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Good components of curves in projective spaces outside the Brill–Noether range

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Abstract: For all integers n, d, g such that $n \geq 4$, $d \geq n+1$, and $(n+2)(d-n-1) \geq n(g-1)$, we define a good (i.e. generically smooth of dimension $(n+1)d+(3-n)(g-1)$ and with the expected number of moduli) irreducible component $A(d,g;n)$ of the Hilbert scheme of smooth and nondegenerate curves in $\mathbb{P}^n$ with degree d and genus g. For most of these (d,g), we prove that a general $X \in A(d,g;n)$ has maximal rank. We cover, in this way, a range of (d,g,n) outside the Brill–Noether range.

Key words: Curve in projective spaces, normal bundle, Hilbert scheme, Hilbert function

1. Introduction

Let $X \subset \mathbb{P}^n$ be any closed subscheme. We recall that X is said to have maximal rank if for all $t \in \mathbb{N}$ the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_X(t))$ has maximal rank, i.e. it is either injective or surjective. Note that X has maximal rank if and only if for each $t \in \mathbb{N}$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$. Assume that X has maximal rank. We know that $h^0(\mathcal{O}_{\mathbb{P}^n}(t)) = \binom{n+t}{n}$. Hence, $h^0(\mathcal{I}_X(t)) = \max\{0, \binom{n+t}{n} - h^0(\mathcal{O}_X(t))\}$ and $h^1(\mathcal{I}_X(t)) = \max\{0, h^0(\mathcal{O}_X(t)) - \binom{n+t}{n}\}$. If X is a curve of degree d and genus g with maximal rank and if $h^1(\mathcal{O}_X(t)) = 0$, then $h^0(\mathcal{I}_X(t)) = \max\{0, \binom{n+t}{n} - td - 1 + g\}$ and $h^1(\mathcal{I}_X(t)) = \max\{0, td+1-g-\binom{n+t}{n}\}$. In this paper we always consider nondegenerate curves $X \subset \mathbb{P}^n$ (hence with $h^0(\mathcal{I}_X(1)) = 0$) with $h^1(\mathcal{O}_X(2)) = 0$ (and so with $h^1(\mathcal{O}_X(t)) = 0$ for all $t \geq 2$). For these curves, if we know that X has maximal rank then we know its Hilbert function. In the range $d < g+n$, i.e. in the range of degrees and genera not covered by [4–6], we always consider linearly normal curves, i.e. curves X with $h^1(\mathcal{I}_X(1)) = 0$.

The problem of the existence of curves in projective space with maximal rank and the related problem of the existence of good components of the Hilbert scheme of curves in $\mathbb{P}^n$ have been considered in a huge number of papers ([1, 2, 4–15, 17, 18, 20–26, 28, 29, 31]). From these papers, it became clear that there is a fundamental difference between the case $n = 3$ and the case $n > 3$.

In this paper for all integers d, g, n such that $n \geq 4$, $d \geq n+1$, $g > 0$ and $(n+2)(d-n-1) \geq n(g-1)$ we construct an irreducible component $A(d,g;n)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ whose general element is a smooth and nondegenerate curve $X \subset \mathbb{P}^n$ with $\deg(X) = d$ and $p_a(X) = g$. We prove that $A(d,g;n)$ is generically smooth and with the expected dimension $(n+1)d+(n-3)(1-g)$ and that (if $g \geq 2$) it has the expected number of moduli in the sense of Sernesi ([30]), i.e. the natural map $A(d,g;n) \to \mathcal{M}_g$ is either

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dominant or its generic fiber has dimension $n^2+2n=\dim \mathrm{Aut}(\mathbb{P}^n)$ (Proposition 3.1). For $d\ge g+n$ we just take as $A(d,g;n)$ the irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ with as general elements the nonspecial nondegenerate smooth curves of degree d and genus g. In the next theorem, N_X denotes the normal bundle of the smooth curve $X\subset \mathbb{P}^n$.

Theorem 1.1 *Fix an integer $n\ge 4$ and a real number $\epsilon>0$. Then there exists an integer $d(n,\epsilon)$ such that for all integers d,g with $d\ge d(n,\epsilon)$ and $0\le g\le (\frac{n+2}{n}-\epsilon)d$ there is a smooth, connected, and nondegenerate curve $X\subset\mathbb{P}^n$ with $\deg(X)=d$, $p_a(X)=g$, $h^1(N_X)=0$, $h^1(\mathcal{O}_X(2))=0$, $X\in A(d,g;n)$ and X has maximal rank.*

The fact that we get a nice curve with maximal rank in a specific component of $\mathrm{Hilb}(\mathbb{P}^n)$ is (in our opinion) interesting, but it is also useful to prove the corresponding statement without the mention of the component. Indeed, to prove the existence of some $X\subset\mathbb{P}^n$ with degree d, genus g, $h^1(\mathcal{O}_X(2))=0$ we will need (for a certain integer k, the critical value of the triple (d,g,n)) to prove the existence of some $X',X''\in A(n,d;g)$ with degree d, genus g, $h^1(\mathcal{O}_{X'}(2))=h^1(\mathcal{O}_{X''}(2))=0$, $h^0(\mathcal{I}_{X'}(k-1))=0$ and $h^1(\mathcal{I}_{X''}(t))=0$ for all $t\ge k$. Since $\dim X''=1$, a standard exact sequence gives $h^i(\mathcal{I}_{X''}(t))=0$ for all $t\ge 0$ and all $i\ge 2$. Since in all cases we need $k\ge 3$, the Castelnuovo–Mumford Lemma shows that to prove $h^1(\mathcal{I}_{X''}(t))=0$ for all $t\ge k$ it is sufficient to prove $h^1(\mathcal{I}_{X''}(k))=0$. Thus, if we know that X, X', and X'' are in the same irreducible component of the Hilbert scheme of $\mathbb{P}^n$, then by the semicontinuity theorem for cohomology we have $h^0(\mathcal{I}_X(t))=0$ for all $t<k$ and $h^1(\mathcal{I}_X(t))=0$ for all $t\ge k$; hence, X has maximal rank.

Question 1.2 *Describe the triples (d,g,n), $n\ge 4$, such that there is a smooth, connected, and nondegenerate curve $X\subset\mathbb{P}^n$ such that $\deg(X)=d$, $p_a(X)=g$ and $h^1(N_X)=0$.*

Question 1.3 *Do we obtain the same triples as in Question 1.2 if we allow integral and nondegenerate curves with mild singularities (e.g., locally complete intersection singularities or nodal singularities)?*

Question 1.4 *Are there (for many of the triples (d,g,n) as in Theorem 1.1) other irreducible components of* $\mathrm{Hilb}(\mathbb{P}^n)$ *whose general element is a smooth curve X with maximal rank? And with the additional restriction that $h^1(\mathcal{O}_X(2))=0$? Or with the restriction that X is linearly normal?*

In section 2 we define the component $A(d,g;n)$ and prove its existence. In Section 3 we prove many properties of $A(d,g;n)$. We prove that it has the expected number of moduli in the sense of Sernesi [30] (Proposition 3.1). We also prove some statements on the intersection of some elements of $A(d,g;n)$ with a hyperplane (in the spirit of [2, 4, 8, 10, 23, 26]) (see Section 4). The last 5 sections are devoted to the proof of Theorem 1.1, the last one containing the numerical lemmas used in the proof.

2. The definition and existence of the component $A(d,g;n)$

For any nodal curve $X\subset\mathbb{P}^n$, let N_X denote its normal sheaf in $\mathbb{P}^n$. The sheaf N_X is locally free, $\mathrm{rank}(N_X)=n-1$ and $\deg(N_X)=(n+1)\deg(X)+(n-3)\chi(\mathcal{O}_X)$.

Let $A\subset\mathbb{P}^n$ be a reduced curve. A line $L\subset\mathbb{P}^n$ is said to be k-secant to A if $|A\cap L|=k$, $\mathrm{Sing}(A)\cap L=\emptyset$, and L is not tangent to A. We give the definition of the component $A(d,g;n)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ quoting inside the definition the lemmas used to prove that $A(d,g;n)$ is well-defined.

Definition 2.1 *Fix an integer* $n \ge 3$.

(a) For all integers $g \ge 0$ *and* $d \ge g+n$, *let* $A(d,g;n)$ *be the irreducible component of* $\mathrm{Hilb}(\mathbb{P}^n)$ *whose general element is a smooth, connected, and nondegenerate curve* $X \subset \mathbb{P}^n$ *such that* $\deg(X) = d$, $p_a(X) = g$, *and* $h^1(\mathcal{O}_X(1)) = 0$.

(b) Fix an integer $g \ge n+1$. *Let* $A(g+n-1,g;n)$ *denote the irreducible component of* $\mathrm{Hilb}(\mathbb{P}^n)$ *whose general element is a smooth, connected, and nondegenerate curve* $X \subset \mathbb{P}^n$ *such that* $\deg(X) = g+n-1$ *and* $p_a(X) = g$, *i.e. whose general element is a linear projection of a general canonically embedded smooth curve* $C \subset \mathbb{P}^{g-1}$ *from* $g-n$ *general points of* C.

(c) Assume $g \ge n+3$ *and fix an integer* d *such that* $d \le g+n-2$ *and*

$$(n+2)(d-n-1) \ge n(g-1). \tag{2.1}$$

There are uniquely determined integers t, y, x *such that* $g = 1+t(n+2)+y$, $d = n+1+tn+x$ *and* $0 \le y \le n+1$. *By the inequality* $g \ge n+3$ *and* (2.1) $t > 0$ *and* $x \ge y$. *Let* $A(d-x, g-y; n)$ *be the only component of* $\mathrm{Hilb}(\mathbb{P}^n)$ *containing the nodal curve* $K \cup D_1 \cup \cdots \cup D_t$, *where* $K \subset \mathbb{P}^n$ *is a linearly normal elliptic curve and* $D_1, \dots, D_t$ *are* t *general rational normal curves with* $\#(D_i \cap K) = n+3$ *for all* i *(Lemma 2.6). Let* $A(d,g;n)$ *be the irreducible component of* $\mathrm{Hilb}(\mathbb{P}^n)$ *containing the nodal curve* $Y \cup L_1 \cup \cdots \cup L_y \cup R_{y+1} \cup \cdots \cup R_{x-y}$, *where* Y *is a general element of* $A(d-x, g-y; n)$, *each* L_i *is a general* 2*-secant line of* Y *and, if* $x > y$, *each* R_i *is a general* 1*-secant line of* Y *(Lemma 2.7).*

Remark 2.2 *Fix* (d,g,n) *for which Definition 2.1 defines* $A(d,g;n)$. *In case (a)* $A(d,g;n)$ *is the component of the Hilbert scheme of* $\mathbb{P}^n$ *containing the nonspecial and nondegenerate smooth curves of degree* d *and genus* g. *In case (b) a general element of* $A(d,g;n)$ *is a linear projection of a canonically embedded smooth curve* $C \subset \mathbb{P}^{g-1}$ *from* $g-1-n$ *general points of* C. *These are the triples* (d,g,n) *such that* $h^1(\mathcal{O}_X(1)) = 1$ *for a general* $X \in A(d,g;n)$. *Case (c) covers all cases with* $h^1(\mathcal{O}_X(1)) \ge 2$ *for a general* $X \in A(d,g;n)$ *for which we are able to prove that* $A(d,g;n)$ *has good properties. Its definition using a linearly normal elliptic curve* $K \subset \mathbb{P}^n$ *is given for its use in Remark 4.4.*

Notation 2.3 *For any hyperplane* $H \subset \mathbb{P}^n$, $n \ge 4$, *we write* $A(d,g;H)$ *instead of* $A(d,g;n-1)$ *to emphasize that all elements of* $A(d,g;H)$ *are contained in* H.

A general element of $A(d,g;H)$ is smooth. When $n = 4$ for $A(d,g;H)$ we may take the irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ defined in [6]. We only need $A(d,g;H)$ when $d \ge g+n-1$ and in this case $A(d,g;H)$ is the irreducible component of $\mathrm{Hilb}(\mathbb{P}^{n-1})$ whose general element is a nonspecial smooth curve spanning H.

Among the extremal components $A(d,g;n)$ (i.e. the ones for which $A(d-1,g;n)$ and $A(d,g+1;n)$ are not defined) there are the ones with $d = 1+n+tn$ and $g = 1+t(n+2)$, $t > 0$, (resp. $d = n+1+tn$ and $g = t(n+2)$, resp. $d = 2n+tn$ and $g = n+1+t(n+2)$) obtained from a smooth degree $n+1$ curve of genus 1 (resp. a smooth rational curve of degree $n+1$, resp. a canonically embedded smooth curve of degree $2n$ and genus $n+1$) adding t times an $(n+3)$-secant rational normal curve and applying each time Lemma 2.5.

Remark 2.4 *Fix a finite set* $S \subset \mathbb{P}^n$, $n \ge 3$, *such that* $|S| = n+3$ *and* S *is in linear general position, i.e. any* $S' \subset S$ *with* $|S'| = n+1$ *spans* $\mathbb{P}^n$. *Let* D *be the unique rational normal curve of* $\mathbb{P}^n$ *containing* S. *Since* N_D

is a direct sum of $n-1$ *line bundles of degree* $n+2$*, we have* $h^1(N_D)=0$ *and* $h^1(N_D(-S))=0$*; hence, the restriction map* $H^0(N_D)\to H^0(N_{D|S})$ *is surjective. Thus, for each rank* $n-1$ *vector bundle* E *on* D *with an injective map* $N_D\hookrightarrow E$ *we have* $h^1(E(-S))=0$*; hence, the restriction map* $H^0(E)\to H^0(E_{|S})$ *is surjective.*

Lemma 2.5 *Let* $Y\subset\mathbb{P}^n$ *be an integral and nondegenerate curve with* $\deg(Y)\neq n$*. Fix* $S\subset Y_{\mathrm{reg}}$ *such that* $|S|=n+3$ *and* S *in linear general position. Let* D *be the only rational normal curve containing* S*. Assume* $Y\cap D=S$ *and that* $X:=Y\cup D$ *is nodal. Then* X *is a connected nodal curve,* $\deg(X)=\deg(Y)+n$*,* $p_a(X)=p_a(Y)+n+2$*,* $h^1(N_X)=0$ *and* X *is smoothable.*

Proof By assumption the curve X is connected, nodal and $p_a(X)=p_a(Y)+n+2$. Thus, N_X is a rank $n-1$ vector bundle on X with degree $(n+1)\deg(X)+2-2p_a(X)$. By [16, §2] the vector bundle $N_{X|Y}$ on Y is obtained from N_Y making $n+3$ positive elementary transformations. Thus, $h^1(N_{X|Y})\le h^1(N_Y)=0$. By [16, §2] the vector bundle $N_{X|D}$ on D is obtained from N_D making $n+3$ positive elementary transformations. By Remark 2.4 we have $h^1(N_{X|Y})=0$ and the restriction map $H^0(N_{X|D})\to H^0(N_{X|S})$ is surjective. Thus, the Mayer–Vietoris exact sequence

$$0\to N_X\to N_{X|Y}\oplus N_{X|D}\to N_{X|S}\to 0 \tag{2.2}$$

gives $h^1(N_X)=0$. Since $h^1(N_Y)=0$, we have $h^1(F)=0$ for each vector bundle F on Y obtained from N_Y making $n+2$ positive elementary transformations. Since $h^1(N_D(-S))=0$ (Remark 2.4), we have $h^1(A)=0$ for each vector bundle A on D obtained from $N_{X|D}$ making $\#S$ negative elementary transformation, one for each $q\in S$ (with the language of [2] for each $q\in S$ take $p_q\in T_qY\setminus\{q\}$; the relevant vector bundle is $N_D[q_1\to p_{q_1}]\cdots[q_{n+3}\to p_{q_{n+3}}]$, where $S=\{q_1,\dots,q_{n+3}\}$). Thus, X is smoothable. □

Lemma 2.6 *Take a smooth* $Y\in A(d,g;n)$ *with* $h^1(N_Y)=0$ *and a rational normal curve* $D\subset\mathbb{P}^n$ *such that* $Y\cup D$ *is nodal and* $1\le|Y\cap D|\le n+3$*. Then* $Y\cup D\in A(d+n,g+|D\cap Y|-1;n)$*.*

Proof As in the proof of Lemma 2.5 we get $h^1(N_{Y\cup D})=0$. Thus, we may assume that Y is a general element of $A(d,g;n)$. Set $x:=|D\cap Y|$.

(a) Assume for the moment $d\le g+n-2$. It is easy to check that $h^1(N_{Y\cup D})=0$ and so $Y\cup D$ is a smooth point of the Hilbert scheme. Thus, it is sufficient to do it for one D to get it for a general D intersecting Y with the prescribed cardinality. We degenerate Y to a defining curve $Y_1\cup D_1\cup\dots\cup D_t\cup R_1\cup\dots\cup R_a\cup L_1\cup\dots\cup L_b$ of $A(d,g;n)$ with Y_1 a linearly normal elliptic curve. The case $|D\cap Y|=n+3$ is done taking a rational normal curve D_{t+1} with $|D_{t+1}\cap Y_1|=n+3$. Thus, it is sufficient to do the cases with $1\le|D\cap Y|\le n+2$. By the case just done it is sufficient to do it for the curve $Y_1\cup R_1\cup\dots\cup R_a\cup L_1\cup\dots\cup L_b$. If either $x\le n+1$ or $b>0$, then we see with another degeneration that we land in a nonspecial case. Now assume $b=0$ and $x=n+2$. It is sufficient to show that $Y_1\cup D\in A(2n,n+2;n)$. This is obvious, because $h^1(N_{Y_1\cup D})=0$, $Y_1\cup D$ is smoothable and in this range the Hilbert scheme of smooth and nondegenerate curves is irreducible.

(b) Now assume $d=g+n-1$ and $d\ge 2n$.

(b1) If $x\le n+1$ we land again in a smoothable curve with $h^1(\mathcal{O}_{Y\cup D}(1))=1$.

(b2) Assume $x=n+2$. We degenerate $Y\cup D$ to a nodal curve $Y_2\cup L_1\cup\dots\cup L_{g-n-1}\cup D_1$ with Y_2 a canonically embedded curve of degree $2n$ and genus $n+1$, D_1 a rational normal curve, $\#(Y_2\cap D_1)=n+3$ and

$L_1, \dots, L_{g-n-1}$ general secant lines of Y_2. By step (b1) it is sufficient to prove that $Y_1 \cup D_1 \in A(3n, 2n+2; n)$. The nodal curve $Y_1 \cup D_1$ is smoothable and $h^1(N_{Y_1 \cup D_1}) = 0$ (Lemma 2.5). In this range of degrees and genera the Hilbert scheme of smooth and nondegenerate curves is irreducible.

(b3) Assume $x = n+3$. We degenerate $Y \cup D$ to a nodal curve $Y_2 \cup L_1 \cup \cdots \cup L_{g-n-1} \cup D_1$ with Y_2 a canonically embedded curve of degree $2n$ and genus $n+1$, D_1 a rational normal curve, $\#(Y_2 \cap D_1) = n+3$ and $L_1, \dots, L_{g-n-1}$ general secant lines of Y_2. By step (a) it is sufficient to prove that $Y_1 \cup D_1 \in A(3n, 2n+3; n)$. The nodal curve $Y_1 \cup D_1$ is smoothable and $h^1(N_{Y_1 \cup D_1}) = 0$ (Lemma 2.5). In this range of degrees and genera the Hilbert scheme of smooth and nondegenerate curves is irreducible

(c) Now assume $d \geq g+n$. If $\#(Y \cap D) \leq n+1$, then $h^1(\mathcal{O}_{Y \cup D}(1)) = 0$, because $h^1(\mathcal{O}_Y(1)) = 0$ by the generality of Y. Now assume $\#(Y \cap D) = n+2$. We get $h^1(\mathcal{O}_{Y \cup D}(1)) \leq 1$. Since any $n+3$ points in linear general position in $\mathbb{P}^n$ are contained in a unique rational normal curve, we may degenerate $Y \cup D$ to the nodal curve $E := Y_1 \cup D_1 \cup L_1 \cup \cdots \cup L_g \cup R_1 \cup \cdots \cup R_{d-g-n}$, where Y_1 and D_1 are rational normal curves, $\#(Y_1 \cap D_1) = n+2$, each L_i is a 2-secant line of Y_1 and each R_i is a 1-secant line of Y_1. Since $Y_1 \cup D_1$ is a limit of canonically embedded curves, we reduce to a case (d', g') with $d' = g'+n-1$ and $d \geq 2n$ done in step (b). Now assume $\#(Y \cap D) = n+3$. Assume $g > 0$. We degenerate $Y \cup D$ to the nodal curve $K \cup D_1 \cup L_1 \cup \cdots \cup L_{g-1} \cup R_1 \cup \cdots \cup R_{d-g}$, where K is a linearly normal elliptic curve, D_1 is a rational normal curve, $\#(K \cap Y_1) = n+3$, each L_i is a general 2-secant line of K and each R_i is a general 1-secant line of K. We apply the case $t = 1$ part (c) of Definition 2.1 and then we add the 2-secant and 1-secant lines (a case fully proved). Now assume $g = 0$. Since $\#(Y \cap D) = n+3$ and D is a rational normal curve, $\deg(Y) > n$. As above it is sufficient to do the case $d = n+1$. We degenerate $Y \cup D$ to the nodal curve $C \cup D_1 \cup L$, where C and D rational normal curves, $\#(C \cap D_1) = n+2$ and L is a line intersecting both C and D_1 at a unique point, which are not in $C \cap D_1$. In this range of degrees and genera the Hilbert scheme of smooth and nondegenerate curves is irreducible. □

Lemma 2.7 *Fix integers d, g, n such that $A(d,g;n)$ is defined. Fix integers $a > 0$ and $1 \leq b \leq a+1$. Take a smooth $Y \in A(d,g;n)$ such that $h^1(N_Y) = 0$ and a smooth rational curve D with $Y \cup D$ nodal,* $\deg(D) = a$ *and $|Y \cap D| = b$. Then $h^1(N_{Y \cup D}) = 0$ and $Y \cup D \in A(d+a, g+b-1; n)$.*

Proof The assertions that $h^1(N_{Y \cup D}) = 0$ and that $Y \cup D$ is smoothable are well-known ([2], [16, Theorem 4.1, Remark 4.1.1]) and easier than the proof of Lemma 2.5. To prove that $Y \cup D \in A(d+a, g+b-1; n)$ we may assume (moving if necessary D) that Y is general in $A(d,g;n)$ and that D is a general rational curve of degree a intersecting Y at b points and quasitransversally (the set of all such D's is an irreducible variety, because $a \geq b-1$). We distinguish the following cases.

First assume $d \geq g+n$. We have $h^1(\mathcal{O}_Y(1)) = 0$. A Mayer–Vietoris exact sequence gives $h^1(\mathcal{O}_{Y \cup D}(1)) = 0$; hence (since $Y \cup D$ is smoothable), $Y \cup D \in A(d+a, g+b-1; n)$.

Now assume $d = g+n-1$. Let $C \subset \mathbb{P}^{g-1}$ be a general canonically embedded curve and $T \subset \mathbb{P}^{g-1}$ a general degree a smooth rational curve such that $C \cup T$ is nodal and $|C \cap T| = b$. Since $h^1(N_C) = 0$ and T is a rational normal curve in its linear span, it is easy to check that $h^1(N_{C \cup T}) = 0$ and that $C \cup T$ is smoothable (to a nonspecial curve if $b \leq a$, to a curve E with $h^1(\mathcal{O}_E(1)) = 1$ if $b = a+1$). Then we use a family of inner projections in the fiber of this smoothing to get that $Y \cup D \in A(d+a, g+b-1; n)$.

Now assume $d \leq g+n-2$. Take a general $Y_1 \in A(d-n, g-n-2; n)$. Let D_2 be a general rational normal curve such that $Y_1 \cup D_2$ is nodal and $|D_2 \cap Y_1| = n+3$. By the definition of $A(d,g;n)$ we may deform $Y_1 \cup D_2$ to the general Y. By induction on g we have $Y_1 \cup D_1 \in A(d-n+a, g-n+b; n)$, where D_1 is a general rational curve with $Y_1 \cup D_1$ nodal and $|Y_1 \cap D_1| = b$. In the deformation $\beta : \mathcal{X} \to \Delta$ of $Y_1 \cup D_2$ to Y (i.e. with Δ irreducible and Y and $Y_1 \cup D_2$ fibers of β, say $Y_1 \cup D_2 = \beta^{-1}(o)$) as we may find (up to a covering of Δ) b sections $s_1, \dots, s_b$ of β such that $\{s_1(o), \dots, s_b(o)\} = Y_1 \cap D_1$. In this way in the deformation of $Y_1 \cup D_2$ to Y we get a deformation of $Y_1 \cup D_1 \cup D_2$ to $Y \cup D_1$. We have $h^1(N_{Y_1 \cup D_1 \cup D_2}) = 0$. Since $Y_1 \cup D_1 \in A(d-n+a, g-n-2+b-1; n)$ by the inductive assumption, we get $Y_1 \cup D_1 \cup D_2 \in A(d+a, g+b-1; n)$. □

Lemma 2.8 *Fix integers d, g, n such that $A(d,g;n)$ is defined. Fix an integer $b \in \{n+2, n+3\}$. If $b = n+3$ assume $d > n$. Fix a smooth $Y \in A(d,g;n)$ such that $h^1(N_Y) = 0$. Let $D \subset \mathbb{P}^n$ be a rational normal curve such that $Y \cup D$ is nodal and $|Y \cap D| = b$. Then $h^1(N_{Y \cup D}) = 0$ and $Y \cup D \in A(d+n, g+b-1; n)$.*

Proof As in Lemma 2.5 we see that $h^1(N_{Y \cup D}) = 0$ and that $Y \cup D$ is smoothable. We use induction on the integer d, the starting point of the induction being the case $(d,g) = (n,0)$ when $b = n+2$ and the cases $(d,g) = (n+1,0)$ and $(d,g) = (n+1,1)$ when $b = n+3$. When $b = n+3$ to start the induction it is sufficient to use the definition of the varieties $A(x,y;n)$ when $x \le y-2$. If $(d,g) = (n,0)$ and $b = n+2$ we use that a general union of 2 rational normal curves with $n+2$ common points is a flat limit of canonically embedded smooth curves of $\mathbb{P}^n$.

(a) Assume $d > g+n$. If $b = n+3$, we may assume $(d,g) \ne (n+1,0)$, since we did the case $(d,g,b) = (n+1,0,n+3)$ as an initial case. We degenerate Y to $Y_1 \cup L$ with Y_1 a general element of $A(d-1,g;n)$ and L a general 1-secant line of Y_1. We add to Y_1 a general rational normal curve containing b points of Y_1 and then we apply the case $a = b = 1$ of Lemma 2.7 to $A(d+n-1, g+b-1; n)$.

(b) Assume $d = g+n$. The case $b = n+3$ follows from the definition of the varieties $A(x,y;n)$; hence, we may assume $b = n+2$. If $g = 0$, i.e. if $(d,g) = (n,0)$ we use again the starting case of the induction. If $g > 0$ we degenerate Y to $Y_1 \cup L$ with Y_1 a general element of $A(d-1, g-1; n)$, L a general 2-secant line of Y_1, then we add a general rational normal curve containing b points of Y_1 and then apply Lemma 2.7 with $a = 1$ and $b = 2$.

(c) Assume $d = g+n-1$; hence, $g \ge n+1$. Since the case $b = n+3$ follows from the definition of $A(g+2n-1, g+n+2; n)$, it is sufficient to do the case $b = n+2$.

First assume $g = n+1$, i.e. assume that Y is canonically embedded. We have $h^1(N_{Y \cup D}) = 0$ and $Y \cup D$ is smoothable ([10, Lemma 2.3]; these facts are easier than the proof of Lemma 2.5.

If $g \ge n+2$, we degenerate Y to $Y_1 \cup L$ with Y_1 of degree $d-1$ and genus $g-1$ and L a 2-secant line of Y.

(d) Now assume $d \le g+n-2$. By the definition of $A(d+n, g+n+2; n)$ we conclude when $b = n+3$. Now assume $b = n+2$. Take a general $Y_1 \in A(d-n, g-n-2; n)$. Let D_2 be a general rational normal curve such that $Y_1 \cup D_2$ is nodal and $|D_2 \cap Y_1| = n+3$. By the definition of $A(d,g;n)$ we may deform $Y_1 \cup D_2$ to the general Y. By induction on g we have $Y_1 \cup D_1 \in A(d, g-1)$, where D_1 is a general rational normal curve with $Y_1 \cup D_1$ nodal and $|Y_1 \cap D_1| = n+2$. In the deformation $\beta : \mathcal{X} \to \Delta$ of $Y_1 \cup D_2$ to Y (with $\beta^{-1}(o) = Y_1 \cup D_1$ for some $o \in \Delta$) we may find (up to a covering) b sections $s_1, \dots, s_b$ of β such that $\{s_1(o), \dots, s_b(o)\} = Y_1 \cap D_1$. In this way in the deformation of $Y_1 \cup D_2$ to Y we get (after a finite covering of the base of the deformation) a deformation of $Y_1 \cup D_1 \cup D_2$ to $Y \cup D_1$. We have $h^1(N_{Y_1 \cup D_1 \cup D_2}) = 0$. Since $Y_1 \cup D_1 \in A(d, g-1; n)$ by the inductive assumption, we get $Y_1 \cup D_1 \cup D_2 \in A(d+n, g+n+1; n)$. □

Lemma 2.9 *Take $Y \in A(x, y; n)$ and a nonspecial curve $D \subset H$ meeting quasitransversally and at a unique point. Set $z := \deg(D)$ and $w := p_a(D)$. Then $Y \cup D \in A(x+z, y+w; n)$.*

Proof We degenerate D to a union $D_1 \cup \cdots \cup D_k \subset H$ of smooth rational curves such that from $D_1 \cup \cdots \cup D_i$ to $D_1 \cup \cdots \cup D_i \cup D_{i+1}$ we may use either Lemma 2.7 or Lemma 2.8 in H. Then we apply k times Lemmas 2.7 or 2.8 first to $Y \cup D_1$ and then adding each time a curve D_i. □

3. The right number of moduli in the sense of Sernesi

We adapt the proof of [10, Proposition 3.1] to prove the following result.

Proposition 3.1 *The irreducible component $A(d, g; n)$, $g \geq 2$, of* Hilb$(\mathbb{P}^n)$ *has the expected number of moduli.*

Proof If $d \geq g+n-1$ we use that in these cases there is a unique irreducible component of Hilb$(\mathbb{P}^n)$ which dominates $\mathcal{M}_g$. Now assume $d \leq g+n-2$. Set $r := \rho(g, n, d) := (n+1)d - ng - n(n+1)$ (the Brill–Noether number). Set $A'(d, g; n) := \{X \in A(d, g; n) \mid X \text{ is nodal and semistable}\}$ and let $p_{d,g} : A(d, g; n) \to \overline{\mathcal{M}}_g$ be the moduli map.

(a) First assume $r \geq 0$. Let x be the only integer such that $d = n + r + xn$ and $g = r + x(n+1)$. Since $d < g+n$, we have $x > 0$. We have $\rho(g, d, n) = \rho(g-n-1, d-n, n)$. By induction on x starting with the case $x = 0$ we may assume that $p_{d-n,g-n-1}$ is dominant. We want to prove that the general fibers of $p_{d-n,g-n-1}$ have the same dimension. Take a general $C \in A(d-n, g-n-1; n)$. In particular C is smooth and $h^1(N_C) = 0$. Take a general $B \subset C$ with $|B| = n+2$ and let D be a general rational normal curve containing C. By Lemma 2.8 the curve $C \cup D$ is nodal, $p_a(C \cup D) = g$ and $C \cup D \in A(d, g; n)$. It is sufficient to prove that the fiber F of $p_{d-n,g-n-1}$ containing C has the same dimension as the fiber F' of $p_{d,g}$ containing $C \cup D$. Let G be the set of all rational normal curves containing B. We have $\dim G = n-1$ and G is an irreducible variety. Fix any ordering of the set $B = \{q_1, \ldots, q_{n+2}\}$. To show that $\dim_{[C]} F = \dim_{[C \cup D]} F'$ it is sufficient to check that there are only finitely many $D', D'' \in G$ such that $p_{d,g}(C \cup D') = p_{d,g}(C \cup D'')$. It is sufficient to show that if $h : D \to D'$ is an isomorphism, then h is uniquely determined by the element of S_{n+2} induced by the permutation σ such that $\sigma(i)$ is the only element of $\{1, \ldots, n+2\}$ such that $q_{\sigma(i)} = h(q_i)$. Since D, D' are rational normal curves, h is induced by an element h' of Aut$(\mathbb{P}^n)$. The projective transformation h' fixes the set B in linear general position and with $|B| = n+2$. Thus, h' is uniquely determined.

(b) Now assume $r < 0$. We need to prove that each irreducible component of a general fiber of $p_{d,g}$ parametrizes projectively equivalent elements of $A(d, g; n)$, i.e. (since in this range a general element of $A(d, g; n)$ has finitely many automorphisms) it is sufficient to prove that a general fiber of $p_{d,g}$ has dimension $n^2 + 2n$. We use induction on the integer d. Since $r < 0$, by the definition of $A(d, g; n)$ there is an integer $t > 0$ and a pair (d', g') such that $d = d' + tn$, $g = g' + t(n+2)$ and either $d' \geq g' + n$ or $g' \geq n+1$ and $d' \geq g + n - 1$. Hence, $A(d-n, g-n-2; n)$ is defined. By the inductive assumption the irreducible component Γ of the fiber of $p_{d-n,g-n-2}$ over $p_{d-n,g-n-2}(Y)$ has dimension $\max\{n^2 + 2n, r + 3n + n^2\}$. We fix a general $S \subset Y$ with $|S| = n+3$ and let $D \subset \mathbb{P}^n$ be the only rational normal curve containing S. Moving Y among the elements of $A(d-n, g-n-2)$ containing S we may assume that $Y \cup D$ is nodal and $Y \cap D = S$. Thus,

$Y \cup D \in A(d, g; n)$. Since $|S| \geq 3$, $Y \cup D$ is stable. Take a general element W of an irreducible component Γ of $p_{d,g}^{-1}(p_{d,g}(Y \cup D))$ containing $Y \cup D$. W is a 2-component nodal curve, say $W = W_1 \cup W_2$ with W_1, W_2 smooth, W_1 of genus $g-n-2$, W_2 of genus 0 and $|W_1 \cap W_2| = n+3$. Assume $\dim \Gamma > 0$. Since $Y \cup D$ is a limit of a family of curves like $W_1 \cup W_2$, we get $\deg(W_1) = d-n$, $\deg(W_2) = n$ and $W_1 \cap W_2$ in linear general position, there is a nonempty open subset Γ' of Γ such that all $W' \in \Gamma'$ are nodal 2-component curves, say $W' = A \cup B$ with A, B smooth, A of genus $g-n-2$ and isomorphic to Y as an abstract curve, B rational and $|A \cap B| = n+3$. Since $h^1(N_Y) = 0$, Y is a smooth point of $\mathrm{Hilb}(\mathbb{P}^n)$. Hence, all A appearing as a nonrational normal curve of an element of Γ' are in $A(d-n, g-n-2; n)$. Since S is in linear general position $A \cap B$ is in linear general position for a general $A \cup B \in \Gamma'$. If $r \leq -n$ we get that $\dim \Gamma' = n^2 + 2n$.

From now on we assume $r \geq -n+1$.

Assume for the moment $g-n-2 \geq 2$. For a fixed (but general) $[Y] \in \mathcal{M}_{g-n-2}$ (seen as an abstract curve) there are ∞^{r+n} degree $d-n$ nondegenerate embeddings of Y in $\mathbb{P}^n$, we have $\deg(W_2) \geq n$. Call $\{u_t\}$, $t \in \Lambda$, this family of embeddings and for each $t \in \Lambda$ call B_t the unique rational normal curve containing $u_t(S)$. Note that B_t is uniquely determined by the set $B_t \cap u_t(Y)$ with cardinality $n+3$. We need to prove that for each $t \in \Lambda$ the set of all $s \in \Lambda$ such that $u_s(Y) \cup B_s \cong u_t(Y) \cup B_t$ (as abstract curves) have finitely many orbits for the action of $\mathrm{Aut}(\mathbb{P}^n)$. Since $g-n-2 \geq 2$, $\mathrm{Aut}(Y)$ is finite and there are only finitely many $S' \subset Y$, such that (Y, S) and (Y, S') give the same element of $\mathcal{M}_{g-n-2,n+3}$. Consider the forgetful map $\varphi : \mathcal{M}_{g-n-2,n+3} \to \mathcal{M}_{g-n-2}$ and set $\Delta := \varphi^{-1}([Y])$. For a general $S \subset Y$ we get an $(n+r)$-dimensional family $\{u_t(S)\}_{t \in \Lambda}$ of subsets of $\mathbb{P}^n$ and, taking a Zariski dense open subset Λ' of Λ instead of Λ, we may assume that each set $\{u_t(S)\}$, $t \in \Lambda'$, has cardinality $n+3$ and it is in linear general position in $\mathbb{P}^n$.

Now assume $g-n-2 \leq 1$. Since $\rho(d-n, g-n-2, n) \leq n-1$ and $d > n$, we have either $g-n-2 \geq 2$ or $(d-n, g-n-2) = (n+1, 1)$. Ordering the points $q_1, \dots, q_{n+3}$ we may see (Y, S) as an element of $\mathcal{M}_{1,n+3}$. We may use (Y, q_1) as an element of the moduli space $\mathcal{M}_{1,1}$. □

4. Intersection with a hyperplane

Lemma 4.1 *There is a smooth linearly normal elliptic curve $Y \subset \mathbb{P}^n$, $n \geq 3$, and a rational normal curve $D \subset \mathbb{P}^n$ such that $|D \cap Y| = n+3$ and $Y \cup D$ is nodal.*

Proof First assume n is odd. Set $e := (n-1)/2$. The line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, e)$ is very ample and $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, e)) = 2e + 2 = n + 1$. Let $j : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^n$ denote the linearly normal embedding induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, e)|$. Fix a general $A \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, e+1)|$ and a general $B \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)|$. A is a smooth rational curve and B is a smooth elliptic curve. For a general (A, B) the curve $A \cup B$ is nodal and $|A \cap B| = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, e+1) \cdot \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2) = 2 + 2e + 2 = n + 3$. Take $Y := j(B)$ and $D := j(A)$.

Now assume $n \geq 4$ even and set $a = n/2 \geq 2$. Let F_1 be the Hirzebruch surface with a minimal self-intersection curve h with $h^2 = -1$. Take as a $\mathbb{Z}$-basis of $\mathrm{Pic}(F_1)$ the curve h and a fiber f of the ruling of F_1. The line bundle $\mathcal{O}_{F_1}(h + af)$ is very ample and $h^0(\mathcal{O}_{F_1}(h + af)) = 2a + 1 = n + 1$. Let $u : F_1 \to \mathbb{P}^n$ be the linearly normal embedding induced by the complete linear system $|\mathcal{O}_{F_1}(h + af)|$. The adjunction formula gives $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h - 3f)$. Fix a general $E \in |\mathcal{O}_{F_1}(h + (a+1)f)|$ and a general $F \in |\mathcal{O}_{F_1}(2h + 3f)|$. The adjunction formula gives that E is a smooth rational curve and that F is an elliptic curve. For a general (E, F), the curve $E \cup F$ is nodal and $|E \cap F| = \mathcal{O}_{F_1}(h + (a+1)f) \cdot \mathcal{O}_{F_1}(2h + 3f) = -2 + 2a + 2 + 3 = n + 3$.

Take $Y := u(F)$ and $D := u(E)$. □

Lemma 4.2 *For every positive integer $t > 0$ there are a smooth linearly normal elliptic curve $Y \subset \mathbb{P}^n$, $n \geq 3$, and rational normal curves $D_1, \dots, D_t \subset \mathbb{P}^n$ such that $D_i \cap D_j = \emptyset$ for all $i \neq j$, $|Y \cap D_i| = n+3$ for all i and $Y \cup D_1 \cup \cdots \cup D_t$ is nodal.*

Proof The case $t = 1$ is true by Lemma 4.1, which gives the existence of a linear normal elliptic curve Y such that for a general $S \subset Y$ with $|S| = n+3$ the only smooth rational normal curve D_S containing S is quasitransversal to Y and $Y \cap D_S = S$. We only need to prove that for a general $(A, B) \subset Y \times Y$ with $|A| = |B| = n+3$ we have $D_A \cap D_B = \emptyset$. Call $\mathcal{S}$ the set of all rational normal curves $D \subset \mathbb{P}^n$ intersecting quasitransversally Y and with $|Y \cap D| = n+3$. We just observed that $\mathcal{S} \neq \emptyset$. Since $\dim Y = 1$, Y is irreducible, a general $A \subset Y$ with $|A| = n+3$ is in linear general position and any $n+3$ points of $\mathbb{P}^n$ in linear general position are contained in a unique rational normal curve, $\mathcal{S}$ is an irreducible variety of dimension $n+3$. Thus, we only need to prove that $D \cap D' = \emptyset$ for a general $(D, D') \in \mathcal{S} \times \mathcal{S}$.

Claim 1: $\cup_{D \in \mathcal{S}} D$ is dense in $\mathbb{P}^n$.

Proof of Claim 1: We use induction on n starting the induction with the case $n = 2$ in which the result is obvious. Assume $n > 2$. Fix a general $(a, b) \in Y \times \mathbb{P}^n$ and call $\ell_a : \mathbb{P}^n \setminus \{a\} \to \mathbb{P}^{n-1}$ the linear projection from a. Let Y_a be the closure of $\ell_a(Y \setminus \{a\})$ in $\mathbb{P}^{n-1}$. The curve Y_a is a linearly normal elliptic curve of $\mathbb{P}^{n-1}$. Fix an open neighborhood $\mathcal{U}$ of b in $\mathbb{P}^n$ such that $a \notin \mathcal{U}$ and call $\mathcal{V} \subset \mathbb{P}^{n-1}$ an open neighborhood of $\ell_a(b)$ contained in the dense set $\ell_a(\mathcal{U})$. Restricting if necessary $\mathcal{U}$ we may assume $\mathcal{U} = \ell_a^{-1}(\mathcal{V})$. By the inductive assumption there is a rational normal curve $D' \subset \mathbb{P}^{n-1}$ such that $|D' \cap Y_a| = n+2$, D' intersects transversally Y_a and $\mathcal{V} \cap D' \neq \emptyset$. Let T be the cone of $\mathbb{P}^n$ with vertex a and base D'. For a general (b, D') we may assume that T is quasitransversal to Y outside a. Thus, it is sufficient to find a rational normal curve $D \subset \mathbb{P}^n$ such that $a \in D$ and $\ell_a(D \setminus \{a\}) \subseteq D'$. Let $\pi : W \to T$ denote minimal desingularization of T. The surface W is isomorphic to the Hirzebruch surface F_{n-1} and π is induced by the complete linear system $|\mathcal{O}_{F_{n-1}}(h + (n-1)f)|$, where $h = \pi^{-1}(a)$ and f is a fiber of the ruling of F_{n-1}. A general $K \in |\mathcal{O}_{F_{n-1}}(h + nf)|$ is smooth and rational and $\pi(K)$ is a rational normal curve. Take $D := \pi(K)$.

Fix a large integer $k \gg 0$. Since Y is a curve, its Hilbert polynomial has degree 1. Thus, for large k we have $h^0(\mathcal{I}_Y(k)) \geq \binom{n+k}{n} - k^2 > (kn-n-1)(kn-n-2)/2$. Set $x := kn-n-2$ and take a general $(D_1, \dots, D_x) \in \mathcal{S}^x$. Since $(D_1, \cup D_x)$ is general, we have $|(Y \cup D_1 \cup \cdots \cup D_i) \cap D_{i+1}| \geq n+3+i$ for $i = 1, \dots, x-1$. Since $h^0(\mathcal{O}_{\mathbb{P}^1}(t)(-Z)) = \max\{0, t+1-\deg(Z)\}$ for any finite set $Z \subset \mathbb{P}^1$ and $\deg(\mathcal{O}_{D_{i+1}}(k)) = kn+1$, we have $h^0(\mathcal{I}_{Y \cup \cdots \cup D_{i+1}}(k)) \geq h^0(\mathcal{I}_{Y \cup \cdots \cup D_i}(k)) - kn - n - 3 - i$ for all $i = 1, \dots, x-1$. We get that a general $D \in \mathcal{S}$ is in the base locus of $|\mathcal{I}_{Y \cup D_1 \cup D_x}(k)|$. Claim 1 implies $h^0(\mathcal{I}_{Y \cup D_1 \cup \cdots \cup D_x}(k)) = 0$. Thus, $h^0(\mathcal{I}_Y(k)) \leq (kn-n-1)(kn-n-2)/2$, a contradiction. □

Remark 4.3 *Let $S \subset \mathbb{P}^n$, $n \geq 2$, be a subset in linear general position and with $|S| = n+3$. It is well-known that S is contained in a unique rational normal curve of $\mathbb{P}^n$. This is obvious for $n = 2$, while if $n > 2$ the rational normal curve C and its uniqueness is obtained in the following way. Fix $o \in S$ and a hyperplane $M \subset \mathbb{P}^n$ such that $o \notin M$. Set $S' := S \setminus \{o\}$ and $B := \ell_o(S')$. Since S is in linear general position, the set B is a finite subset of M with cardinality $n+2$ and in linear general position in M. By the inductive assumption there is a unique rational normal curve D of M containing B. Let $T \subset \mathbb{P}^n$ be the cone with* *vertex o and base D. It is obvious that if C exists, then $C \subset T$. The minimal desingularization of T is isomorphic to the Hirzebruch surface F_{n-1}, i.e. (calling h the section of the ruling of F_{n-1} and f a fiber of its ruling) the complete linear system $|\mathcal{O}_{F_{n-1}}(h+(n-1)f|$ has no base points and it induces a birational morphism $\pi : F_{n-1} \to T$ with $\pi(h) = \{o\}$ and π inducing an isomorphism between $F_{n-1}\setminus h$ and $T\setminus\{o\}$. Since $S' \subset T \setminus \{o\}$, $A := \pi^{-1}(S')$ has cardinality $n+2$. No two of the points of S' are contained in a line of T, because S is in linearly normal position. The rational normal curves $C \subset T$ are the images of the irreducible elements of $|\mathcal{O}_{F_{n-1}}(f)|$ and each such element contains o. Thus, it is sufficient to prove that $|\mathcal{I}_A(h+nf)|$ is a singleton, $\{Y\}$, and that Y is irreducible. Since $h^0(\mathcal{O}_{F_{n-1}}(h+nf)) = h^0(\mathcal{O}_{\mathbb{P}^1}(n)) + h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = n+3$, we have $|\mathcal{I}_A(h+nf)| \neq \emptyset$. Since S is in linear general position, no two of the points of A are contained in the same element of $|\mathcal{O}_{F_{n-1}}(f)|$.*

Claim: *A general $M \in |\mathcal{I}_A(h+nf)|$ is irreducible.*

Proof of the claim: *Assume that a general $M \in |\mathcal{I}_A(h+nf)|$ is reducible. There is $F \in |\mathcal{O}_{F_{n-1}}(f)|$ such that $A = G+F$ for some $G \in |\mathcal{O}_{F_{n-1}}(h+(n-1)f)|$ containing a subset $A' \subseteq A$ with $|A'| \ge n+1$ and no two of the points of A are contained in the same element of $|\mathcal{O}_{F_{n-1}}(h+(n-1)f)|$. Since $|\mathcal{I}_{A'}(h+nf))| = \emptyset$, we get a contradiction.*

Remark 4.4 *Let $K \subset \mathbb{P}^n$, $n \ge 3$, be a linearly normal elliptic curve. Let $\mathcal{S}$ be the set of all $S \subset K$ such that $|S| = n+3$ and S is in linear general position. For each $S \in \mathcal{S}$ let D_S be the unique rational normal curve containing S. For any $q \in \mathbb{P}^n \setminus K$ set $\mathcal{A}(q) := \{S \in \mathcal{S} \mid q \in D_S\}$. Let $\mathcal{B}$ denote the set of all $q \in \mathbb{P}^n \setminus K$ such that $\mathcal{A}(q) = \emptyset$. Let $\mathcal{E}$ denote the set of all $q \in \mathbb{P}^n \setminus K$ such that $\dim \mathcal{A}(q) \ge 5$. For any $o \in \mathbb{P}^n$ let $\ell_o : \mathbb{P}^n \setminus \{o\} \to \mathbb{P}^{n-1}$ denote the linear projection from o. We identify $\mathbb{P}^{n-1}$ with a hyperplane $M_o \subset \mathbb{P}^n$ such that $o \notin M_o$.*

Claim 1: *Assume $n = 3$. The set $\mathcal{B}$ is formed by the 4 points $\{o_1, o_2, o_3, o_4\}$ of $\mathbb{P}^3 \setminus K$ which are the vertices of the quadric cones containing K and $\dim \mathcal{A}(q) = 4$ for all $q \in \mathbb{P}^3 \setminus (K \cup \mathcal{B})$.*

Proof of Claim 1: *There are exactly 4 points $q \in \mathbb{P}^3 \setminus K$ (call it $o_1, \dots, o_4$) such that the linear projection from q induces a $2:1$ map onto a smooth conic. Call $T_1, \dots, T_4$ the quadric cones containing K and with vertex $o_1, \dots, o_4$, respectively. Fix $q \in \mathbb{P}^3 \setminus (T_1 \cup T_2 \cup T_3 \cup T_4)$. Since K is the complete intersection of 2 quadrics and $q \notin K$, there is a unique quadric, Q, containing $K \cup \{q\}$. Since $q \notin T_1 \cup T_2 \cup T_3 \cup T_4$, Q is smooth. We have $K \in |\mathcal{O}_Q(2,2)|$. Since Q is homogeneous and there is a smooth $C \in |\mathcal{O}_Q(2,1)|$, a general $C \in |\mathcal{I}_q(2,1)|$ is smooth. By Bertini's theorem and the assumption $q \notin K$ for a general $C \in |\mathcal{I}_q(2,1)|$ the scheme $K \cap C$ is smooth, i.e. it is formed by 6 points. Since C is a rational normal curve, $K \cap C$ is formed by 6 points in linear general position. Thus, $C \in \mathcal{A}(q)$. We get an irreducible subset of $\mathcal{A}(q)$ with dimension 4 and another one is obtained taking the elements of $|\mathcal{I}_q(1,2)|$. To prove that $\dim \mathcal{A}(q) = 4$ it is sufficient to prove that each element of $\mathcal{A}(q)$ is contained in Q. Now take an arbitrary $D \in \mathcal{A}(q)$. If $\deg(C \cap K) > 6$, then $C \subset Q$ by Bezout. Thus, we may assume $\deg(K \cap C) = 6$, i.e. $p_a(K \cup C) = 6$ and $K \cup C$ is nodal. Since $\omega_{K\cup C|K} \cong \mathcal{O}_K(K \cap C)$ and $\deg(\omega_{K\cup C|C}) = 4$, duality gives $h^1(\mathcal{O}_{K\cup C}(2)) = 0$. Since $p_a(C \cup D) \ge 1+6-1 = 6$, $\deg(D \cup C) = 7$ and $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, Riemann–Roch gives $h^0(\mathcal{I}_{C\cup K}(2)) \neq 0$. Since Q is the only quadric containing $K \cup \{q\}$, we get $C \subset Q$. Now assume $q \in T_i$ for some i, but $q \neq o_i$. T_i is the only quadric surface containing $K \cup \{q\}$. We conclude using the desingularization F_2 of T_i discussed in Remark 4.3; with the notation of Remark 4.3 it is sufficient to observe that a general element of $|\mathcal{I}_p(h+3f)|$, $p \in F_2 \setminus h$, is* *irreducible. First, use the inductive assumption and then use Lemma 4.3.*

Claim 2: *We have* $\dim \mathcal{E} \le n-2$*.*

Proof of Claim 2: *Assume* $\dim \mathcal{E} \ge n-1$*. Since* $\dim \mathcal{S} = n+3$ *and* $\mathcal{S}$ *is irreducible, we get* $\mathcal{B} = \mathbb{P}^n \setminus (K \cup \mathcal{E})$ *contradicting the fact that* $\dim \mathcal{B} \le n-1$ *proved as Claim 1 during the proof of Lemma 4.2.*

This is the key lemma (an adaptation of [10, Lemma 5.2]) for the proof of Theorem 1.1.

Lemma 4.5 *Fix an integer* $t \ge 2$*, a hyperplane* $H \subset \mathbb{P}^n$*, a linearly normal elliptic curve* $K \subset \mathbb{P}^n$ *and a reduced scheme* $Y \subset H$ *such that* $\dim Y \le 1$ *and* $h^0(H, \mathcal{I}_{Y,H}(t-1)) > n$*. Then there exists a rational normal curve* $D \subset \mathbb{P}^n$ *such that* D *intersects transversally* H *and quasitransversally* K*,* $Y \cap D = \emptyset$*,* $|K \cap D| = n+3$ *and* $h^0(H, \mathcal{I}_{Y \cup (H \cap D),H}(t)) = h^0(H, \mathcal{I}_{Y,H}(t)) - n$*.*

Proof Fix a degree $t-1$ hypersurface T of H. Let $\mathcal{S}$ be the set of all $S \subset K$ such that $|S| = n+3$ and S is in linear general position. The algebraic set $\mathcal{S}$ is irreducible and $\dim \mathcal{S} = n+3$. For each $S \in \mathcal{S}$ let D_S be the only rational normal curve of $\mathbb{P}^n$ containing S. We know that $K \cup D_S$ is nodal and with arithmetic genus $n+3$ for a general $S \in \mathcal{S}$. Thus, $D_S \cap K = S$ for a general $S \in \mathcal{S}$. Taking S not containing any point of $K \cap H$ we get $D \cap H \cap K = \emptyset$.

Claim: For a general $S \in \mathcal{S}$ the curve D_S is transversal to H.

Proof of the claim: We allow the case $n = 2$ and use induction on n to prove the claim. In the case $n = 2$ Claim 1 is true, because the pencil of conics through 4 points of $\mathbb{P}^2$, no 3 of them collinear, has the 4 points as scheme-theoretic base locus. Now assume $n > 2$ and that for a general $S \subset K$ there is $q \in S$ such that D_S and K are tangent at q. Since $\mathcal{S}$ is irreducible, a monodromy argument shows that K and D_S are tangent at all points of S. Since K is a linearly normal elliptic curve, the linear projection of K from q maps K isomorphically onto a linearly normal elliptic curve. Apply the inductive assumption to get a contradiction.

By Claim 2 of Remark 4.4 the set of all $S \in \mathcal{S}$ containing at least one point of $T \setminus T \cap K \cap H$ has dimension at most $n-2$. Thus, for a general $S \in \mathcal{S}$ the set $D_S \cap H$ is formed by n points (Claim 1), say $p_1, \dots, p_n$, none of them contained in T. Since D_S is a rational normal curve, the points $p_1, \dots, p_n$ span H. Call $\mathcal{A}$ the n-dimensional linear subspace of $|\mathcal{I}_Y(n)|$ formed by the hypersurfaces $T \cup M$ with M a hyperplane of H. It is sufficient to prove that $p_1, \dots, p_n$ gives n independent conditions to $\mathcal{A}$. This is true, because $p_1, \dots, p_n$ are linearly independent. □

5. Preliminaries for the proof of Theorem 1.1

We recall the following result ([2, 25, 31]).

Lemma 5.1 *Fix integers* m, d, g, s *such that* $m \ge 3$*,* $g \ge 0$*,* $d \ge g+m$ *and* $0 \le s(m-1) \le (m+1)d + (m-1)(1-g)$*. Exclude the following cases:*

$$(d, g, s) \in \{(5, 2, 3), (6, 2, 4), (7, 2, 5)\}.$$

Let $S \subset \mathbb{P}^m$ *be a general subset with cardinality* s*. Then there exists a smooth, connected, and nondegenerate curve* $X \subset \mathbb{P}^m$ *such that* $S \subset X$*,* $\deg(X) = d$*,* $p_a(X) = g$*, and* $h^1(\mathcal{O}_X(1)) = 0$*.*

For more in the Brill–Noether range (instead of just the nonspecial range), see [25].

Lemma 5.2 *Fix integers m,d,g,s such that $s \ge 0$ and $A(d-s,g;m)$ is defined. Let $H \subset \mathbb{P}^m$ be a hyperplane. Fix a general $S \subset H$ such that $|S| = s$. Then there exists $X \in A(d,g;m)$ such that X is smooth, $h^1(N_X) = 0$, X intersects transversally H and $S \subset X$.*

Proof Fix $Y \in A(d-s,g;m)$ such that Y is smooth and $h^1(N_Y) = 0$. By Bertini's theorem (or the definition of $A(d-s,g,m)$) a general hyperplane section of Y is formed by $d-s$ points. Instead of Y we take the curve $h(Y)$ with h a general element of $\mathrm{Aut}(\mathbb{P}^m)$. Thus, we may assume that H is transversal to Y. Moving S to another general subset of H with cardinality s we may assume $S \cap (Y \cap H) = \emptyset$. We order the points $p_1,\dots,p_s$ of S. Let D_i, $1 \le i \le s$, be a general line containing p_i and 1-secant to Y. Set $W := Y \cup D_1 \cup \cdots \cup D_s$. Each D_i meets H only at p_i; hence, $\deg(W) = d$. For general S and general $D_1,\dots,D_s$ we have $p_a(W) = g$. Applying s times Lemma 2.7 we get $W \in A(d,g;m)$. Since W contains S and intersects transversally H, a general element of $A(d,g;m)$ contains s general points of H and intersects transversally H. □

We often use the following lemma, called lemme d'Horace in the original source ([18]) and sometimes called the Horace Lemma.

Lemma 5.3 *Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix reduced schemes $Y, D \subset \mathbb{P}^n$ such that $D \subset H$ and no irreducible component of Y is contained in H. Fix any $k \in \mathbb{N}$. We have an exact sequence*

$$0 \to \mathcal{I}_Y(k-1) \to \mathcal{I}_{Y\cup D}(k) \to \mathcal{I}_{(Y\cap H)\cup D,H}(k) \to 0$$

(called the residual exact sequence of H). Thus,

$$h^0(\mathcal{I}_{Y\cup D}(k)) \le h^0(\mathcal{I}_Y(k-1)) + h^0(H, \mathcal{I}_{(Y\cap H)\cup D,H}(k)),$$

$$h^1(\mathcal{I}_{Y\cup D}(k)) \le h^1(\mathcal{I}_Y(k-1)) + h^0(H, \mathcal{I}_{(Y\cap H)\cup D,H}(k)).$$

Lemma 5.4 *Fix nonnegative integers $n,d,d',g',x,s,t,g,e,k,t,t',\delta,w$ satisfying the following conditions:*

(i) $n \ge 4$, $x > 0$, $k \ge 4$, $0 \le w \le \delta$, $0 \le s \le n-2$;

(ii) $d' \ge g' + n - 1$;

(iii) $0 \le s \le \lfloor (nd + (n-2)(1-g))/(n-1) \rfloor$;

(iv) $(w+x)(n-2) \le nd' + (n-1)(1-g')$;

(v) $g = 1 + t(n+2) + s$.

Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix $T \subset H$ such that $T = T' \cup T''$, $T' \cap T'' = \emptyset$, T' is a union of t' disjoint lines $T_1, \dots T_{t'}$ and T'' is a closed subscheme with $\dim T'' \le 1$. Assume the existence of $D \in A(d',g';H)$ such that $D \cap T = \emptyset$, $h^1(H, \mathcal{I}_{D\cup T,H}(k-2)) = 0$ and $h^0(H, \mathcal{I}_{D\cup T,H}(k-2)) \ge n + 1 + (t-1)n$. Then there exist $Y \in A(d,g;n)$, δ disjoint lines $D_1, \dots D_\delta$, and $D' \in W(d',g';H)$ such that

1. *$D_i \cap D' \ne \emptyset$ if and only if $1 \le i \le w$;*
2. *$|T'_i \cap Y| = 1$ for all $i = 1,\dots,t'$;*

3. $|Y \cap D'| = x$;

4. $h^0(H, \mathcal{I}_{(D' \cup T \cup Y \cup D_1 \cup \cdots \cup D_\delta) \cap H, H}(k)) = \max\{0, h^0(H, \mathcal{I}_{D' \cup T, H}(k)) - (d - x - t') - (\delta - w)\}$;

5. $Y \cup D' \cup T' \in A(d + d' + t', g + g' + x - 1; n)$.

Proof Let $P \subset H$ be a general subset with cardinality $w + x$. By (iv) and Lemma 5.1 there is a nonspecial $Y' \in A(d', g'; H)$ such that $P \subset Y'$ and $h^1(N_{Y'}) = 0$. We may assume $Y' \cap T = \emptyset$ (use $h(T)$, with h a general element of $\mathrm{Aut}(H)$ instead of T and apply Kleiman's Bertini theorem [19]). By semicontinuity we may assume $h^1(\mathcal{I}_{D' \cup T}(a)) = 0$ for all $a \geq k - 1$. Fix $p \in P$. Let $C \subset \mathbb{P}^n$ be a general linearly normal elliptic curve such that $p \in C$ and C intersects transversally H. Take $s + 2$ general 2-secant lines to C. Take t general rational normal curves $D_1, \dots, D_t$ such that $|D_i \cap C| = n + 3$ for all i and $D_i \cap D_j = \emptyset$ for all $i \neq j$ (Lemma 4.2). We have $E := C \cup D_1 \cup \cdots \cup D_t \in A(n + 1 + tn, 1 + t(n + 2); n)$. Applying t times Lemma 4.5 we get $h^1(H, \mathcal{I}_{D' \cup T \cup (Y \cup H), H}(k)) = 0$. Then we add $x - 1$ lines 1-secant to C, each of them containing a different point of $P \setminus \{p\}$. Then we add δ further 1-secants to C; we add w lines $D_1, \dots, D_w$ through the remaining points of P and $\delta - w$ general lines $D_{w+1}, \dots, D_\delta$. Note that (5) follows from Lemmas 2.7 and 2.8. □

Remark 5.5 *In all quotations of Lemma 5.4 we will have*

$$\chi(\mathcal{I}_{Y \cup T, H}(k)) + d + \delta - x + I = \binom{n + k - 1}{n - 1} \tag{5.1}$$

with $I = 0$, except in Section 9. Thus, to check that $h^0(H, \mathcal{I}_{Y \cup T, H}(k)) \geq 2n + t(n - 1)$ to apply Lemma 5.4 it is sufficient to check the following inequality:

$$d + \delta + 2n + t(n - 1) \leq \binom{n + k - 1}{n - 1} + x - I \tag{5.2}$$

6. The assertion $B(k)$

For all integers $k \geq 2$ set

$$b_k := n! k^{n-2} \tag{6.1}$$

For all integer $m \geq 3$ and $k \geq 2$ define the integers $g_{k,m}$ and $f_{k,m}$ by the following relations

$$k(g_{k,m} + m) + f_{k,m} = \binom{m + k}{m}, \quad 0 \leq f_{k,m} \leq k - 2 \tag{6.2}$$

Note that

$$-1 - m + \binom{m + k}{m}/k \leq g_{k,m} \leq \binom{m + k}{m}/(k - 2) \tag{6.3}$$

Since $f_{k,m} \leq k - 2$, (6.3) gives $g_{k,m} \geq f_{k,m}$; hence, $A(g_{k,m} + m, g_{k,m} - f_{k,m}; m)$ is well-defined and its general element is nonspecial.

Remark 6.1 *By [4, 5, 8] (respectively for $m = 4$, $m = 3$ and $m > 4$) a general $T \in A(g_{k,m} + m, g_{k,m} - f_{k,m}; m)$ satisfies $h^i(\mathbb{P}^m, \mathcal{I}_T(k)) = 0$, $i = 0, 1$.*

Fix an integer $a \geq 2n+6$ (depending on n) such that for all $k > a-2$ the following inequalities hold:

$$g_{k,n} \geq n(n+2+k) + (n+2)b_k \tag{6.4}$$

$$\binom{n+k-1}{n}/(k-3) \geq 1 + \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1} + b_k \tag{6.5}$$

$$nk(b_k - b_{k-1} + 2k) \leq \binom{n+k-1}{n-1} - \binom{n+k-1}{n}/(k-2) - b_k + b_{k-1} \tag{6.6}$$

$$2\binom{n+a-1}{n}/(a-2) \geq n^2 + n + 1 + b_{a-1} \tag{6.7}$$

The integer a exists because $\binom{t}{m}$ (as a function of t) is a degree m polynomial with $t^m/m!$ as its leading term. For instance, the left hand side of (6.5) is a degree $n-1$ polynomial in k with $1/n!$ as its leading coefficient, while the right hand side is a degree $n-2$ polynomial in k.

For all integers $k \geq 2$ we define the integers d_k and g_k satisfying the following relation

$$kd_k + 1 - g_k = \binom{n+k}{n} \tag{6.8}$$

in the following way. For $k \leq a-1$ set $d_k := g_{k,n} + n$ and $g_k := g_{k,n} - f_{k,n}$ (we have $g_{k,n} \geq f_{k,n}$ by Lemma 9.1). Set $c_{a-1} := (n+2)(d_{a-1} - n - 1) - n(g_{a-1} - 1)$. We have $c_{a-1} \geq b_{a-1}$ (Lemma 9.2). For all integers $k \geq a$ set $c_k := b_k - b_{a-1} + c_{a-1}$. Note that $b_k - b_{k-1} = c_k - c_{k-1}$ for all $k \geq a$. Fix an integer $k \geq a$ and assume defined the pairs $(d_i, g_i) \in \mathbb{N}^2$ for all $i \leq k-1$. We also assume that for $a-1 \leq i \leq k-1$ (6.8) for i instead of k is satisfied and that for $a-1 \leq i \leq k-1$ the following inequalities are satisfied

$$c_i \leq (n+2)(d_i - n - 1) - n(g_i - 1) \leq c_i + (i-a+1)(i-a+2)n/2 \tag{6.9}$$

By the definition of c_{a-1} for $i = a-1$ the two inequalities in (6.9) are equalities for $i = a-1$.

Notation 6.2 *Let z be the maximal integer such that:*

$$(k - \frac{n+2}{n})z \leq \binom{n+k-1}{n-1} - d_{k-1} - b_k + b_{k-1} \tag{6.10}$$

Set $d_k := d_{k-1} + z$ and $g_k := kd_k + 1 - \binom{n+k}{n}$.

The maximality of the integer z in (6.10) gives

$$0 \leq \binom{n+k-1}{n-1} - c_k + c_{k-1} - (k - \frac{n+2}{n})z \leq k \tag{6.11}$$

By the definition of the integer g_k (6.8) is satisfied. Taking the difference between (6.8) and the same equation for the integer $k-1$ we get

$$d_{k-1} + k(d_k - d_{k-1}) + g_{k-1} - g_k = \binom{n+k-1}{n-1} \tag{6.12}$$

Claim 1: The inequalities in (6.9) are satisfied for $i = k$.

Proof of Claim 1: Since the inequalities in (6.9) are satisfied for $i = k-1$, it is sufficient to prove that

$$c_k - c_{k-1} \le (n+2)(d_k - d_{k-1}) - n(g_k - g_{k-1}) \le c_k - c_{k-1} + kn \tag{6.13}$$

This is true by Lemma 9.3.

For $i = k$ the first inequality in (6.9) shows that the variety $A(d_k, g_k; n)$ is defined for all $k \ge 2$.

For each integer $k > 0$ we define the Assertion $B(k)$ in the following way.

Assertion $B(k)$**:** We have $h^i(\mathcal{I}_X(k)) = 0$, $i = 0, 1$, for a general $X \in A(d_k, g_k; n)$.

Lemma 6.3 *Assertion $B(k)$ is true for all positive integers k.*

Proof If $k < a$, then use [4, 5] respectively for the case $n = 4$ and the case $n \ge 5$. Now assume $k \ge a$ and that $B(k-1)$ is true. Fix a general $Y \in A(d_{k-1}, g_{k-1}; n)$. By $B(k-1)$ and the semicontinuity theorem for cohomology we have $h^i(\mathcal{I}_Y(k-1)) = 0$. Take z, d_k and g_k as in Notation 6.2 and set $s := n + g_k - g_{k-1} - z$.

(a) Assume $s > 0$, i.e. assume $z < n + g_k - g_{k-1}$. Fix a general $S \subset H$ such that $|S| = s$. Since $(n+2)(d_{k-1} - s - n - 1) \ge n(g_{k-1} - 1)$, $A(d_{k-1} - s, g_{k-1}; n)$ is defined. We have $d_{k-1} - s = d_{k-1} - n - g_k + g_{k-1} + z$. By Lemma 9.5 $A(d_{k-1} - s; g_{k-1}; n)$ is well-defined. By Lemma 5.2 we may assume that Y intersects transversally H and that $S \subset Y \cap H$. We have $s(n-2) \le nz + (n-2)(1 - z - 1 + n) = 2z + n^2 - 2n$ by Lemma 9.6. Thus, by Lemma 5.2, i.e. by [2], there is a nonspecial $D \in A(z, z-n+1; H)$ containing S. For a general S we may also assume that D is general in $A(z, z-n+1; H)$; hence, by [4–6], respectively in the cases $n-1 = 4$, $n-1 = 3$ and $n-1 > 4$, the curve D has maximal rank. By Lemma 9.7 we have $(k-2)z + 1 - z + n - 1 \le \binom{n+k-3}{n-1}$. Thus, $h^1(H, \mathcal{I}_{D,H}(k-2))) = 0$. By Lemmas 2.7 and 2.8 we may degenerate Y to a curve $K \cup D_1 \cdots \cup D_s$, where $K = E \cup T \cup L_1 \cup \cdots \cup L_{s-n-1}$, E is a general element of $A(d_{k-1} - sn - s, g_{k-1} - 1 - s(n+2); n)$, T a linearly normal elliptic curve meeting E quasitransversally at a unique point, $L_1, \dots, L_{s-n-1}$ general lines intersecting T, each D_i a general rational normal curve of $\mathbb{P}^n$ intersecting T quasitransversally at exactly $n+3$ points. For a general T we may assume that T intersects transversally H and that any n of the points of $T \cap H$ span H. Since any two sets of $n+1$ points of H are projectively equivalent, we may assume that $T \cap H$ are $n+1$ general points of H. Thus, for general lines $L_1, \dots, L_{s-n-1}$ intersecting H the s points $(T \cup L_1 \cup \cdots \cup L_{s-n-1}) \cap H$ is a general union of s points of H. Thus, we may take $(T \cup L_1 \cup \cdots \cup L_{s-n-1}) \cap H = S$. Thus, $(T \cup L_1 \cup \cdots \cup L_{s-n-1} \cup D) \cap H = D$ as schemes. Recall that $h^1(H, \mathcal{I}_D(k-2)) = 0$. Set $W := T \cup L_1 \cup \cdots \cup L_{s-n-1} \cup D_1 \cup \cdots \cup D_s$. We may smooth $W \cup E$ to a general $Y' \in A(d_{k-1}, g_{k-1}; n)$. Moving S (and the curve D) along this smoothing we get that $Y' \cup D$ is a connected nodal curve of degree d_k and arithmetic genus g_k such that $Y' = \mathrm{Res}_H(Y' \cup D)$ satisfies $h^i(\mathcal{I}_{Y'}(k-1)) = 0$, $i = 0, 1$. We have $h^i(H, \mathcal{I}_{(Y' \cup D) \cap H, H}(k)) = 0$, $i = 0, 1$, by (6.5) and Lemma 5.4; to apply Lemma 5.4 we need the inequality $(k-2)z + 1 - z + n + (n+1) + n(t-1) \le \binom{n+k-3}{n-1}$ which is true by Lemma 9.7. The residual exact sequence of H gives $h^i(\mathcal{I}_{Y' \cup D}(k)) = 0$, $i = 0, 1$. Lemma 2.9 give $Y' \cup D \in A(d_k, g_k; n)$, concluding the inductive proof.

(b) Assume $s \le 0$, i.e. assume $z \ge n + g_k - g_{k-1}$. We take one point, P, instead of S, and take as D a general $D \in A(z, g_k - g_{k-1}; H)$ containing P. □

7. Assertion $A(k)$

Fix a real number $\epsilon > 0$. To prove Theorem 1.1 we need to find an integer d_0 (depending on ϵ and n) such that for all $(d,g) \in \mathbb{N}^2$ with $d \geq d_0$ and $g \leq (\frac{n+2}{n} - \epsilon)d$ the component $A(d,g;n)$ is defined and a general $X \in A(d,g;n)$ has maximal rank.

Definition 7.1 *For any $(d,g) \in \mathbb{N}^2$ such that $d < g+n$, $2d+1-g > \binom{n+2}{2}$ and $A(d,g;n)$ is defined (i.e. $(n+2)(d-n-1) \geq n(g-1)$) the* critical value *of (d,g) is the minimal integer $x \geq 2$ such that $xd+1-g \leq \binom{x+n}{n}$.*

Definition 7.2 *Fix an integer $h \geq a+2$ (depending on n and ϵ) such that for all integers $k \geq h-2$ we have*

$$d_k < \frac{(n+\epsilon/2)g_k}{n+2}, \tag{7.1}$$

$$g_{k+5}\Big(k+4-\frac{1}{n+\epsilon/2}\Big) \leq g_k\Big(k+4-\frac{1}{n+\epsilon}\Big), \tag{7.2}$$

The existence of h is obvious, because $\lim_{k\to+\infty} \binom{k}{n}/k^n = 1/n!$ and we have $\lim_{k\to+\infty} d_k/g_k = n/(n+2)$ and $\lim_{k\to+\infty} g_{k+1}/g_k = 1$ by Lemma 9.4.

Definition 7.3 *Set $k(\epsilon) := h+6$.*

Fix an integer $v \geq k(\epsilon)$ and $(d,g) \in A(d,n;g)$ such that $d < g+n$, (d,g) has critical value v and $(n+2)(d-n-1) \geq (n+\epsilon)(g-1)$. Since $h \geq a$, we have $g_{a-1} \leq g$. Let m be the maximal integer $k \geq a-1$ such that $g_k \leq g$; g_k is well-defined, because $g \geq g_{a-1}$ and $g_i < g_{i+1}$ for all i (Lemma 9.3). By Lemma 9.8 we have $m \leq v-6$. If $g - g_m < m$ set $u := m$, $d'_u := d_u$, $v_u := g - g_u$; note that $0 \leq v_u < m$.

Assume $g - g_u \geq m$. In this case we set $u := m+1$ and define the integers d'_u and v_u by the relations

$$ud'_u + 1 - g + v_u = \binom{n+u}{n}, \ 0 \leq v_u < u. \tag{7.3}$$

Remark 7.4 *Since $g_{m+1} > g$, when $u = m+1$ we have $d'_u \leq d_u$ and*

$$u(d_u - d'_u) = g_u - g + v_u. \tag{7.4}$$

Lemma 7.5 *We have $h^i(\mathcal{I}_W(u)) = 0$, $i = 0,1$, for a general $W \in A(d'_u, g-v_u; n)$.*

Proof If $u = m$, then the lemma is the case $k = u$ of Lemma 6.3. Now assume $u = m+1$ and set $z' := d'_u - d_m$, $s := 1+g-v_u-g_m-(z'-n+1)$. By [4, 5, 8] there is $D \in A(z', z'-n+1; H)$ with maximal rank.

(a) Assume $s > 0$. By [4–6] (respectively for the case $n = 5$, $n = 4$ and $n > 5$) a general $D \in A(z', z'-n+1; H)$ has maximal rank. We apply Lemma 5.4 with $d = d_m$, $g = g_m$, $d' = z$, $g' = z-n+1$, $k = u$, $e = t' = \delta = w = 0$, $T = \emptyset$, x and t determined by the following inequalities: $0 \leq x \leq n+1$, $g_m = n+1+x+t(n+2)$. We need to check the assumptions of Lemma 5.4, i.e. $t \geq 0$ and

$h^0(H, \mathcal{I}_{D,H}(u-2)) \ge n+1+(t-1)n$. We explain the numerology behind Lemma 5.4. To compare the set-up of the lemma we are proving with the one of Lemma 6.3 we write $k := u$, $\mu_k := g - v_u$ and $z' := d'_u - d_{k-1}$. In the set-up of Lemma 6.3 we set $z := d_k - d_{k-1}$. Thus, we have the equality

$$d_{k-1} + kz + g_{k-1} - g_k = \binom{n+k-1}{n-1} \tag{7.5}$$

We called Y a curve in $\mathbb{P}^n$ with $\deg(Y) = d_{k-1}, p_a(Y) = g_{k-1}$ and $h^i(\mathcal{I}_X(k-1)) = 0$, $i = 0,1$. We had $\sharp(Y \cap D) = s$ and $g_k - g_{k-1} = z - n + 1 + s - 1$. In the set-up of the lemma we need to prove that the curve Y we have for the degree $k-1$ is the same curve as the one in Lemma 6.3. We set $s' := \sharp(Y \cap D)$. We have $\mu_k = g_{k-1} + z - n + 1 + s' - 1$. Thus,

$$d_{k-1} + kz' + g_{k-1} - \mu_k = \binom{n+k-1}{n-1} \tag{7.6}$$

Since $\mu_k \le g_k$, we have $z' \le z$ and $s' \le s$. The inequality $s' \ge 0$ is true by the definition of u. Since $s' \le s$, the check for Y is Lemma 9.5. Moreover, to check that $h^0(H, \mathcal{I}_{D'}(k-2)) \ge 2n + n(t-1)$ we have the same t as the one in Lemma 6.3. Both D' and D have maximal rank with $D' \in A(z', z-n+1; H)$, $D \in A(z, z-n+1; H)$ and $z' \le z$. Thus, $h^0(H, \mathcal{I}_{D'}(k-2)) \ge h^0(H, \mathcal{I}_D(k-2)) \ge n+1+(t-1)n$. Now we check that we may find D' passing through s' general points of H. Since $\mu_k = s' - 1 + g_{k-1} + z' - n + 1$ and $g_k = s - 1 + g_{k-1} + z - n + 1$, subtracting (7.6) from (7.5) we get

$$(k-1)(z - z') = s - s' \tag{7.7}$$

Since $k > n$ and $z \ge z' \ge n$ if (z, s) satisfies the assumptions of Lemma 5.2 for $m = n-1$, then (z', s) satisfies the same assumption.

(b) Assume $s \le 0$. In this case instead of D we add a nonspecial curve of degree z' with lower genus and/or meeting in a smaller number of points the curve Y. To quote Lemma 5.4 we only need to use that $(u-2)z' + 1 + n + 1 + (t-1)n \le \binom{n+u-3}{n-1}$, which is true, because $z' \le d_k - d_{k-1}$ and the integer t is the same as the one appearing in the proof of Lemma 6.3. □

Definition 7.6 *For every integer $j \ge u$ define the integers a_j and x_j by the relations*

$$ja_j + 1 - g + x_j = \binom{n+j}{n}, \ 0 \le x_j < j. \tag{7.8}$$

In particular $a_u = d'_u$ and $x_u = v_u$. Taking the difference between (7.8) and the same equation for the integer $j-1$ we get

$$a_j + j(a_j - a_{j-1}) + x_j - x_{j-1} = \binom{n+j-1}{n-1} \tag{7.9}$$

for all $j > u$.

Consider the following assertion $A(j)$ defined for every integer $j > u$.

Assertion $A(j)$, $j > u$: There is $Y = Z \cup T$ with $Z \in A(a_j - x_j, g; n)$, T a union of x_j disjoint lines, $Z \cap T = \emptyset$ and $h^i(\mathcal{I}_Y(j)) = 0$, $i = 0,1$.

Note that $A(a_j - x_j, g; n)$ is defined, because $a_j - x_j > a_{j-1}$ for all $j > u$ by Lemma 9.9. Lemma 9.9, induction on j and the definition of c_{a-1} give

$$(n+2)(a_j - j - 1) \geq n(g-1) + c_j \tag{7.10}$$

Lemma 7.7 *$A(u+1)$ is true.*

Proof Fix a hyperplane $H \subset \mathbb{P}^n$. By Lemma 9.9 we have $a_{u+1} - x_{u+1} - a_u \geq 0$. By Lemma 7.5 there is $Y \in A(d'_u, g - v_u; n)$ with $h^i(\mathcal{I}_Y(u)) = 0$, $i = 0, 1$. We add in H a curve $A \cup B \subset H$ with A smooth and rational, $\deg(A) = a_{u+1} - x_{u+1} - a_u$, A containing exactly one point of $Y \cap H$, B a union of x_{u+1} disjoint lines and $A \cap B = \emptyset$. By [17] (case $n = 3$) and [7] (case $n > 3$) we may assume that $A \cup B$ has maximal rank in H. We take Y transversal to H, with $\sharp(A \cap Y) = 1$ and with $B \cap Y = \emptyset$. Write $p_a(Y) = g - v_u = 1 + (n+2)t + w$, $d'_u = n + 1 + nt + w + w'$ with $w' \geq 0$. To apply Lemma 5.4 (in the set-up of Remark 5.5) it is sufficient to use that $h^0(H, \mathcal{I}_{A\cup B}(u-1)) \geq n + 1 + (t-1)n$ (Lemma 9.10). We conclude by Lemma 5.3. □

Lemma 7.8 *$A(j)$ is true for all $j \geq u + 2$.*

Proof Assume by induction that $A(j-1)$ is true and take $Y = Z \cup T$ satisfying $A(j-1)$.

(a) Assume $x_j \geq x_{j-1}$. In this case the only difference with respect to the proof of Lemma 7.7 is that now we take $x = 2$ and $d' = a_j - a_{j-1}$. To check the condition on h^0 in Lemma 5.4 we use Remark 5.5 and the proof of Lemma 9.10, i.e. the proof of Lemma 9.5 using (7.9) instead of (6.12) and that $ng_{k-1} - (n+2)d_{k-1} \geq ng - (n+2)a_{j-1}$.

(b) Assume $x_j < x_{j-1}$. By Lemma 9.10 we have $a_j - a_{j-1} \geq n + j \geq n + 1 + x_{j-1} - x_j$. Let $F \subset H$ be a smooth rational curve with maximal rank ([4–6, 18] passing though $1 + x_{j-1} + x_j$ general points of H, one on Z and the remaining ones in different lines of T. We may apply Lemma 5.4 for the reasons explained in step (a). □

8. End of the proof of Theorem 1.1

For all integers $k \geq 2$ set $\gamma_k := 1 - g_k + \lfloor (n+2)(d_k - n - 1)/n \rfloor$. The integer γ_k is the maximal integer such that $A(d_k, g_k + \gamma_k; n)$ is defined.

Remark 8.1 *By Lemma 9.4 we have* $\lim_{k\to+\infty} \gamma_k / k^{n-1} = 0$.

Note that $a_{u+4} \leq d - u - n$ by Lemma 9.8. Let σ be the maximal integer such that $a_\sigma + \sigma \leq d$. Thus, $d < a_{\sigma+1} + \sigma + 1$.

Remark 8.2 *If $d > a_{\sigma+1}$ the critical value v of (d, g) is $\sigma + 2$, while if $d \leq a_{\sigma+1}$ we have $v = \sigma + 1$.*

Lemma 8.3 *Assume $d > a_{\sigma+1}$. Then there is $X \in A(d, g; n)$ such that $h^1(\mathcal{I}_X(\sigma+2)) = 0$.*

Proof The proof is divided into two steps. We first prove an assertion similar to $A(\sigma+1)$ for a connected curve $X \in A(a_{\sigma+1}, g; n)$. Then in step (b) we add a smooth rational curve $A \subset H$ with $\deg(A) = d - a_{\sigma+1}$, $X \cup A \in A(d, g; n)$ and $h^1(\mathcal{I}_{X\cup A}(\sigma+1)) = 0$.

(a) Take $Y = Z \cup T$ satisfying $A(\sigma)$ and intersecting transversally H. Take a smooth rational curve $F \subset H$ such that $\deg(F) = a_{\sigma+1} - a_\sigma$ and containing exactly one point of each connected component of Y. We use Lemmas 5.3 and 5.4 and Remark 5.5 with $I > 0$; to apply Remark 5.5 we observe that when $I > 0$ the inequalities used in the proof of Lemma 9.5 are better by I.

Deform $Y \cup F$ to a smooth $X \in A(a_{\sigma+1}, g; n)$ intersecting transversally H.

(b) By Lemma 9.9 we have $d < a_{\sigma+2}$. Take the union of X and a smooth rational curve $A \subset H$ such that $\deg(A) = a_{\sigma+1} - a_\sigma$ containing exactly one point of $Y \cap H$. By the definition of σ we have $d - a_{\sigma+1} \le \sigma$. Thus, $\deg(A)$ is much smaller than the integer $\deg(F)$ used in step (a). We apply Lemma 5.4 and Remark 5.5 with $I > 0$; the inequalities needed here are easier than the ones used in step (a). □

Lemma 8.4 *Assume $a_\sigma + \sigma \le d \le a_{\sigma+1}$. Then there is $X \in A(d, g; n)$ such that $h^1(\mathcal{I}_X(\sigma+1)) = 0$.*

Proof We start with a curve C satisfying $A(\sigma)$ and hence with $h^0(\mathcal{I}_C(\sigma)) = 0$. We add a smooth rational curve D with $\deg(D) = d - a_\sigma \ge 0$, meeting C at a unique point and quasitransversally. We have $C \cup D \in A(d, g; n)$ by Lemmas 2.7 and 2.8. □

Proof [Proof of Theorem 1.1:] By Remark 8.1 and Lemma 9.4 Theorem 1.1 follows from the irreducibility of $A(d, g; n)$ and Lemma 8.3 (case $d > a_{\sigma+1}$) and Lemma 8.4 (case $a_\sigma + \sigma \le d \le a_{\sigma+1}$). □

9. Numerical lemmas

We will often silently use that as a polynomial in t the polynomial function $\binom{t}{m}$, $m \ge 0$, has degree m and $t^m/m!$ as its leading term.

Lemma 9.1 *For each integer $k \ge 2$ we have $g_{k,n} \ge f_{k,n}$*

Proof Since $f_{k,n} \le k - 2$, we have $g_{k,n} = \lfloor (\binom{n+k}{n} - 1 - kn)/(k-1) \rfloor$. Thus, it is sufficient to check the easy inequality $\binom{n+k}{n} \ge 1 + kn + (k-1)(k-2)$. □

Lemma 9.2 *We have $c_{a-1} \ge b_{a-1}$.*

Proof We have $g_{a-1,n} \le d_{a-1,n} - n$. Since $c_{a-1} = (n+2)(d_{a-1,n} - n - 1) - n(g_{a-1,n} - 1)$ we have $c_{a-1} \ge 2d_{a-1,n} - n^2 - 2n - 1$. Since

$$(a-1)d_{a-1,n} + 1 - g_{a-1,n} = \binom{n+a-1}{n}$$

and $g_{a-1,n} \le d_{a-1,n} - n$, we have

$$(a-2)d_{a-1,n} + 1 - n \ge \binom{n+a-1}{n} \tag{9.1}$$

Use (6.7). □

Lemma 9.3 *Fix an integer $k \ge a$. For every integer $k \ge a$ we have:*

(i) $d_k - d_{k-1} \geq n(b_k - b_{k-1} + 2k)$ *;*

(ii) $g_k > g_{k-1}$ *;*

(iii) $c_k - c_{k-1} \leq (n+2)(d_k - d_{k-1}) - n(g_k - g_{k-1}) \leq c_k - c_{k-1} + nk$ *.*

Proof We use induction on k. We do not write down the initial case of the induction, i.e. the case $k = a$, because the inductive step works verbatim for $k = a$, just using that $g_{a-1} \leq d_{a-1} - n$ and in particular $g_{a-1} < 2d_{a-1}$. We also use that for the integer $k-1$ the variety $A(d_{k-1}, g_{k-1}; n)$ is defined, which is true if we assume (iii) for the integer $k-1$. Set $z := d_k - d_{k-1}$. By (6.10) and the inequality $n \geq 4$ it is sufficient to prove that

$$nk(b_k - b_{k-1} + 2k) \leq \binom{n+k-1}{n-1} - d_{k-1} - b_k + b_{k-1} \tag{9.2}$$

Since $g_{k-1} < 2d_{k-1}$, (6.8) for the integer $k-1$ gives $d_{k-1} \leq \binom{n+k-1}{n}/(k-2)$. Hence, it is sufficient to use the inequality (6.6). Since $d_{k-1} \leq \binom{n+k-1}{n}/(k-2) \leq \binom{n+k-1}{n-1}$ and $(k-1)d_{k-1} + 1 - g_{k-1} = \binom{n+k-1}{n}$ by (6.8) for the integer $k-1$, we have $kd_{k-1} + 1 - g_{k-1} \leq \binom{n+k}{n}$. Since $d_k > d_{k-1}$, (6.8) gives $g_k > g_{k-1}$.

Part (iii) is the case $i = k$ of (6.9) proved before the definition of $B(k)$. □

Lemma 9.4 *We have*

$$\lim_{k \to +\infty} d_k/g_k = \frac{n}{n+2}, \tag{9.3}$$

$$\lim_{k \to +\infty} d_{k+1}/d_k = g_{k+1}/g_k = 1. \tag{9.4}$$

$$\lim_{k \to +\infty} k^{n-1}/d_k = k!\ .$$

Proof By part (ii) of Lemma 9.3 we have $\lim_{k \to +\infty} g_k = +\infty$. Part (iii) of Lemma 9.3 gives (9.3). By (9.3) the two equalities in (9.4) are equivalent. We also see that $\lim_{k \to +\infty} g_k/k^{n-1} = (n+2)/(n!)n$, which implies the second equality in (9.4). □

Lemma 9.5 *In the set-up of the proof of Lemma 6.3 we have* $(n+2)(d_{k-1} - n - g_k + g_{k-1} + z) \geq n(g_{k-1} - 1)$*, i.e.* $A(d_{k-1} - n - g_k + g_{k-1} + z, g_{k-1}; n)$ *is well-defined.*

Proof We have $n + g_k - g_{k-1} - z \leq 2z/n$. Since $(n+2)(d_{k-1} - n - 1) \geq n(g_{k-1} - 1) + c_{k-1}$, it is sufficient to observe that $2z/n \leq c_{k-1}$ by (6.10) and the inequality $c_{k-1} \geq b_{k-1}$. □

Lemma 9.6 *In the set-up of the proof of Lemma 6.3 we have* $s(n-2) \leq 2z + n(n-2)$*, where* $s := n + g_k - g_{k-1} - z$*.*

Proof By the definition of s the lemma is true if and only if

$$(n-2)(g_k - g_{k-1}) \leq nz \tag{9.5}$$

Since $(n+2)z \geq n(g_k - g_{k-1}) + (n+2)(b_k - b_{k-1})$ and $n/(n-2) \geq (n+2)/n$, (9.5) is true. □

Lemma 9.7 *In the notation of the proof of Lemma 6.3 we have* $(k-2)z+1-z+n+(n+1)+n(t-1) \le \binom{n+k-3}{n-1}$.

Proof Look at (6.12). We have $g_k - g_{k-1} \le (n+2)(d_k - d_{k-1})/n - (b_k + b_{k-1})/n$; hence, $(k - \frac{n+2}{n})z \le \binom{n+k-1}{n-1} - (b_k - b_{k-1})/n$. We have $d_{k-1} \ge 1+nt+b_{k-1}/n$. Thus, it is sufficient to have $b_k/n \ge \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}$, which is true by the definition of b_k and the inequality $a \ge n+7$. □

Lemma 9.8 *We have* $m \le v-6$.

Proof Assume $m \ge v-5$. Thus, $m \ge h$ and $g_{v-5} \le g$. Since (d,g) has critical value v, we have

$$vd + 1 - g \le vd_v + 1 - g_v. \tag{9.6}$$

By (9.6), (7.1), and (7.2) we get a contradiction. □

Lemma 9.9 *For every integer* $j > u$ *we have* $a_j - x_j > a_{j-1}$ *and (7.10) is true.*

Proof Assume $a_j - a_{j-1} \le x_j$. Since $x_j \le j-1$ and $x_{j-1} \ge 0$, (7.9) gives $a_j + j^2 - 1 \le \binom{n+j-1}{n-1}$, contradicting the inequality $a_j \le d_j$. Since $j(d_j - a_j) \le g_j - j$ and $(n+2)(d_j - d_{a-1}) \ge n(g_j - g_{j-1}) + b_j - b_{a-1}$, we get (7.10). □

Lemma 9.10 *In the set-up of the proof of Lemma 7.7 we have* $h^0(H, \mathcal{I}_{A\cup B}(u-1)) \le n+1+(t-1)n$.

Proof We mimic the proof of Lemma 9.7 using (7.9) instead of (6.12). In the set-up of Lemma 7.7 we have $z := \deg(A \cup B) = a_{u+1} - a_u$. Since $x_{u+1} \le u$ and $x_u \ge 0$, (7.9) gives $kz \le \binom{n+u}{n-1} - u - a_u$. We use that $a_u \ge n+1+nt+c_u/n$ as in the proof of Lemma 9.5. □

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