4$ will appear if we perform the following operation. Choose some $i \in [k]$ and move 1 immediately to the left of ℓ_{hi}, then move the sequence $\ell_{h1}...1\ell_{hi}...\ell_{hk_h}$ to the right of 0. In this way we obtain the sequence $m_1\ell_{11}...\ell_{1k_1}...0\ell_{h1}...1\ell_{hi}...\ell_{hk_h}$, whose normalization is a permutation, which we denote by $\pi^{[i]}$. Note that $\ell(\pi') = k$. Hence, operation (i) and (iii) produce $k+1$ children with label k from a node with label k.

(iv) Finally, we have a last possibility to transform π and prevent an occurrence of $1-32-4$ to appear. Move 1 back to the right of ℓ_{hk_h}, then move the sequence $\ell_{h1}...\ell_{hk_h}1$ to the right of 0. In this way we obtain the sequence $m_1\ell_{11}...\ell_{1k_1}...0\ell_{h1}...\ell_{hk_h}1$ and normalize it to a permutation, which we denote by $\pi^{[k+1]}$. Note that in this case, unlike in the previous one, we have $\ell(\pi^{[k+1]}) = k+1$. Hence, this operation will produces a unique child with label $k+1$ from a node with label k. Note that this last possibility could actually be regarded as a special case of (iii) if we let π terminate with an additional empty increasing sequence $\ell_{(h+1)k_{h+1}}$.

Note that all permutations defined in (iii) and (iv) have type $(1,2)$.

Example 2.1.2. *Let us take* $\pi = (5, 9, 14, 10, 12, 1, 2, 7, 13, 6, 11, 3, 8, 4)$ *as a working example to illustrate some of the previous constructions. Note that in this case* π *has the form* $m_1\ell_{11}\ell_{12}m_2\ell_{21}\ell_{22}\ell_{23}\ell_{24}$ *where* $m_1 = 5$, $\ell_{12} = (9,14)$ *and* $\ell_{12} = (10,12)$, *while* $m_2 = 1$, $\ell_{21} = (2,7,13)$, $\ell_{22} = (6,11)$, $\ell_{23} = (3,8)$ *and* $\ell_{24} = (4)$, *in particular* $\ell(\pi) = 4$. *First we insert a* 0 *at the end of* π *to obtain the sequence* $(5, 9, 14, 10, 12, 1, 2, 7, 13, 6, 11, 3, 8, 4, 0)$.

(i) We start by constructing $\pi^{(4)}$. *We move the sequence* $(2, 7, 13, 6, 11, 3, 8, 4)$ *to the right of* 0, *thus obtaining the sequence* $(5, 9, 14, 10, 12, 1, 0, 2, 7, 13, 6, 11, 3, 8, 4)$, *whose normalization is given by* $\pi^{(4)} = (6, 10, 15, 11, 13, 2, 1, 3, 8, 14, 7, 12, 4, 9, 5)$.

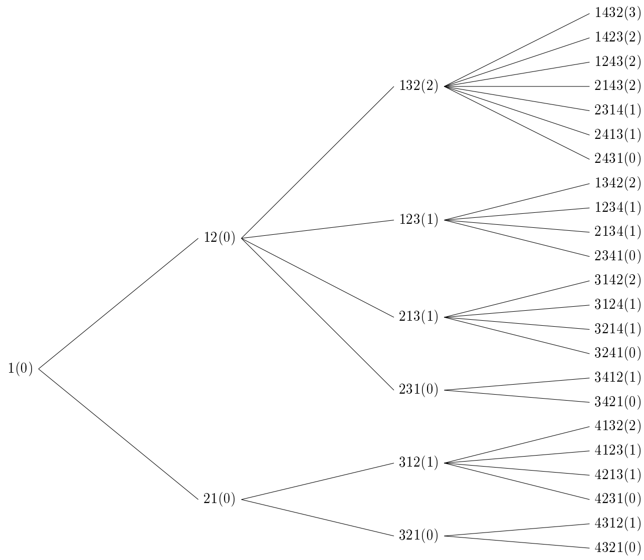


Figure 2.2: The first 3 levels of the generating tree for permutations avoiding the pattern $1-32-4$, where each node $\pi \in \mathcal{S}(1-32-4)$ is labeled by $(\ell(\pi))$.

(ii) *Now let us construct the permutation* $\pi^{(2,2)}$. *We move the suffix* $(6,11)(3,8)(4)$ *of the list* $(2,7,13)(6,11)(3,8)(4)$, *except for its* 2^{nd} *element* $(3,8)$, *to the right of* 0, *thus obtaining the sequence* $(5,9,14,10,12,1,2,7,13,3,8,0,6,11,4)$, *whose normalization is given by* $\pi^{(2,2)} = (6,10,15,11,13,2,3,8,14,4,9,1,7,12,5)$.

(iii) *Let us now construct* $\pi^{[3]}$. *We move the sequence* $(1,2,7,13,6,11,3,8,4)$ *to the right of* 0 *and move* 1 *immediately to the left of* $(3,8)$, *thus obtaining the sequence* $(5,9,14,10,12,0,2,7,13,6,11,1,3,8,4)$, *whose normalization is given by* $\pi^{[3]} = (6,10,15,11,13,1,3,8,14,7,12,2,4,9,5)$.

(iv) *Finally we construct* $\pi^{[4]}$. *We move the whole sequence* $(1,2,7,13,6,11,3,8,4)$ *to the right of* 0 *and move* 1 *to the right of* (4), *thus obtaining the sequence* $(5,9,14,10,12,0,2,7,13,6,11,3,8,4,1)$, *whose normalization is given by* $\pi^{[3]} = (6,10,15,11,13,,1,3,8,14,7,12,4,9,5,2)$.

Now we are in a position to state and prove the main result.

Theorem 2.1.2. *Suppose* $\pi \in \mathcal{S}(1-32-4)$ *and* $k = \ell(\pi)$.

(i) *If* $k = 0$, *then* $\vartheta(\pi) = \{\pi^{(0)}, \pi^{[1]}\}$.

(ii) *If* $k \geq 1$, *then* $\vartheta(\pi) = \{\pi^{(k)}, \pi^{(i,j)}, \pi^{[p]} : \ 0 \leq i \leq k-1, 1 \leq j \leq i+1, 1 \leq p \leq k+1\}$.

(iii) *The map* ℓ *is an* $\Omega-$*labelling of* $\mathcal{T}_\vartheta$.

Proof. It is mere routine to check that $\rho(\pi^{(0)}) = \rho(\pi^{[1]}) = \pi$ when $k = 0$ and that $\rho(\pi^{(k)}) = \rho(\pi^{(i,j)}) = \rho(\pi^{[p]}) = \pi$ when $k \geq 1$, $0 \leq i \leq k-1, 1 \leq j \leq i+1$ and $1 \leq p \leq k+1$. Conversely, assume $\sigma \in \vartheta(\pi)$ and write σ in the form $m_1\ell_{11}...\ell_{1k_1}...m_h\ell_{h1}...\ell_{hk_h}$ as in Equation (2.4). Suppose first that σ has type $(2,1)$. If 2 and 1 occur consecutively in π, then it is immediate to see that $\sigma = \pi^{(k)}$. Otherwise, it is also fairly easy to check that $\sigma = \pi^{(i,j)}$ where $i = \ell(\sigma)$, while $j = 1$ in case $\max(\ell_{h1}) < \max(\ell_{(h-1)k_{h-1}})$ and $j = \max\{j \in \{2,3,...,i+1\} : \max(\ell_{h(j-1)}) > \max(\ell_{(h-1)k_{h-1}})\}$ otherwise. Suppose now σ has type $(1,2)$. Then it is also easy to check that $\sigma = \pi^{[p]}$ where p is the unique $p \in [k_h]$ such that 2 occurs in the increasing sequence ℓ_{hp}. This proves (i) and (ii). Finally, (iii) holds because we know that $\ell(\pi^{(0)}) = 0$ and $\ell(\pi^{[1]}) = 1$ when $k = 0$, while $\ell(\pi^{(i,j)}) = i$, $\ell(\pi^{(k)}) = \ell(\pi^{[p]}) = k$ and $\ell(\pi^{[k+1]}) = k+1$ when $k \geq 1$, $0 \leq i \leq k-1, 1 \leq j \leq i+1$ and $1 \leq p \leq k$. □

2.1.4 Conclusion and further work

Concerning enumeration of permutations avoiding $1-32-4$, although the recursive formula found by Callan provides quite a simple and elegant way to count them, a closed formula would clearly be a more satisfactory answer. In this regard, the generating function $u(z)$ of permutations avoiding $1-32-4$ can be recursively described as a continued fraction $u(z) = 1 - z(U(0) - z)$ where $U(n) = 1 - z^n - z/U(n+1)$ for every $n \in \mathbb{N}$ (see [S]). However, this description can hardly be considered a closed formula. The succession rule describing permutations avoiding $1-32-4$ can also be translated into a functional equation for the generating function $u(z,t)$ of these permutations (where z keeps track of the length and t keeps track of the label), as explained at the very end of Section 1.0.4. This equation is actually a linear PDE of the form

$$(1-t)zt^2\frac{\partial u}{\partial t}(z,t) + ((1-t)^2(1-zt) + zt)u(z,t) = zt(1-t)^2 + ztu(z,1). \qquad (2.5)$$

However, we do not know whether this equation provides enough information to find some closed form expression for $u(z,t)$. Further research in this direction could improve our understanding of sequence A113227.

As for bijective issues, using the generating tree describing permutations avoiding $1-32-4$, it is likely that one can recursively construct some bijection with increasing trees having increasing leaves (preserving the respective labels). However, despite the efforts spent in this direction, we could not find a bijection admitting a reasonably simple description yet. Hopefully, a closer study of the structure of these permutations will suggest alternative approaches to simplify Callan's bijection.

As for a more generic issue, we observe that permutations avoiding the classical pattern 1324 form a subclass of permutations avoiding the vincular pattern $1-32-4$. Hence, it might be interesting to investigate in which cases the ECO operator ϑ fails to expand a permutation avoiding the classical pattern 1324 to another permutation avoiding the same pattern, causing an occurrence of 1324 to appear.

Finally, the fast recurrence for permutations avoiding $1-32-4$ provided in [C] suggests to look for nonobvious recurrences counting other similar patterns. For instance,

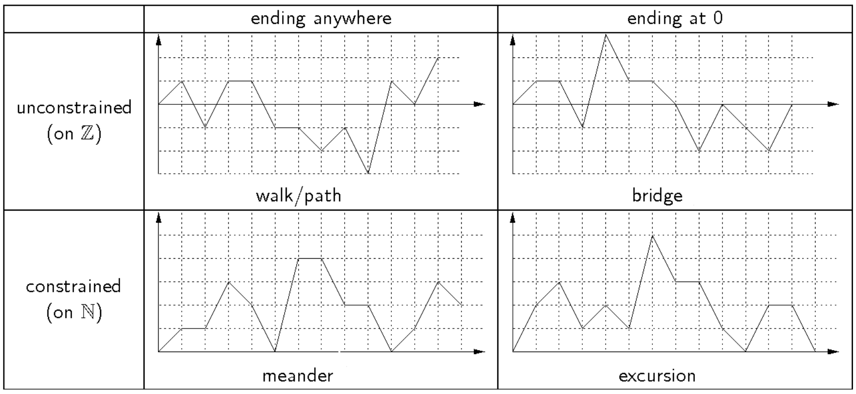


Figure 2.3: A graphical representation of simple walks, meanders and excursions.

note that a kind of structural description for permutations avoiding $12-34$ is given in [E], hence there may be hope to find a nice generating tree and deduce some reasonable recurrence relation, just like we did in the case of permutations avoiding $1-23-4$.

2.2 Lattice path pattern avoidance

In this section we continue the study of pattern avoidance in the context of lattice paths.

In Section 2.2.2 we extend to Schröder paths a method already developed in [BBFGPW] by Bacher, Ferrari, Bernini, Gunby, Pinzani and West to recursively compute the generating function of Dyck paths avoiding a single pattern.

In Section 2.2.3 we propose a deterministic finite state automata approach to the study of pattern avoidance in the context of lattice paths, based on a work by Asinowski, Bacher, Banderier and Gittemberger [ABBG].

We begin by fixing some preliminary notations and introducing the most general setting of walks, meanders and excursions. Let $\mathcal{S}$ be a finite set and, as usual, denote by $\mathcal{S}^*$ the language generated by $\mathcal{S}$, equipped with the subword order. For a language $\mathcal{L}$ in the alphabet $\mathcal{S}$ and $\pi \in \mathcal{S}^*$, we denote by $\mathcal{L}(\pi)$ the set of all $\sigma \in \mathcal{L}$ such that $\pi \not\leq \sigma$.

Suppose now $\mathcal{S}$ is a subset of $\mathbb{Z}^2$. In this context, the elements of $\mathcal{S}$ will be called *steps* and any element of $\mathcal{S}^*$ will be called a *walk* with steps in $\mathcal{S}$. This terminology is justified by the fact that every walk $\sigma_1\sigma_2...\sigma_n$ with steps in $\mathcal{S}$ can be visualized as a directed polygonal line in the lattice $\mathbb{Z}^2$ starting at the origin $(0,0)$ and formed by successively appending the vectors $\sigma_1, \sigma_2, ..., \sigma_n$. Steps of the form $(1, v)$ for some $v \in \mathbb{Z}$ are called *simple* and a walk with simple steps is called *simple*. Let now σ be a walk with steps in $\mathcal{S}$. The *final height* of σ, denoted by $\mathsf{h}(\sigma)$, is 0 when $\sigma = \varepsilon$ and it is $\sum_{i=1}^{|\sigma|} y_i$ otherwise, where $\sigma_i = (x_i, y_i)$ for every $i \in [|\sigma|]$. The following two classes of walks have a great combinatorial significance. We say that σ is a *meander* when $\mathsf{h}(\sigma_1...\sigma_i) \geq 0$ for every $i \in [|\sigma|]$. The set of all meanders with steps in $\mathcal{S}$ will be denoted by $\mathcal{S}^{\geq 0}$. We say that

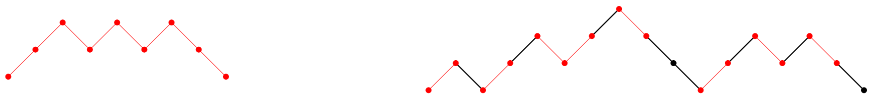


Figure 2.4: The Dyck path $UUDUDUDD$ on the left is a pattern of the Dyck path $UDUUDUUDDDUUDUDD$ on the right.

σ is an *excursion* when it is a meander and in addition $\mathsf{h}(\sigma_1 \ldots \sigma_{|\sigma|}) = 0$. The set of all excursions with steps in $\mathcal{S}$ will be denoted by $\mathcal{S}^0$.

Observe that, given a finite set of steps $\mathcal{S}$ and a pattern $\pi \in \mathcal{S}^*$, enumerating the class $\mathcal{S}^*(\pi)$ of walks with steps in $\mathcal{S}$ avoiding π is a quite elementary problem admitting a concise answer, as it does not depend on the fact that $\mathcal{S}$ is a subset of $\mathbb{Z}^2$. Instead the same problem for the classes $\mathcal{S}^{\geq 0}(\pi)$ and $\mathcal{S}^0(\pi)$ of meanders and excursions with steps in $\mathcal{S}$ avoiding π, respectively, is an open question in its full generality. In fact, although this notion of pattern for meanders and excursions is inherited by their description as words, the conditions a word has to obey in order to belong to either of these classes make the resulting posets highly nontrivial, and fully justify our approach, consisting of the study of their properties independently of their relationship with the full word pattern poset.

Example 2.2.1. *We recall some remarkable examples of lattice paths that are widely studied in literature.*

(i) *When $\mathcal{S} = \{(1, \pm 1)\}$, step $(1, 1)$ is ususally called* Up *step and it is denoted by U, while step $(1, -1)$ is usually called* Down *step and it is denoted by D. Furthermore, meanders with steps in $\mathcal{S}$ are called* Dyck prefixes *and excursions with steps in $\mathcal{S}$ are called* Dyck paths. *The class of Dyck paths is denoted by $\mathcal{D}$. Since every Dyck path has even length, we usually prefer to consider as its size the semilength rather than the length, in which case it is well known that $|\mathcal{D}_n| = C_n$ for every $n \in \mathbb{N}$.*

(ii) *When $\mathcal{S} = \{(1, 0), (1, \pm 1)\}$, the step $(1, 0)$ is usually called the* Horizontal *step and it is denoted by H. Furthermore, meanders with steps in $\mathcal{S}$ are called* Motzkin prefixes *and excursions with steps in $\mathcal{S}$ are called* Motzkin paths. *It is well known that Motzkin paths are enumerated by the Motzkin numbers sequence, which form sequence A001006 in [S].*

(iii) *When $\mathcal{S} = \{(2, 0), (1, \pm 1)\}$, the step $(2, 0)$ is usually called the* Double Horizontal *step and it is denoted by H_2 or simply by H when no risk of confusion with Motzkin paths arises. Furthermore, meanders with steps in $\mathcal{S}$ are called* Schröder prefixes *and excursions with steps in $\mathcal{S}$ are called* Schröder paths. *It is well known that Schröder paths are enumerated by the (large) Schröder numbers, which form sequence A006318 in [S].*

2.2.1 Previous work

In [BBFGPW], the authors developed a method to compute the generating functions of Dyck paths avoiding a single pattern in a recursive fashion. From these generating functions, one can then get the exact enumeration of the associated Dyck paths. In the following we shortly review this approch, which will be extended to Schröder paths in Section 2.2.2. Let P be a Dyck path and $\Delta_P(x)$ be the generating function of Dyck paths avoiding the pattern P, where x takes into account the length rather than the semilength (thus, $\Delta_P(x)$ is an even power series). We also define intermediate generating functions: let $C_P(x, y)$ be the bivariate generating function of smallest Dyck prefixes containing the pattern P (i.e., such that no proper prefix contains P), where x takes into account the length and y the final height. Note that, for what concerns $C_P(x, y)$, the pattern P can be any Dyck prefix (rather than a Dyck path). Finally, we denote with ε the empty path. The following is one of the main results contained in [BBFGPW].

Theorem 2.2.1. *The generating function $C_P(x, y)$ satisfies the following recurrence formulas:*

$$\begin{aligned} C_\varepsilon(x, y) &= 1; \\ C_{PU}(x, y) &= \frac{yC_P(x, y) - xC_P(x, x)}{y - x} xy; \\ C_{PD}(c, y) &= \frac{xy^{-1}}{1 - xy} C_P(x, y) - xy^{-1} C_P(x, 0). \end{aligned}$$

Moreover, the generating function $\Delta_P(x)$ is given by:

$$\begin{aligned} \Delta_\varepsilon(x) &= 0; \\ \Delta_{PU}(x) &= \Delta_P(x) + C_P(x, x); \\ \Delta_{PD}(x) &= \Delta(x) + C_P(x, 0). \end{aligned}$$

2.2.2 Schröder paths avoiding a single pattern

Let P be a Schröder prefix, which in the sequel will be interpreted as a *pattern*. Denote with $C_P(x, y)$ the bivariate generating function of the smallest Schröder prefixes containing P, where x keeps track of the length and y keeps track of the final height. Here "smallest" indicates that there must not be any proper prefix containing P. For instance, choosing $P = UH$, the generic smallest Schröder prefix containing P is obtained by concatenating a sequence of letters H with a nonempty Dyck prefix followed by an H. Thus, recalling the expression of the bivariate generating function $\mathcal{DP}(x, y) = \frac{2}{1-2xy+\sqrt{1-4x^2}}$ of Dyck prefixes, we get

$$C_P(x, y) = \frac{1}{1 - x^2} \left(\frac{2}{1 - 2xy + \sqrt{1 - 4x^2}} - 1 \right) x^2.$$

The following result gives a recursive procedure to compute $C_P(x, y)$. In the sequel $\mathrm{Cat}(x) = C(x^2)$ will denote the generating function of Dyck paths enumerated according to the length, rather than the semilength.

Proposition 2.2.1. *For any given Schröder prefix P, we have:*

$$C_\epsilon(x,y) = 1, \tag{2.6}$$

$$C_{PU}(x,y) = \frac{xy}{(1-x^2)(x-y(1-x^2))}\left(xC_P\left(x, \frac{x}{1-x^2}\right) - y(1-x^2)C_P(x,y)\right), \tag{2.7}$$

$$C_{PH}(x,y) = \frac{x^2}{y-x-xy^2}\left(yC_P(x,y) - x\mathrm{Cat}(x)C_P(x, x\mathrm{Cat}(x))\right), \tag{2.8}$$

$$C_{PD}(x,y) = \frac{x}{y}\left(\frac{1}{1-xy-x^2}C_P(x,y) - \frac{1}{1-x^2}C_P(x,0)\right). \tag{2.9}$$

Proof. Clearly, the only smallest Schröder prefix containing the empty word ϵ is the empty word itself, which gives $C_\epsilon(x,y) = 1$.

Let now Q be a smallest Schröder prefix containing PU and denote with Q' the smallest prefix of Q containing P. Moreover, we indicate with h the final height of Q'. Then, the prefix Q can be factorized as

$$Q = Q'B^{(h)}U, \tag{2.10}$$

where $B^{(h)}$ is a path starting at height h (which is the height of the final point of Q') using only H steps and D steps and not crossing the x axis (i.e., the final height i of $B^{(h)}$ is such that $0 \le i \le h$). Clearly, the path $B^{(h)}$ is the reverse of a path A starting at the origin, using only U steps and H steps, with final height less than or equal to h. It is not difficult to compute the bivariate generating function $A(x,y)$ of such paths A, where x and y track the length and the final height of A, respectively. Indeed, such a pattern A, if not empty, can be obtained either by taking an H step followed by a pattern of the same kind or by taking a U step followed by a pattern of the same kind. This leads to the functional equation:

$$A(x,y) = 1 + x^2A(x,y) + xyA(x,y)$$

(recall that H has length 2, while U has length 1). Therefore,

$$A(x,y) = \frac{1}{1-xy-x^2} = \frac{1}{1-x^2}\sum_{n\ge 0}\left(\frac{x}{1-x^2}\right)^n y^n\ , \tag{2.11}$$

If $B^{(h)}(x,y)$ denotes the generating function of the paths of the form $B^{(h)}$ (where x and y have the same role as in $A(x,y)$), using essentially the same argument as above, we have

$$B^{(h)}(x,y) = \sum_{i=0}^{h}\left[y^{h-i}\right]A(x,y)y^i;$$

hence, in terms of generating functions, relation (2.10) can be translated as follows:

$$C_{PU}(x,y) = \left(\sum_{h\geq 0} \left[y^h\right] C_p(x,y)B^{(h)}(x,y) \right) xy = \left(\sum_{h\geq 0} \left[y^h\right] C_p(x.y) \sum_{i=0}^{h} \left[y^{h-i}\right] A(x,y)y^i \right) xy \ . \quad (2.12)$$

We note that, referring to (2.10), the term $\sum_{h\geq 0} \left[y^h\right] C_p(x,y)$ in (2.12) records the prefix Q', while the term xy tracks the step U. By using (2.11) for the coefficient $\left[y^{h-i}\right] A(x,y)$, expression (2.12) can be reduced to (2.7).

Similarly, let now Q be a smallest Schröder prefix containing PH and let Q' be the smallest prefix of Q containing P. We denote with $D_F^{(h)}$ a Dyck factor starting at height h and with $D_F^{(h)}(x,y)$ the bivariate generating function for such paths $D_F^{(h)}$. We have:

$$Q = Q' D_F^{(h)} H \ .$$

Therefore,

$$C_{PH}(x,y) = \sum_{h\geq 0} \left(\left[y^h\right] C_p(x,y) D_F^{(h)}(x,y) \right) x^2 \ . \quad (2.13)$$

As far as $D_F^{(h)}(x,y)$ is concerned, denoting with E a generic Dyck prefix and with D_i a Dyck path, we observe that a Dyck factor $D_F^{(h)}$ can be factorized as $D_F^{(h)} = (D_1 D)(D_2 D) \dots (D_r D)E$, with $0 \leq r \leq h$, where the first D step reaching height $h-i$ is explicitly recorded, for each $i = 1, 2, \dots, r$. In other words, $D_F^{(h)}$ is a Dyck prefix E starting at height h (in the case $r = 0$), or it is made by a Dyck path D_1 followed by a D step followed by a Dyck prefix (in the case $i = 1$), or it is the concatenation of $D_1 D$ and $D_2 D$ followed by a Dyck prefix (in the case $i = 2$), and so on up to $i = r$. From the above construction, in terms of generating functions we have:

$$\begin{aligned} D_F^{(h)}(x,y) &= \mathcal{DP}(x,y)y^h + \mathrm{Cat}(x)x\mathcal{DP}(x,y)y^{h-1} + \\ &\quad \mathrm{Cat}^2(x)x^2\mathcal{DP}(x,y)y^{h-2} + \ldots + \mathrm{Cat}^h(x)x^h\mathcal{DP}(x,y)y^{h-h} \\ &= \mathcal{DP}(x,y) \sum_{i=0}^{h} x^i y^{h-i} \mathrm{Cat}^i(x) \ , \end{aligned}$$

leading to

$$D_F^{(h)}(x,y) = \frac{2}{1 - 2xy + \sqrt{1-4x^2}} \cdot \frac{y^{h+1} - x^{h+1}\mathrm{Cat}^{h+1}(x)}{y - x\mathrm{Cat}(x)} \ . \quad (2.14)$$

Replacing (2.14) into (2.13) we obtain:

$$C_{PH}(x,y) = \frac{2x^2}{\left(1-2xy+\sqrt{1-4x^2}\right)(y-x\mathrm{Cat}(x))} \sum_{h\geq 0} \left[y^h\right] C_P(x,y)\left(y^{h+1}-x^{h+1}\mathrm{Cat}(x)^{h+1}(x)\right) ,$$

which (recalling that $\mathrm{Cat}(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$) boils down to (2.8).

Finally, let Q a smallest Schröder prefix containing PD and let Q' be the smallest prefix of Q containing P. Then

$$Q = Q'AD$$

with the restriction that, if Q' ends at height $h = 0$, then $A \neq H^r$, for any $r \geq 0$ (otherwise Q would not be a Schröder prefix, since it would end with D). In terms of generating functions, we then have:

$$\begin{aligned} C_{PD}(x,y) &= \left(\sum_{h\geq 0}\left[y^h\right] C_P(x,y)y^h A(x,y)\right) xy^{-1} - \left[y^0\right] C_P(x,y)\sum_{i\geq 0} x^{2i}xy^{-1} = \\ &= \frac{x}{y}A(x,y)C_P(x,y) - \frac{x}{y}C_P(x,0)\frac{1}{1-x^2} = \\ &= \frac{x}{y}\left(\frac{1}{1-xy-x^2}C_P(x,y) - \frac{1}{1-x^2}C_P(x,0)\right), \end{aligned}$$

which is equal to (2.9). □

Let $\Delta_P(x)$ be the generating function of Schröder paths avoiding P with respect to the length. The following result gives a recursive procedure to compute $\Delta_P(x)$, where P is a Schröder pattern.

Proposition 2.2.2. *For any Schröder pattern P, the generating function $\Delta_P(x)$ is given by:*

$$\Delta_\epsilon(x) = 0 \tag{2.15}$$

$$\Delta_{PD}(x) = \Delta_P(x) + C_P(x,0)\frac{1}{1-x^2} \tag{2.16}$$

$$\Delta_{PH}(x) = \Delta_P(x) + \mathrm{Cat}(x)\cdot C_P\left(x, x\mathrm{Cat}(x)\right) \tag{2.17}$$

$$\Delta_{PU}(x) = \Delta_P(x) + \frac{1}{1-x^2}C_P\left(x, \frac{x}{1-x^2}\right) . \tag{2.18}$$

Proof. Every Schröder path contains the empty path ϵ, hence $\Delta_\epsilon(x) = 0$. Let Q be a Schröder path avoiding PD. There are two cases: either Q avoids P, and such paths Q are counted by means of the generating function $\Delta_P(x)$. In the remaining case (i.e., the path Q contains P but avoids PD), let Q' be the smallest prefix of Q containing P. Obviously, Q' cannot be followed by any D step in any position, otherwise the path Q

would contain PD, against the hypothesis. Thus the only possibility is that Q' has final height equal to 0 and is followed by a certain number of consecutive H steps. In other words Q can be factorized as follows.

$$Q = Q'H^i \ ,$$

with $i \geq 0$. In terms of generating functions, the above argument leads to:

$$\Delta_{PD}(x) = \Delta_P(x) + \left[y^0\right] C_P(x,y) \sum_{i \geq 0} x^{2i} = \Delta_P(x) + C_P(x,0)\frac{1}{1-x^2} \ .$$

which is equation (2.16).

Now suppose that Q avoids PH. If Q also avoids P, then, as in the previous case, we obtain the generating function $\Delta_P(x)$. Otherwise, Q can be decomposed as its smallest prefix Q' containing P, ending at height $h \geq 0$, followed by a path starting from height h, using only U and D steps and ending on the x axis. This path is easily seen to be the reverse of a Dyck prefix having final height h, hence:

$$\Delta_{PH}(x) = \Delta_P(x) + \sum_{h \geq 0} \left(\left[y^h\right] C_P(x,y) \left[y^h\right] \mathcal{DP}(x,y) \right) \ . \qquad (2.19)$$

Since we have

$$\left[y^h\right] \mathcal{DP}(x,y) = \frac{2}{1+\sqrt{1-4x^2}} \left(\frac{2x}{1+\sqrt{1-4x^2}} \right)^h \ ,$$

then, replacing in (2.19), and observing that $\frac{2}{1+\sqrt{1-4x^2}} = \mathrm{Cat}(x)$, we get:

$$\Delta_{PH}(x) = \Delta_P(x) + \mathrm{Cat}(x) \sum_{h \geq 0} \left[y^h\right] C_P(x,y) \left(x\mathrm{Cat}(x)\right)^h$$

which is equivalent to (2.17).

Finally, let Q be a Schröder path avoiding PU. If Q contains P, as usual let Q' be the smallest prefix of Q containing P. The path Q can be written as Q', which ends at height $h \geq 0$, followed by a path starting from height h and using only D and H steps. This latter path is the reverse of a path A starting at the origin, using only U steps and H steps and ending at height h. We have already computed the bivariate generating function $A(x,y)$ of such paths in the proof of Proposition 2.2.1 (where y keeps track of the final height), hence:

$$\Delta_{PU}(x) = \Delta_P(x) + \sum_{h \geq 0} \left(\left[y^h\right] C_P(x,y) \left[y^h\right] A(x,y) \right) \ .$$

Since $\left[y^h\right] A(x,y) = \frac{1}{1-x^2}\left(\frac{x}{1-x^2}\right)^h$, we obtain:

$$\Delta_{PU}(x) = \Delta_P(x) + \frac{1}{1-x^2}\sum_{h\geq 0}\left[y^h\right] C_P(x,y)\left(\frac{x}{1-x^2}\right)^h ,$$

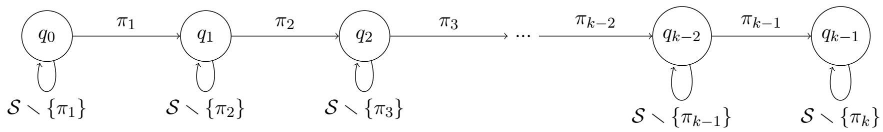
which is (2.18).

□

As an immediate consequence of Proposition 2.2.2, we observe that for every Schröder pattern P the generating function $\Delta_P(x)$ is a rational function of x and $\mathrm{Cat}(x)$, in particular it is algebraic over $\mathbb{Z}(x)$.

2.2.3 A general automata approach

In this section we develop a deterministic finite state automata approach to the study of pattern avoidance in the context of simple lattice paths. We begin from the following general observation. Let $\mathcal{S}$ be any set, $\pi \in \mathcal{S}^* \smallsetminus \{\varepsilon\}$ and $k = |\pi|$. Then it is easy to check that $\mathcal{S}^*(\pi)$ is a regular language accepted by a deterministic finite state automaton M with k states defined as follows. Let $Q = \{q_0, q_1, ..., q_{k-1}\}$ denote the set of states of M, then the transition partial function $f : Q \times \mathcal{S} \longrightarrow Q$ of M is defined by setting $f(q_{i-1}, \pi_i) = q_i$ and $f(q_{i-1}, \alpha) = q_{i-1}$ when $\alpha \in \mathcal{S} \setminus \{\pi_i\}$ and $i \in [k]$. This automaton is represented by the following directed multigraph



Let now $\mathcal{L}$ denote a language in the alphabet $\mathcal{S}$. For every $i \in [k]$ we denote by $\mathcal{L}_i(\pi)$ the set of all $\sigma \in \mathcal{L}(\pi)$ such that $\pi_1\pi_2...\pi_{i-1} \leq \sigma$ (where by convention $\pi_0 = \varepsilon$), but $\pi_1...\pi_i \not\leq \sigma$. Note that for every $i \in [k]$ the set $\mathcal{L}_i(\pi)$ consists precisely of the words in $\mathcal{L}(\pi)$ that teminates in state q_{i-1} of the automaton M.

We return now to simple lattice paths. Suppose $\mathcal{S}$ is a subset of $\mathbb{Z}^2$ consisting of simple steps. In this case, we will identify a simple step $(1, v)$ with the integer v for every $v \in \mathbb{Z}$. Denote by $\mathcal{L}(t, u)$ the generating function

$$\mathcal{L}(t,u) = \sum_{\sigma\in\mathcal{L}} t^{|\sigma|} u^{\mathsf{h}(\sigma)}$$

where t keeps track of the length and u keeps track of the final height. Note that clearly $\mathcal{L}(\pi, t, u) = \sum_{i=1}^{k} \mathcal{L}_i(\pi, t, u)$. Following [ABBG], we will take advantage of the description of $\mathcal{S}^*(\pi)$ as a regular language accepted by the automaton M to derive a system of functional equations for $\mathcal{L}_1(\pi, t, u), ..., \mathcal{L}_k(\pi, t, u)$ when $\mathcal{L} \in \{\mathcal{S}^*, \mathcal{S}^{\geq 0}\}$. To this purpose, we set

$$\lambda_A(u) = \sum_{\alpha\in A} u^{\alpha}$$

for every subset A of $\mathcal{S}$ and we denote by $A(\pi,u)$ the transition matrix of M, i.e. the square matrix defined by setting $A_{ii}(\pi,u)=\lambda_{\mathcal{S}\smallsetminus\{\pi_i\}}(u)$ when $i\in[k]$, while $A_{(i-1)i}(\pi,u)=u^{\pi_{i-1}}$ when $i\in[k]$ and $i\geq 2$ and $A_{ij}(\pi,u)=0$ otherwise. Thus $A(\pi,u)$ has the form

$$A(\pi,u)=\begin{pmatrix}\lambda_{\mathcal{S}\smallsetminus\{\pi_1\}}(u) & u^{\pi_1} & 0 & 0 & 0 & \dots & 0 & 0\\ 0 & \lambda_{\mathcal{S}\smallsetminus\{\pi_2\}}(u) & u^{\pi_2} & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \lambda_{\mathcal{S}\smallsetminus\{\pi_3\}}(u) & u^{\pi_3} & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \lambda_{\mathcal{S}\smallsetminus\{\pi_4\}}(u) & u^{\pi_4} & \dots & 0 & 0\\ 0 & 0 & 0 & 0 & \lambda_{\mathcal{S}\smallsetminus\{\pi_5\}}(u) & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \dots & u^{\pi_{k-2}} & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & \lambda_{\mathcal{S}\smallsetminus\{\pi_{k-1}\}}(u) & u^{\pi_{k-1}}\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \lambda_{\mathcal{S}\smallsetminus\{\pi_k\}}(u)\end{pmatrix}$$

Now we are in a position to state the main result for walks with steps in $\mathcal{S}$ avoiding the pattern π. For the sake of simplicity, in general we will denote with $\underline{\mathcal{L}(t,u)}$ the row vector $(\mathcal{L}_1(t,u) \ \ \dots \ \ \mathcal{L}_k(t,u))$ and we will use the standard notation e_1 for the row vector $(1 \ \ 0 \ \ \dots \ \ 0)$.

Proposition 2.2.3. *The following system of functional equations holds*

$$\underline{\mathcal{S}^*(\pi,t,u)}=e_1+t\underline{\mathcal{S}^*(\pi,t,u)}A(\pi,u). \tag{2.20}$$

Proof. Suppose $j\in[k]$ and $j\geq 2$. Pick now $\sigma\in\mathcal{S}^*-\{\varepsilon\}$ and write $\sigma=\tau\alpha$ for some $(\tau,\alpha)\in\mathcal{S}^*\times\mathcal{S}$. When $\tau\in\mathcal{S}^*_j(\pi)$, then $\sigma\in\mathcal{S}^*_j(\pi)$ if and only if $(\tau,\alpha)\in\mathcal{S}^*_j(\pi)\times(\mathcal{S}-\{\pi_j\})$, otherwise, when $\tau\notin\mathcal{S}^*_j(\pi)$, then $\sigma\in\mathcal{S}^*_j(\pi)$ if and only if $(\tau,\alpha)\in\mathcal{S}^*_{j-1}(\pi)\times\{\pi_{j-1}\}$. It follows that

$$\begin{aligned}\mathcal{S}^*_j(\pi,t,u) &= \sum_{\sigma\in\mathcal{S}^*_j(\pi)} t^{|\sigma|}u^{\mathsf{h}(\sigma)} = \sum_{\substack{\tau\in\mathcal{S}^*_j(\pi)\\ \alpha\in\mathcal{S}-\{\pi_j\}}} t^{|\tau|+1}u^{\mathsf{h}(\tau)+\alpha} + \sum_{\tau\in\mathcal{S}^*_{j-1}(\pi)} t^{|\tau|+1}u^{\mathsf{h}(\tau)+\pi_{j-1}}\\ &= t\left[\sum_{\tau\in\mathcal{S}^*_j(\pi)} t^{|\tau|}u^{\mathsf{h}(\tau)} \sum_{\alpha\in\mathcal{S}-\{\pi_j\}} u^{\alpha} + \sum_{\tau\in\mathcal{S}^*_{j-1}(\pi)} t^{|\tau|}u^{\mathsf{h}(\tau)}u^{\pi_{j-1}}\right]\\ &= t[\mathcal{S}^*_j(\pi,t,u)\lambda_{\mathcal{S}-\{\pi_j\}}(u)+\mathcal{S}^*_{j-1}(\pi,t,u)u^{\pi_{j-1}}]\\ &= t[\mathcal{S}^*_j(\pi,t,u)A_{jj}(\pi,u)+\mathcal{S}^*_{j-1}(\pi,t,u)A_{(j-1)j}(\pi,u)]\\ &= t\sum_{i=1}^{k}\mathcal{S}^*_i(\pi,t,u)A_{ij}(\pi,u).\end{aligned}$$

Similarly one can easily prove that

$$\mathcal{S}^*_1(\pi,t,u)=1+t\mathcal{S}^*_1(\pi,t,u)A_{11}(\pi,u).$$

This completes the proof. □

In other words, Equation (2.20) reads as follows: "A walk avoiding π is either the empty walk or it consists of a step added to a walk avoiding π in such a way that no occurrence of π appears (in which case $t\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u)$ describes all the possibilities)".

Now Equation (2.20) can be actually solved to find a closed form for $\mathcal{S}^*(\pi,t,u)$. We stress that, although enumeration of $\mathcal{S}^*(\pi)$ is a quite standard result, our method actually allows to refine the enumeration according to the final height of the walks. The following two lemmas immediately allow to find an explicit solution to Equation (2.20) (the proof of the first lemma is a fairly trivial exercise and is therefore omitted).

Lemma 2.2.1. *Suppose R is a commutative ring, $A \in M(k,R)$ and $v \in R^k$. If $v = e_1 + vA$ and $1_k - A \in \mathrm{GL}(k,R)$, then*

$$\sum_{i=1}^{k} v_i = (1 \quad 0 \quad \dots \quad 0)(1_k - A)^{-1}(1 \quad 1 \quad \dots \quad 1)^{\mathsf{T}}.$$

Lemma 2.2.2. *Let R be a commutative ring and A be a matrix in $M(k,R)$ such that $A_{ij} = 0$ when $i \neq j$ and $i \neq j-1$ for every $(i,j) \in [k]^2$, i.e. A has the form*

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & \dots & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & \dots & 0 & 0 \\ 0 & 0 & 0 & a_{44} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{(k-2)(k-1)} & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{(k-1)(k-1)} & a_{(k-1)k} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{kk} \end{pmatrix}$$

Then

(i) $\det(A) = \prod_{i=1}^{k} a_{ii}$

(ii) *$A \in \mathrm{GL}(k,R)$ if and only if $a_{ii} \in R^{\times}$ for every $i \in [k]$*

(iii) *If $A \in \mathrm{GL}(k,R)$, then*

$$(1 \quad 0 \quad \dots \quad 0)A^{-1}(1 \quad 1 \quad \dots \quad 1)^{\mathsf{T}} = \sum_{j=1}^{k}(-1)^{j+1}\prod_{i=1}^{j}\frac{a_{(i-1)i}}{a_{ii}}$$

where we agree that $a_{01} = 1$.

Proof. Claim (i) and (ii) are trivial; for claim (iii) note that

$$\begin{aligned}(1 \quad 0 \quad \dots \quad 0)A^{-1}(1 \quad 1 \quad \dots \quad 1)^{\mathsf{T}} &= (1 \quad 0 \quad \dots \quad 0)(\textstyle\sum_{j=1}^{k} A^{-1}_{1j} \quad \dots \quad \sum_{j=1}^{k} A^{-1}_{kj})^{\mathsf{T}} \\ &= \sum_{j=1}^{k} A^{-1}_{1j} = \sum_{j=1}^{k}(-1)^{j+1}\frac{\det(A:j,1)}{\det(A)}\end{aligned}$$

Note that, when $j \in [k]$, then $(A : j, 1)$ has the form

$$(A : j, 1) = \begin{pmatrix} U_j & 0 \\ 0 & L_j \end{pmatrix}$$

where

$$U_j = \begin{pmatrix} a_{12} & 0 & \dots & 0 & 0 \\ a_{22} & a_{23} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{(j-2)(j-1)} & 0 \\ 0 & 0 & \dots & a_{(j-1)(j-1)} & a_{(j-1)j} \end{pmatrix} \quad \text{and}$$

$$L_j = \begin{pmatrix} a_{(j+1)(j+1)} & a_{(j+1)(j+2)} & \dots & 0 & 0 \\ 0 & a_{(j+2)(j+2)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{(k-1)(k-1)} & a_{(k-1)k} \\ 0 & 0 & \dots & 0 & a_{kk} \end{pmatrix}$$

are possibly empty, hence applying iteratively Laplace theorem starting from the top row of A one finds

$$\det(A : j, 1) = \det(U_j)\det(L_j) = \prod_{i=2}^{j} a_{(i-1)i} \prod_{i=j+1}^{k} a_{ii}$$

where we agree that the determinant of an empty matrix is 1 and we agree to assign value 1 to empty products. Therefore

$$\frac{\det(A : j, 1)}{\det(A)} = \prod_{i=1}^{j} \frac{a_{(i-1)i}}{a_{ii}}$$

where $a_{01} = 1$, completing the proof. □

Theorem 2.2.2. *The following formula holds*

$$\mathcal{S}^*(\pi, t, u) = \frac{1}{1 - t\lambda_{\mathcal{S}-\{\pi_1\}}(u)} + \frac{1}{1 - t\lambda_{\mathcal{S}-\{\pi_1\}}(u)} \sum_{j=2}^{k} t^{j-1} \prod_{i=2}^{j} \frac{u^{\pi_{i-1}}}{1 - t\lambda_{\mathcal{S}-\{\pi_i\}}(u)}.$$

in particular

$$\mathcal{S}^*(\pi, t, 1) = \frac{1}{dt - 1}\left[\left(\frac{t}{1-(d-1)t}\right)^k - 1\right]$$

where $d = |\mathcal{S}|$.

Proof. It follows immediately from Proposition 2.5 applying Lemmas 2.2.1 and 2.2.2 to the matrix $1_k - tA(\pi, u)$. □

Now we deal with the case of meanders avoiding a single pattern. The next proposition provides a system of functional equations analogous to Equation (2.20) for walks avoiding a single pattern. To this purpose, given any ring R, we denote by $[u^{<0}]$ the unique $R-$linear operator $[u^{<0}] : R[u, u^{-1}] \longrightarrow R[u^{-1}]$ such that

$$[u^{<0}]u^k = \begin{cases} u^k & k < 0 \\ 0 & k \geq 0 \end{cases}$$

so that

$$[u^{<0}]f(u) = \sum_{k<0} f_k u^k$$

for every $f(u) \in R[u, u^{-1}]$.

Proposition 2.2.4. *The following system of functional equations holds*

$$\underline{\mathcal{S}^{\geq 0}(\pi, t, u)} = e_1 + t\underline{\mathcal{S}^{\geq 0}(\pi, t, u)}A(\pi, u) - t[u^{<0}]\underline{\mathcal{S}^{\geq 0}(\pi, t, u)}A(\pi, u) \tag{2.21}$$

Proof. Suppose $j \in [k]$ and $j \geq 2$. Pick $\sigma \in \mathcal{S}^{\geq 0} - \{\varepsilon\}$ and write $\sigma = \tau\alpha$ for some $(\tau, \alpha) \in \mathcal{S}^{\geq 0} \times \mathcal{S}$. When $\tau \in \mathcal{S}_j^{\geq 0}(\pi)$, then $\sigma \in \mathcal{S}_j^{\geq 0}(\pi)$ if and only if $(\tau, \alpha) \in \mathcal{S}_j^{\geq 0}(\pi) \times (\mathcal{S} - \{\pi_j\})$ and $\mathsf{h}(\tau) + \alpha \geq 0$, otherwise, when $\tau \notin \mathcal{S}_j^{\geq 0}(\pi)$, then $\sigma \in \mathcal{S}_j^{\geq 0}(\pi)$ if and only if $(\tau, \alpha) \in \mathcal{S}_{j-1}^{\geq 0}(\pi) \times \{\pi_{j-1}\}$ and $\mathsf{h}(\tau) + \pi_{j-1} \geq 0$. It follows that

$$\begin{aligned} \mathcal{S}_j^{\geq 0}(\pi, t, u) = \sum_{\sigma \in \mathcal{S}_j^{\geq 0}(\pi)} t^{|\sigma|} u^{\mathsf{h}(\sigma)} &= \sum_{\substack{\tau \in \mathcal{S}_j^{\geq 0}(\pi) \\ \alpha \in \mathcal{S} - \{\pi_j\} \\ \mathsf{h}(\tau) + \alpha \geq 0}} t^{|\tau|+1} u^{\mathsf{h}(\tau)+\alpha} + \sum_{\substack{\tau \in \mathcal{S}_{j-1}^{\geq 0}(\pi) \\ \mathsf{h}(\tau) + \pi_{j-1} \geq 0}} t^{|\tau|+1} u^{\mathsf{h}(\tau)+\pi_{j-1}} \\ &= t \sum_{i=1}^{k} \mathcal{S}_i^{\geq 0}(\pi, t, u) A_{ij}(\pi, u) - t[u^{<0}] \sum_{i=1}^{k} \mathcal{S}_i^{\geq 0}(\pi, t, u) A_{ij}(\pi, u) \end{aligned}$$

because

$$
\begin{aligned}
&\sum_{\substack{\tau\in\mathcal{S}_j^{\geq 0}(\pi)\\ \alpha\in\mathcal{S}-\{\pi_j\}}} t^{|\tau|+1}u^{\mathsf{h}(\tau)+\alpha} + \sum_{\tau\in\mathcal{S}_{j-1}^{\geq 0}(\pi)} t^{|\tau|+1}u^{\mathsf{h}(\tau)+\pi_{j-1}} &=\\
&t\left[\sum_{\tau\in\mathcal{S}_j^{\geq 0}(\pi)} t^{|\tau|}u^{\mathsf{h}(\tau)} \sum_{\alpha\in\mathcal{S}-\{\pi_j\}} u^{\alpha} + \sum_{\tau\in\mathcal{S}_{j-1}^{\geq 0}(\pi)} t^{|\tau|}u^{\mathsf{h}(\tau)}u^{\pi_{j-1}}\right] &=\\
&t[\mathcal{S}_j^{\geq 0}(\pi,t,u)\lambda_{\mathcal{S}-\{\pi_j\}}(u) + \mathcal{S}_{j-1}^{\geq 0}(\pi,t,u)u^{\pi_{j-1}}] &=\\
&t[\mathcal{S}_j^{\geq 0}(\pi,t,u)A_{jj}(\pi,u) + \mathcal{S}_{j-1}^{\geq 0}(\pi,t,u)A_{(j-1)j}(\pi,u)] &=\\
&t\sum_{i=1}^{k}\mathcal{S}_i^{\geq 0}(\pi,t,u)A_{ij}(\pi,u).
\end{aligned}
$$

Similarly one can easily prove that

$$\mathcal{S}_1^{\geq 0}(\pi,t,u) = 1 + t\mathcal{S}_1^{\geq 0}(\pi,t,u)A_{11}(\pi,u) - t[u^{<0}]\mathcal{S}_1^{\geq 0}(\pi,t,u)A_{11}(\pi,u).$$

This completes the proof. □

Again, in other words, Equation (2.21) reads as follows: "A meander avoiding π is either the empty meander or it consists of a step added to a meander avoiding π in such a way that no occurrence of π appears (in which case $t\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u)$ describes all the possibilities), except that the steps that would take the walk below below the $x-$axis are to be taken out (whence the operator $[u^{<0}]$)".

Unlike in the case of walks, the appearance of the term involving operator $[u^{<0}]$ makes it harder to find a closed form solution for Equation (2.21). However, at least in the remarkable case when -1 is the only negative step (and thus in classical cases such as Dyck and Motzkin paths), the next lemma shows that Equation (2.21) takes a particularly simple form, which can be solved using a kernel-type method in a recursive fashion.

Lemma 2.2.3. *Suppose $\mathcal{S}\cap\mathbb{Z}_{<0} = \{-1\}$ and $B(\pi,t,u) = u(1_k - tA(\pi,u))$, then*

$$\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}B(\pi,t,u) = ue_1 + \underline{\mathcal{S}^{\geq 0}(\pi,t,0)}B(\pi,t,0). \tag{2.22}$$

Proof. Indeed, note that Equation (2.21) can be rewritten as follows

$$\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}B(\pi,t,u) = ue_1 - tu[u^{<0}]\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u). \tag{2.23}$$

On the other hand, since by assumption $\mathcal{S}\cap\mathbb{Z}_{<0} = \{-1\}$, it follows that actually $u[u^{<0}]\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u) \in \mathbb{Z}[[t]]^k$, hence setting $u = 0$ in Equation (2.23) we deduce that in fact $-tu[u^{<0}]\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u) = \underline{\mathcal{S}^{\geq 0}(\pi,t,0)}B(\pi,t,0)$ and the result follows again by Equation (2.23). □

As mentioned above, the particularly nice form of Equation (2.22) suggests to exploit a kernel-type method. Roughly speaking, the unknown term $\underline{\mathcal{S}^{\geq 0}(\pi,t,0)}B(\pi,t,0)$ in the right hand side of Equation (2.22) does not depend on u, hence there is hope to compute it explicitly and solve the equation, provided we find a particular value of u for which the left hand side of Equation (2.22) vanishes. Actually, it does not suffice to apply this method once, but we need to apply it row by row to recursively solve Equation (2.22). The next proposition contains the recursive formulas that allow us to compute the solution.

Proposition 2.2.5. *Suppose* $\mathcal{S}\cap\mathbb{Z}_{<0}=\{-1\}$, $B(\pi,t,u)=u(1_k-tA(\pi,u))$, $i\in[k]$ *and* $u_i(t)\in\mathbb{Z}[[t]]$ *is a root of* $u(1-t\lambda_{\mathcal{S}-\{\pi_i\}}(u))$. *Then*

(i)

$$\mathcal{S}_1^{\geq 0}(\pi,t,u)=\frac{1}{u(1-t\lambda_{\mathcal{S}-\{\pi_1\}}(u))}\left[u-(1-\delta_1)u_1(t)\right],$$

where $\delta_1=1$, *when* $\pi_1=-1$, *and* $\delta_1=0$ *otherwise. Furthermore, if* $\pi\in\mathcal{S}^{\geq 0}$, *then*

$$\mathcal{S}_1^{\geq 0}(\pi,t,u)=\frac{u-u_1(t)}{u(1-t\lambda_{\mathcal{S}-\{\pi_1\}}(u))};$$

(ii)

$$\mathcal{S}_i^{\geq 0}(\pi,t,u)=\frac{t}{u(1-t\lambda_{\mathcal{S}-\{\pi_i\}}(u))}\left[u^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u)-u_i(t)^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u_i(t))\right],$$

when $i\geq 2$.

Proof. Note that (i) follows immediately from Equation (2.22). For (ii), suppose $i\geq 2$, then the i^{th} row of Equation (2.22) reads as follows

$$\begin{aligned}-tu^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u)+u(1-t\lambda_{\mathcal{S}-\{\pi_i\}}(u))\mathcal{S}_i^{\geq 0}(\pi,t,u)=\\ -\delta_{i-1}t\mathcal{S}_{i-1}^{\geq 0}(\pi,t,0)-(1-\delta_{i-1})t\mathcal{S}_i^{\geq 0}(\pi,t,0)\end{aligned}$$

where $\delta_{i-1}=1$ when $\pi_{i-1}=-1$ and $\delta_{i-1}=0$ when $\pi_{i-1}\neq -1$. On the other hand, by assumption

$$\begin{aligned}-tu_i(t)^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u_i(t)) &= -tu_i(t)^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u_i(t))+\\ &\qquad u_i(t)(1-t\lambda_{\mathcal{S}-\{\pi_i\}}(u_i(t)))\mathcal{S}_i^{\geq 0}(\pi,t,u_i(t))\\ &= -\delta_{i-1}t\mathcal{S}_{i-1}^{\geq 0}(\pi,t,0)-(1-\delta_{i-1})t\mathcal{S}_i^{\geq 0}(\pi,t,0)\end{aligned}$$

and the claim follows. □

We end this section by applying Proposition 2.2.5 to the classical cases of Dyck and Motzkin paths. We stress the fact that, although the bivariate generating functions involved have actually a slightly different meaning, this method closely resembles the recursive method contained in [BBFGPW] to compute the generating function of Dyck paths avoiding a single pattern.

The Dyck case. Suppose $\mathcal{S} = \{\pm 1\}$ and $i \in [k]$. If $\pi_i = 1$, then $u(1 - t\lambda_{\mathcal{S}-\{\pi_i\}}(u)) = u(1 - tu^{-1}) = u - t$ and one can choose $u_i(t) = t$. If $\pi_i = -1$, then $u(1 - t\lambda_{\mathcal{S}-\{\pi_i\}}(u)) = u(1 - tu)$ and one can choose $u_i(t) = 0$. When π is a Dyck path, it follows that

$$\mathcal{S}_1^{\geq 0}(\pi, t, u) = 1$$

and

$$\mathcal{S}_i^{\geq 0}(\pi, t, u) = \begin{cases} \frac{t}{u-t}\left[u^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, u) - t^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, t)\right] & \pi_i = 1 \\ \frac{t}{u(1-tu)}u^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, u) & \pi_i = -1 \end{cases}$$

when $i \geq 2$.

The Motzkin case. Suppose $\mathcal{S} = \{0, \pm 1\}$ and $i \in [k]$. If $\pi_i = 0$, then $u(1 - t\lambda_{\mathcal{S}-\{\pi_1\}}(u)) = u(1 - t(u^{-1} + u)) = -tu^2 + u - t$, hence one can choose

$$u_i(t) = tC(t^2) = \frac{1 - \sqrt{1 - 4t^2}}{2t}.$$

If $\pi_i = 1$, then $u(1 - t\lambda_{\mathcal{S}-\{\pi_1\}}(u)) = u(1 - t(1 + u^{-1})) = (1 - t)u - t$ and one can choose

$$u_i(t) = \frac{t}{1 - t}.$$

If $\pi_i = -1$, then $u(1 - t\lambda_{\mathcal{S}-\{\pi_1\}}(u)) = u(1 - t(1 + u))$ and one can choose $u_i(t) = 0$. When π is a Motzkin path, it follows that

$$\mathcal{S}_1^{\geq 0}(\pi, t, u) = \begin{cases} \frac{u - tC(t^2)}{-tu^2 + u - t} = -\frac{1}{tu + t^2C(t^2) - 1} & \pi_1 = 0 \\ \frac{1}{1-t} & \pi_1 = 1 \end{cases}$$

and

$$\mathcal{S}_i^{\geq 0}(\pi, t, u) = \begin{cases} \frac{t}{-tu^2+u-t}\left[u^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, u) - (tC(t^2))^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, tC(t^2))\right] & \pi_i = 0 \\ \frac{t}{(1-t)u-t}\left[u^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, u) - \left(\frac{t}{1-t}\right)^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}\left(\pi, t, \frac{t}{1-t}\right)\right] & \pi_i = 1 \\ \frac{t}{u(1-t(1+u))}u^{\pi_{i-1}+1}\mathcal{S}_{i-1}^{\geq 0}(\pi, t, u) & \pi_i = -1 \end{cases}$$

when $i \geq 2$.

2.2.4 Non-simple walks

In this last section we develop a slight generalization of the automata approach contained in Section 2.2.3, which will allow us to deal with the case of non-simple walks. Let us suppose again that $\mathcal{S}$ is a subset of $\mathbb{Z}^2$. For every $v \in \mathbb{Z}^2$ we will denote by $x(v)$ the first component of v and by $y(v)$ the second component of v, thus $v = (x(v), y(v))$. For every $\sigma \in \mathcal{S}^*$, the *final length* of σ, denoted by $\ell(\sigma)$, is 0 when $\sigma = \varepsilon$ and it is $\sum_{i=1}^{|\sigma|} x(v_i)$ otherwise. Now, generalizing the notation in Section 2.2.3, we denote by $\mathcal{L}(t,u)$ the generating function

$$\mathcal{L}(t,u) = \sum_{\sigma \in \mathcal{L}} t^{\ell(\sigma)} u^{\mathsf{h}(\sigma)}$$

where t keeps track of the final length and u keeps track of the final height. Furthermore, we set

$$\lambda_A(t,u) = \sum_{\alpha \in A} t^{x(\alpha)} u^{y(\alpha)}$$

and finally

$$A(\pi,t,u) = \begin{pmatrix} \lambda_{\mathcal{S}-\{\pi_1\}}(t,u) & t^{x(\pi_1)}u^{y(\pi_1)} & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_{\mathcal{S}-\{\pi_2\}}(t,u) & t^{x(\pi_2)}u^{y(\pi_2)} & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_{\mathcal{S}-\{\pi_3\}}(t,u) & t^{x(\pi_3)}u^{y(\pi_3)} & \dots & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\mathcal{S}-\{\pi_4\}}(t,u) & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & t^{x(\pi_{k-2})}u^{y(\pi_{k-2})} & 0 \\ 0 & 0 & 0 & 0 & \dots & \lambda_{\mathcal{S}-\{\pi_{k-1}\}}(t,u) & t^{x(\pi_{k-1})}u^{y(\pi_{k-1})} \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_{\mathcal{S}-\{\pi_k\}}(t,u) \end{pmatrix}$$

Then, by means of similar computations, we find results analogous to those in Section 2.2.3, in particular we recover the following propositions for meanders with non-simple steps.

Proposition 2.2.6. *The following system of functional equations holds*

$$\underline{\mathcal{S}^{\geq 0}(\pi,t,u)} = e_1 + t\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u) - t[u^{<0}]\underline{\mathcal{S}^{\geq 0}(\pi,t,u)}A(\pi,u) \qquad (2.24)$$

Proposition 2.2.7. *Suppose $\mathcal{S} \cap (\mathbb{Z} \times \mathbb{Z}_{<0}) = \{(\alpha,-1)\}$ for some $\alpha \in \mathbb{Z}$, $B(\pi,t,u) = u(1_k - A(\pi,t,u))$, $i \in [k]$ and $u_i(t) \in \mathbb{Z}[[t]]$ is a root of $u(1 - \lambda_{\mathcal{S}-\{\pi_i\}}(t,u))$. Then*

(i)

$$\mathcal{S}_1^{\geq 0}(\pi,t,u) = \frac{1}{u(1-\lambda_{\mathcal{S}-\{\pi_1\}}(t,u))}\left[u - (1-\delta_1)t^{x(\alpha)}u_1(t)\right]$$

where $\delta_1 = 1$ when $y(\pi_1) = -1$ and $\delta_1 = 0$ otherwise. Furthermore, if $\pi \in \mathcal{S}^{\geq 0}$, then

$$\mathcal{S}_1^{\geq 0}(\pi,t,u) = \frac{u - t^{x(\alpha)}u_1(t)}{u(1-\lambda_{\mathcal{S}-\{\pi_1\}}(t,u))}$$

(ii)

$$\mathcal{S}_i^{\geq 0}(\pi,t,u) = \frac{t^{x(\pi_{i-1})}}{u(1-\lambda_{\mathcal{S}-\{\pi_i\}}(t,u))}\left[u^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u) - u_i(t)^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u_i(t))\right]$$

when $i \geq 2$.

We end this section by applying Proposition 2.2.7 to the classical case of Schröder paths, which has been discussed from a slightly different perspective in Section 2.2.2.

The Schröder case. Suppose $\mathcal{S} = \{(2,0),(1,\pm 1)\}$ and let $i \in [k]$. If $\pi_i = (2,0)$, then $u(1-\lambda_{\mathcal{S}-\{\pi_1\}}(t,u)) = u(1-t(u^{-1}+u)) = -tu^2+u-t$, hence one can choose

$$u_i(t) = tC(t^2) = \frac{1-\sqrt{1-4t^2}}{2t}.$$

If $\pi_i = (1,1)$, then $u(1-\lambda_{\mathcal{S}-\{\pi_1\}}(t,u)) = u(1-t(t+u^{-1})) = (1-t^2)u-t$ and one can choose

$$u_i(t) = \frac{t}{1-t^2}.$$

If $\pi_i = (1,-1)$, then $u(1-\lambda_{\mathcal{S}-\{\pi_1\}}(t,u)) = u(1-t(t+u))$ and one can choose $u_i(t) = 0$. When π is a Schröder path, it follows that

$$\mathcal{S}_1^{\geq 0}(\pi,t,u) = \begin{cases} \frac{u-tC(t^2)}{-tu^2+u-t} = -\frac{1}{tu+t^2C(t^2)-1} & \pi_1 = (2,0) \\ \frac{1}{1-t^2} & \pi_1 = (1,1) \end{cases}$$

and

$$\mathcal{S}_i^{\geq 0}(\pi,t,u) = \begin{cases} \frac{t^{x(\pi_{i-1})}}{-tu^2+u-t}\left[u^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u) - (tC(t^2))^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,tC(t^2))\right] & \pi_i = (2,0) \\ \frac{t^{x(\pi_{i-1})}}{(1-t^2)u-t}\left[u^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u) - \left(\frac{t}{1-t^2}\right)^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}\left(\pi,t,\frac{t}{1-t^2}\right)\right] & \pi_i = (1,1) \\ \frac{t^{x(\pi_{i-1})}}{u(1-t(t+u))}u^{y(\pi_{i-1})+1}\mathcal{S}_{i-1}^{\geq 0}(\pi,t,u) & \pi_i = (1,-1) \end{cases}$$

when $i \geq 2$.

2.2.5 Conclusion and further work

The enumerative combinatorics of the excursion pattern poset still remains largely unknown. In this section we have provided a couple of quite general methods to enumerate some interesting classes of excursions avoiding a single pattern, however these methods are recursive in nature and it is not clear if a more concise answer could be available as in the case of consecutive patterns (see [ABBG]). Furthermore, we don't know how to deal with the case of simple meanders involving negative steps other than -1, as in this case the related functional equations become more and more complicated and kernel type methods do not seem to prove useful. Moreover, avoidance of two or more patterns in these posets has not been investigated yet and it provides a further interesting issue for future works. As for asymptotics, the asymptotic behaviour of excursions avoiding a single pattern is known only in the case of Dyck paths (see [BBFGPW]), but it is likely that similar results can be obtained for Motzkin and Schröder paths.

2.3 Matching pattern avoidance

In this section we continue the study of classes of pattern avoiding matchings initiated by Bloom and Elizalde [BE], Chen, Deng Du, Stanley and Yan [CDDSY], Jelinek and Mansour [JM]. Let $n \in \mathbb{N}^*$ and set as usual $[n] = \{1,2,...,n\}$. A *matching* of $[2n]$ is a

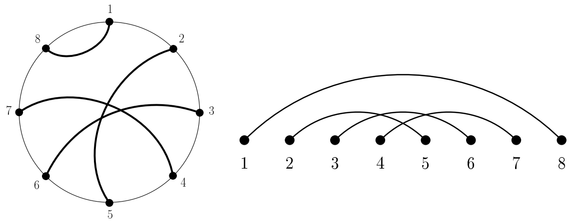


Figure 2.5: The circular chord diagram representing the matching $\{\{1,8\},\{2,5\},\{3,6\},\{4,7\}\}$ on the set $[8]$ and the corresponding linear chord diagram.

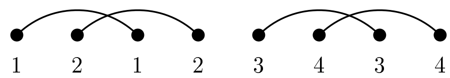


Figure 2.6: Encoding the matching $\{\{1,3\},\{2,4\},\{5,7\},\{6,8\}\}$ in the sequence 12123434.

partition of $[2n]$ into blocks having two elements. Note that a matching of $[2n]$ is the same as a graph on $[2n]$ such that every vertex has degree one, hence we will borrow some standard terminology from graph theory, as well as the usual representation of graphs using diagrams consisting of dots and lines. In particular, every matching of $[2n]$ will be represented either by a circular or by a linear chord diagram, as shown in Figure 2.5. Let τ be a matching of $[2n]$. The integer n, i.e. the number of edges of τ, will be called the *order* of τ and will be denoted by $|\tau|$. The set of all matchings will be denoted by $\mathcal{M}$ and the set of all matchings of order n will be denoted by $\mathcal{M}_n$. Given $e \in \tau$, the integers $\min(e)$ and $\max(e)$ will be called the *left vertex* and the *right vertex* of e respectively. Given a subset S of τ and $e \in S$, we will say that e is the *leftmost* (respectively *rightmost*) edge of S when $\min(e) \leq \min(f)$ (respectively $\max(e) \geq \max(f)$) for every $f \in S$. Following [JM], we will represent τ by means of the unique integer sequence $\tilde{\tau} \in [n]^{2n}$ such that $\tilde{\tau}_{\min(e)} = \tilde{\tau}_{\max(e)}$ and $\tilde{\tau}_{\min(e)} < \tilde{\tau}_{\min(f)}$ for every $e, f \in \tau$ such that $\min(e) < \min(f)$. Using this encoding, the vertices of τ are represented by the elements of $\tilde{\tau}$ and two vertices of τ are connected by an edge when the corresponding components of $\tilde{\tau}$ are equal (see Figure 2.6). In the following, we will always identify matchings with their corresponding integer sequences. Let σ and τ be matchings. The matching $\sigma(\tau + |\sigma|)$ will be called the *juxtaposition* of σ and τ (where $\tau + |\sigma|$ denotes the sequence obtained from τ by adding $|\sigma|$ to each of its elements). Its linear chord diagram can be indeed represented by juxtaposing the linear chord diagrams of σ and τ, respectively. The matching $1(\tau+1)1$ will be called the *lifting* of τ. Its linear chord diagram can be represented by nesting the linear chord diagram of σ into an additional edge. The matching obtained from the sequence $\tau_n ... \tau_2\tau_1$ by suitably renaming its elements so to obtain a valid matching will be called the *reversal* of τ and denoted by $\overline{\tau}$. Its linear chord diagram can be represented

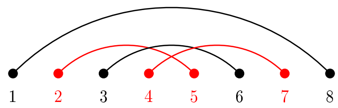


Figure 2.7: (2,4,5,7) is an occurrence of the perfect matching $\{\{1,3\},\{2,4\}\}$ in the perfect matching $\{\{1,8\},\{2,5\},\{3,6\},\{4,7\}\}$.

by reflecting the linear chord diagram of τ along a vertical line.

Given $k \in \mathbb{N}^*$, let σ be a matching of $[2k]$ and $i = (i_1, ..., i_k) \in [2n]^{2k}$. We say that i is an *occurrence* of σ in τ when $i_1 < i_2 < ... < i_{2k}$ and $\{i_p, i_q\} \in \tau$ if and only if $\{p, q\} \in \sigma$, for every $p, q \in [2k]$ (see Figure 2.7). We say that σ is a *pattern* of τ, and write $\sigma \leq \tau$, when there is an occurrence of σ in τ, and that τ *avoids* σ otherwise. The relation $\leq$ is a partial order turning the set of all matchings into a poset, which we will call the *matching pattern poset*. If S is a set of matchings, the class of all matchings avoiding every pattern in S will be denoted by $\mathcal{M}(S)$, the class of all matchings in $\mathcal{M}(S)$ of order n will be denoted by $\mathcal{M}_n(S)$ and the generating function of the sequence $\{|\mathcal{M}_n(S)|\}_{n\in\mathbb{N}}$ will be denoted by $\mathcal{M}(S, z)$. We say that σ and τ are *Wilf-equivalent* when $\mathcal{M}(\sigma, z) = \mathcal{M}(\tau, z)$.

In Section 2.3.2 and Section 2.3.3 we investigate classes of the form $\mathcal{M}(\sigma(\tau + |\sigma|))$ and $\mathcal{M}(1(\sigma+1)1, \chi, \overline{\chi})$, providing a general approach which yields enumerative formulas for some patterns σ, τ and χ. Following a recursive approach already described in [JM], we reduce the enumeration of $\mathcal{M}(\sigma(\tau + |\sigma|))$ to the enumeration of a specific class of matchings $\mu(\sigma)$ (depending on σ) and the class $\mathcal{M}(\tau)$, finding an explicit answer for the prefix $\sigma = 1212$. Moreover, we introduce a suitable pattern $\chi = 123132$ to relate the generating function of $\mathcal{M}(1(\sigma+1)1, \chi, \overline{\chi})$ to the generating function of $\mathcal{M}(\sigma, \chi, \overline{\chi})$.

In Section 2.3.4 we introduce the notion of *unlabeled matching*, which is an equivalence class of matchings having the same unlabeled circular chord diagram. This seems a reasonable and combinatorially meaningful way to collect patterns. As a first result concerning unlabeled pattern avoidance, we provide enumerative formulas for two classes of matchings avoiding an unlabeled pattern of order three, as well as a bijection between matchings avoiding a certain unlabeled pattern and ternary trees.

Finally, Section 3.2 is a first step towards the study of the enumerative combinatorics of the intervals in the matching pattern poset, and Section 2.3.5 provides some hints for further work.

2.3.1 Previous work

Given a permutation σ of $[n]$, we can construct a matching of $[2n]$ by connecting the vertices $\{1, ..., n\}$ with the vertices $\{n + 1, ..., 2n\}$ in the order prescribed by σ, thus obtaining the matching corresponding to the integer sequence $12...n\sigma_1...\sigma_n$. A matching of this kind will be called a *permutational matching* and it is immediate to notice that a matching of $[2n]$ is permutational if and only if it avoids the pattern 1122, so that $|\mathcal{M}_n(1122)| = n!$. Two remarkable examples of permutational matchings are

123...n123...n and 123...nn...321, which will be called the *totally crossing* and the *totally nesting* matching of $[2n]$ respectively. It is easy to see that the map sending every permutation to the corresponding permutational matching is a poset embedding, hence we can regard the permutation pattern poset as a subposet of the matching pattern poset. Throughout this section we will denote by $C_n = \frac{1}{n+1}\binom{2n}{n}$ the n^{th} Catalan number and by $C(z)$ the generating function of Catalan numbers. As for other enumerative results on pattern avoidance, it is well known that noncrossing matchings have a Catalan structure, therefore $|\mathcal{M}_n(1212)| = C_n$, and it is also well known that nonnesting matchings are counted by the same sequence, that is $|\mathcal{M}_n(1221)| = C_n$. More surprisingly, it was proved in [CDDSY] that the matchings 123...k123...k and 123...kk...321 are Wilf-equivalent for every $k \in \mathbb{N}^*$. No closed formula for the number of matchings avoiding these patterns is available in general, although it was proved in [GB] that $|\mathcal{M}_n(123123)| = C_nC_{n+2} - C_{n+1}^2$. Furthermore, Wilf-equivalences between several classes of patterns are established in [JM] through bijective methods; for instance, as an immediate consequence of Lemmas 3.7 and 3.10 in that paper, one can deduce the following useful fact.

Proposition 2.3.1. *If σ and σ' are two Wilf-equivalent matchings and τ and τ' are two Wilf-equivalent matchings, then $\sigma(\tau + |\sigma|)$ and $\sigma'(\tau' + |\sigma'|)$ are Wilf-equivalent.*

Moreover, the same paper also contains an enumerative result which reduces the enumeration of $\mathcal{M}(11(\sigma+1))$ to the enumeration of $\mathcal{M}(\sigma)$ in a recursive fashion. Finally, aside for the classes of patterns mentioned above, the only (up to Wilf-equivalence) further class of matchings avoiding a small pattern has been enumerated in [BE], proving that

$$\mathcal{M}(123132, z) = \frac{54z}{1 + 36z - (1 - 12z)^{\frac{3}{2}}}.$$

In the same paper, some enumerative results are given for most of the classes of matchings avoiding a pair of permutational patterns of order three. Nevertheless, enumerating all the remaining classes of matchings avoiding a single patterns of order three remains an open problem and it is likely to be a hard one. Indeed, it is suggested in [BE] that enumeration of the class of matchings avoiding the pattern 123231 could be related to the enumeration of the class of permutations avoiding 1324, which is considered to be a very hard problem.

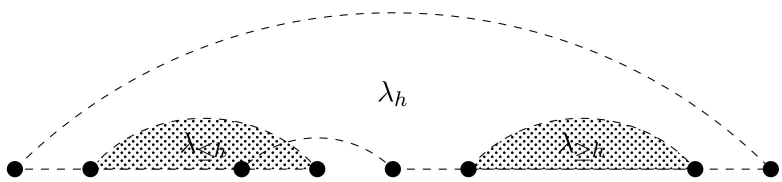
2.3.2 Avoiding the juxtaposition of two patterns

Let σ and τ be two matchings. In this section we investigate the class of matchings avoiding the juxtaposition of σ and τ. To this purpose, we define a set of matchings depending on σ. Let $n \in \mathbb{N}$ and λ be a matching of order n. We will say that λ *minimally contains* σ when it contains σ and the matching obtained from λ by deleting its rightmost edge does not contain σ. Denote by $\mu(\sigma)$ the set of matchings minimally containing σ, by $\mu_n(\sigma)$ the set of elements in $\mu(\sigma)$ with order n and by $\mu(\sigma, z)$ the corresponding generating function. Generalizing an approach already used in [JM], the following formula allows us to relate the enumeration of $\mathcal{M}(\sigma(\tau+|\sigma|))$ to the enumeration of $\mathcal{M}(\sigma)$, $\mu(\sigma)$ and $\mathcal{M}(\tau)$.

Proposition 2.3.2. *Let σ and τ be matchings and $n \in \mathbb{N}$, with $n \geq |\sigma|$. Then*

$$|\mathcal{M}_n(\sigma(\tau+|\sigma|))| = |\mathcal{M}_n(\sigma)| + \sum_{\ell=|\sigma|}^{n} \sum_{k=0}^{n-\ell} \binom{2\ell+k-1}{k} \binom{2n-2\ell-k}{k} k! |\mu_\ell(\sigma)| |\mathcal{M}_{n-\ell-k}(\tau)| \tag{2.25}$$

Proof. Given $\lambda \in \mathcal{M}_n(\sigma(\tau+|\sigma|))$, then either $\lambda \in \mathcal{M}_n(\sigma)$ or σ is a pattern of λ. From now on we assume that the latter case occurs, since the former one is taken into account by the first summand in the right hand side of (2.25). For $h \in [2n]$, we denote by $\lambda_{\leq h}$ the pattern of λ consisting of all the edges of λ with both vertices smaller than or equal to h, by $\lambda_{\geq h}$ the pattern of λ consisting of all the edges of λ with both vertices bigger than or equal to h and finally by λ_h the pattern of λ consisting of all the edges of λ that are neither in $\lambda_{\leq h}$ nor in $\lambda_{\geq h}$. Note that an edge of λ belongs to λ_h if and only if its left vertex is smaller than or equal to h and its right vertex is bigger than or equal to h.



Now let h denote the smallest integer such that $\lambda_{\leq h}$ contains an occurrence of σ (of course such an h exists because $\lambda_{\leq 2n} = \lambda$) and let ℓ be the order of $\lambda_{\leq h}$. Then, by definition, $\lambda_{\leq h} \in \mu_\ell(\sigma)$ and $\ell \in \{|\sigma|, ..., n\}$, hence there are $|\mu_\ell(\sigma)|$ possible choices for $\lambda_{\leq h}$. Furthermore, $\lambda_h \in \mathcal{M}_k(1122)$ for some $k \in \{0, ..., n-\ell\}$, hence there are $|\mathcal{M}_k(1122)| = k!$ possible choices for λ_h. Moreover, $\lambda_{\geq h} \in \mathcal{M}_{n-\ell-k}(\tau)$ because $\lambda \in \mathcal{M}_n(\sigma(\tau+|\sigma|))$, hence there are $|\mathcal{M}_{n-\ell-k}(\tau)|$ possibile choices for $\lambda_{\geq h}$. Finally, notice that $h = 2\ell + k$ and that the vertex h necessarily belongs to $\lambda_{\leq h}$, hence the left vertices of the edges of λ_h can be chosen among the vertices of λ smaller than h in $\binom{2\ell+k-1}{k}$ ways. Similarly, the right vertices of the edges of λ_h can be chosen among the vertices of λ bigger than h in $\binom{2n-2\ell-k}{k}$ ways. This explains the factors in the remaining summands of the right hand side of (2.25) and concludes the proof. □

Unfortunately, Formula (2.25) is not very informative, as enumerating $\mu(\sigma)$ is often as difficult as enumerating $\mathcal{M}(\sigma)$ itself. Nevertheless, one might still hope that this task can be achieved for some special prefixes σ. For instance, note that, for $\sigma = 11$ and $n \in \mathbb{N}^*$, we easily recover the formula

$$|\mathcal{M}_n(11(\tau+1))| = \sum_{k=1}^{n} k! \binom{2n-k-1}{k-1} |\mathcal{M}_{n-k}(\tau)|$$

which can be found in [JM]. The next proposition shows that the prefix $\sigma = 1212$ can be also succesfully addressed.

Proposition 2.3.3. *Let $n \in \mathbb{N}$, with $n \geq 2$, then*

(i)

$$|\mu_n(1212)| = \sum_{k=0}^{n-2} (2k+1) C_k C_{n-k-2} + C_k |\mu_{n-k-1}(1212)|; \tag{2.26}$$

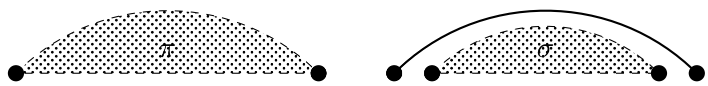
(ii)

$$\mu(1212, z) = \frac{C(z) - 1}{(1 - 2zC(z))(1 - zC(z))}; \tag{2.27}$$

(iii) $|\mu_n(1212)|$ *is the* $(n-1)^{th}$ *term of sequence* A002054 *in* [*S*], *i.e.*

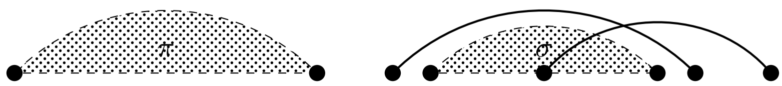
$$|\mu_n(1212)| = \binom{2n-1}{n-2}.$$

Proof. (i) Let $\lambda \in \mu_n(1212)$ and let $\hat{\lambda}$ denote the matching obtained from λ by removing its rightmost edge, so that $\hat{\lambda} \in \mathcal{M}_{n-1}(1212)$. Using the standard decomposition of non-crossing matchings, we can write $\hat{\lambda} = \pi(|\pi|+1)(\sigma + |\pi| + 1)(|\pi|+1)$, where $\pi \in \mathcal{M}_{n-k-2}(1212)$ and $\sigma \in \mathcal{M}_k(1212)$ for some $k \in \{0, ..., n-2\}$, as shown in the picture below:



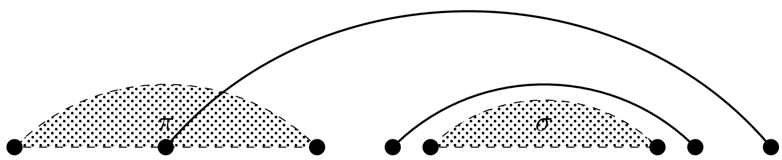
Since $1212 \leq \lambda$, there are two cases:

- The rightmost edge of λ crosses the rightmost edge of $\hat{\lambda}$, as shown in the picture below:



  In this case the left vertex of the rightmost edge of λ can be inserted in all the possible places between the vertices of the rightmost edge of $\hat{\lambda}$, that are $2k+1$ possible places, and therefore there are $(2k+1)C_k C_{n-k-2}$ possible choices for λ.

- The rightmost edge of λ crosses an edge of π, as shown in the picture below:



In this case, the pattern of λ consisting of π and the rightmost edge of λ minimally contains 1212, hence there are $C_k|\mu_{n-k-1}(1212)|$ possible choices for λ.

Now summing over $k \in \{0, ..., n-2\}$ we find (2.26).

(ii) It follows from (i) that

$$\mu(1212, z) = \sum_{n\geq 2}\left(\sum_{k=0}^{n-2}(2k+1)C_k C_{n-k-2}\right) z^n + \sum_{n\geq 2}\left(\sum_{k=0}^{n-2} C_k|\mu_{n-k-1}(1212)|\right) z^n$$

Now

$$\begin{aligned}\sum_{n\geq 2}\left(\sum_{k=0}^{n-2}(2k+1)C_k C_{n-k-2}\right) z^n &= z^2\sum_{n\geq 0}\left(\sum_{k=0}^{n}(2k+1)C_k C_{n-k}\right) z^n \\ &= 2z^2\sum_{n\geq 0}\left(\sum_{k=0}^{n} kC_k C_{n-k}\right) z^n + z^2\sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k C_{n-k}\right) z^n \\ &= z^2(2zC'(z)C(z) + C(z)^2)\end{aligned}$$

On the other hand, we have $C(z) = 1 + zC(z)^2$, therefore

$$2zC(z)C'(z) + C(z)^2 = (zC(z)^2)' = (C(z)-1)' = C'(z)$$

hence

$$C'(z) = \frac{C(z)^2}{1-2zC(z)}$$

and finally

$$\sum_{n\geq 2}\left(\sum_{k=0}^{n-2}(2k+1)C_k C_{n-k-2}\right) z^n = \frac{z^2C(z)^2}{1-2C(z)} = \frac{z(C(z)-1)}{1-2zC(z)}.$$

Similarly, we get

$$\begin{aligned}\sum_{n\geq 2}\left(\sum_{k=0}^{n-2} C_k|\mu_{n-k-1}(1212)|\right) z^n &= z^2\sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k|\mu_{n-k+1}(1212)|\right) z^n \\ &= z^2C(z)\sum_{n\geq 0}|\mu_{n+1}(1212)|z^n \\ &= zC(z)\mu(1212, z)\end{aligned}$$

Summing up, we have

$$\mu(1212, z) = \frac{z(C(z)-1)}{1-2zC(z)} + zC(z)\mu(1212, z),$$

hence

$$\mu(1212, z) = \frac{z(C(z)-1)}{(1-2zC(z))(1-zC(z))}.$$

(iii) The generating function for sequence A002054 can be found in [S] and is given by

$$f(z) = \frac{zC(z)^3}{1-2zC(z)}.$$

On the other hand

$$zC(z)^3(1-zC(z)) = C(z)(C(z)-1)(1-zC(z)) =$$

$$C(z)(C(z)-1-zC(z)^2+zC(z)) = zC(z)^2 = C(z)-1,$$

hence $\mu(1212, z) = zf(z)$, thus proving (2.27). □

Unfortunately, we have not been able to provide a neat combinatorial argument to explain the appearance of the binomial coefficient in Proposition 2.3.3. However, observe that, as a byproduct, we also find the following identity:

$$\sum_{k=1}^{n-1}\sum_{i=1}^{k}\sum_{\substack{\alpha\in(\mathbb{N}^*)^k\\|\alpha|=n}}(2\alpha_i-1)C_{\alpha_1-1}...C_{\alpha_k-1} = \binom{2n-1}{n-2}$$

which holds for every $n \in \mathbb{N}$ such that $n \geq 2$. Indeed, the left hand side of the above equation counts all matchings in $\mu_n(1212)$ by deleting the rightmost edge, then counting the resulting 1212−avoiding matchings according to the number of factors. As an immediate consequence of Proposition 2.3.1, 2.3.2 and 2.3.3 we deduce the following.

Theorem 2.3.1. *Let $\sigma \in \{1212, 1221\}$ and let τ be a matching. Then, for $n \geq 2$,*

$$|\mathcal{M}_n(\sigma(\tau+2))| = C_n + \sum_{\ell=2}^{n}\sum_{k=0}^{n-\ell}\binom{2\ell-1}{\ell-2}\binom{2\ell-1+k}{k}\binom{2(n-\ell)-k}{k}k!|\mathcal{M}_{n-\ell-k}(\tau)|.$$

Specializing τ in Theorem 2.3.1, we are able to enumerate a couple of new classes of matchings avoiding a single pattern (see also Figure 2.3.2).

Corollary 2.3.1. *Let $n \in \mathbb{N}$ with $n \geq 2$ and $\sigma \in \{1212, 1221\}$.*

(i) If $\tau \in \{1212, 1221\}$, then

$$|\mathcal{M}_n(\sigma(\tau+2))| = C_n + \sum_{\ell=2}^{n}\sum_{k=0}^{n-\ell}\binom{2\ell-1}{\ell-2}\binom{2\ell+k-1}{k}\binom{2n-2\ell-k}{k}k!C_{n-\ell-k}.$$

(ii) If $\tau \in \{123123, 123321\}$, then

$$|\mathcal{M}_n(\sigma(\tau+2))| = C_n + \sum_{\ell=2}^{n}\sum_{k=0}^{n-\ell}\binom{2\ell-1}{\ell-2}\binom{2\ell+k-1}{k}\binom{2n-2\ell-k}{k}k!(C_{n-\ell-k}C_{n-\ell-k+2}-C_{n-\ell-k+1}^2).$$

n	$\mathcal{M}_n(12123434)$	$\mathcal{M}_n(1212345345)$
1	1	1
2	3	3
3	15	15
4	104	105
5	910	944
6	9503	10341
7	114317	133132
8	1547124	1961919
9	23169162	32441303
10	379308106	592718236

Figure 2.8: The first terms of the sequences of Corollary 2.3.1. This sequences are not recorded in [S].

2.3.3 Avoiding the lifting of a pattern

In this section we investigate classes of matchings avoiding the lifting of a given matching σ. The enumeration of such classes seems to be a hard problem in general, since a special instance of it is the enumeration of matchings avoiding the pattern 123231, which is the lifting of 1212, and it was remarked at the beginning of Section 2.3 that this is likely to be a hard problem. However, if we impose additional constraints, namely the avoidance of a special pattern χ and its reversal $\overline{\chi}$, the description of the structure of matchings avoiding the lifting of σ becomes more accessible. We start by fixing some preliminary definitions. Let e and f be any two edges of σ. We say that e is *nested* in f when $\min(f) < \min(e)$ and $\max(e) < \max(f)$. We say that e is a *nested edge* when it is nested in some edge of σ and that e is a *top edge* otherwise. The pattern of σ consisting of all the nested edges of σ will be called the *core* of σ and the pattern of σ consisting of all the top edges of σ will be called the *roof* of σ. Note that, by definition, the roof of σ is a nonnesting matching. We say that a matching is *connected* when it is nonempty and it is not the juxtaposition of two nonempty matchings. Let S be a set of matchings and $n \in \mathbb{N}$, we denote by $\mathcal{M}^*(S)$ the class of all connected matchings, by $\mathcal{M}^*_n(S)$ the set of matchings in $\mathcal{M}^*(S)$ of order n and by $\mathcal{M}^*(S,z)$ the generating function of $\mathcal{M}^*(S)$. In the following we will make some use of the so-called symbolic method, borrowing some standard constructions and notations from [FS], such as disjoint union, cartesian product and composition of combinatorial classes (in particular, the operator Seq), which will allow us to easily translate combinatorial descriptions into generating functions.

Remark 2.3.1. *Note that, by definition, $\mathcal{M}(S) = \mathrm{Seq}(\mathcal{M}^*(S))$ and therefore $\mathcal{M}(S,z) = \frac{1}{1-\mathcal{M}^*(S,z)}$. In particular, $C(z) = \mathcal{M}(1221,z) = \frac{1}{1-\mathcal{M}^*(1221,z)}$, which leads to*

$$\mathcal{M}^*(1221,z) = \frac{C(z)-1}{C(z)} = \frac{zC(z)^2}{C(z)} = zC(z)$$

which means that, for every $n \in \mathbb{N}^*$, there are C_{n-1} connected nonnesting matchings of order n.

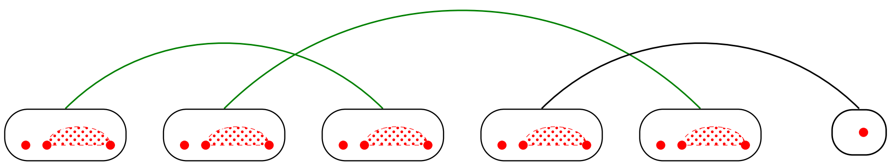
We are now in a position to state and prove the main result of this section.

Theorem 2.3.2. *Let σ be a connected matching and set $\chi = 123132$, so that $\overline{\chi} = 123213$. Then*

$$\mathcal{M}(1(\sigma+1)1, \chi, \overline{\chi}, z) = \frac{1}{1 - z\mathcal{M}(\sigma, \chi, \overline{\chi}, z)C(z\mathcal{M}(\sigma, \chi, \overline{\chi}, z)^2)}.$$

Proof. Let $n \in \mathbb{N}^*$, $\lambda \in \mathcal{M}^*_n(1(\sigma+1)1, \chi, \overline{\chi})$ and m be the order of its roof. The matching λ is required to avoid both χ and $\overline{\chi}$, that are the matchings represented by the following linear chord diagrams:

This means that every nested edge of λ is forced to never cross a top edge of λ. Therefore the core of λ can be decomposed as the juxtaposition of $2m-1$ (possibly empty) matchings $\lambda_1, \ldots, \lambda_{2m-1} \in \mathcal{M}(\sigma, \chi, \overline{\chi})$, moreover the occurrences of these factors in λ are separated by the vertices of the top edges of λ. Conversely, every matching constructed as above belongs to the class $\mathcal{M}^*(1(\sigma+1)1, \chi, \overline{\chi})$, because σ is connected and so no occurrence of σ can show up by juxtaposing two patterns in the class $\mathcal{M}(\sigma, \chi, \overline{\chi})$. Thus $\mathcal{M}^*_n(1(\sigma+1)1, \chi, \overline{\chi})$ is the set of matchings obtained by choosing some $m \in \mathbb{N}^*$ and a matching in $\mathcal{M}^*_m(1221)$, then replacing its edges other than the rightmost one with $(\{\bullet\!\frown\!\bullet\} \times \mathcal{M}^*(\sigma, \chi, \overline{\chi})^2)$–structures and the rightmost edge with a $(\{\bullet\!\frown\!\bullet\} \times \mathcal{M}^*(\sigma, \chi, \overline{\chi}))$–structure. An instance of this decomposition is illustrated in the following figure when the roof is 121323.



It follows that the combinatorial class $\mathcal{M}^*(1(\sigma+1)1, \chi, \overline{\chi})$ is isomorphic to the combinatorial class

$$\{\bullet\!\frown\!\bullet\} \times \mathcal{M}(\sigma, \chi, \overline{\chi}) \times \sum_{m \geq 1} \mathcal{M}^*_m(1221) \times (\{\bullet\!\frown\!\bullet\} \times \mathcal{M}(\sigma, \chi, \overline{\chi})^2)^{m-1}$$

and this isomorphism immediately translates into the following expression for the generating function

$$\begin{aligned}\mathcal{M}^*(1(\sigma+1)1,\chi,\overline{\chi},z) &= z\mathcal{M}(\sigma,\chi,\overline{\chi},z)\sum_{m\geq 1}[z^m](zC(z))(z\mathcal{M}(\sigma,\chi,\overline{\chi},z)^2)^{m-1}\\ &= z\mathcal{M}(\sigma,\chi,\overline{\chi},z)\sum_{m\geq 0}C_m(z\mathcal{M}(\sigma,\chi,\overline{\chi},z)^2)^m\\ &= z\mathcal{M}(\sigma,\chi,\overline{\chi},z)C(z\mathcal{M}(\sigma,\chi,\overline{\chi},z)^2).\end{aligned}$$

Now the claim follows from the above Remark. □

Note that, at least in principle, iterating Theorem 2.3.2 allows us to find expressions for the generating function of $\mathcal{M}(12...k(\sigma+k)k...21,\chi,\overline{\chi})$ in terms of the generating function of $\mathcal{M}(\sigma,\chi,\overline{\chi})$, for every $k\in\mathbb{N}^*$. As an immediate application, we are able to compute the generating function of two classes of matchings avoiding three patterns of order three.

Corollary 2.3.2. *The following equality holds*

$$\mathcal{M}(123231,123132,123213,z)=\mathcal{M}(123321,123132,123213,z)=\frac{1}{1-zC(z)C(C(z)-1)}$$

and $|\mathcal{M}_n(123231,123132,123213)| = |\mathcal{M}_n(123321,123132,123213)|$ *is the* n^{th} *term of sequence* A125188 *in* [*S*].

Proof. Let $\sigma\in\{1212,1221\}$, then it follows from Theorem 2.3.2 that

$$\mathcal{M}(1(\sigma+1)1,\chi,\overline{\chi},z)=\frac{1}{1-z\mathcal{M}(\sigma,\chi,\overline{\chi},z)C(\mathcal{M}(\sigma,\chi,\overline{\chi},z))}$$

where $\chi=123132$ and $\overline{\chi}=123213$. Moreover, $\mathcal{M}(\sigma,\chi,\overline{\chi},z)=\mathcal{M}(\sigma,z)=C(z)$ and the first claim follows. The generating function for sequence A125188 can be found in [S] and is given by

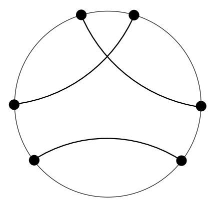
$$f(z)=\frac{1+zC(z)-\sqrt{1-zC(z)-5z}}{2z(1+C(z))}$$

Applying the change of variable $y=zC(z)$, so that $z=y(1-y)$ and $C(z)=\frac{1}{1-y}$, some routine computations show that $f(z)=\frac{1}{1-zC(z)C(C(z)-1)}$, hence the second claim also follows. □

Sequence A125188 counts Dumont permutations of the first kind avoiding the patterns 2413 and 4132, but we have not been able to find any bijection with our classes of pattern avoiding matchings. Note that, for $\sigma\in\{1212,1221\}$, iterating Theorem 2.3.2 allows to prove that $\mathcal{M}(12...k(\sigma+k)k...21,\chi,\overline{\chi},z)$ is an algebraic function of $C(z)$, hence it is itself algebraic, for every $k\in\mathbb{N}^*$.

2.3.4 Unlabeled pattern avoidance

In this section we introduce the notion of unlabeled matching, which provides a way to collect patterns that are combinatorially equivalent, in a sense that is specified below. Given $n \in \mathbb{N}^*$, let γ_n denote the $2n-$cycle $(1\ 2\ 3\ ...\ 2n)$ on $[2n]$ and let σ and τ be two matchings of order n. We say that σ and τ are *ciclically equivalent* when there exists $k \in [2n]$ such that $\{i, j\} \in \sigma$ if and only if $\{\gamma_n^k(i), \gamma_n^k(j)\} \in \tau$, for every $i, j \in [2n]$. In other words, two matchings are ciclically equivalent when they have the same unlabeled circular chord diagram. An equivalence class of matchings is called an *unlabeled matching.* For instance, $[112323] = \{112323, 123231, 123312, 121233, 121332, 122313\}$. Thus, an unlabeled matching can be represented by an unlabeled circular chord diagram; for instance, the unlabeled matching [112323] can be represented by the following unlabeled chord diagram



Note that a matching avoids an unlabeled pattern if and only if its circular chord diagram avoids the unlabeled chord diagram of the pattern.

The unlabeled matchings of order 2 are exactly $[1122] = \{1122, 1221\}$ and $[1212] = \{1212\}$. Note that, for every $n \in \mathbb{N}^*$, a matching λ of order n avoids [1122] if and only if it is permutational and nonnesting, hence $\mathcal{M}_n([1122]) = \{123...n123...n\}$. We thus have $\mathcal{M}([1212], z) = C(z)$ and $\mathcal{M}([1122], z) = \frac{1}{1-z}$. The unlabeled matchings of order 3 are exactly five, namely:

$$[112323] = \{112323, 123231, 123312, 121233, 121332, 122313\},$$

$$[123132] = \{121323, 123213, 121323\},$$

$$[123321] = \{123321, 122133, 112332\},$$

$$[112233] = \{112233, 122331\},$$

$$[123123] = \{123123\}.$$

Clearly $|\mathcal{M}_n([123123])| = C_n C_{n+2} - C_{n+1}^2$. In this section we will work out explicit formulas to enumerate $\mathcal{M}([112323])$ and $\mathcal{M}([123132])$.

Proposition 2.3.4. *The generating function of matchings avoiding the unlabeled pattern* [112323] *is given by*

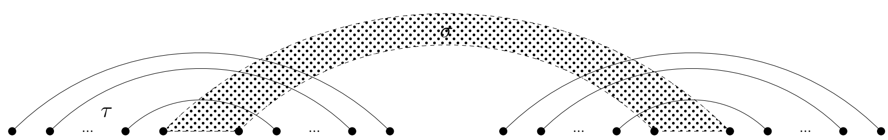
$$\mathcal{M}([112323], z) = C(z) + \frac{z^2}{(1-z)^2(1-2z)}$$

As a consequence, its coefficients have the following closed form:

$$|\mathcal{M}_n([112323])| = C_n + 2^n - n - 1,$$

for $n \geq 2$.

Proof. Clearly the noncrossing matchings in $\mathcal{M}_n([112323])$ are counted by the Catalan number C_n, hence it remains to count the crossing matchings in $\mathcal{M}_n([112323])$. Let λ be a crossing matching in $\mathcal{M}_n([112323])$. Let σ denote the pattern of λ consisting of all the edges intersecting the leftmost edge of λ and let τ denote the pattern of λ consisting of all the remaining edges. Note that σ is nonempty, otherwise, since λ is assumed to be crossing, there would be a pair of crossing edges that do not cross the leftmost edge of λ, thus forming an occurrence of [112323]. Assume that σ contains k edges, where $k \in [n-1]$. Observe that σ has to be permutational, because an occurrence of 1122 in σ should have at least one edge which does not cross the leftmost edge of λ, against the definition of σ. Moreover, since σ avoids [112323], the corresponding permutation has to avoid both the permutation patterns 231 and 312, therefore there are $|\mathcal{S}_k(231, 312)| = 2^{k-1}$ possible choices for σ. Furthermore, τ must be noncrossing, otherwise any pair of crossing edges of τ together with the leftmost edge of λ would form an occurrence of [112323]. Finally, using a similar argument, we deduce that each edge of τ has to cross all the edges of σ. We can thus conclude that τ is the juxtaposition of two totally crossing matchings of order $n - k$ such that the leftmost one is nonempty. Hence there are exactly $n - k$ possibile choices for τ. In other words, λ has the form illustrated by the following linear chord diagram



From this characterization of $\mathcal{M}_n([112323])$, it follows that

$$|\mathcal{M}_n([112323])| = C_n + \sum_{k=1}^{n-1}(n-k)2^{k-1},$$

hence

$$\begin{aligned}\mathcal{M}([112323],z) &= C(z) + \sum_{n\geq 2}\sum_{k=1}^{n-1}(n-k)2^{k-1}z^n \\ &= C(z) + z^2\sum_{n\geq 0}\sum_{k=0}^{n}(n-k+1)2^k z^n \\ &= C(z) + z^2\left(\sum_{n\geq 0}(n+1)z^n\right)\left(\sum_{n\geq 0}2^n z^n\right) \\ &= C(z) + \frac{z^2}{(1-z)^2(1-2z)}.\end{aligned}$$

Finally we can compute the partial fraction decomposition of $\frac{z^2}{(1-z)^2(1-2z)}$ to find

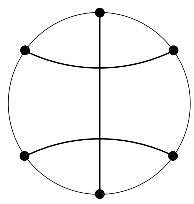
$$\begin{aligned}\frac{z^2}{(1-z)^2(1-2z)} &= z^2\left[\frac{-3+2z}{(1-z)^2} + \frac{4}{1-2z}\right] \\ &= z^2\left[-3\sum_{n\geq 0}(n+1)z^n + 2z\sum_{n\geq 0}(n+1)z^n + 4\sum_{n\geq 0}2^n z^n\right] \\ &= \sum_{n\geq 2}(2^n-n-1)z^n,\end{aligned}$$

which proves the claim.

□

The sequence enumerating $\mathcal{M}([112323])$ begins $1, 1, 1, 3, 9, 25, 68, 189, ...$ and it is not recorded in [S], however it is worth noting that $2^n - n - 1$ is the n^{th} Eulerian number (sequence A000295 in [S]).

Our last result concerns the unlabeled pattern [123132], which is represented by the following unlabeled chord diagram:



It turns out that matchings avoiding [123132] have a ternary tree structure and the following discussion is in fact devoted to describe a bijection between this class of matchings and the class of ternary trees. Recall from Section 1.0.3 that the class of ternary

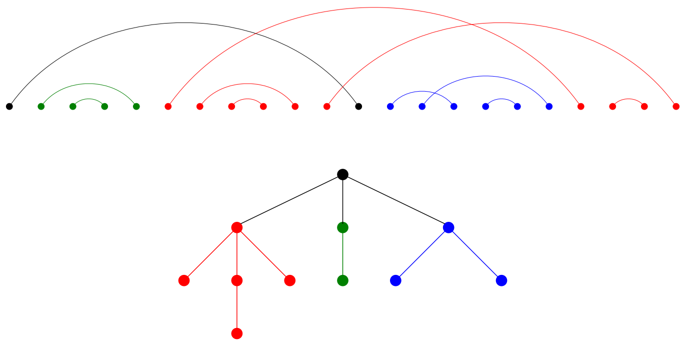


Figure 2.9: The linear chord diagram of a matching with semilength 11 avoiding the unlabeled pattern [123132] and the corresponding 3−ary tree.

trees $\mathcal{T}_3$ is isomorphic to the class $\{\emptyset\} + \{\bullet\} \times (\mathcal{T}_3)^3$ and that ternary trees with n edges are counted by the 3−Catalan numbers $\frac{1}{2n+1}\binom{3n}{n}$.

Now we recursively define a map $\varphi : \{\emptyset\}+\{\bullet\}\times(\mathcal{T}_3)^3 \longrightarrow \mathcal{M}([123132])$ as follows. Set $\varphi(\emptyset) = \emptyset$; furthermore, for every $(T_1, T_2, T_3) \in (\mathcal{T}_3)^3$, let $\varphi(\bullet, T_1, T_2, T_3)$ be the matching whose linear chord diagram Γ is constructed as follows:

1. Denote with $\Gamma^{(i)}$ the linear chord diagram of $\varphi(T_i)$, for every $i \in \{1, 2, 3\}$.

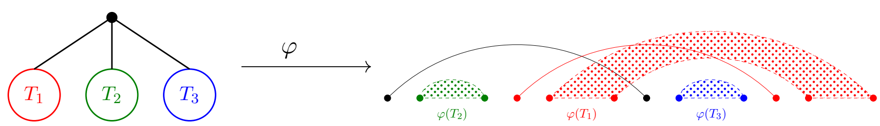
2. If $\Gamma^{(1)}$ is empty, then:

   1. draw a vertex ℓ' on the left of a vertex r' and connect them with an edge;
   2. draw $\Gamma^{(2)}$ between ℓ' and r' and $\Gamma^{(3)}$ to the right of r'.

3. If $\Gamma^{(1)}$ is nonempty, then:

   1. let ℓ and r denote the left and right vertex of the leftmost edge of $\Gamma^{(1)}$, respectively; draw two vertices ℓ' and r' to the left of ℓ and r, respectively, and connect them with an edge.
   2. draw Γ_2 between ℓ' and ℓ and Γ_3 between r' and r.

In other words, the map φ can be represented by the following diagram:



Conversely, define recursively a map $\psi : \mathcal{M}([123132]) \longrightarrow \mathcal{T}_3$ as follows. Set $\psi(\emptyset) = \emptyset$. For every $\lambda \in \mathcal{M}([123132]) \setminus \{\emptyset\}$, let $\psi(\lambda)$ be the ternary tree defined as follows:

1. Suppose that the leftmost edge of λ does not cross any other edge. In this case, denote by λ_2 the pattern of λ consisting of all the edges of λ which are nested below the leftmost edge of λ and denote by λ_3 the pattern of λ consisting of all the remaining edges of λ other than the leftmost edge. We then define $\psi(\lambda) = (\bullet, \emptyset, \psi(\lambda_2), \psi(\lambda_3))$.

2. Suppose that the leftmost edge ℓ of λ crosses some other edge of λ and let ℓ' denote the leftmost edge of λ among those crossed by ℓ. Let λ_2 denote the pattern of λ consisting of all $e \in \lambda$ such that $\min(\ell) < \min(e) < \max(e) < \min(\ell')$ and let λ_3 denote the pattern of λ consisting of all $e \in \lambda$ such that $\max(\ell) < \min(e) < \max(e) < \max(\ell')$. Finally, let λ_1 denote the pattern of λ consisting of all the remaining edges of λ other than ℓ. We then define $\psi(\lambda) = (\bullet, \psi(\lambda_1), \psi(\lambda_2), \psi(\lambda_3))$.

Proposition 2.3.5. *The maps φ and ψ are well defined mutually inverse bijections. In particular*

$$|\mathcal{M}_n([123132])| = \frac{1}{2n+1}\binom{3n}{n},$$

for every $n \in \mathbb{N}$.

Proof. The main thing we have to prove is that φ is well defined. Denote by Λ the unlabeled chord diagram of [123132]. Given $T = (\bullet, T_1, T_2, T_3) \in \{\bullet\} \times (\mathcal{T}_3)^3$, we now prove (by induction hypotesis on the number of nodes of T) that $\varphi(T) \in \mathcal{M}([123132])$. Using the same notation as in the definition of φ, we first observe that (by iduction hypothesis) there is no occurrence of Λ in $\Gamma^{(1)}$. Furthermore, the edge $\{\ell', r'\}$ cannot be involved in any occurrence of Λ. Suppose in fact that Λ_0 is an occurrence of Λ involving $\{\ell', r'\}$. If $\Gamma^{(1)}$ is nonempty, then it is not difficult to realize that the leftmost edge $\{\ell, r\}$ of $\Gamma^{(1)}$ cannot occur in Λ_0 (this is due to the choice of the specific pattern Λ). Thus we can replace $\{\ell', r'\}$ with $\{\ell, r\}$ in Λ_0 to get an occurrence of Λ in $\Gamma^{(1)}$, which is a contradiction. On the other hand, if $\Gamma^{(1)}$ is empty, it is easy to check that $\{\ell', r'\}$ cannot belong to any occurrence of Λ in $\varphi(T)$. Finally, no edge in $\Gamma^{(2)}$ or $\Gamma^{(3)}$ can be involved in an occurrence of Λ, because both $\varphi(T_2)$ and $\varphi(T_3)$ avoid [123132] (by induction) and each of the edges of their chord diagrams does not cross any of the remaining edges of $\varphi(T)$. To conclude, it suffices to prove that φ and ψ are mutually inverse, which is immediate by their construction. □

2.3.5 Conclusion and further work

The enumerative combinatorics of the matching pattern poset remains still largely unknown. Although some major efforts to enumerate pattern avoiding matchings have already been spent, as mentioned in Section 2.3.1, the enumeration of most classes of matchings avoiding a single pattern of order three is still lacking.

To this regard, in Section 2.3.2 and 2.3.3 we studied classes of matchings avoiding the juxtaposition and the lifting of patterns, finding some interesting enumerative results. However, as for the juxtaposition of two patterns, we did not succed so far in extending this results to prefixes other than 11 and 1212, so that it could be interesting to investigate other similar prefixes, such as 123123. As for the lifting of a pattern, ECO method and generating trees could prove to be fruitful tools for enumeration purposes, therefore we believe such an approach deserves to be explored further.

In Section 2.3.4, we have also introduced the notion of unlabeled pattern, and we have enumerated matchings avoiding the unlabeled patterns [123123], [112323] and [123132], respectively. However we did not succeed in finding a formula for the number of matchings avoiding the remaining two unlabeled patterns of order three, namely [123321] and [112233], although matchings in the former class seem to have a rather neat combinatorial structure.

Chapter 3

Structure of intervals

3.1 Intervals in the Dyck pattern poset

In this section, we initiate the study of the structure of intervals in Dyck pattern poset. Recall from Section 2.2 that a Dyck path can be represented as a word on the alphabet $\{U, D\}$ such that each prefix has at least as many U's as D's and the total number of U's and D's is the same. Such words are commonly called *Dyck words*. In the following we will frequently switch between paths and words, and in particular we will use the same notations when no confusion is likely to arise. Recall also that, given two Dyck paths P, Q, we say that $P \leq Q$ when P is a subword of Q (i.e. there exists a subset of the letters of Q which, read from left to right, are equal to P). In this case, we also say that P is a *pattern* of Q, and any subword of Q which is equal to P is called an *occurrence* of P is Q. So, for instance, $UUDD \leq UDUDUD$, whereas $UUDDUD$ and $UUDUUUDDDD$ are incomparable. The Dyck pattern poset has a minimum, which is the path UD, and has no maximum; moreover, it is graded, the rank of an element being its semilength.

In [BFPW] and [BBFGPW] some enumerative properties of the Dyck pattern poset have been investigated, mainly focusing on pattern avoidance questions. Here we start the analysis of the enumerative combinatorics of the intervals of this poset.

Given any poset, one of the most natural aspect to investigate is the structure of its intervals. This has been done in several combinatorially interesting posets, such as for Tamari lattices [CCP, F], the Bruhat order [T], the consecutive pattern poset [EM], to cite just a few. From this point of view, a fairly general problem is that of counting (saturated) chains (here "saturated" means that the chain cannot be extended except that at the beginning and at the end). Special instances of this problem are the enumeration of the elements and of the covering relations of the interval (which are saturated chains of length 0 and 1, respectively). Another important quantity associated to a (combinatorially interesting) poset is the Möbius function of its intervals. For our purposes, we can define the Möbius function $\mu : \mathcal{P}^2 \to \mathbb{C}$ of the poset $\mathcal{P}$ in the following recursive way (for $x \leq y$):

$$\begin{cases} \mu(x,x) = 1, \\ \mu(x,y) = -\sum_{x \le z < y} \mu(x,z), \qquad \text{when } x < y. \end{cases}.$$

In the present section we analyze a few types of *initial intervals* $[UD, P]$ in the Dyck pattern poset. More specifically, we first consider the case in which $P = (UD)^n$, for some $n \in \mathbb{N}$, for which we are able to determine the cardinality of the interval, also refined by rank. Then we examine in detail the case in which P has exactly two peaks: here we find formulas both for the cardinality of the interval (and also in this case we have a refined version for ranks) and for the number of covering relations. We find also a nice bijection between Dyck paths having two peaks inside $[UD, P]$ and squares fitting inside a rectangle of appropriate dimensions. Finally, we give also a complete description of the Möbius function of such intervals. We remark that the computation of the Möbius function of the Dyck pattern poset is still open for general intervals, and the results contained in the present section are the first ones for this poset. In the last section, together with some proposals for further work, we also provide some additional results on the Möbius function which suggests that the Dyck pattern intervals have a nice structure that certainly deserves to be better investigated.

We close this Introduction by fixing the main notations we are using throughout the section.

Given a poset $\mathcal{P}$ and a nonnegative integer ℓ, a *saturated chain of length* ℓ in $\mathcal{P}$ is a sequence $(x_0, x_1, ..., x_\ell)$ of $\ell + 1$ elements of $\mathcal{P}$ such that $x_0 \prec x_1 \prec ... \prec x_\ell$, where $\prec$ denotes the covering relation of $\mathcal{P}$. For a finite poset $\mathcal{P}$, denote with $s_\ell(\mathcal{P})$ the number of saturated chains of length ℓ of $\mathcal{P}$. In particular, $s_0(\mathcal{P})$ is the number of elements of $\mathcal{P}$, and $s_1(\mathcal{P})$ is the number of edges of the Hasse diagram of $\mathcal{P}$ (which is also the number of coverings relations in $\mathcal{P}$). When $\mathcal{P}$ is graded and $\ell, k \in \mathbb{N}$, the number of saturated chains of length ℓ whose top element has rank k will be denoted $s_\ell^{(k)}(\mathcal{P})$. Therefore $s_\ell(\mathcal{P}) = \sum_{k \ge \ell} s_\ell^{(k)}(\mathcal{P})$. In particular, $s_0^{(k)}(\mathcal{P})$ is the number of elements of $\mathcal{P}$ having rank k.

Given $x \in \mathcal{P}$, we write $\Delta(x)$ for the number of elements in $\mathcal{P}$ covered by x. Moreover, $\Delta_t(\mathcal{P})$ will denote the number of $x \in \mathcal{P}$ such that $\Delta(x) = t$. As a consequence, we have that $s_1(\mathcal{P}) = \sum_{t \ge 0} t \cdot \Delta_t(\mathcal{P})$.

In the following sections, we will be especially concerned with intervals in $\mathcal{P}$, in particular with the lower closure of an element $x \in \mathcal{P}$. Given $x \in \mathcal{P}$, the *lower closure* of x is the set of all $y \in \mathcal{P}$ such that $y \le x$.

3.1.1 The lower closure of $(UD)^n$

Our first goal is to find an explicit formula for the number of elements of the interval $[UD, (UD)^n]$, for $n \in \mathbb{N}$. Recall that, given $n, k \in \mathbb{N}$, the *Narayana number* $N_{n,k}$ is defined as the number of Dyck paths of semilength n and having k peaks (a peak of a Dyck path P is an occurrence of the Dyck path UD as a consecutive pattern in P). It is well known that $N_{0,0} = 1$, $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n, k \ge 1$ and $N_{n,k} = 0$ in the remaining cases.

Suppose now P is a Dyck path and denote with $\mathsf{asc}(P)$ the number of ascents of P, where an *ascent* of a Dyck path is a maximal consecutive substrings of P of the form U^m, for some $m > 0$. It is clear that $\mathsf{asc}(P)$ also counts the number of peaks of P, in particular $N_{n,k}$ also counts the number of Dyck paths of semilength n with k ascents. The next lemma characterizes Dyck paths in the interval $[UD, (UD)^n]$ in terms of the number of ascents.

Lemma 3.1.1. *Let $n > 0$, $k \in \{1, ..., n\}$ and P be a Dyck path of semilength k. Then $P \leq (UD)^n$ if and only if $\mathsf{asc}(P) \geq 2k - n$.*

In order to prove this lemma, it is convenient to regard it as a special case of the following slight generalization.

Lemma 3.1.2. *For any positive integers $n, m, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$, set $\alpha = \sum_{i=1}^m \alpha_i$ and $\beta = \sum_{i=1}^m \beta_i$. Then the string $U^{\alpha_1} D^{\beta_1} U^{\alpha_2} D^{\beta_2} \cdots U^{\alpha_m} D^{\beta_m}$ is a substring of $(UD)^n$ if and only if $\alpha + \beta - n \leq m \leq n$.*

Proof. Set $P = U^{\alpha_1} D^{\beta_1} U^{\alpha_2} D^{\beta_2} ... U^{\alpha_m} D^{\beta_m}$.

Suppose first that P is a substring of $(UD)^n$. Then, for each step of P not belonging to a peak, we need one factor UD from $(UD)^n$ in order to embed P into $(UD)^n$. Instead, each peak of P needs just one factor from $(UD)^n$. Thus the total number of factor UD of $(UD)^n$ must be at least the sum of the two above quantities, that is

$$n \geq \left(\sum_{i=1}^m (\alpha_i - 1) + \sum_{i=1}^m (\beta_i - 1) \right) + m = \alpha + \beta - m,$$

which implies the desired inequality.

Suppose now that $n \geq \alpha + \beta - m$. We look for an occurrence of P in $(UD)^n$. It is not hard to see that each step of P not belonging to a peak, as well as each peak of P, requires precisely one factor UD from $(UD)^n$. Thus, for any i, in order to embed $U^{\alpha_i} D^{\beta_i} = U^{\alpha_i - 1}(UD) D^{\beta_i - 1}$ into $(UD)^n$ we need $(\alpha_i - 1) + 1 + (\beta_i - 1) = \alpha_i + \beta_i - 1$ factors UD. Therefore, we can embed P into $(UD)^n$ provided that n is at least $\sum_{i=1}^m (\alpha_i + \beta_i - 1) = \alpha + \beta - m$, which is the hypothesis.

□

Now as an immediate consequence of this lemma we can deduce an explicit formula for $s_0^{(k)}([UD, (UD)^n])$ and $s_0([UD, (UD)^n])$ when $n \in \mathbb{N}$ and $k \in \{1, ..., n\}$.

Proposition 3.1.1. *Let $n > 0$ and $k \in \{1, ..., n\}$, then*

(i)

$$s_0^{(k)}([UD, (UD)^n]) = \sum_{m=\max\{1, 2k-n\}}^{k} N_{k,m}$$

$n\backslash k$	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	1	2	1						
4	1	2	4	1					
5	1	2	5	7	1				
6	1	2	5	13	11	1			
7	1	2	5	14	31	16	1		
8	1	2	5	14	41	66	22	1	
9	1	2	5	14	42	116	127	29	1

Table 3.1: Triangle associated with sequence A137940

(ii)

$$s_0([UD,(UD)^n]) = \sum_{k=1}^{n} \sum_{m=\max\{1,2k-n\}}^{k} N_{k,m}. \tag{3.1}$$

In Figure 3.1 the Hasse diagram of the interval $[UD,(UD)^5]$ is depicted.

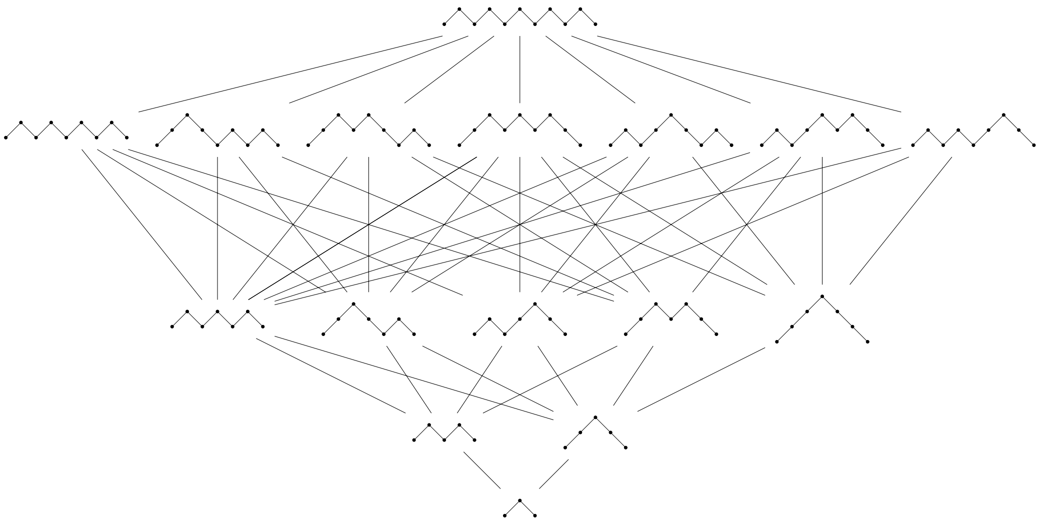


Figure 3.1: The Hasse diagram of the interval $[UD,(UD)^5]$ in the Dyck pattern poset

Sequence $(s_0^{(k)}([UD,(UD)^n]))_{n\geq k\geq 0}$ appears as A137940 in [S]. The first few lines of the associated triangle are recorded below (Table 3.1). As a consequence of the last proposition, and since the Narayana array is symmetric, the k-th diagonal of Table 1 contains the sum of the first k columns of the Narayana array. Therefore, $s_0^{(k)}([UD,(UD)^n])$ counts the number of Dyck paths of semilength k having at most $n-k+1$ peaks.

The sequence $(s_0([UD,(UD)^n]))_{n\geq 0}$ of the sizes of the interval $[UD,(UD)^n]$ starts 1,2,4,8,16, 33,70,152,337 and is not recorded in [S]; however, it is the sequence of the partial sums of A004148 of [S], called "generalized Catalan numbers" and counting, among other things, peak-less Motzkin paths with respect to the length. In fact, it is not difficult to find a bijective explanation.

Proposition 3.1.2. *There is a bijection between $[UD,(UD)^n]$ and the set of peak-less Motzkin paths of length at most n.*

Proof. Given a peak-less Motzkin path of length $k\leq n$, just replace each of its level step with a peak: the resulting Dyck path is easily seen to be in $[UD,(UD)^n]$. Also, it is not difficult to realize that such a map is indeed a bijection. □

3.1.2 The lower closure of a Dyck path with two peaks

Size of the interval

Let a,b,h be nonnegative integers. Denote with $Q_{a,b}^{(h)}=U^{a+h}D^aU^bD^{b+h}$ the generic Dyck paths with two peaks. Our first goal is to find an explicit formula for the cardinality of $\left[UD,Q_{a,b}^{(h)}\right]$, depending on a,b and h. Without loss of generality, we can suppose $b\geq a\geq 1$.

It is clear that each path in the above interval has either one peak or two peaks. The generic path with only one peak in $\left[UD,Q_{a,b}^{(h)}\right]$ has height j, with $1\leq j\leq b+h$. Therefore there are $b+h$ such paths. From now on, we will focus on the remaining elements of our interval, i.e. paths having exactly two peaks.

We start with the case $h=0$, that is $Q_{a,b}^{(0)}=U^aD^aU^bD^b$: this is the generic *nonelevated* Dyck path with two peaks.

Proposition 3.1.3. *Denote with $\varphi_0(a,b)$ the number of Dyck paths having exactly two peaks in the interval $[UD,Q_{a,b}^{(0)}]$. Then*

$$\varphi_0(a,b)=\frac{a(a+1)(3b-a+1)}{6}\ . \tag{3.2}$$

Proof. The generic Dyck path P with two peaks in $[UD,Q_{a,b}^{(0)}]$ can be constructed as follows. Start by choosing k up steps from the first run of $Q_{a,b}^{(0)}$, for k running from 1 to a. Then choose t down steps from the second run: here it must be $1\leq t\leq k$, since otherwise P would have some points of negative height. Similarly, the up steps chosen from the third run of $Q_{a,b}^{(0)}$ cannot be too many, otherwise there would remain too few down steps in the last run to complete the Dyck path. Specifically, it is possible to select s up steps from the third run of $Q_{a,b}^{(0)}$, with $k-t+s\leq b$ (the quantity $k-t+s$ being the height of the path P at the end of the last ascending run), and so $1 \leq s \leq b-k+t$. Summing up, we get

$$\varphi_0(a,b) = \sum_{k=1}^{a} \sum_{t=1}^{k} \sum_{s=1}^{b-k+t} 1$$

which reduces to

$$\varphi_0(a,b) = \frac{a(a+1)(3b-a+1)}{6},$$

as desired. □

Sequence $(\varphi_0(a,b))_{1\leq a\leq b}$ is A082652 in [S], where an interpretations in terms of squares inside a $a \times b$ rectangular grid is provided. We will discuss the connections with our combinatorial setting at the end of the present section.

We are now ready to express the number of Dyck paths having exactly two peaks lying below $Q_{a,b}^{(h)}$.

Proposition 3.1.4. *Denote with $\varphi_h(a,b)$ the number of Dyck paths having exactly two peaks in the interval $[UD, Q_{a,b}^{(h)}]$. Then*

$$\varphi_h(a,b) = \varphi_0(a,b) + hab \ . \tag{3.3}$$

Proof. Denote with $[UD, Q_{a,b}^{(\ell)}]_2$ the set of Dyck paths in $[UD, Q_{a,b}^{(\ell)}]$ having exactly two peaks. For each $i = 1, \ldots, h$, define $\mathcal{C}_i = [UD, Q_{a,b}^{(i)}]_2 \setminus [UD, Q_{a,b}^{(i-1)}]_2$. It is not hard to see that a Dyck path $U^\alpha D^\beta U^\gamma D^\delta \in \mathcal{C}_i$ if and only if

- $(\alpha, \beta, \gamma, \delta) \leq (i+a, a, b, i+b)$ and
- $\alpha = i+a$ or $\delta = i+b$.

For $i = h$, the above definitions immediately imply that

$$\varphi_h(a,b) = \varphi_{h-1}(a,b) + |\mathcal{C}_h|$$

Iterating this argument, we get:

$$\varphi_h(a,b) = \varphi_0(a,b) + \sum_{i=1}^{h} |\mathcal{C}_i|. \tag{3.4}$$

Now observe that, if $\Gamma \in \mathcal{C}_i$, then $\Gamma = U^i P D^i$, for some Dyck path P with two peaks. This is due to the fact that, if (w.l.o.g.) the first ascending run of Γ has length $i+a$, the first descending run of Γ terminates at height $\geq i$ (because $\Gamma \in [UD, Q_{a,b}^{(h)}]_2$). Of course an analogous argument holds if the last descending run of Γ has length $i+b$. As a consequence, for each $i = 1, \ldots, h$, we can define a function from $\mathcal{C}_i$ to $\mathcal{C}_0 = \{P = U^\alpha D^\beta U^\gamma D^\delta \in [UD, Q_{a,b}^{(0)}]_2 \,|\, \alpha = a \text{ or } \delta = b\}$ which maps $\Gamma = U^i P D^i$ into P. It is easy

$Q_{4,6}^{(0)} \quad \geq \quad U^4D^2U^3D^5 \quad =$

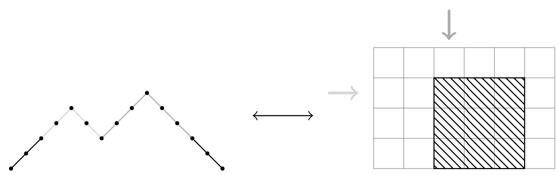


Figure 3.2: The path $U^4D^2U^3D^5 \leq Q_{4,6}^{(0)}$ and the associated square inside the grid 4×6

to see that such a function is well defined (i.e. $P \in \mathcal{C}_0$) and is bijective. Thus formula (3.4) can be rewritten as

$$\varphi_h(a,b) = \varphi_0(a,b) + h|\mathcal{C}_0|. \tag{3.5}$$

Finally, in order to enumerate $\mathcal{C}_0$, we observe that a path $P \in \mathcal{C}_0$ is uniquely determined by the length s of its first descending run and the length t of its last ascending run, where $1 \leq s \leq a$ and $1 \leq t \leq b$. In fact, the unique path of $\mathcal{C}_0$ corresponding to a legal choice of s and t is $U^x Q_{s,t}^{(0)} D^x$, where $x = \min\{a-s, b-t\}$. Hence we have that $|\mathcal{C}_0| = ab$. This leads immediately to formula (3.3), as desired. □

As we have already noticed, the triangular array determined by $(\varphi_0(a,b))_{1 \leq a \leq b}$ is sequence A082652 in [S], which counts the numbers of squares that can be found in an $a \times b$ rectangular grid. We now describe a bijection between the set of such squares and the set $[UD, Q_{a,b}^{(0)}]_2$ of Dyck paths having two peaks less than or equal to $Q_{a,b}^{(0)}$. To this aim, we encode the Dyck path $U^{k+i}D^iU^jD^{j+k} \in [UD, Q_{a,b}^{(0)}]_2$ with the triple $(i,j;k)$. Observe that $(i,j;k)$ represents a legal Dyck path in $[UD, Q_{a,b}^{(0)}]_2$ if and only if $i,j \geq 1$, $i+k \leq a$ and $j+k \leq b$. Moreover, a triple $(i,j;k)$ satisfying the above conditions also uniquely determines a square inside an $a \times b$ rectangular grid of unit cells as follows. Label the rows of the grid with integers $1,2,\ldots,a$ from top to bottom and the columns with integers $1,2,\ldots,b$ from left to right. A unit cell lying at the intersection between row i and column i will be said in *position* (i,j). A square inside the grid can be characterized by providing the position (i,j) of its topmost and leftmost cell and the length $k+1$ of its side. Notice that the triple $(i,j;k)$ determines (in a unique way) a square if and only if $i,j \geq 1$, $i+k \leq a$ and $j+k \leq b$. We have thus found the same encoding both for Dyck paths in $[UD, Q_{a,b}^{(0)}]_2$ and for squares inside an $a \times b$ rectangular grid, which gives the desired bijection. In Figure 3.2 the path encoded by the triple $(2,3;2)$ is mapped into the corresponding square.

The above bijection can be exploited to transport the order structure on Dyck paths, so to describe it in terms of squares in a grid. In the next lemma, we use the above encoding into triples to give an alternative presentation of the pattern order inside $[UD, Q_{a,b}^{(0)}]_2$.

Lemma 3.1.3. *Set $\Theta = \{(i,j;k) \,|\, i,j \geq 1, i+k \leq a, j+k \leq b$. Define a partial order on Θ by setting $(\alpha,\beta;\gamma) \sqsubseteq (i,j;k)$ when $(\alpha,\beta,\gamma) \leq (i,j,\min\{i+k-\alpha, j+k-\beta\}$, where $\leq$ is the usual coordinatewise order on $\mathbb{N}^3$. Then $(\alpha,\beta;\gamma) \sqsubseteq (i,j;k)$ if and only if $(\alpha,\beta;\gamma) \leq (i,j;k)$ in the Dyck pattern poset, i.e. $U^{\gamma+\alpha}D^{\alpha}U^{\beta}D^{\beta+\gamma} \leq U^{k+i}D^iU^jD^{j+k}$.*

Proof. The fact that $U^{\gamma+\alpha}D^{\alpha}U^{\beta}D^{\beta+\gamma} \leq U^{k+i}D^{i}U^{j}D^{j+k}$ is equivalent to the system of inequalities

$$\begin{cases} \alpha+\gamma \leq i+k \\ \alpha \leq i \\ \beta \leq j \\ \beta+\gamma \leq j+k \end{cases}$$

since the two paths have the same number of peaks, and so the steps of each run of the smallest path must be selected from the steps of the corresponding run of the largest one. The above inequalities can be equivalently written as $(\alpha,\beta,\gamma) \leq (i,j,\min\{i+k-\alpha, j+k-\beta\}$, which is exactly $(\alpha,\beta;\gamma) \sqsubseteq (i,j;k)$. □

Using the above lemma and bijection, the pattern order on $[UD, Q^{(0)}_{a,b}]_2$ can now be expressed as a partial order on the set of squares inside an $a \times b$ rectangular grid. Denote with $\underline{xy|}$ the rectangle having a pair of opposite corners in positions $(1,1)$ and (x,y). Take two squares Q, Q' in the grid whose topmost and leftmost cells are in positions (α,β) and (i,j), respectively, and whose sides have length $\gamma+1$ and $k+1$; hence they are encoded by the triples $Q = (\alpha,\beta;\gamma)$ and $Q' = (i,j;k)$. The partial order on Θ defined in the above lemma can be read off on squares in the following way: $Q \leq Q'$ when the topmost and leftmost cell of Q is in the rectangle $\underline{ij|}$ and the opposite cell is in the rectangle $\underline{(i+k)(j+k)|}$.

Suppose now $r \in \mathbb{N}$ and $2 \leq r \leq a+b$. We will refine our previous enumerative result by counting the number of elements in $[UD, Q^{(h)}_{a,b}]$ with semilength r. This clearly gives the rank distribution of the elements of $[UD, Q^{(h)}_{a,b}]$. As an example, Figure 3.3 shows the interval $[UD, Q^{(1)}_{2,3}]$.

Proposition 3.1.5. *The number of elements of $[UD, Q^{(h)}_{a,b}]$ having semilength r is given by*

$$s_0^{(r)}[UD, Q^{(h)}_{a,b}] = \sum_{i=\max\{1,r-b-h\}}^{\min\{a,r-1\}} (\min\{b, r-i\} - \max\{1, r-a-h\} + 1) + [r \leq b+h] \quad , \tag{3.6}$$

where the notation $[\Omega]$ denotes the characteristic function of the property Ω, that is $[\Omega] = 1$ if Ω is true and $[\Omega] = 0$ if Ω is false.

Proof. We can clearly limit ourselves to considering paths having two peaks, since paths with just one peak inside $[UD, Q^{(h)}_{a,b}]$ are easily counted (we have exactly one such path for any rank $r \leq b+h$).

Using an approach similar to that of Proposition 3.1.3, we observe that a generic Dyck path $P = U^{k+i}D^{i}U^{j}D^{j+k} \in [UD, Q^{(h)}_{a,b}]_2$ of semilength r can be constructed as follows. Start by choosing $k+i$ up steps from the first run of $Q^{(h)}_{a,b}$, with $k+i \leq a+h$.

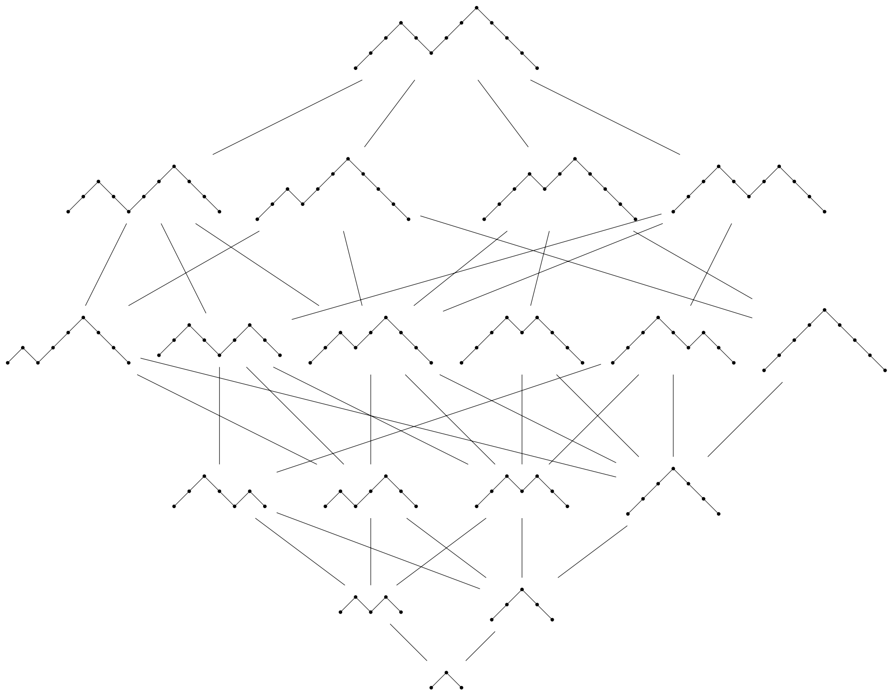


Figure 3.3: The Hasse diagram of the interval $[UD, Q^{(1)}_{2,3}]$ in the Dyck pattern poset.

Then choose i down steps from $Q^{(h)}_{a,b}$ for the second run of P, so that $1 \leq i \leq a$, since in the second run of $Q^{(h)}_{a,b}$ there are exactly a down steps. For the third run of P, it must be $1 \leq j \leq b$ (since the third run of $Q^{(h)}_{a,b}$ has length b) and $j+k \leq b+h$ (since the height of P at the end of the third run cannot exceed $b+h$, which is the maximum possible length of the fourth run of P). Finally, we have $i+j+k=r$, so that the following relations hold:

$$\begin{cases} i+k \leq a+h \\ 1 \leq i \leq a \\ 1 \leq j \leq b \\ j+k \leq b+h \\ i+j+k = r \ . \end{cases}$$

From the last relation, we have $i+k=r-j$ and $j+k=r-i$, so that $i \leq r-j$ and $j \leq r-i$. Therefore, after some manipulations, the above inequalities can be equivalently written as follows, in order to make as explicit as possible the ranges of i and j (note that once i and j are fixed, the term k is uniquely determined):

$$\begin{cases} j \geq r-a-h \\ 1 \leq i \leq a \\ 1 \leq j \leq b \\ i \geq r-b-h \\ i \leq r-j \leq r-1 \\ j \leq r-i \\ k = r-i-j \ . \end{cases}$$

The last inequalities can be condensed in

$$\begin{cases} \max\{1, r-b-h\} \leq i \leq \min\{a, r-1\} \\ \max\{1, r-a-h\} \leq j \leq \min\{b, r-i\} \\ k = r-i-j. \end{cases}$$

The above computations immediately lead to a formula for the rank distribution of the elements of $[UD, Q_{a,b}^{(h)}]$:

$$s_0^{(r)}[UD, Q_{a,b}^{(h)}] = [r \leq b+h] + \sum_{i=\max\{1,r-b-h\}}^{\min\{a,r-1\}} \ \sum_{j=\max\{1,r-a-h\}}^{\max\{b,r-i\}} 1$$

$$= \sum_{i=\max\{1,r-b-h)\}}^{\min\{a,r-1\}} (\min\{b, r-i\} - \max\{1, r-a-h\} + 1) + [r \leq b+h] \quad .$$

□

Formula 3.6 is not very easy to read as it is written. However, if we assign specific values to a, b or h, some much nicer expressions can be obtained. For instance, in the particular case $h = 0$, depending on the value of the semilength r, we get what follows.

- If $2 \leq r \leq a+1$, then $\min\{a, r-1\} = r-1$, $\max\{1, r-b\} = 1$, $\min\{r-i, b\} = r-i$, and $\max\{1, r-a\} = 1$. Therefore

$$s_0^{(r)}[UD, Q_{a,b}^{(0)}]_2 = \sum_{i=1}^{r-1} (r-i) = \sum_{i=1}^{r-1} i \quad .$$

- If $a+1 \leq r \leq b$, then $\min\{a, r-1\} = a$, $\max\{1, r-b\} = 1$, $\min\{r-i, b\} = r-i$, and $\max\{1, r-a\} = r-a$. Therefore

$$s_0^{(r)}[UD, Q_{a,b}^{(0)}]_2 = \sum_{i=1}^{a} (r-i-r+a+1) = \sum_{i=1}^{a} (a+1-i) = \sum_{i=1}^{a} i \quad .$$

- If $b+1 \le r \le a+b$, then $\min\{a, r-1\} = a$, $\max\{1, r-b\} = r-b$, $\min\{r-i, b\} = r-i$, and $\max\{1, r-a\} = r-a$. Therefore

$$s_0^{(r)}[UD, Q_{a,b}^{(0)}]_2 = \sum_{i=r-b}^{a} (r-i-r+a+1) = \sum_{i=r-b}^{a} (a+1-i) = \sum_{i=1}^{a+b-r+1} i \quad .$$

Setting $m = \min\{r-1, a, a+b-r+1\}$, it is not difficult to show that $2 \le r \le a+1$ ($a+1 \le r \le b$, $b+1 \le r \le a+b$, respectively) if and only if $m = r-1$ ($m = a$, $m = a+b-r+1$, respectively). As a consequence

$$s_0^{(r)}[UD, Q_{a,b}^{(0)}]_2 = \sum_{i=1}^{m} i = \binom{m+1}{2} \quad ,$$

hence

$$s_0^{(r)}[UD, Q_{a,b}^{(0)}] = \binom{m+1}{2} + [r \le b] \quad .$$

Enumeration of covering relations

In this section we work out a formula for the number $s_1([UD, Q_{a,b}^{(0)}])$ of edges in the Hasse diagram of the interval $[UD, Q_{a,b}^{(0)}]$; as usual, we can assume w.l.o.g. that $1 \le a \le b$. For this purpose, we will use the following lemma.

Lemma 3.1.4. *Let $(i,j,k) \in \mathbb{N}^3$. Recall that we denote with $\Delta(Q_{i,j}^{(h)})$ the number of paths covered by $Q_{i,j}^{(h)}$. Then*

$$\Delta(Q_{i,j}^{(h)}) = \begin{cases} 1 & (i,j,k) \in \{(1,1,0)\} \\ 2 & (i,j,k) \in \{(i,1,0), (1,j,0), (1,1,k) \,|\, i,j \ge 2,\ k \ge 1\} \\ 3 & (i,j,k) \in \{(i,j,0), (i,1,k), (1,j,k) \,|\, i,j \ge 2,\ k \ge 1\} \\ 4 & (i,j,k) \in \{(i,j,k) \,|\, i,j \ge 2,\ k \ge 1\} \end{cases} \tag{3.7}$$

Proof. A Dyck path having two peaks can cover at most 4 paths, obtained by removing one up step from one of the two ascending runs and one down step from one of the two descending runs. However, in some cases different choices can lead to the same path. Using Proposition 2.1 of [BBFGPW], we get exactly the formula given in the statement. □

The above lemma, together with the following formula

$$s_1([UD, Q_{a,b}^{(0)}]) = \sum_{n \ge 0} n \cdot \Delta_n([UD, Q_{a,b}^{(0)}]), \tag{3.8}$$

where $\Delta_n([UD, Q_{a,b}^{(0)}]) = \{P \in [UD, Q_{a,b}^{(0)}] \,|\, \Delta(P) = n\}$, allows to find the desired enumeration.

Clearly we have $\Delta(P) = 1$ if and only if either $P = (UD)^2$ or $P = U^iD^i$, for some $i \in \{2, ..., b\}$, hence $\Delta_1([UD, Q^{(0)}_{a,b}]) = 1 + (b-1) = b$.

Now take $(i, j, k) \in \mathbb{N}^3$, with $i, j \geq 2$ and $k \geq 1$. Recalling relations (3.7), we are able to compute $\Delta_n([UD, Q^{(0)}_{a,b}])$ for $n = 2, 3, 4$.

When $n = 2$, we have that

$$\begin{cases} Q^{(0)}_{i,1} \leq Q^{(0)}_{a,b} \text{ if and only if } 2 \leq i \leq a; \\ Q^{(0)}_{1,j} \leq Q^{(0)}_{a,b} \text{ if and only if } 2 \leq j \leq b; \\ Q^{(k)}_{1,1} \leq Q^{(0)}_{a,b} \text{ if and only if } 1 \leq k \leq a-1. \end{cases}$$

Therefore it follows from Lemma 3.1.4 that $\Delta_2([UD, Q^{(0)}_{a,b}]) = (a-1)+(b-1)+(a-1) = 2a + b - 3$.

When $n = 3$, we have that

$$\begin{cases} Q^{(0)}_{i,j} \leq Q^{(0)}_{a,b} \text{ if and only if } 2 \leq i \leq a,\ 2 \leq j \leq b; \\ Q^{(k)}_{i,1} \leq Q^{(0)}_{a,b} \text{ if and only if } 2 \leq i \leq a-k,\ 1 \leq k \leq a-1; \\ Q^{(k)}_{1,j} \leq Q^{(0)}_{a,b} \text{ if and only if } 2 \leq j \leq b-k,\ 1 \leq k \leq a-1. \end{cases}$$

Again using Lemma 3.1.4 and some standard computations, we get

$$\begin{aligned} \Delta_3([UD, Q^{(0)}_{a,b}]) &= (a-1)(b-1) + \sum_{k=1}^{a-1} ((a-k-1) + (b-k-1)) \\ &= (a-1)(b-1) + (a+b-2)(a-1) - a(a-1) \\ &= (a-1)(2b-3). \end{aligned}$$

Finally, when $n = 4$, we have that $Q^{(k)}_{i,j} \leq Q^{(0)}_{a,b}$ if and only if $1 \leq k \leq a-1$, $2 \leq i \leq a-k$ and $2 \leq j \leq b-k$; thus we obtain

$$\begin{aligned} \Delta_4([UD, Q^{(0)}_{a,b}]) &= \sum_{k=1}^{a-1} (a-k-1)(b-k-1) \\ &= \sum_{k=1}^{a-1} \left((a-1)(b-1) - (a+b-2)k + k^2\right) \\ &= (a-1)^2(b-1) - (a+b-2)\binom{a}{2} + \frac{a(a-1)(2a-1)}{6}. \end{aligned}$$

Using formula (3.8), we thus have the following.

Proposition 3.1.6. *The number of covering relation in the interval* $[UD, Q^{(0)}_{a,b}]$ *is given by*

$$s_1\left(\left[UD, Q^{(0)}_{a,b}\right]\right) = -\frac{1}{3}(2a^3 - 6a^2b + a - 3b + 3). \tag{3.9}$$

3.1.3 The Möbius function

To conclude our analysis of the intervals $[UD, Q_{a,b}^{(h)}]$ we now completely determine their Möbius function. Notice that the Möbius function of intervals of the form $[UD, U^n D^n]$ is trivial:

- if $n = 1$, the Möbius function is 1;
- if $n = 2$, the Möbius function is -1;
- otherwise, the Möbius function is 0.

The next proposition shows that $\mu(UD, Q_{a,b}^{(h)})$ is almost always 0. As usual, we keep on assuming that $a \leq b$.

Proposition 3.1.7. *If at least one among h and $b - a$ is strictly bigger than 1, then $\mu(UD, Q_{a,b}^{(h)}) = 0$.*

Proof. We use induction on the semilength $r = a + b + h$ of $Q_{a,b}^{(h)}$.

It is easy to see that, if the hypothesis of the proposition is satisfied, then $r \geq 4$. Moreover, if $r = 4$, then either $a = b = 1$ and $h = 2$ or $a = 1$, $b = 3$ and $h = 0$. In such cases, the maximum of $\mu(UD, Q_{a,b}^{(h)})$ is either U^3DUD^3 or UDU^3D^3, respectively. and it is immediate to verify that $\mu(UD, U^3DUD^3) = \mu(UD, UDU^3D^3) = 0$.

Now suppose that $r > 4$. Consider the longest Dyck path $Q_{i,j}^{(k)} < Q_{a,b}^{(h)}$ such that $k, j - i \leq 1$. Such a path can be explicitly described as follows:

- if $h \leq 1$, then necessarily $a + 1 < b$, so we set $k = h$, $i = a$ and $j = a + 1$;
- if $h > 1$ and $a = b$, then we set $k = 1$ and $i = j = a$;
- if $h > 1$ and $a < b$, then we set $k = 1$, $i = a$ and $j = a + 1$.

From the above construction, it is clear that $Q_{i,j}^{(k)}$ is the longest path in the interval $[UD, Q_{a,b}^{(h)}]$ for which the hypothesis of the proposition does not hold. In other words, if $Z < Q_{a,b}^{(h)}$ is a Dyck path which does not satisfies the hypothesis of the proposition, i.e. such that the absolute value of the difference of the lengths of every pair of consecutive runs is ≤ 1, then $Z \leq Q_{i,j}^{(k)}$.

Now take a path $Z \nleq Q_{i,j}^{(k)}$, $Z \neq Q_{a,b}^{(h)}$. By the inductive hypothesis, we then have that $\mu(UD, Z) = 0$. So we can now compute the Möbius function of $[UD, Q_{a,b}^{(h)}]$ as follows:

$$\mu(UD, Q_{a,b}^{(h)}) = -\sum_{Z \leq Q_{i,j}^{(k)}} \mu(UD, Z) - \sum_{\substack{Z < Q_{a,b}^{(h)} \\ Z \nleq Q_{i,j}^{(k)}}} \mu(UD, Z).$$

In the r.h.s, the first sum is 0 by definition of Möbius function, and the second sum is 0 as well, since each of its summand is 0. This concludes the proof. □

Now, to conclude the computation of the Möbius function when the maximum of the interval has exactly two peaks, we have to analyze a few remaining cases.

Proposition 3.1.8. *If $h, b-a \leq 1$, then we are in one of the following cases:*

- $\mu(UD, Q^{(1)}_{a,a+1}) = -1$ *(with $a \geq 1$);*
- $\mu(UD, Q^{(0)}_{a,a+1}) = 1$ *(with $a \geq 1$);*
- $\mu(UD, Q^{(0)}_{a,a}) = -2$, *if $a \geq 2$, moreover, $\mu(UD, Q^{(0)}_{1,1}) = -1$;*
- $\mu(UD, Q^{(1)}_{a,a}) = 2$, *if $a \geq 2$, moreover, $\mu(UD, Q^{(1)}_{1,1}) = 1$.*

Proof. All the small cases (namely, when $a \leq 2$) can be easily checked with a simple computation. When $a > 2$, we can proceed by induction on the semilength of the top of the interval. So, suppose for instance that the top of the interval is $Q^{(1)}_{a,a+1}$ (the first of the above listed cases). Then, among the paths covered by $Q^{(1)}_{a,a+1}$, there is $Q^{(1)}_{a,a}$, and we can write:

$$\mu(UD, Q^{(1)}_{a,a+1}) = -\sum_{Z \leq Q^{(1)}_{a,a}} \mu(UD, Z) - \sum_{\substack{Z \leq Q^{(1)}_{a,a+1} \\ Z \not\leq Q^{(1)}_{a,a}}} \mu(UD, Z).$$

The first sum of the r.h.s is of course 0. To evaluate the second sum, we need to find all paths $Z \leq Q^{(1)}_{a,a+1}, Z \not\leq Q^{(1)}_{a,a}$ such that the absolute value of the difference between the lengths of any two consecutive runs is at most one (otherwise, thanks to the previous proposition, the contribution to the above sum is 0). It is not difficult to realize that, in the case under consideration, the only path with the required properties is $Q^{(0)}_{a-1,a}$. By induction, we know that $\mu(UD, Q^{(0)}_{a-1,a}) = 1$, and so we can conclude that $\mu(UD, Q^{(1)}_{a,a+1}) = -1$, as desired. The three remaining cases can be dealt with using analogous arguments. □

3.1.4 Again on the Möbius function and further work

The combinatorics of the intervals of the Dyck pattern poset is still largely unknown. We have just provided the first results in this directions, concerning the enumerative combinatorics of specific intervals (cardinality and covering relations), as well as the computation of the Möbius function in a special case. Concerning this last topic, we can prove some further results, which give some insight on this important invariant.

First of all, the absolute value of the Möbius function of the Dyck pattern poset is unbounded. This is a consequence of the following.

Proposition 3.1.9. *For all $n \geq 2$, $\mu((UD)^{n-1}, (UD)^{n+1}) = \binom{n}{2}$.*

Proof. Since the interval $I_n = [(UD)^{n-1}, (UD)^{n+1}]$ has rank 2, its Möbius function is simply given by the number of its elements covering the minimum (or covered by the maximum) minus 1. Any path greater than $(UD)^{n-1}$ has at least $n-1$ peaks. Any path smaller than $(UD)^{n+1}$ has at most n peaks; moreover, if it has exactly n peaks, it is necessarily the path $(UD)^n$. In order to count the paths inside I_n having exactly $n-1$ peaks, we observe that they can be obtained from $(UD)^{n-1}$ by just adding a new up step to one of the peaks and a new down step to one of the peaks as well (possibly the same one), in such a way that the resulting path is still Dyck. It is not difficult to realize that this is equivalent to choosing a multiset having 2 elements out of a set having $n-1$ elements, which can be done in $\binom{n}{2}$ ways. □

We can also determine the maximum value of the Möbius function on intervals of rank 2.

Proposition 3.1.10. *For all $n \geq 1$, $\mu(U(UD)^{n-1}D, U(UD)^{n+1}D) = n^2$, and this is the maximum value attained by μ on intervals of rank 2.*

Proof. We start by observing that, for a given path Q of semilength n, the maximum number of paths covering it is n^2+1. In fact, if Q has k factors having semilengths $f_1, f_2, \dots f_k$, respectively, then, using Proposition 2.2 of [BFPW], we get that the number of paths covering Q is

$$1 + \sum_i f_i^2 + \sum_{i<j} f_i f_j = 1 + \left(\sum_i f_i \right)^2 - \sum_{i<j} f_i f_j = 1 + n^2 - \sum_{i<j} f_i f_j.$$

The maximum of the above quantity is indeed n^2+1 and is attained when $\sum_{i<j} f_i f_j$ (which corresponds to having only one factor). To finish the proof, it will be enough to show that, for the interval in the statement of the proposition, all the paths covering the lower path are also covered by the upper path. This can be done quite easily, by means of a case-by-case analysis (just insert in the lower path a U and a D in all possible places and show that the resulting path is still covered by the upper path). □

Notice that the above proof does not show that the interval in the statement of the proposition is the unique interval of rank 2 attaining the maximum of the Möbius function.

Some computations also suggest the following conjecture:

Conjecture 3.1.1. *The maximum absolute value of the Möbius function on intervals of rank 3 is $(2n+1) \cdot n^2$, attained by the interval $[U(UD)^{n-1}D, U(UD)^{n+2}D]$.*

Another intriguing conjecture, again supported by computational evidence, is the following:

Conjecture 3.1.2. *The Möbius function is alternating, meaning that it is ≥ 0 on intervals of even rank and ≤ 0 on intervals of odd rank.*

The above conjecture suggests that the intervals of the Dyck pattern poset are all shellable. Therefore is would be interesting to find a suitable labeling to confirm this fact.

Concerning the enumerative combinatorics of the intervals in the Dyck pattern poset, we still have to understand what happens in most of the cases. Counting elements and covering relations in the case of initial intervals in which the maxiumum has exactly three peaks could be a good starting point. Moreover, there are some asymptotic issues that seems to be rather interesting. Indeed, some computations suggests that the size of an interval of fixed rank whose minimum has semilength n is polynomial in n (when n tends to infinity). Finally, we remark that this kind of investigations, which has already been pursued for many combinatorially interesting posets (as we recalled in the Introduction), seems to still lack for the permutation pattern poset.

3.2 Intervals in the matching pattern poset

Another important topic that deserves to be investigated is the combinatorics of the intervals of the matching pattern poset. In this sense, typical questions concern the counting of elements or, more generally, the enumeration of (saturated) chains of a given interval. Another important line of research is the computation of the Möbius function. These are problems that have been classically studied for many combinatorial posets, such as Bruhat orders [T] and Tamari lattices [CCP, F]. In this section we just scratch the surface of this vast subject, by proposing a couple of relatively simple results concerning the enumeration of intervals of the form $[\frown, \tau]$, when τ has a specific form. In particular, in all the cases we will consider τ will be noncrossing.

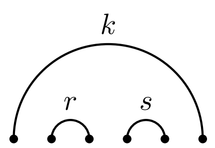
3.2.1 The lower closure of matchings with small edges

Given a matching τ, we say that an edge of τ is *small* whenever its vertices are consecutive integers. If $\tau(n,k)$ is a noncrossing matching of size n having k small edges, what is the cardinality of the interval $[\frown, \tau(n,k)]$? This may be a difficult problem in general. Here we address only a few very simple cases.

First of all, it is immediate to see that:

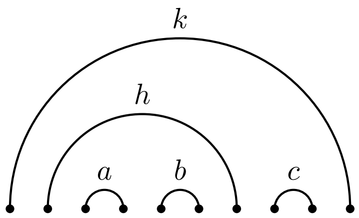
- $|[\frown, \tau(n,0)]| = 0$, for all $\tau(n,0)$ (since there are no noncrossing matchings having no small edges);
- $|[\frown, \tau(n,1)]| = n$, for all $\tau(n,1)$ (since, in this case, the interval is a chain having n elements, which are all totally nesting matchings).

When $k = 2$, the generic noncrossing matching having 2 small edges has the following form:



where an edge labeled x stands for a totally nesting matching having x edges. In words, the above matching is the juxtaposition of two totally nesting matchings having r and s edges, respectively, enclosed in a totally nesting matching having k edges. In order to have easy inline notations, such a matching will be denoted $\mathbf{k}(\mathbf{r};\mathbf{s})$. Assuming w.l.o.g. that $r \geq s$, it is easy to see that $[\frown, \mathbf{k}(\mathbf{r};\mathbf{s})]$ contains $r+k$ matchings having 1 small edge and $rs(k+1)$ matchings having 2 small edges. Therefore $|[\frown, \mathbf{k}(\mathbf{r};\mathbf{s})]| = r+k+rs(k+1)$.

When $k = 3$, again w.l.o.g., the generic matching $\tau(n, 3)$ has the form



Similarly as before, we denote the above matching with $\mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})$. We can count the elements of $[\frown, \mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})]$ with respect to the number of small edges.

- In order to count the number χ_1 of matchings having one small edge, we have to understand how many edges the largest totally nesting matching smaller than $\mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})$ has. To construct such a matching, we take the k external edges, and add the largest number between c and $h+\max(a,b)$. Therefore $\chi_1 = \max(k+h+a, k+h+b, k+c)$.

- Matching having two edges can be obtained in two different ways from $\mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})$. First, we can remove the totally nesting matchings having c, thus obtaining the matching $(\mathbf{h}+\mathbf{k})(\mathbf{a};\mathbf{b})$, which has $ab(h+k+1)$ matchings with 2 small edges below. The second option is to remove one of the two totally nesting matchings with a and b edges, and precisely the smaller one, thus obtaining the matching $\mathbf{k}((\max(\mathbf{a},\mathbf{b})+\mathbf{h});\mathbf{c})$, which has $(\max(a,b)+h)c(k+1)$ matchings with 2 small edges below. However, there are matchings in common in the two above cases, which causes an overcount. Indeed, the matchings which can be obtained in both the above cases are precisely those lying below $\mathbf{k}(\mathbf{a};\min(\mathbf{b},\mathbf{c}))$ and having 2 small edges, which are $a \cdot \min(b,c) \cdot (k+1)$. From the above consideration, we can write the total number χ_2 of elements of the interval $[\frown, \mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})]$ having 2 small edges, which is $\chi_2 = ab(h+k+1) + (\max(a,b)+h)c(k+1) - a \cdot \min(b,c) \cdot (k+1)$.

- Finally, the total number χ_3 of matchings in $[\frown, \mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})]$ having 3 small edges is immediate to compute, and we get $\chi_3 = abc(h+1)(k+1)$.

Summing up the above contribution, we then find the desired closed expression for $|[\frown, \mathbf{k}(\mathbf{h}(\mathbf{a};\mathbf{b});\mathbf{c})]|$.

3.2.2 The lower closure of another class of matchings

Our last example concerns a class of noncrossing matchings defined in a recursive fashion. Before introducing them, we state an easy, but useful, lemma whose proof is left to the reader.

Lemma 3.2.1. *Let σ and τ be any matchings. Then the following are equivalent:*

(i) $\sigma \leq \tau$;

(ii) $\overset{\frown}{\sigma} \leq \overset{\frown}{\tau}\ \frown$

(iii) $\sigma \frown \leq \overset{\frown}{\tau \frown}$

Set $\tau_0 = \emptyset$. For every $n > 0$, define:

- $\tau_{2n-1} = \tau_{2n-2} \frown$, and
- $\tau_{2n} = \overset{\frown}{\tau_{2n-1}}$.

Denote with f_n the cardinality of the interval $[\frown, \tau_n]$ and with $f_{n,k}$ the number of elements having k edges of the same interval, for $k > 0$. In particular, it is clear that $f_{n,k} = 0$ for $n < k$ and whenever $n \leq 0$ or $k \leq 0$ (actually, when $n = k = 0$, we set $f_{n,k} = 0$ by convention). In the next proposition we give closed formulas for such quantities.

Proposition 3.2.1. *Let $n > 0$ and $0 < k \leq n$, and denote with φ_n the n-th Fibonacci number. Then:*

(i) $f_{n,k} = \sum_{i=0}^{n-1} \binom{k-1}{n-k-1}$;

(ii) $f_n = \varphi_{n+2} - 1$.

Proof. We use the following notations: $A_{n,k}$ is the set of all matchings in $[\frown, \tau_{2n}]$ having k edges, $B_{n,k}$ is the set of all matchings in $[\frown, \tau_{2n-1}]$ having k edges, and $C_{n,k}$ is the set of all matchings of the form $\overset{\frown}{\sigma}$, with $\sigma \in B_{n,k-1}$. We then have that $f_{2n,k} = |A_{n,k}| = |B_{n,k}| + |C_{n,k}| - |B_{n,k} \cap C_{n,k}|$. By definition, we have $|B_{n,k}| = f_{2n-1,k}$ and clearly $|C_{n,k}| = f_{2n-1,k-1}$. Furthermore, as a consequence of Lemma 3.2.1 and of the specific shape of the matchings under consideration, the set $B_{n,k} \cap C_{n,k}$ is precisely the set of matchings of the form $\overset{\frown}{\sigma}$, with $\sigma \in B_{n-1,k-1}$, hence $|B_{n,k} \cap C_{n,k}| = f_{2n-3,k-1}$. We thus get the recurrence relation $f_{2n,k} = f_{2n-1,k} + f_{2n-1,k-1} - f_{2n-3,k-1}$. Using a completely similar argument, we can also prove the analogous recurrence $f_{2n-1,k} = f_{2n-2,k} + f_{2n-2,k-1} - f_{2n-4,k-1}$. Summing up, we thus have the following recurrence relation, which holds for all $n, k \geq 2$:

$$f_{n,k} = f_{n-1,k} + f_{n-1,k-1} - f_{n-3,k-1}. \tag{3.10}$$

Together with the starting condition $f_{1,1} = 1$, formula (3.10) allows us to compute the generating function $F(x,y) = \sum_{n,k\geq 0} f_{n,k}x^n y^k$. Indeed, using standard arguments, our recurrence translates into the functional equation

$$F(x,y) = xy + xF(x,y) + xyF(x,y) - x^3F(x,y),$$

which gives

$$F(x,y) = \frac{xy}{1 - x - xy + x^3y}.$$

It turns out that $F(x,y) = xyG(x,y)$, where $G(x,y)$ is the generating function given in [S] for the number triangle A004070: from there, we deduce the desired closed form given in (i) for $f_{n,k}$. Moreover, denoting with $\Phi(x) = \sum_{n\geq 0} \varphi_n x^n$ the generating function of Fibonacci numbers, it is easy to see that

$$\Phi(x) - \frac{x}{1-x} = \frac{x}{1-x-x^2} - \frac{x}{1-x} = x^2 f(x,1),$$

which proves (ii). □

3.2.3 Conclusion and further work

In Section 3.2 we have started the investigation of the combinatorial structure of intervals in the matching pattern poset, with special emphasis on enumerative issues. However, all important general questions concerning this topic are completely unanswered yet. How many elements does a generic interval contain? How many (saturated) chain of fixed length? What is the Möbius function? In which cases an interval has a (possibly distributive) lattice structure? Notice that the subposet of noncrossing matchings is isomorphic to the pattern order on 231-avoiding permutations (this is rather easy to show, see also [AB]). This can be useful, for instance, in the computation of the Möbius function, since the results developed in [BJJS] can be applied. However, it is possible (and maybe likely) that the specific combinatorial structure of matchings may help in finding neater formulas.

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