

SHAPE PROGRAMMING FOR NARROW RIBBONS OF NEMATIC ELASTOMERS

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ABSTRACT. Using the theory of Γ -convergence, we derive from three-dimensional elasticity new one-dimensional models for non-Euclidean elastic ribbons, i.e. ribbons exhibiting spontaneous curvature and twist. We apply the models to shape-selection problems for thin films of nematic elastomers with twist and splay-bend texture of the nematic director. For the former, we discuss the possibility of helicoid-like shapes as an alternative to spiral ribbons.

1. INTRODUCTION

Shape morphing systems are common in Biology. They are used to control locomotion in unicellular organisms [8, 7] and to produce controlled motions in plants [12, 16, 19, 31, 5]. Differential swelling and shrinkage processes, partially hindered by fibers, lead to dynamical conformation changes which are essential in the life of many botanical systems [28]. Inspired by Nature, many attempts have been reported in the recent literature to synthesize artificial shape-morphing systems based on synthetic soft materials [29, 24, 20] and the interest in these phenomena is steadily growing.

A useful tool has emerged in the mathematical literature to describe the mechanics of shape programming, namely, *non-Euclidean* structures (non-Euclidean plates and rods). These are elastic structures described by functionals which are minimised when exhibiting nonzero curvature. The relevant energy functionals are often postulated on the basis of physical intuition [22], but in more recent attempts they are derived from three dimensional models [27, 23, 3], through rigorous dimension reduction based on the theory of Γ -convergence, following the approach pioneered in [18].

Liquid crystal elastomers provide an interesting model system for the study of shape programming. They are polymeric materials that respond to external stimuli (temperature, light, electric fields) by changing shape [33, 13, 4, 2, 1] and are typically manufactured as thin films [9, 14, 6, 11, 10]. Suitable textures of the nematic director imprinted at fabrication lead to thin structures with tunable and controllable spontaneous curvature, see [25, 26, 32]. In particular, for the *twist* geometry (nematic director always parallel to the mid-plane of the film and rotating by $\pi/2$ from the bottom to the top surface of the film) it has been observed both experimentally and computationally [26, 30] that, depending on the aspect ratio of the mid-plane, either spiral ribbons (this is the case of large width over length aspect ratio) or helicoid-like shapes (this is the case of small width over length aspect ratio) emerge spontaneously.

In this paper, we provide a rigorous mathematical description of thin structures of nematic elastomers (in the case of twist and splay-bend geometry) where the minimisers of the deduced energy functionals reproduce the experimentally observed minimum energy configurations. Our analysis stems from the combination of two main results: first, we use the 3D-to-2D dimension reduction result in [3] for (non-Euclidean) thin films of nematic elastomers, and, secondly, we use a non-Euclidean

version of the 2D-to-1D dimension reduction result of [17], where a corrected version of the well-known Sadowsky functional is derived for the mechanical description of inextensible elastic ribbons. The reader is also referred to [21, 15] and all other papers in the same special issue of the Journal of Elasticity for more material on the mechanics of elastic ribbons.

Our results show that the technique of rigorous dimensional reduction based on Γ -convergence, far from being just a mathematical exercise, can provide a tool to *derive*, rather than *postulate*, the functional form and the material parameters (elastic constants, spontaneous curvature and twist, etc...) for dimensionally reduced models of thin structures. We concentrate our discussion on liquid crystal elastomers, but clearly our method is applicable to more general systems, whenever differential spontaneous distortions in the cross section induce spontaneous curvature and twist of the mid-line of the rod.

The starting point of the subsequent analysis is a family of non-Euclidean plate models defined on a narrow strip of width ε cut out from the plane and forming an angle θ with the horizontal axis (see (2.4) below). In Section 2 we set-up our 2D model and show that minor modifications of the results of [17] allow us to derive in the limit as $\varepsilon \downarrow 0$ the 1D model defined in (2.11)–(2.12). Again in Section 2, we observe that examples of our starting 2D theory are given by twist and splay-bend nematic elastomer sheets, as obtained from 3D nonlinear elasticity in [3]. The limiting 1D theory, which is a non-Euclidean rod theory, is then explicitly computed for twist and splay-bend nematic elastomers in Section 3 and 4, respectively. In particular, the explicit expression of the limiting energy densities associated with the rods through their flexural strains around the width axis and the torsional strains, and the corresponding minimisers, are given in Proposition 3.2 and Lemma 3.3 (in the twist case) and in Proposition 4.1 and Lemma 4.2 (in the splay-bend case).

2. A NON-EUCLIDEAN SADOWSKY FUNCTIONAL

Let ω be an open planar domain of \mathbb{R}^2 . In the framework of a nonlinear plate theory [18], we consider the bending energy

$$\int_{\omega} \left\{ c_1 |A_{\hat{v}}(\hat{z}) - \bar{A}|^2 + c_2 \operatorname{tr}^2(A_{\hat{v}}(\hat{z}) - \bar{A}) + \bar{e} \right\} d\hat{z} \quad (2.1)$$

associated with a developable surface $\hat{v}(\omega)$, \hat{v} being a deformation from ω to \mathbb{R}^3 . In the previous expression, $A_{\hat{v}}(\cdot) \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ denotes the second fundamental form of $\hat{v}(\omega)$, $c_1 > 0$ and $c_2 > 0$ are material constants, and $\bar{A} \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ and \bar{e} represent a characteristic target curvature tensor and a characteristic nonnegative energy constant, respectively. Moreover, the notation $\operatorname{tr}^2 A$ stands for the square of the trace of A . We recall that $A_{\hat{v}}$ can be expressed as $(\nabla \hat{v})^T \nabla \hat{v}$, where $\hat{v} = \partial_{z_1} \hat{v} \wedge \partial_{z_2} \hat{v}$. In [3], the two-dimensional energy (2.1) has been rigorously derived from a three dimensional model for thin films of nematic elastomers with *splay-bend* and *twist* orientation of the nematic directors along the thickness and the following explicit formulas have been obtained for \bar{A}

$$\bar{A}_S = k \operatorname{diag}(-1, 0), \quad \bar{A}_T = k \operatorname{diag}(-1, 1), \quad k := \frac{6\eta_0}{\pi^2 h_0}, \quad (2.2)$$

and for \bar{e}

$$\bar{e}_S = \mu(1 + \lambda) \left(\frac{\pi^4 - 12}{32} \right) \frac{\eta_0^2}{h_0^2}, \quad \bar{e}_T = \mu \left(\frac{\pi^4 - 4\pi^2 - 48}{8\pi^4} \right) \frac{\eta_0^2}{h_0^2}. \quad (2.3)$$

Here and throughout the paper we use the indices “ S ” and “ T ” for the quantities related to the splay-bend and the twist case, respectively. In the previous formulas,

η_0 is a positive dimensionless parameter quantifying the magnitude of the spontaneous strain variation along the (small) thickness h_0 of the film, μ is the elastic shear modulus, and $\lambda + 2\mu/3$ is the bulk modulus. Also, the material constants appearing in (2.1) are given by $c_1 = \mu/12$ and $c_2 = \lambda\mu/12$. We refer the reader to [3] for a detailed description of the three-dimensional model and of the splay-bend and twist nematic director fields. In Sections 3 and 4 we specialize our results to the case where the curvature tensor \bar{A} is of the form (2.2), while in the rest of this section we focus on a general energy density of type (2.1).

We cut out of the planar region ω a narrow strip

$$S_\varepsilon^\theta := \left\{ z_1 \mathbf{e}_1^\theta + z_2 \mathbf{e}_2^\theta : z_1 \in (-\ell/2, \ell/2), z_2 \in (-\varepsilon/2, \varepsilon/2) \right\} \subset \omega, \quad 0 \leq \theta < \pi,$$

with

$$\mathbf{e}_i^\theta = R_\theta \mathbf{e}_i, \quad i = 1, 2, \quad R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and consider the energy (2.1) restricted to the strip S_ε^θ , namely

$$\hat{\mathcal{E}}_\varepsilon^\theta(\hat{v}) := \int_{S_\varepsilon^\theta} \left\{ c_1 |A_{\hat{v}}(\hat{z}) - \bar{A}|^2 + c_2 \operatorname{tr}^2(A_{\hat{v}}(\hat{z}) - \bar{A}) + \bar{e} \right\} d\hat{z}, \quad (2.4)$$

where $\hat{v} : S_\varepsilon^\theta \rightarrow \mathbb{R}^3$ is a deformation such that $\hat{v}(S_\varepsilon^\theta)$ is a developable surface. We are interested in examining the behaviour of the minimisers of the functionals $\hat{\mathcal{E}}_\varepsilon^\theta$ in the limit of vanishing width, i.e. $\varepsilon \downarrow 0$. Notice that using the function $v : S_\varepsilon \rightarrow \mathbb{R}^3$ defined in the unrotated strip $S_\varepsilon := S_\varepsilon^0$ as $v(z) = \hat{v}(R_\theta z)$, we have that $v(S_\varepsilon)$ is developable and $\hat{\mathcal{E}}_\varepsilon^\theta(\hat{v})$ can be rewritten as

$$\begin{aligned} \hat{\mathcal{E}}_\varepsilon^\theta(\hat{v}) &= \int_{S_\varepsilon} \left\{ c_1 |A_{\hat{v}}(R_\theta z) - \bar{A}|^2 + c_2 \operatorname{tr}^2(A_{\hat{v}}(z) - \bar{A}) + \bar{e} \right\} dz \\ &= \int_{S_\varepsilon} \left\{ c_1 |A_v(z) - R_\theta^T \bar{A} R_\theta|^2 + c_2 \operatorname{tr}^2(A_v(z) - \bar{A}) + \bar{e} \right\} dz, \end{aligned}$$

where in the second equality we have used the fact that $A_{\hat{v}}(R_\theta z) = R_\theta A_v(z) R_\theta^T$. Now, introducing a suitable rescaling and setting

$$\mathcal{E}_\varepsilon^\theta(v) := \frac{1}{\varepsilon} \int_{S_\varepsilon} \left\{ c_1 |A_v(z) - \bar{A}^\theta|^2 + c_2 \operatorname{tr}^2(A_v(z) - \bar{A}) + \bar{e} \right\} dz, \quad (2.5)$$

with

$$\bar{A}^\theta := R_\theta^T \bar{A} R_\theta, \quad (2.6)$$

we have that $\hat{\mathcal{E}}_\varepsilon^\theta(\hat{v}) = \varepsilon \mathcal{E}_\varepsilon^\theta(v)$. Having this identification in mind, from now on we always deal with the functional $v \mapsto \mathcal{E}_\varepsilon^\theta(v)$. Expanding the integrand we obtain the following general form of the bending energy

$$\mathcal{E}_\varepsilon^\theta(v) = \frac{1}{\varepsilon} \int_{S_\varepsilon} \left\{ c |A_v(z)|^2 + L^\theta(A_v(z)) \right\} dz, \quad (2.7)$$

where

$$L^\theta(A_v) := -2c_1 A_v \cdot \bar{A}^\theta - 2c_2 \operatorname{tr} A_v \operatorname{tr} \bar{A}^\theta + c_1 |\bar{A}^\theta|^2 + c_2 \operatorname{tr}^2 \bar{A}^\theta + \bar{e}, \quad (2.8)$$

and the notation is meant to remind the reader that, as a function of the matrix A , L^θ is linear. In the above expression, we have set $c = c_1 + c_2$ and we have made use of the fact that

$$\operatorname{tr}^2 A = |A|^2 + 2 \det A,$$

for all $A \in \mathbb{R}_{\operatorname{sym}}^{2 \times 2}$. In fact, since v is an isometry of the planar strip S_ε , the Gaussian curvature associated with $v(S_\varepsilon)$ vanishes, i.e.

$$\det A_v(z) = 0.$$

The natural function space for v is the space of $W^{2,2}$ isometries of S_ε defined as

$$W_{\text{iso}}^{2,2}(S_\varepsilon, \mathbb{R}^3) := \left\{ v \in W^{2,2}(S_\varepsilon, \mathbb{R}^3) : \partial_i v \cdot \partial_j v = \delta_{ij} \right\}.$$

In order to express the energy over the fixed domain

$$S = I \times \left(-\frac{1}{2}, \frac{1}{2} \right), \quad I := \left(-\frac{\ell}{2}, \frac{\ell}{2} \right),$$

we change variables and define the rescaled version $y : S \rightarrow \mathbb{R}^3$ of v , given by

$$y(x_1, x_2) = v(x_1, \varepsilon x_2).$$

The following procedure is rather standard and we use the notation of [17] as, in the sequel, our proofs will be largely based on this paper. By introducing the scaled gradient

$$\nabla_\varepsilon \cdot = (\partial_1 \cdot | \varepsilon^{-1} \partial_2 \cdot)$$

we obtain that $\nabla_\varepsilon y(x_1, x_2) = \nabla v(x_1, \varepsilon x_2)$ and y belongs to the space of scaled isometries of S defined as

$$W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3) := \left\{ y \in W^{2,2}(S, \mathbb{R}^3) : |\partial_1 y| = |\varepsilon^{-1} \partial_2 y| = 1, \partial_1 y \cdot \partial_2 y = 0 \text{ a.e. in } S \right\}.$$

Similarly, we may define the scaled unit normal to $y(S)$ by

$$n_{y,\varepsilon} = \partial_1 y \wedge \varepsilon^{-1} \partial_2 y$$

and the scaled second fundamental form associated to $y(S)$ by

$$A_{y,\varepsilon} = \begin{pmatrix} n_{y,\varepsilon} \cdot \partial_1 \partial_1 y & \varepsilon^{-1} n_{y,\varepsilon} \cdot \partial_1 \partial_2 y \\ \varepsilon^{-1} n_{y,\varepsilon} \cdot \partial_1 \partial_2 y & \varepsilon^{-2} n_{y,\varepsilon} \cdot \partial_2 \partial_2 y \end{pmatrix}.$$

With this definition, $A_{y,\varepsilon}(x_1, x_2) = A_v(x_1, \varepsilon x_2)$ and $\mathcal{E}_\varepsilon^\theta(v) = \mathcal{J}_\varepsilon^\theta(y)$, where the functional

$$\mathcal{J}_\varepsilon^\theta(y) := \int_S \left\{ c |A_{y,\varepsilon}|^2 + L^\theta(A_{y,\varepsilon}) \right\} dx \quad (2.9)$$

is defined over the space $W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$ of scaled isometries of S .

Lemma 2.1 (Compactness). *Suppose $(y_\varepsilon) \subset W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$ satisfy*

$$\sup_\varepsilon \mathcal{J}_\varepsilon^\theta(y_\varepsilon) < \infty.$$

Then, up to a subsequence and additive constants, there exist a deformation $y \in W^{2,2}(I, \mathbb{R}^3)$ and a orthonormal frame $(d_1 | d_2 | d_3) \in W^{1,2}(I, \text{SO}(3))$ fulfilling

$$d_1 = y' \quad \text{and} \quad d'_1 \cdot d_2 = 0 \quad \text{a.e. in } I,$$

and such that

$$y_\varepsilon \rightharpoonup y \quad \text{in} \quad W^{2,2}(S, \mathbb{R}^3), \quad \nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1 | d_2) \quad \text{in} \quad W^{1,2}(S, \mathbb{R}^{3 \times 2}).$$

Moreover, for some $\gamma \in L^2(S, \mathbb{R}^3)$,

$$A_{y,\varepsilon} \rightharpoonup \begin{pmatrix} d'_1 \cdot d_3 & d'_2 \cdot d_3 \\ d'_2 \cdot d_3 & \gamma \end{pmatrix} \quad \text{in} \quad L^2(S, \mathbb{R}_{\text{sym}}^{2 \times 2}).$$

Proof. Note that

$$\begin{aligned} \mathcal{J}_\varepsilon^\theta(y_\varepsilon) &= \int_S \left\{ c_1 |A_{y_\varepsilon, \varepsilon} - \bar{A}^\theta|^2 + c_2 \text{tr}^2(A_{y_\varepsilon, \varepsilon}(x) - \bar{A}^\theta) + \bar{e} \right\} dx \\ &\geq \int_S c_1 |A_{y_\varepsilon, \varepsilon}(x) - \bar{A}^\theta|^2 dx. \end{aligned}$$

But, since \bar{A}^θ is constant, this implies that $\|A_{y_\varepsilon, \varepsilon}\|_{L^2(S, \mathbb{R}^3)}^2$ is bounded uniformly in ε and the proof is identical to the proof of Lemma 2.1 in [17]. \square

In order to state the Γ -convergence result, we define

$$\mathcal{A} := \left\{ (d_1, d_2, d_3) : (d_1 | d_2 | d_3) \in W^{1,2}(I, \text{SO}(3)), d'_1 \cdot d_2 = 0 \text{ a.e. in } I \right\}, \quad (2.10)$$

and the functional $\mathcal{J}^\theta : \mathcal{A} \rightarrow \mathbb{R}$ by

$$\mathcal{J}^\theta(d_1, d_2, d_3) := \int_I \bar{Q}^\theta(d'_1 \cdot d_3, d'_2 \cdot d_3) dx_1, \quad (2.11)$$

where

$$\bar{Q}^\theta(\alpha, \beta) := \min_{\gamma \in \mathbb{R}} \left\{ c|M|^2 + 2c|\det M| + L^\theta(M) : M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \right\}, \quad (2.12)$$

and $L^\theta(M)$ is defined according to (2.8). We recall that the constraint $d'_1 \cdot d_2 = 0$ means that the narrow strip does not bend within its plane or, equivalently, that there is no flexure around the direction of d_3 .

Theorem 2.2 (Γ -convergence). *The functionals $\mathcal{J}_\varepsilon^\theta$ Γ -converge to \mathcal{J}^θ as $\varepsilon \rightarrow 0$ in the following sense:*

- (1) (*Γ -lim inf inequality*) for every sequence $(y_\varepsilon) \subset W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$, $y \in W^{2,2}(I, \mathbb{R}^3)$ and $(d_1, d_2, d_3) \in \mathcal{A}$ with $y' = d_1$ a.e. in I , $y_\varepsilon \rightharpoonup y$ in $W^{2,2}(S, \mathbb{R}^3)$ and $\nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1 | d_2)$ in $W^{1,2}(S, \mathbb{R}^{3 \times 2})$, we have

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon^\theta(y_\varepsilon) \geq \mathcal{J}^\theta(d_1, d_2, d_3);$$

- (2) (*recovery sequence*) for every $(d_1, d_2, d_3) \in \mathcal{A}$ there exists $(y_\varepsilon) \subset W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$ and (up to an additive constant) y satisfying $y' = d_1$ such that $y_\varepsilon \rightharpoonup y$ in $W^{2,2}(S, \mathbb{R}^3)$, $\nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1 | d_2)$ in $W^{1,2}(S, \mathbb{R}^{3 \times 2})$, and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon^\theta(y_\varepsilon) = \mathcal{J}^\theta(d_1, d_2, d_3).$$

In proving the existence of a recovery sequence, we will need the following lemma which is a slight variation of [17, Lemma 3.1].

Lemma 2.3. *For every $M \in L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ there exists a sequence $(M_n) \subset L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ satisfying $\det M_n = 0$ a.e. in I and for all $n \in \mathbb{N}$ such that $M_n \rightharpoonup M$ in $L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ and*

$$\int_I [c|M_n|^2 + L^\theta(M_n)] dx_1 \longrightarrow \int_I [c|M|^2 + 2c|\det M| + L^\theta(M)] dx_1.$$

Proof. Without loss of generality, we may assume that $\det M \neq 0$. We first consider the case M constant and diagonal, i.e.

$$M = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2.$$

Set

$$\mu = \frac{|\lambda_1|}{|\lambda_1| + |\lambda_2|} \in (0, 1)$$

to find that

$$|M|^2 + 2|\det M| = \lambda_1^2 + \lambda_2^2 + 2|\lambda_1 \lambda_2| = \frac{\lambda_1^2}{\mu} + \frac{\lambda_2^2}{1-\mu}.$$

Denoting by $\chi : \mathbb{R} \rightarrow \{0, 1\}$ the 1-periodic extension of the characteristic function of the interval $(0, \mu)$ and defining $M_n : I \rightarrow \mathbb{R}_{\text{sym}}^{2 \times 2}$ by

$$M_n(x_1) = \chi(nx_1) \frac{\lambda_1}{\mu} e_1 \otimes e_1 + (1 - \chi(nx_1)) \frac{\lambda_2}{1-\mu} e_2 \otimes e_2$$

we infer that $\det M_n = 0$ and $M_n \xrightarrow{*} M$ in $L^\infty(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$, since $\chi(n \cdot) \xrightarrow{*} \theta$ in $L^\infty(I)$. Also,

$$\begin{aligned} \int_I c|M_n|^2 + L^\theta(M_n) \, dx_1 &= \int_I c \frac{\lambda_1^2}{\mu} + c \frac{\lambda_2^2}{1-\mu} + L^\theta(M_n) \, dx_1 \\ &= \int_I c|M|^2 + 2c|\det M| + L^\theta(M_n) \, dx_1 \\ &\rightarrow \int_I c|M|^2 + 2c|\det M| + L^\theta(M) \, dx_1 \end{aligned}$$

as $n \rightarrow \infty$, since L^θ is affine. If $M \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ is not diagonal, yet constant, we can find an orthogonal matrix R such that $R^T M R$ is diagonal and apply the above argument.

If M is instead piecewise constant, we apply the same argument to each interval on which M is constant and, for general $M \in L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ we approximate M in the strong topology of $L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ by a sequence (M_k) of piecewise constant maps. We may then apply the above argument to each M_k to obtain a sequence $(M_{k,n})$ with the required properties and such that

$$\|M_{k,n}\|_{L^2}^2 = \int_I |M_{k,n}|^2 \, dx_1 = \int_I |M_k|^2 + 2|\det M_k| \, dx_1 \leq 2\|M_k\|_{L^2}^2,$$

i.e. the sequence $(M_{k,n})$ is bounded in L^2 . Noting that the weak topology in L^2 is metrisable on bounded sets and taking a diagonal sequence, we conclude the proof. \square

Proof of Theorem 2.2.

(1) (Γ -lim inf inequality) Let $(y_\varepsilon) \subset W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$, $y \in W^{2,2}(I, \mathbb{R}^3)$ and $(d_1, d_2, d_3) \in \mathcal{A}$ with $y' = d_1$ a.e. in I , $y_\varepsilon \rightharpoonup y$ in $W^{2,2}(S, \mathbb{R}^3)$ and $\nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1 | d_2)$ in $W^{1,2}(S, \mathbb{R}^3)$. We may assume that $\liminf_\varepsilon \mathcal{J}_\varepsilon^\theta(y_\varepsilon) < \infty$ as, otherwise the result follows trivially, and by passing to a subsequence that $\sup_\varepsilon \mathcal{J}_\varepsilon^\theta(y_\varepsilon) < \infty$.

Lemma 2.1 now states that $A_{y_\varepsilon, \varepsilon} \rightharpoonup A$ in $L^2(S, \mathbb{R}_{\text{sym}}^{2 \times 2})$ where

$$A = \begin{pmatrix} d'_1 \cdot d_3 & d'_2 \cdot d_3 \\ d_2 \cdot d_3 & \gamma \end{pmatrix}.$$

Set $A^\varepsilon := A_{y_\varepsilon, \varepsilon}$ and recall that for every 2×2 , symmetric matrix M , it holds that $\text{tr}^2 M = |M|^2 + 2 \det M$. Since $\det A^\varepsilon = 0$ we infer that $|A^\varepsilon|^2 = \text{tr}^2 A^\varepsilon$. Also note that

$$\text{tr}^2 A^\varepsilon = (A_{11}^\varepsilon - A_{22}^\varepsilon)^2 + 4A_{11}^\varepsilon A_{22}^\varepsilon = (A_{11}^\varepsilon - A_{22}^\varepsilon)^2 + 4(A_{12}^\varepsilon)^2.$$

Splitting $S = S^+ \cup S^-$ where $S^+ := \{x \in S : \det A(x) \geq 0\}$ and $S^- := \{x \in S : \det A(x) < 0\}$, we find that

$$\begin{aligned} \liminf_\varepsilon \mathcal{J}_\varepsilon^\theta(y_\varepsilon) &= \liminf_\varepsilon \left\{ \int_{S^+} c \text{tr}^2 A^\varepsilon \, dx + \int_{S^-} c (A_{11}^\varepsilon - A_{22}^\varepsilon)^2 + 4c (A_{12}^\varepsilon)^2 \, dx \right\} \\ &\quad + \lim_\varepsilon \int_S L^\theta(A^\varepsilon) \, dx \\ &\geq \int_{S^+} c \text{tr}^2 A \, dx + \int_{S^-} c (A_{11} - A_{22})^2 + 4c (A_{12})^2 \, dx + \int_S L^\theta(A) \, dx \end{aligned}$$

since, with respect to the weak L^2 topology, the first two functionals are lower semicontinuous by convexity and the third functional is continuous by linearity. Next, note that

$$(A_{11} - A_{22})^2 + 4(A_{12})^2 = |A|^2 - 2 \det A$$

so that

$$\begin{aligned}
 \liminf_{\varepsilon} \mathcal{J}_{\varepsilon}^{\theta}(y_{\varepsilon}) &\geq c \int_{S^+} (|A|^2 + 2 \det A) \, dx + c \int_{S^-} (|A|^2 - 2 \det A) \, dx \\
 &\quad + \int_S L^{\theta}(A) \, dx \\
 &= \int_S [c|A|^2 + 2c|\det A| + L^{\theta}(A)] \, dx \\
 &\geq \mathcal{J}^{\theta}(d_1, d_2, d_3)
 \end{aligned}$$

(2) (recovery sequence) Fix $(d_1, d_2, d_3) \in \mathcal{A}$ and $y \in W^{2,2}(I, \mathbb{R}^3)$ such that $y' = d_1$ a.e. in I . Define $R := (y'|d_2|d_3) \in \text{SO}(3)$ a.e. in I and

$$M = \begin{pmatrix} y'' \cdot d_3 & d_2' \cdot d_3 \\ d_2' \cdot d_3 & \gamma \end{pmatrix}$$

where $\gamma \in L^2(I)$ is chosen such that

$$\bar{Q}^{\theta}(y'' \cdot d_3, d_2' \cdot d_3) = \bar{Q}^{\theta}(M_{11}, M_{12}) = c|M|^2 + 2c|\det M| + L^{\theta}(M).$$

Through Lemma 2.3 we find a sequence $(M_n) \subset L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ such that $\det M_n = 0$, $M_n \rightharpoonup M$ in $L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ and, as $n \rightarrow \infty$,

$$\int_I [c|M_n|^2 + L^{\theta}(M_n)] \, dx_1 \longrightarrow \int_I [c|M|^2 + 2c|\det M| + L^{\theta}(M)] \, dx_1.$$

We now proceed exactly as in [17]. Let $\lambda_n = \text{tr} M_n$. Since M_n is symmetric and $\det M_n = 0$, we may find $\beta_n(x_1) \in (-\pi/2, \pi/2]$ such that

$$M_n = \begin{pmatrix} \cos \beta_n & -\sin \beta_n \\ \sin \beta_n & \cos \beta_n \end{pmatrix} \begin{pmatrix} \lambda_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \beta_n & \sin \beta_n \\ -\sin \beta_n & \cos \beta_n \end{pmatrix},$$

which is well-defined by setting $\beta_n(x_1) = 0$ when $\lambda_n(x_1) = 0$. As in [17], we may assume that $\lambda_n \in L^{\infty}(I)$ and, by mollifying, we find $\lambda_{n,k} \in C^{\infty}(\bar{I})$ and $\beta_{n,k} \in C^{\infty}(\bar{I})$ with the property that

- $|\beta_{n,k}| < \pi/2$ (note the strict inequality) for all $x_1 \in \bar{I}$;
- $\lambda_{n,k} \rightarrow \lambda_n$ in $L^p(I)$ as $k \rightarrow \infty$ and for all $p < \infty$;
- $\beta_{n,k} \rightarrow \beta_n$ in $L^p(I)$ as $k \rightarrow \infty$ and for all $p < \infty$.

Next, set

$$M_{n,k} = \begin{pmatrix} \cos \beta_{n,k} & -\sin \beta_{n,k} \\ \sin \beta_{n,k} & \cos \beta_{n,k} \end{pmatrix} \begin{pmatrix} \lambda_{n,k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \beta_{n,k} & \sin \beta_{n,k} \\ -\sin \beta_{n,k} & \cos \beta_{n,k} \end{pmatrix},$$

and note that $\det M_{n,k} = 0$ for all k, n and $M_{n,k} \rightarrow M_n$ in $L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ as $k \rightarrow \infty$. Thus, by extracting a diagonal sequence, we find $\lambda^j, \beta^j \in C^{\infty}(\bar{I})$ with $|\beta^j| < \pi/2$ on \bar{I} and for

$$\begin{aligned}
 M^j &= \begin{pmatrix} \cos \beta^j & -\sin \beta^j \\ \sin \beta^j & \cos \beta^j \end{pmatrix} \begin{pmatrix} \lambda^j & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \beta^j & \sin \beta^j \\ -\sin \beta^j & \cos \beta^j \end{pmatrix} \\
 &= \lambda^j \begin{pmatrix} \cos^2 \beta^j & \sin \beta^j \cos \beta^j \\ \sin \beta^j \cos \beta^j & \sin^2 \beta^j \end{pmatrix}
 \end{aligned}$$

it holds that $\det M^j = 0$ for all j , and as $j \rightarrow \infty$, $M^j \rightharpoonup M$ in $L^2(I, \mathbb{R}_{\text{sym}}^{2 \times 2})$ as well as

$$\int_I [c|M^j|^2 + L^{\theta}(M^j)] \, dx_1 \longrightarrow \int_I [c|M|^2 + 2c|\det M| + L^{\theta}(M)] \, dx_1.$$

Extend β^j smoothly to \mathbb{R} maintaining the constraint $|\beta^j| < \pi/2$ and for $t^j := \pi/2 + \beta^j$, define

$$\tilde{b}^j(\xi_1) = \cos t^j(\xi_1)e_1 + \sin t^j(\xi_1)e_2 \text{ and } \Phi^j(\xi_1, \xi_2) = \xi_1 e_1 + \xi_2 \tilde{b}^j(\xi_1),$$

noting that there exists ε^j such that for all $\varepsilon \leq \varepsilon^j$ the map $(\Phi^j)^{-1} : S_\varepsilon \rightarrow \mathbb{R}^2$ is well defined (see [17]).

Define $R^j : I \rightarrow \text{SO}(3)$ as the solution to the ODE

$$(R^j)' = R^j \begin{pmatrix} 0 & 0 & -M_{11}^j \\ 0 & 0 & -M_{12}^j \\ M_{11}^j & M_{12}^j & 0 \end{pmatrix}$$

with initial data $R^j(0) = R(0) = (y'(0)|d_2(0)|d_3(0))$ and set

$$d_k^j(s) = R^j(s)e_k, \quad k = 1, 2, 3 \text{ and } y^j(s) = y(0) + \int_0^s d_1^j.$$

Then $y^j \rightarrow y$ in $W^{2,2}(I, \mathbb{R}^3)$ and by the ODE we infer that

$$\begin{aligned} (d_1^j)' \cdot d_2^j &= 0, \\ (d_2^j)' \cdot d_3^j &= M_{12}^j = -\lambda^j \sin t^j \cos t^j, \\ (d_1^j)' \cdot d_3^j &= M_{11}^j = \lambda^j \sin^2 t^j. \end{aligned}$$

Define

$$\begin{aligned} b^j(\xi_1) &= \cos t^j(\xi_1)d_1^j(\xi_1) + \sin t^j(\xi_1)d_2^j(\xi_1), \\ v^j(\xi_1, \xi_2) &= y^j(\xi_1) + \xi_2 b^j(\xi_1), \\ u^j(x_1, x_2) &= v^j \left((\Phi^j)^{-1}(x_1, x_2) \right). \end{aligned}$$

Following [17], one then obtains that $(\nabla u^j)(\nabla u^j) = \mathbb{I}$, $\nabla u^j(\cdot, 0) = (d_1^j|d_2^j)$ and that $A_{u^j}(\cdot, 0) = M^j(\cdot)$. For $\varepsilon > 0$ small enough, the maps $y_\varepsilon^j : S \rightarrow \mathbb{R}^3$ defined by $y_\varepsilon^j(x_1, x_2) = u^j(x_1, \varepsilon x_2)$ are well-defined scaled C^2 isometries of S such that

$$\nabla_\varepsilon y_\varepsilon^j \rightarrow \nabla u^j(\cdot, 0) = (d_1^j|d_2^j) \text{ strongly in } W^{1,2}(S, \mathbb{R}^{3 \times 2}) \text{ as } \varepsilon \rightarrow 0$$

and since $A_{u^j}(x_1, 0) = M^j(x_1)$, we also get that

$$A_{y_\varepsilon^j, \varepsilon} \rightarrow M^j \text{ strongly in } L^2(S, \mathbb{R}_{\text{sym}}^{2 \times 2}) \text{ as } \varepsilon \rightarrow 0.$$

Hence, we find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon^\theta(y_\varepsilon^j) &= \lim_{\varepsilon \rightarrow 0} \int_S [c|A_{y_\varepsilon^j, \varepsilon}|^2 + L^\theta(A_{y_\varepsilon^j, \varepsilon})] \, dx \\ &= \int_S [c|M^j|^2 + L^\theta(M^j)] \, dx \\ &\xrightarrow{j \rightarrow \infty} \int_S [c|M|^2 + 2c|\det M| + L^\theta(M)] \, dx \\ &= \int_S \bar{Q}^\theta(d'_1 \cdot d_3, d'_2 \cdot d_3) \, dx_1 = \mathcal{J}^\theta(d_1, d_2, d_3). \end{aligned}$$

The proof can then be finished by taking diagonal sequences. \square

3. THE TWIST CASE

In the case where the (physical) energy (2.4) is derived from a three-dimensional model with a twist-type nematic director field imprinted in the thickness of an elastomeric thin film, the characteristic quantities \bar{A} and \bar{e} present in (2.4) are given by \bar{A}_T and \bar{e}_T in formulas (2.2) and (2.3). Correspondingly, the θ -dependent target curvature tensor \bar{A}^θ defined in (2.6) becomes

$$\bar{A}_T^\theta = k \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} =: k \begin{pmatrix} -a_\theta & b_\theta \\ b_\theta & a_\theta \end{pmatrix}, \quad \theta \in [0, \pi), \quad (3.1)$$

and $\text{tr} \bar{A}_T^\theta = \text{tr} \bar{A}_T = 0$. Moreover, the functional $\mathcal{E}_\varepsilon^\theta$ defined in (2.7) in this case reads

$$\mathcal{E}_{\varepsilon,T}^\theta(v) := \frac{1}{\varepsilon} \int_{S_\varepsilon} \left\{ c |A_v(z)|^2 + L_T^\theta(A_v(z)) \right\} dz,$$

with

$$L_T^\theta(A_v) := -2c_1 A_v \cdot \bar{A}_T^\theta + 2c_1 k^2 + \bar{e}_T.$$

Also, the energy $\mathcal{J}_\varepsilon^\theta$ defined in the rescaled configuration S (see (2.9)) is

$$\mathcal{J}_{\varepsilon,T}^\theta(y) := \int_S \left\{ c |A_{y,\varepsilon}(x)|^2 + L_T^\theta(A_{y,\varepsilon}(x)) \right\} dx, \quad (3.2)$$

for every $y \in W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$. We recall that $\hat{\mathcal{E}}_\varepsilon^\theta(\hat{v}) = \varepsilon \mathcal{E}_\varepsilon^\theta(v) = \varepsilon \mathcal{J}_\varepsilon^\theta(y)$, where $\hat{v} : S_\varepsilon^\theta \rightarrow \mathbb{R}^3$, $v : S_\varepsilon \rightarrow \mathbb{R}^3$, $v : S \rightarrow \mathbb{R}^3$ are isometries and are related to each other via the following relations

$$v(z) = \hat{v}(R_\theta z), \quad y(x_1, x_2) = v(x_1, \varepsilon x_2).$$

Lemma 2.1 and Theorem 2.2 apply in particular for the functionals (3.2). As an easy consequence of the compactness and the Γ -convergence results, via standard arguments of the theory of Γ -convergence, the following corollary holds. In order to state it, we define $\mathcal{J}_T^\theta : \mathcal{A} \rightarrow \mathbb{R}$ as

$$\mathcal{J}_T^\theta(d_1, d_2, d_3) := \int_I \bar{Q}_T^\theta(d'_1 \cdot d_3, d'_2 \cdot d_3) dx_1, \quad (3.3)$$

where \mathcal{A} is the class of orthonormal frames defined in (2.10) and \bar{Q}_T^θ is defined as in (2.12) with L_T^θ in place of L^θ (see also above).

Corollary 3.1. *If $(y_\varepsilon) \subset W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$ is a sequence of minimisers of $\mathcal{J}_{\varepsilon,T}^\theta$, then, up to a subsequence, we have that there exist $y \in W^{2,2}(I, \mathbb{R}^3)$ and a minimiser $(d_1|d_2|d_3) \in \mathcal{A}$ of \mathcal{J}_T^θ with $d_1 = y'$ such that*

$$y_\varepsilon \rightharpoonup y \text{ in } W^{2,2}(S, \mathbb{R}^3), \quad \nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1|d_2) \text{ in } W^{1,2}(S, \mathbb{R}^{3 \times 2}),$$

and

$$A_{y,\varepsilon} \rightharpoonup \begin{pmatrix} d'_1 \cdot d_3 & d'_2 \cdot d_3 \\ d'_2 \cdot d_3 & \gamma \end{pmatrix} \text{ in } L^2(S, \mathbb{R}_{\text{sym}}^{2 \times 2}), \quad \text{for some } \gamma \in L^2(S, \mathbb{R}^3). \quad (3.4)$$

Moreover,

$$\min_{W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)} \mathcal{J}_{\varepsilon,T}^\theta \longrightarrow \min_{\mathcal{A}} \mathcal{J}_T^\theta. \quad (3.5)$$

Notice that thanks to [3, Lemma 3.8] and Lemma 3.3 below, the minimum of $\mathcal{J}_{\varepsilon,T}^\theta$ in $W_{\text{iso},\varepsilon}^{2,2}(S, \mathbb{R}^3)$ and the minimum of \mathcal{J}_T^θ in \mathcal{A} can be computed explicitly, so that the convergence in (3.5) can be checked by hand. The minimizing sequences and the minimizer of \mathcal{J}_T^θ can be computed as well, together with γ in (3.4) (see the proof of Theorem 2.2 (2)). Therefore, also the convergence of the minimizing sequences can be checked by hand.

The following proposition gives the explicit expression of \bar{Q}_T^θ .

Proposition 3.2. *\bar{Q}_T^θ is a continuous function given by*

$$\bar{Q}_T^\theta(\alpha, \beta) = \begin{cases} 4c_1 k (a_\theta \alpha - b_\theta \beta) + c_1 k^2 \left(2 - \frac{c_1}{c} a_\theta^2 \right) + \bar{e}_T, & \text{in } \mathcal{D}_T \\ 4(c\beta^2 - c_1 k b_\theta \beta) + c_1 k^2 \left(2 - \frac{c_1}{c} a_\theta^2 \right) + \bar{e}_T, & \text{in } \mathcal{U}_T \\ c \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} + 2c_1 k \left(a_\theta \frac{\alpha^2 - \beta^2}{\alpha} - 2b_\theta \beta \right) + 2c_1 k^2 + \bar{e}_T, & \text{in } \mathcal{V}_T, \end{cases}$$

where a_θ and b_θ are defined in (3.1), and

$$\begin{aligned}\mathcal{D}_T &:= \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \frac{c_1}{c} k a_\theta \alpha > \beta^2 + \alpha^2 \right\}, \\ \mathcal{U}_T &:= \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \frac{c_1}{c} k a_\theta \alpha \leq \beta^2 - \alpha^2 \right\}, \\ \mathcal{V}_T &:= \mathbb{R}^2 \setminus (\mathcal{D}_T \cup \mathcal{U}_T).\end{aligned}$$

Proof. For a matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

the expression for \bar{Q}_T^θ in (2.12) (with L_T^θ in place of L^θ) becomes

$$\bar{Q}_T^\theta(\alpha, \beta) = \min_{\gamma \in \mathbb{R}} f(\gamma),$$

where

$$f(\gamma) := c(\alpha^2 + 2\beta^2 + \gamma^2) + 2c|\alpha\gamma - \beta^2| + 2c_1 k a_\theta (\alpha - \gamma) - 4c_1 k b_\theta \beta + 2c_1 k^2 + \bar{e}_T.$$

Note that if $\alpha = 0$, f reduces to the differentiable function

$$f(\gamma) = 4c\beta^2 + c\gamma^2 - 2c_1 k a_\theta \gamma - 4c_1 k b_\theta \beta + 2c_1 k^2 + \bar{e}_T$$

and it is minimised at $\gamma = \frac{c_1}{c} k a_\theta$, i.e. for all $\beta \in \mathbb{R}$

$$\bar{Q}_T^\theta(0, \beta) = 4(c\beta^2 - c_1 k b_\theta \beta) + c_1 k^2 \left(2 - \frac{c_1}{c} a_\theta^2\right) + \bar{e}_T.$$

Next assume that $\alpha \neq 0$. If $\gamma = \beta^2/\alpha$,

$$f(\beta^2/\alpha) = c \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} + 2c_1 k \left(a_\theta \frac{\alpha^2 - \beta^2}{\alpha} - 2b_\theta \beta \right) + 2c_1 k^2 + \bar{e}_T$$

and for any $\gamma \neq \beta^2/\alpha$, the function f is differentiable with

$$f'(\gamma)/2 = c\gamma + c\alpha \operatorname{sgn}(\alpha\gamma - \beta^2) - c_1 k a_\theta,$$

which vanishes at

$$\gamma = \frac{c_1}{c} k a_\theta - \alpha \operatorname{sgn}(\alpha\gamma - \beta^2).$$

If $\alpha\gamma > \beta^2$, then $\gamma_1 = \frac{c_1}{c} k a_\theta - \alpha$ is the critical point and this can only be true in the regime

$$\frac{c_1}{c} k a_\theta \alpha > \beta^2 + \alpha^2.$$

In this case, we compute

$$f(\gamma_1) = 4c_1 k (a_\theta \alpha - b_\theta \beta) + c_1 k^2 \left(2 - \frac{c_1}{c} a_\theta^2\right) + \bar{e}_T.$$

Similarly, for $\alpha\gamma < \beta^2$, we find that $\gamma_2 = \frac{c_1}{c} k a_\theta + \alpha$ is the critical point which can only be true in the regime

$$\frac{c_1}{c} k a_\theta \alpha < \beta^2 - \alpha^2$$

and then

$$f(\gamma_2) = 4(c\beta^2 - c_1 k b_\theta \beta) + c_1 k^2 \left(2 - \frac{c_1}{c} a_\theta^2\right) + \bar{e}_T.$$

On the other hand, in the regime

$$\beta^2 - \alpha^2 \leq \frac{c_1}{c} k a_\theta \alpha \leq \beta^2 + \alpha^2,$$

a straightforward computation shows that $f'(\gamma) < 0$ if $\gamma < \beta^2/\alpha$ and $f'(\gamma) > 0$ if $\gamma > \beta^2/\alpha$. Hence, in this regime, and with $\alpha \neq 0$, the minimum value of f is achieved at $\gamma = \beta^2/\alpha$ so that

$$\bar{Q}_T^\theta(\alpha, \beta) = f(\beta^2/\alpha) = c \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} + 2c_1 k \left(\bar{a}_\theta \frac{\alpha^2 - \beta^2}{\alpha} - 2b_\theta \beta \right) + 2c_1 k^2 + \bar{e}_T$$

To compute \bar{Q}_T^θ in the respective regimes of values of $ka_\theta \alpha$, one needs to understand whether the value of f at its respective local minima γ_1 and γ_2 is lower than $f(\beta^2/\alpha)$. We compute

$$\begin{aligned} f(\gamma_1) - f(\beta^2/\alpha) &= -\frac{c_1^2}{c}k^2a_\theta^2 + 4c_1ka_\theta\alpha - c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} - 2c_1ka_\theta\frac{\alpha^2 - \beta^2}{\alpha} \\ &= -\frac{c_1^2}{c}k^2a_\theta^2 - c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} + 2c_1ka_\theta\frac{\alpha^2 + \beta^2}{\alpha} \\ &= -c\left\{\frac{c_1^2}{c^2}k^2a_\theta^2 + \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} - 2\frac{c_1}{c}ka_\theta\frac{\alpha^2 + \beta^2}{\alpha}\right\} \\ &= -c\left[\frac{\alpha^2 + \beta^2}{\alpha} - \frac{c_1}{c}ka_\theta\right]^2 \leq 0 \end{aligned}$$

with equality if and only if

$$\frac{c_1}{c}ka_\theta\alpha = \beta^2 + \alpha^2.$$

Similarly,

$$\begin{aligned} f(\gamma_2) - f(\beta^2/\alpha) &= -\frac{c_1^2}{c}k^2a_\theta^2 + 4c\beta^2 - c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} - 2c_1ka_\theta\frac{\alpha^2 - \beta^2}{\alpha} \\ &= -\frac{c_1^2}{c}k^2a_\theta^2 - 2c_1ka_\theta\frac{\alpha^2 - \beta^2}{\alpha} - c\frac{(\alpha^2 - \beta^2)^2}{\alpha^2} \\ &= -c\left\{\frac{c_1^2}{c^2}k^2a_\theta^2 + \frac{(\alpha^2 - \beta^2)^2}{\alpha^2} + 2\frac{c_1}{c}ka_\theta\frac{\alpha^2 - \beta^2}{\alpha}\right\} \\ &= -c\left[\frac{\alpha^2 - \beta^2}{\alpha} + \frac{c_1}{c}ka_\theta\right]^2 \leq 0 \end{aligned}$$

with equality if and only if

$$\frac{c_1}{c}ka_\theta\alpha = \beta^2 - \alpha^2.$$

Hence, we deduce the result. Note that these computations show that \bar{Q}_T^θ is continuous. \square

Notice that when $k = 0$ (and $c = 1$), modulo the constant \bar{e}_T the expression for \bar{Q}_T^θ in Proposition 3.2 reduces to expression (1.5) in [17], namely,

$$\bar{Q}(\alpha, \beta) := \begin{cases} 4\beta^2 & \text{if } \alpha^2 \leq \beta^2 \\ \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} & \text{if } \alpha^2 > \beta^2. \end{cases} \quad (3.6)$$

Indeed, in this case

$$\mathcal{D}_T = \emptyset, \quad \mathcal{U}_T = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha^2 \leq \beta^2\}. \quad (3.7)$$

For $k > 0$, it is natural to distinguish the case $\theta = \pi/4$ (and, similarly, the case $\theta = 3\pi/4$) from all the other cases. Indeed, we have $a_{\pi/4} = 0$ and $b_{\pi/4} = 1$, so that \mathcal{D}_T and \mathcal{U}_T are again given by (3.7), whereas

$$\bar{Q}_T^{\pi/4}(\alpha, \beta) = \begin{cases} 4(c\beta^2 - c_1k\beta) + 2c_1k^2 + \bar{e}_T, & \text{if } \alpha^2 \leq \beta^2 \\ c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} - 4c_1k\beta + 2c_1k^2 + \bar{e}_T, & \text{if } \alpha^2 > \beta^2. \end{cases}$$

Observe that for all $\theta \in [0, \pi/2) \setminus \{\pi/4, 3\pi/4\}$, setting $\rho := c_1ka_\theta/(2c)$, we have that \mathcal{D}_T coincides with the (open) disk $(\alpha - \rho)^2 + \beta^2 < \rho^2$ and \mathcal{U}_T with the (closed) region inside the hyperbola $(\alpha + \rho)^2 - \beta^2 = \rho^2$.

We want to examine the minimisers of \bar{Q}_T^θ . Observe that in the case $k = 0$ the above function $(\alpha, \beta) \mapsto \bar{Q}(\alpha, \beta)$ is minimised by $(0, 0)$. When instead $k > 0$,

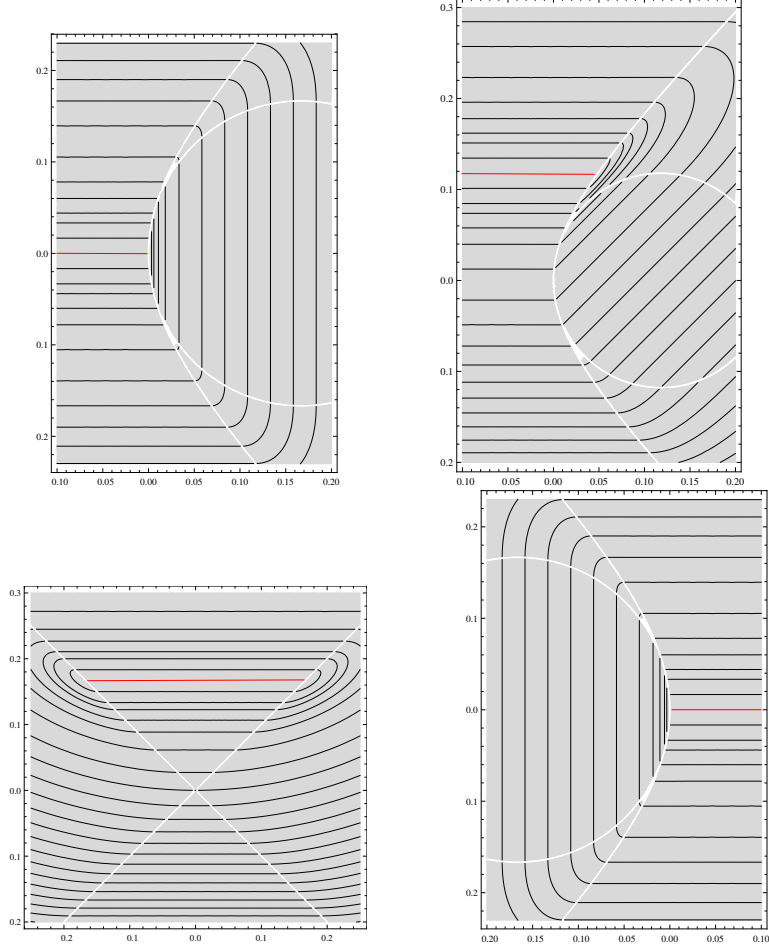


FIGURE 1. Phase diagrams with level curves of \overline{Q}_T^θ , where the white lines emphasize the boundary of \mathcal{V}_T and the red lines the set of minimisers. The pictures from top left to bottom right correspond to the cases $\theta = 0$, $\theta = \pi/8$, $\theta = \pi/4$, and $\theta = \pi/2$, respectively.

we have that the minimisers of \overline{Q}_T^θ lie on a segment. This is the content of the following lemma.

Lemma 3.3. *For every $0 \leq \theta < \pi$, \overline{Q}_T^θ attains its minimum value precisely on the segment $[\alpha_{\theta,1}^T, \alpha_{\theta,2}^T] \times \{\beta_\theta\}$, where*

$$\beta_\theta := \frac{k c_1}{2c} \sin 2\theta,$$

and

$$\alpha_{\theta,1}^T := -\frac{k c_1}{2c} (1 + \cos 2\theta), \quad \alpha_{\theta,2}^T := \frac{k c_1}{2c} (1 - \cos 2\theta).$$

Moreover,

$$\min_{\mathbb{R}^2} \overline{Q}_T^\theta = c_1 k^2 \left(2 - \frac{c_1}{c}\right) + \bar{e}_T. \quad (3.8)$$

Notice that the segment $[\alpha_{\theta,1}^T, \alpha_{\theta,2}^T] \times \{\beta_\theta\}$ is a subset of \mathcal{U}_T connecting the two branches of the hyperbola $\frac{c_1}{c} k a_\theta \alpha \leq \beta^2 - \alpha^2$ (a degenerate hyperbola in the case $k = 0$ or $\theta \in \{\pi/4, 3\pi/4\}$).

Proof. Consider the nontrivial case $k \neq 0$. A straightforward computation shows that the points $(\alpha, \beta) \in (\alpha_{\theta,1}^T, \alpha_{\theta,2}^T) \times \{\beta_\theta\}$, which lie in the interior of \mathcal{U}_T , are local minimisers, and that \bar{Q}_T^θ evaluated at each of these points gives the value (3.8). At the same time, when $\mathcal{D}_T \neq \emptyset$, we have that $\nabla \bar{Q}_T^\theta(\alpha, \beta) \neq 0$ for every $(\alpha, \beta) \in \mathcal{D}_T$, because a_θ and b_θ can never vanish simultaneously. Moreover, a point (α, β) lying in the interior of \mathcal{V}_T is a critical point of \bar{Q}_T^θ iff

$$c\alpha - c\frac{\beta^4}{\alpha^3} + c_1 k a_\theta \frac{\beta^2}{\alpha^2} + c_1 k a_\theta = 0, \quad (3.9)$$

$$c\frac{\beta^3}{\alpha^2} + c\beta - c_1 k a_\theta \frac{\beta}{\alpha} - c_1 k b_\theta = 0. \quad (3.10)$$

Now, observe that in the case $\beta \neq 0$, multiplying the second equation by β/α and adding the first yields

$$c(\alpha^2 + \beta^2) + c_1 k (a_\theta \alpha + b_\theta \beta) = 0.$$

At the same time, equation (3.9) is equivalent to

$$c(\beta^2 - \alpha^2) - c_1 k a_\theta \alpha = 0.$$

Summing up the last two equations gives $\beta = c_1 k b_\theta / (2c)$ and in turn $\alpha = \alpha_{\theta,1}^T$ or $\alpha = \alpha_{\theta,2}^T$, when $b_\theta \neq 0$. In the case $b_\theta = 0$, a similar argument gives that $(-c_1 k / c, 0)$ is the solution to (3.9)–(3.10). In any case, we have obtained that the solutions of (3.9)–(3.10) lie in $\partial\mathcal{U}_T$. Hence, there are no critical points of \bar{Q}_T^θ in the interior of \mathcal{U}_T . Other straightforward computations show that the values of \bar{Q}_T^θ on $\partial\mathcal{U}_T \setminus \{(\alpha_{\theta,1}^T, \beta_\theta), (\alpha_{\theta,2}^T, \beta_\theta)\}$ are strictly smaller than the local minimum, therefore the local minimisers are indeed global. This concludes the proof of the lemma. \square

Using the above lemma we can now find the minimisers and the minimum of our limiting functional (3.3). Indeed, minimising the integrand pointwise, we have that

$$\begin{aligned} \min_{\mathcal{A}} \mathcal{J}_T^\theta &= \ell \min_{\mathbb{R}^2 \times \mathbb{R}^2} \bar{Q}_T^\theta = \ell \left[c_1 k^2 \left(2 - \frac{c_1}{c} \right) + \bar{e}_T \right] \\ &= \frac{\mu \ell}{\pi^4} \left[3 \left(\frac{1 + 2\lambda}{1 + \lambda} \right) + \frac{\pi^4 - 4\pi^2 - 48}{8} \right] \frac{\eta_0^2}{h_0^2}, \end{aligned}$$

where in the second equality we have used (3.8) and in the last one the constants c_1 , c , k , and \bar{e}_T have been substituted with their expressions given in terms of the parameters of the 3D model. The set $(\alpha, \beta) \in [\alpha_{\theta,1}^T, \alpha_{\theta,2}^T] \times \{\beta_\theta\}$ of the minimisers of \bar{Q}_T^θ is given by

$$\left[\frac{3\eta_0}{\pi^2(1+\lambda)h_0}(-a_\theta - 1), \frac{3\eta_0}{\pi^2(1+\lambda)h_0}(1 - a_\theta) \right] \times \left\{ \frac{3\eta_0}{\pi^2(1+\lambda)h_0} b_\theta \right\}.$$

Hence, a minimiser of \mathcal{J}_T^θ is any $(d_1, d_2, d_3) \in \mathcal{A}$ such that $d'_1 \cdot d_3$ and $d'_2 \cdot d_3$ are constant and satisfy

$$d'_1 \cdot d_3 \in \left[\frac{3\eta_0}{\pi^2(1+\lambda)h_0}(-a_\theta - 1), \frac{3\eta_0}{\pi^2(1+\lambda)h_0}(1 - a_\theta) \right], \quad d'_2 \cdot d_3 = \frac{3\eta_0}{\pi^2(1+\lambda)h_0} b_\theta.$$

Notice that when α and β are real constants, the problem

$$(d_1, d_2, d_3) \in \mathcal{A}, \quad d'_1 \cdot d_3 = \alpha, \quad d'_2 \cdot d_3 = \beta, \quad (3.11)$$

has always a solution. Indeed, identifying (d_1, d_2, d_2) with the rotation matrix $Q = (d_1|d_2|d_2)$, it is standard to see that finding a solution of (3.11) consists in solving

$$Q'(s) = Q(s) \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & -\beta \\ \alpha & \beta & 0 \end{pmatrix},$$

for some fixed $Q(0) \in \text{SO}(3)$. Also, once $s \mapsto d_1(s)$ is given, the mid-line curve is given by

$$r(s) = r(0) + \int_0^s d_1(\sigma) d\sigma$$

is fixed up to a translation.

For the convenience of the reader, let us make explicit the condition for which \mathcal{J}_T^θ is minimised in two cases:

$$\begin{aligned} \theta = 0: \quad & (d'_1 \cdot d_3, d'_2 \cdot d_3) \in \left[-\frac{6\eta_0}{\pi^2(1+\lambda)h_0}, 0 \right] \times \{0\}, \\ \theta = \pi/4: \quad & (d'_1 \cdot d_3, d'_2 \cdot d_3) \in \left[-\frac{3\eta_0}{\pi^2(1+\lambda)h_0}, \frac{3\eta_0}{\pi^2(1+\lambda)h_0} \right] \times \left\{ \frac{3\eta_0}{\pi^2(1+\lambda)h_0} \right\}. \end{aligned}$$

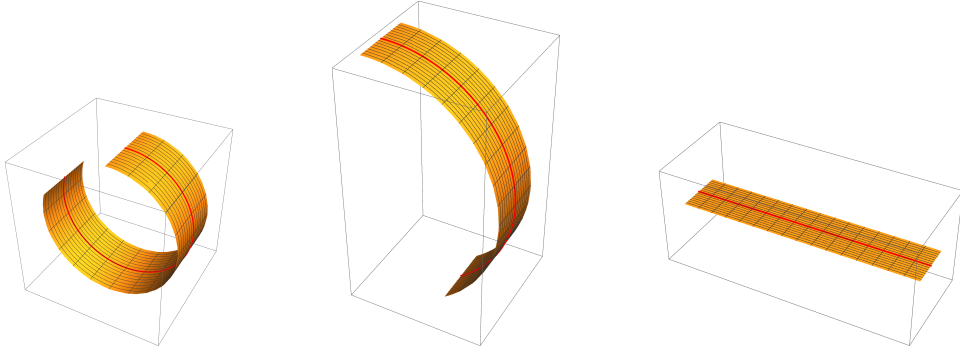


FIGURE 2. Minimal energy configurations for the cases $\theta = 0$ and $\theta = \pi/2$, with a decreasing value of $|\alpha| = |d'_1 \cdot d_3|$ from left to right, and with $\beta = d'_2 \cdot d_3 = 0$. The mid-line is shown in red.

We note that the minimisers predicted by our limiting model are in agreement with [29, Figure 3], where minimum-energy configurations of self-shaping synthetic systems with oriented reinforcement are shown.

4. THE SPLAY-BEND CASE

In the splay-bend case, the θ -dependent target curvature tensor \bar{A}^θ defined in (2.6) is

$$\bar{A}_S^\theta = k \begin{pmatrix} -\cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & -\sin^2 \theta \end{pmatrix}$$

and $\det \bar{A}_S^\theta = \det \bar{A}_S = 0$. The functional defined in (2.7) becomes

$$\mathcal{E}_{\varepsilon, S}^\theta(v) = \frac{1}{\varepsilon} \int_{S_\varepsilon} \left\{ c|A_v(z)|^2 + L_S^\theta(A_v(z)) \right\} dz,$$

with

$$L_S^\theta(A_v) := -2c_1 A_v \cdot \bar{A}^{\theta, SB} + 2c_2 k \operatorname{tr} A_v + ck^2 + \bar{e}_S.$$

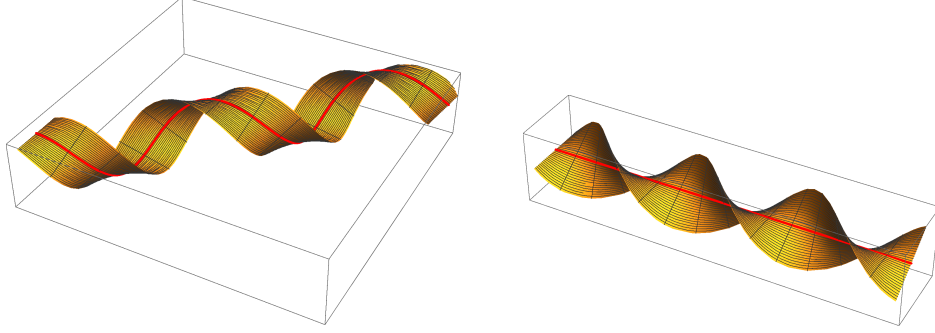


FIGURE 3. A selection of minimal energy configurations for the cases $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$. Both of them correspond to some positive and constant $\beta = d'_2 \cdot d_3$ and some constant $\alpha = d'_1 \cdot d_3$. In particular, the second configuration corresponds to $\alpha = 0$ and its mid-line (shown in red) is a straight line. In the other configuration the mid-line is a helix.

Also, the rescaled energy (2.9) is now

$$\mathcal{J}_{\varepsilon, S}^{\theta}(y) := \int_S \left\{ c|A_{y, \varepsilon}(x)|^2 + L_S^{\theta}(A_{y, \varepsilon}(x)) \right\} dx,$$

for every $y \in W_{\text{iso}, \varepsilon}^{2,2}(S, \mathbb{R}^3)$, and $\mathcal{J}_S^{\theta} : \mathcal{A} \rightarrow \mathbb{R}$ is defined as

$$\mathcal{J}_S^{\theta}(d_1, d_2, d_3) := \int_I \bar{Q}_S^{\theta}(d'_1 \cdot d_3, d'_2 \cdot d_3) dx_1, \quad (4.1)$$

where \bar{Q}_S^{θ} is given by (2.12), with L_S^{θ} in place of L^{θ} . The counterpart of Corollary 3.1 holds for the splay-bend case: it is sufficient to replace $\mathcal{J}_{\varepsilon, T}^{\theta}$ and \mathcal{J}_T^{θ} by $\mathcal{J}_{\varepsilon, S}^{\theta}$ and \mathcal{J}_S^{θ} , respectively, in the statement of Corollary 3.1.

Note that \bar{A}_S^{θ} can be alternatively written as

$$\bar{A}_S^{\theta} = \frac{1}{2} (\bar{A}_T^{\theta} - \mathbb{I})$$

and in turn

$$L_S^{\theta}(A) = -c_1 A \cdot \bar{A}_T^{\theta} + k(c + c_2) \text{tr} A + ck^2 + \bar{e}_S.$$

Hence, setting $d_{\theta} = c_1 a_{\theta} - c - c_2$, we have that

$$\bar{Q}_S^{\theta}(\alpha, \beta) = \min_{\gamma \in \mathbb{R}} f(\gamma),$$

where

$$f(\gamma) := c(\alpha^2 + 2\beta^2 + \gamma^2) + 2c|\alpha\gamma - \beta^2| - kd_{\theta}(\alpha + \gamma) + 2kc_1(a_{\theta}\alpha - b_{\theta}\beta) + ck^2 + \bar{e}_S,$$

recalling that $a_{\theta} := \cos 2\theta$ and $b_{\theta} := \sin 2\theta$. This will be useful in the proof of the following proposition.

Proposition 4.1. \bar{Q}_S^{θ} is a continuous function given by

$$\bar{Q}_S^{\theta}(\alpha, \beta) = \begin{cases} 2c_1 k(a_{\theta}\alpha - b_{\theta}\beta) + ck^2 \left(1 - \frac{d_{\theta}^2}{4c^2}\right) + \bar{e}_S, & \text{in } \mathcal{D}_S \\ 4c\beta^2 - 2kd_{\theta}\alpha + 2c_1 k(a_{\theta}\alpha - b_{\theta}\beta) + ck^2 \left(1 - \frac{d_{\theta}^2}{4c^2}\right) + \bar{e}_S, & \text{in } \mathcal{U}_S \\ c \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} - kd_{\theta} \frac{\alpha^2 + \beta^2}{\alpha} + 2c_1 k(a_{\theta}\alpha - b_{\theta}\beta) + ck^2 + \bar{e}_S, & \text{in } \mathcal{V}_S. \end{cases}$$

where $d_\theta := c_1 a_\theta - c - c_2$, and

$$\begin{aligned}\mathcal{D}_S &:= \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \frac{k}{2c} d_\theta \alpha > \beta^2 + \alpha^2 \right\}, \\ \mathcal{U}_S &:= \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \frac{k}{2c} d_\theta \alpha \leq \beta^2 - \alpha^2 \right\}, \\ \mathcal{V}_S &:= \mathbb{R}^2 \setminus (\mathcal{D}_S \cup \mathcal{U}_S).\end{aligned}$$

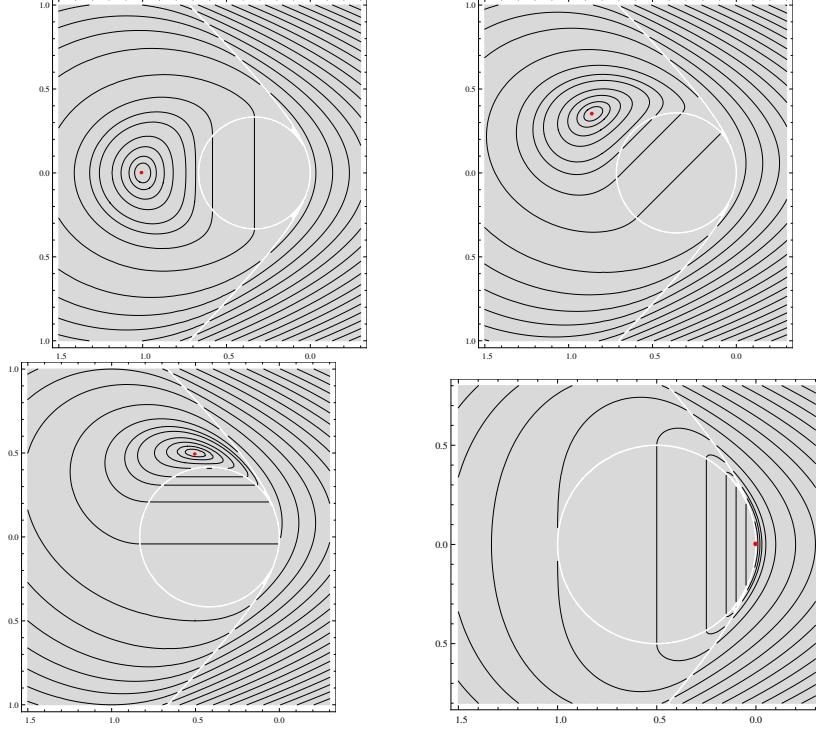


FIGURE 4. Phase diagrams with level curves of \overline{Q}_S^θ , where the white lines emphasize the boundary of \mathcal{V}_S and the red dots are the minimisers. The pictures from top left to bottom right correspond to the cases $\theta = 0$, $\theta = \pi/8$, $\theta = \pi/4$, and $\theta = \pi/2$, respectively.

Remark 4.1. Setting

$$\overline{Q}_{S,1}^\theta := 2c_1 k (a_\theta \alpha - b_\theta \beta) + ck^2 \left(1 - \frac{d_\theta^2}{4c^2}\right) + \bar{e}_S,$$

we have that $\overline{Q}_S^\theta = \overline{Q}_{S,1}^\theta$ in \mathcal{D}_S , that

$$\overline{Q}_S^\theta = \overline{Q}_{S,2}^\theta(\alpha, \beta) := 4c\beta^2 - 2kd_\theta\alpha + \overline{Q}_{S,1}^\theta(\alpha, \beta) \quad \text{in } \mathcal{U}_S,$$

and that

$$\overline{Q}_S^\theta = c \left(\frac{\alpha^2 + \beta^2}{\alpha} - \frac{kd_\theta}{2c} \right)^2 + \overline{Q}_{S,1}^\theta(\alpha, \beta) = c \left(\frac{\beta^2 - \alpha^2}{\alpha} - \frac{kd_\theta}{2c} \right)^2 + \overline{Q}_{S,2}^\theta(\alpha, \beta) \quad \text{in } \mathcal{V}_S.$$

Since the squares in this expression vanishes on $\partial\mathcal{D}_S$ and $\partial\mathcal{U}_S$, respectively, this shows in particular that \overline{Q}_S^θ is continuous.

Proof of Proposition 4.1. The proof is almost identical to that for \overline{Q}_T^θ . For $\alpha = 0$, $f(\gamma)$ reduces to the differentiable function

$$f(\gamma) = 4c\beta^2 + c\gamma^2 - 2kd_\theta\gamma - 2c_1kb_\theta\beta + ck^2 + \bar{e}_S,$$

which is minimised at $\gamma = \frac{k}{2c}d_\theta$ and hence, for all $\beta \in \mathbb{R}$,

$$\overline{Q}_S^\theta(0, \beta) = 4c\beta^2 - 2c_1kb_\theta\beta + ck^2 - \frac{k^2d_\theta^2}{4c} + \bar{e}_S.$$

If $\alpha \neq 0$ and $\gamma = \beta^2/\alpha$,

$$f(\beta^2/\alpha) = c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} - kd_\theta\frac{\alpha^2 + \beta^2}{\alpha} + 2c_1k(a_\theta\alpha - b_\theta\beta) + ck^2 + \bar{e}_S$$

whereas, for $\gamma \neq \beta^2/\alpha$, f is a differentiable function with

$$f'(\gamma) = 2c\gamma + 2c\alpha \operatorname{sgn}(\alpha\gamma - \beta^2) - kd_\theta.$$

The critical points are then given by

$$\gamma_1 = \frac{k}{2c}d_\theta - \alpha, \text{ in the regime } \frac{k}{2c}d_\theta\alpha > \alpha^2 + \beta^2$$

and by

$$\gamma_2 = \frac{k}{2c}d_\theta + \alpha, \text{ in the regime } \frac{k}{2c}d_\theta\alpha < \beta^2 - \alpha^2.$$

The respective values of f are

$$f(\gamma_1) = 2c_1k(a_\theta\alpha - b_\theta\beta) + ck^2 - \frac{k^2d_\theta^2}{4c} + \bar{e}_S$$

and

$$f(\gamma_2) = 4c\beta^2 - 2kd_\theta\alpha + 2c_1k(a_\theta\alpha - b_\theta\beta) + ck^2 - \frac{k^2d_\theta^2}{4c} + \bar{e}_S.$$

It is easy to compute that

$$\begin{aligned} f(\gamma_1) - f(\beta^2/\alpha) &= -\frac{1}{4c}k^2d_\theta^2 - c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} + kd_\theta\frac{\alpha^2 + \beta^2}{\alpha} \\ &= -c\left[\frac{\alpha^2 + \beta^2}{\alpha} - \frac{k}{2c}d_\theta\right]^2 \leq 0 \end{aligned}$$

with equality if and only if $\frac{k}{2c}d_\theta\alpha = \alpha^2 + \beta^2$, i.e. in the regime \mathcal{D}_S ,

$$\overline{Q}_S^\theta(\alpha, \beta) = f(\gamma_1).$$

Similarly, we find that

$$\begin{aligned} f(\gamma_2) - f(\beta^2/\alpha) &= -\frac{1}{4c}k^2d_\theta^2 + 4c\beta^2 - 2kd_\theta\alpha - c\frac{(\alpha^2 + \beta^2)^2}{\alpha^2} + kd_\theta\frac{\alpha^2 + \beta^2}{\alpha} \\ &= -\frac{1}{4c}k^2d_\theta^2 - c\frac{(\alpha^2 - \beta^2)^2}{\alpha^2} + kd_\theta\frac{\beta^2 - \alpha^2}{\alpha} \\ &= -c\left[\frac{\beta^2 - \alpha^2}{\alpha} - \frac{k}{2c}d_\theta\right]^2 \leq 0 \end{aligned}$$

with equality if and only if $\frac{k}{2c}d_\theta\alpha = \beta^2 - \alpha^2$, i.e. in the regime $\frac{k}{2c}d_\theta < \beta^2 - \alpha^2$,

$$\overline{Q}_S^\theta(\alpha, \beta) = f(\gamma_2).$$

At the same time, the calculations above also establish the Remark following Proposition 4.1. To conclude the proof, a straightforward computation shows that $f'(\gamma) < 0$ if $\gamma < \beta^2/\alpha$ and $f'(\gamma) > 0$ if $\gamma > \beta^2/\alpha$. Hence, in the regime $\alpha \neq 0$ and

$$\beta^2 - \alpha^2 \leq \frac{k}{2c}d_\theta\alpha \leq \alpha^2 + \beta^2$$

the minimum value of f is achieved at $\gamma = \beta^2/\alpha$ and

$$\overline{Q}_S^\theta(\alpha, \beta) = f(\beta^2/\alpha).$$

□

We now focus on the minimisers of \overline{Q}_S^θ . As for the twist case, when $k = 0$ and up to additive and multiplicative constants, the function \overline{Q}_S^θ reduces to the function defined in (3.6), which is minimised at $(0, 0)$. When instead $k > 0$, differently from the twist case we have that for every θ the minimiser of \overline{Q}_S^θ is a (θ -dependent) single point, in view of the following lemma.

Lemma 4.2. *For every $0 \leq \theta < \pi$, \overline{Q}_S^θ is minimised precisely at $(\alpha_\theta^S, \beta_\theta^S)$, where*

$$\alpha_\theta^S := -\frac{k}{2}(1 + \cos 2\theta), \quad \beta_\theta^S := \frac{k}{2} \sin 2\theta.$$

Moreover,

$$\min_{\mathbb{R}^2} \overline{Q}_S^\theta = \bar{e}_S. \quad (4.2)$$

Proof. Consider the nontrivial case $k \neq 0$. A straightforward computation shows that there are no critical points of \overline{Q}_S^θ in the interior of \mathcal{D}_S or \mathcal{U}_S , for any value of θ . Next, we look for critical points in the interior of \mathcal{V}_S . Differentiating the first of the two expressions for \overline{Q}_S^θ in \mathcal{V}_S given in Remark 4.1 and setting $\nabla \overline{Q}_S^\theta = 0$ yields

$$2c \left(\frac{\alpha^2 + \beta^2}{\alpha} - \frac{kd_\theta}{2c} \right) \frac{\beta^2 - \alpha^2}{\alpha^2} = 2c_1 k a_\theta \quad (4.3)$$

$$2c \left(\frac{\alpha^2 + \beta^2}{\alpha} - \frac{kd_\theta}{2c} \right) \frac{\beta}{\alpha} = c_1 k b_\theta. \quad (4.4)$$

We first examine the cases $\theta = 0$ and $\theta = \pi/2$. In these cases we have $b_\theta = 0$ and $|a_\theta| = 1$, so that from (4.4) we get $\beta = 0$, since $(\alpha^2 + \beta^2)/\alpha \neq (kd_\theta)/(2c)$ in the interior of \mathcal{V}_S . Setting $\beta = 0$ in (4.3) we then get $\alpha = -\frac{k}{2c}(c_1 a_\theta + c + c_2)$. Now, an easy calculation shows that in the case $\theta = 0$ the point (α_0^S, β_0^S) lies in the interior of \mathcal{V}_S and it is the global minimiser of \overline{Q}_S^θ . On the other hand, in the case $\theta = \pi/2$ another easy computation show that the the point $(-kc_2/c, 0)$ lies in \mathcal{D}_S and it is thus not a critical point of our function in the interior of \mathcal{V}_S . Then, comparing the values of $\overline{Q}_S^{\pi/2}$ on the boundaries of \mathcal{D}_S and \mathcal{U}_S , we find that the minimum is achieved at $(\alpha_{\pi/2}^S, \beta_{\pi/2}^S) = (0, 0)$. Moreover, it can be readily checked that (4.2) holds for $\theta = 0$ and $\theta = \pi/2$.

Consider now an arbitrary $\theta \in (0, \pi) \setminus \{\pi/2\}$ and note that in this case $b_\theta \neq 0$ and $|a_\theta| < 1$. As before, we search for critical points of \overline{Q}_S^θ in the interior of \mathcal{V}_S . From (4.4) we get in particular $\beta \neq 0$. Therefore, we may divide (4.3) by (4.4) getting

$$\frac{\beta^2 - \alpha^2}{\alpha} = 2 \frac{a_\theta}{b_\theta} \beta, \quad (4.5)$$

and in turn that $|\alpha| = |a_\theta \alpha - b_\theta \beta|$. Hence, either $\alpha = a_\theta \alpha - b_\theta \beta$ or $\alpha = b_\theta \beta - a_\theta \alpha$. Suppose that the former case holds true or, equivalently, that

$$\frac{\beta}{\alpha} = \frac{a_\theta - 1}{b_\theta}. \quad (4.6)$$

Before proceeding, consider the second expression for \overline{Q}_S^θ in \mathcal{V}_S given in Remark 4.1, namely

$$\overline{Q}_S^\theta = c \left(\frac{\beta^2 - \alpha^2}{\alpha} - \frac{kd_\theta}{2c} \right) + \overline{Q}_{S,2}^\theta(\alpha, \beta).$$

Differentiating it with respect to β and setting $\partial_\beta \bar{Q}_S^\theta = 0$ yields

$$2c \left(\frac{\beta^2 - \alpha^2}{\alpha} - \frac{kd_\theta}{2c} \right) \frac{\beta}{\alpha} = c_1 k b_\theta - 4c\beta.$$

This expression, coupled with (4.5) and (4.6), easily gives

$$(\alpha, \beta) = \frac{k}{2} \left(\frac{b_\theta^2}{a_\theta - 1}, b_\theta \right).$$

Recalling the definition of a_θ and b_θ , this point coincides with $(\alpha_\theta^S, \beta_\theta^S)$ defined in the statement, and lies in the interior of \mathcal{V}_S . Supposing now $\alpha = b_\theta \beta - a_\theta \alpha$ and proceeding similarly returns the point

$$-\frac{kc_2}{2c} \left(\frac{b_\theta^2}{a_\theta + 1}, b_\theta \right),$$

which belongs to \mathcal{D}_S . Hence, the only critical point in the interior of \mathcal{V}_S is $(\alpha_\theta^S, \beta_\theta^S)$. Other computations show that this is indeed the global minimiser of \bar{Q}_S^θ and that (4.2) holds true. \square

In view of the above lemma, we have that the minimum of our limiting functional (4.1) is

$$\min_A \mathcal{J}_S^\theta = \ell \min_{\mathbb{R}^{2 \times 2}} \bar{Q}_S^\theta = \ell \bar{e}_S = \mu \ell (1 + \lambda) \left(\frac{\pi^4 - 12}{32} \right) \frac{\eta_0^2}{h_0^2},$$

where in the second equality we have used (4.2) and in the last one the constant \bar{e}_S has been replaced by its expression given in terms of the 3D parameters (the first expression in (2.3)). Also, the minimisers of \mathcal{J}_T^θ are all $(d_1, d_2, d_3) \in \mathcal{A}$ such that

$$\begin{aligned} (d'_1 \cdot d_3, d'_2 \cdot d_3) &= \frac{k}{2} \left(-1 - \cos 2\theta, \sin 2\theta \right) \\ &= \frac{3\eta_0}{\pi^2 h_0} \left(-1 - \cos 2\theta, \sin 2\theta \right), \quad \theta \in [0, \pi). \end{aligned}$$

For the convenience of the reader, in what follows we enlist some cases:

$$\begin{aligned} \theta = 0 : \quad & (d'_1 \cdot d_3, d'_2 \cdot d_3) = \frac{3\eta_0}{\pi^2 h_0} (-1, 0); \\ \theta = \pi/8 : \quad & (d'_1 \cdot d_3, d'_2 \cdot d_3) = \frac{3\eta_0}{\pi^2 h_0} \left(-\frac{\sqrt{2} + 2}{2}, \frac{\sqrt{2}}{2} \right); \\ \theta = \pi/4 : \quad & (d'_1 \cdot d_3, d'_2 \cdot d_3) = \frac{3\eta_0}{\pi^2 h_0} (-1, 1); \\ \theta = \pi/2 : \quad & (d'_1 \cdot d_3, d'_2 \cdot d_3) = (0, 0). \end{aligned}$$

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The authors declare that they have no conflict of interest.

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