Finite element approximation of the spectrum of the curl operator in a multiply-connected domain

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Abstract In this paper we are concerned with two topics: the formulation and analysis of the eigenvalue problem for the curl operator in a multiply-connected domain, and its numerical approximation by means of finite elements. We prove that the curl operator is self-adjoint on suitable Hilbert spaces, all of them being contained in the space for which $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on the boundary. Additional conditions must be imposed when the physical domain is not topologically trivial: we show that a viable choice is the vanishing of the line integrals of $\boldsymbol{v}$ on suitable homological cycles lying on the boundary. A saddle-point variational formulation is devised and analyzed, and a finite element numerical scheme is proposed. It is proved that eigenvalues and eigenfunctions are efficiently approximated, and some numerical results are presented in order to test the performance of the method.

Keywords curl operator · eigenvalues and eigenfunctions · finite element approximation · force-free fields · Beltrami fields

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1 Introduction

In electromagnetism, the Lorentz law states that the density of the magnetic force is given by $\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B}$, where $\boldsymbol{J}$ is the current density and $\boldsymbol{B}$ is the magnetic induction. For eddy current approximation $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$, therefore the Lorentz law reads $\boldsymbol{F} = \mathbf{curl}\,\boldsymbol{H} \times \boldsymbol{B}$.

For linear isotropic media a dependence of the form $\boldsymbol{B} = \mu\boldsymbol{H}$ is assumed, the scalar function μ being the magnetic permeability. In this case a magnetic field satisfying $\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H}$, with λ a scalar function, produces a vanishing magnetic force: $\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B} = \lambda\boldsymbol{H} \times \mu\boldsymbol{H} = \mathbf{0}$. For this reason this kind of fields are called force-free fields.

In fluid dynamics, force-free fields are often called Beltrami fields, and a Beltrami field $\boldsymbol{u}$, when it is also divergence-free and tangential to the boundary, is a steady solution of the Euler equations for incompressible inviscid flows (with pressure given by $p = -\frac{|\boldsymbol{u}|^2}{2}$).

A magnetic field satisfying the equation $\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H}$ with a constant λ is called a linear force-free field. These fields appear in important physical problems: for instance, studying problems arising in plasma physics, in [23] it has been proved that a magnetic field $\boldsymbol{H}$ which minimizes the magnetic energy with fixed helicity has to satisfy the equation $\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H}$ for some constant λ (and thus is a linear force-free field). Moreover, in the presence of dissipation the linear force-free fields in non-stationary magnetohydrodynamics are the natural asymptotic configurations. In fact, as proved in [13], only linear force-free magnetic fields remain force-free as time changes.

The reader interested in additional information about the physical problems related to this kind of fields is referred, e.g., to [8,9] and the references therein. From now on we focus on the mathematical aspects of this problem.

A linear force-free field is an eigenfunction of the curl operator:

$$\mathbf{curl}\,\boldsymbol{u} = \lambda\boldsymbol{u}\,.$$

Natural additional conditions, from both the mathematical and physical point of view, are $\operatorname{div}\boldsymbol{u} = 0$ inside and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on the boundary of the physical domain in which the magnetic field is confined. It is also known that these conditions are not sufficient to give a well-posed spectral problem if the physical domain has a non-trivial topology (namely, it is not simply-connected). For arriving at a well-posed spectral problem, the main mathematical point is to devise a domain of definition of the curl operator such that its restriction to that domain is self-adjoint.

Let $\Omega \subset \mathbb{R}^3$ be a bounded open connected set with Lipschitz continuous boundary Γ (either smooth or polyhedral) and outer unit normal vector $\boldsymbol{n}$. The way for defining self-adjoint realizations of the curl operator and the analysis of the associated eigenvalue problem have since many years attracted the attention of many researchers. The starting point is clearly the following Green's formula

$$\int_\Omega (\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w} - \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{w}) = \int_\Gamma \boldsymbol{v}\times\boldsymbol{n}\cdot\boldsymbol{w}\,,$$

that is valid for any regular enough fields $\boldsymbol{v}$ and $\boldsymbol{w}$. The choices of the domain of definition of a symmetric realization of the curl operator are driven by the need of obtaining $\int_\Gamma \boldsymbol{v} \times \boldsymbol{n} \cdot \boldsymbol{w} = 0$.

It is clear that the curl operator is symmetric when acting on vector fields with vanishing tangential components on Γ, namely, satisfying $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. However, it is well-known that this condition is too strong for the spectral problem (for a simple proof see Remark 1; see also [8, pp. 5638–5639], [24, Lemma 3]).

Instead, in a series of papers it has been shown that the curl operator is self-adjoint when restricted to a suitable domain of definition (see [19], [20], [14], [25], [21], [12]). Let us start describing the situation for a simply-connected domain Ω: in this case it is sufficient to assume that the vector fields $\boldsymbol{v}$ in the domain satisfy the boundary condition $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ. Since for an eigenfunction $\boldsymbol{u}$ of the curl operator associated to a non-zero eigenvalue the condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ implies $\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0$, it is thus natural to consider the spectral problem

$$\begin{aligned} &\mathbf{curl}\, \boldsymbol{u} = \lambda \boldsymbol{u} && \text{in } \Omega \\ &\operatorname{div} \boldsymbol{u} = 0 && \text{in } \Omega \\ &\boldsymbol{u} \cdot \boldsymbol{n} = 0 && \text{on } \Gamma\,. \end{aligned} \tag{1}$$

Notice that in a simply-connected domain Ω the unique field satisfying $\mathbf{curl}\, \boldsymbol{u} = \boldsymbol{0}$, $\operatorname{div} \boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ is $\boldsymbol{u} = \boldsymbol{0}$, so $\lambda = 0$ is not an eigenvalue of this problem. The numerical approximation of (1) for a simply-connected domain was analyzed in [22].

When Ω is not simply-connected the condition $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$ is not enough to obtain a self-adjoint **curl** and it is necessary to consider a smaller domain of definition of the curl operator. In particular, it is known that if Ω is not simply-connected the set of eigenvalues of (1) is the whole complex plane (see [25, Theorem 2]).

In [19], [20], [14], [25], [21] it is proved that the curl operator is self-adjoint when acting on vector fields $\boldsymbol{v}$ such that $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ and $\mathbf{curl}\, \boldsymbol{v} \perp \boldsymbol{\mathcal{K}}_T$, where $\boldsymbol{\mathcal{K}}_T$ is the space of the so-called harmonic Neumann fields $\boldsymbol{h}$ satisfying $\mathbf{curl}\, \boldsymbol{h} = \boldsymbol{0}$ in Ω, $\operatorname{div} \boldsymbol{h} = 0$ in Ω and $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ (this space is finite dimensional, its dimension being the first Betti number of Ω; in particular, it is trivial for a simply-connected domain Ω). The numerical approximation of the eigenvalues and eigenfunctions of this problem have been studied in [15].

Differently from what reported in [25, Remark 2], the condition $\mathbf{curl}\, \boldsymbol{v} \perp \boldsymbol{\mathcal{K}}_T$ is not essential, but only sufficient for the proof that the curl operator is self-adjoint. In this respect, an in-depth analysis has been recently presented in [12] using the theory of unbounded operators in Hilbert spaces. The authors, by incorporating in problem (1) additional conditions related to the first homology group of Γ and by using the tools of differential forms, propose suitable choices of the domain of definition that lead to self-adjoint realizations of the curl operator.

Following the analysis in [12], let us show how this family of well-posed eigenvalue problems can be described. We first need to recall some geometrical results (see, e.g., [7, Sect. 6]; see also [4]). If the first Betti number of Ω is equal to $g > 0$, then the first Betti number of Γ is equal to $2g$ and it is possible to consider $2g$ non-bounding cycles on Γ, $\{\gamma_j\}_{j=1}^g \cup \{\gamma'_j\}_{j=1}^g$, that are (representative of) the generators of the first homology group of Γ and such that:

- $\{\gamma_j\}_{j=1}^g$ are (representative of) the generators of the first homology group of $\Omega' = B \setminus \overline{\Omega}$, being B an open ball containing $\overline{\Omega}$ (the tangent vector on γ_j is denoted by $\mathbf{t}_j$);
- $\{\gamma'_j\}_{j=1}^g$ are (representative of) the generators of the first homology group of $\overline{\Omega}$ (the tangent vector on γ'_j is denoted by $\mathbf{t}'_j$);
- in Ω there exist g 'cutting' surfaces $\{\Sigma_j\}_{j=1}^g$, that are connected orientable Lipschitz surfaces satisfying $\Sigma_j \subset \Omega$ and $\partial\Sigma_j \subset \Gamma$, such that every curl-free vector in Ω has a global potential in the 'cut' domain $\Omega^0 := \Omega \setminus \bigcup_{j=1}^g \Sigma_j$; each surface Σ_j satisfies $\partial\Sigma_j = \gamma_j$, 'cuts' the corresponding cycle γ'_j and does not intersect the other cycles γ'_i for $i \neq j$;
- in Ω' there exist g 'cutting' surfaces $\{\Sigma'_j\}_{j=1}^g$, that are connected orientable Lipschitz surfaces satisfying $\Sigma'_j \subset \Omega'$ and $\partial\Sigma'_j \subset \Gamma$, such that every curl-free vector in Ω' has a global potential in the 'cut' domain $(\Omega')^0 := \Omega' \setminus \bigcup_{j=1}^g \Sigma'_j$; each surface Σ'_j satisfies $\partial\Sigma'_j = \gamma'_j$, 'cuts' the corresponding cycle γ_j, and does not intersect the other cycles γ_i for $i \neq j$.

In particular we can assume that $\gamma_i \cap \gamma_j = \emptyset$ and $\gamma'_i \cap \gamma'_j = \emptyset$ if $i \neq j$, while γ_i intersects γ'_i just at a point P_i. Moreover, $\overline{\Sigma_j}$ and $\overline{\Sigma'_j}$ intersect the boundary Γ in a transversal way, namely, the unit normal unit $\boldsymbol{n}_j$ to $\overline{\Sigma_j}$ is not parallel to $\boldsymbol{n}$ on $\partial\Sigma_j \subset \Gamma$, and similarly for Σ'_j.

Let us also note the statement concerning the 'cutting' surfaces Σ_j does not mean that the 'cut' domain Ω^0 is simply connected nor that it is homologically trivial: an example in this sense is furnished by $\Omega = Q \setminus K$, where Q is a cube and K is the trefoil knot (see [4]).

In [12] it has been shown that the curl operator turns out to be self-adjoint in the space of vector fields $\boldsymbol{v}$ with $\mathbf{curl}\,\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ and such that $\oint_{\gamma_i} \boldsymbol{v} \cdot \mathbf{t}_i = 0$ for $1 \leq i \leq g_1$ and $\oint_{\gamma'_j} \boldsymbol{v} \cdot \mathbf{t}'_j = 0$ for $g_1 + 1 \leq j \leq g$, where $0 \leq g_1 \leq g$. (The precise meaning of $\oint_{\gamma'_j} \boldsymbol{v} \cdot \mathbf{t}'_j$ and $\oint_{\gamma_i} \boldsymbol{v} \cdot \mathbf{t}_i$ for vector fields satisfying the condition $\mathbf{curl}\,\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ will be clarified in the next section.)

It is worth noting that, when $\mathbf{curl}\,\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, the choice $g_1 = g$, namely, $\oint_{\gamma_i} \boldsymbol{v} \cdot \mathbf{t}_i = 0$ for $1 \leq i \leq g_1 = g$, is equivalent to $\mathbf{curl}\,\boldsymbol{v} \perp \boldsymbol{\mathcal{K}}_T$ (see Remark 2).

We also underline that, as reported in [9], the most interesting physical case is the one given by the choice $g_1 = 0$, i.e., the additional conditions are given by $\oint_{\gamma'_j} \boldsymbol{v} \cdot \mathbf{t}'_j = 0$ for $1 \leq j \leq g$. In fact, in this case the eigenfunction associated to the eigenvalue of minimum absolute value realizes the minimum of the magnetic energy with fixed helicity.

While the numerical approximation of the eigenvalues and eigenfunctions of (2) in the case $g_1 = g$ was already studied in [15], up to now a complete analysis of the variational formulation of the eigenvalue problem and its numerical approximation for $g_1 = 0$ were not available, and are presented in our paper for the first time.

Summing up, we consider the following eigenvalue problem:

Problem 1. Find $\lambda \in \mathbb{C}$ and $\boldsymbol{u} \in L^2(\Omega)^3$, $\boldsymbol{u} \neq \boldsymbol{0}$, such that

$$\begin{aligned} &\mathbf{curl}\boldsymbol{u} = \lambda \boldsymbol{u} && \text{in } \Omega\,, \\ &\mathrm{div}\boldsymbol{u} = 0 && \text{in } \Omega\,, \\ &\boldsymbol{u}\cdot\boldsymbol{n} = 0 && \text{on } \Gamma\,, \\ &\oint_{\gamma_i} \boldsymbol{u}\cdot\boldsymbol{t}_i = 0 && 1 \le i \le g_1\,, \\ &\oint_{\gamma'_j} \boldsymbol{u}\cdot\boldsymbol{t}'_j = 0 && g_1+1 \le j \le g\,, \end{aligned} \tag{2}$$

where $0 \le g_1 \le g$.

Notice that, from the first and third equations, a solution of Problem 1 satisfies $\mathbf{curl}\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ, hence the last two conditions have a meaning.

Aim of this work is to furnish a sound variational formulation of this general problem and to devise an efficient finite element numerical approximation method. In the next section we present some preliminaries results in order to give a precise meaning to the additional topological conditions. In Section 3 we prove the well-posedness of a mixed variational formulation of this spectral problem. Then in Section 4 we study the finite element approximation of the eigenvalues and the eigenspaces. Finally in Section 5 we present some numerical experiments which allow us to check the theoretical results and to assess the performance of the method.

Remark 1 It is easy to show that the curl operator with the boundary condition $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}$ has no eigenvalues $\lambda \neq 0$. In fact, suppose that we have $\boldsymbol{u} \neq \boldsymbol{0}$ such that

$$\begin{aligned} &\mathbf{curl}\,\boldsymbol{u} = \lambda\boldsymbol{u} \text{ in } \Omega\,, \\ &\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma \end{aligned}$$

for $\lambda \neq 0$. Let $a > 0$ be a number so large that $\Omega \subset Q_a = [-a\frac{\pi}{2}, a\frac{\pi}{2}]^3$, and let $\widetilde{\boldsymbol{u}}$ be the extension of $\boldsymbol{u}$ in Q_a by setting value $\boldsymbol{0}$ outside Ω. Since $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, it follows that $\mathbf{curl}\,\widetilde{\boldsymbol{u}}$ is globally defined in Q_a; moreover $\mathbf{curl}\,\widetilde{\boldsymbol{u}} = \lambda\widetilde{\boldsymbol{u}}$ in Q_a, and therefore $\mathrm{div}\,\widetilde{\boldsymbol{u}} = 0$ in Q_a. Due to the relation $-\Delta = \mathbf{curl}\,\mathbf{curl} - \nabla\mathrm{div}$ we also obtain $-\Delta\widetilde{\boldsymbol{u}} = \mathbf{curl}\,\mathbf{curl}\,\widetilde{\boldsymbol{u}} = \lambda\mathbf{curl}\,\widetilde{\boldsymbol{u}} = \lambda^2\widetilde{\boldsymbol{u}}$ in Q_a. Noting that $\widetilde{\boldsymbol{u}} = \boldsymbol{0}$ on ∂Q_a, we see that λ^2 is an eigenvalue of the Laplace operator in the cube Q_a with homogeneous Dirichlet boundary condition. These eigenvalues are well-known, and are given by N/a^2 for suitable positive integers N. Repeating the same argument for the cube $Q_{2^{1/4}a}$, it follows that λ^2 is also equal to $M/(\sqrt{2}a^2)$ for a suitable positive integer M, and this is not possible as $\sqrt{2}$ is not a rational number.

2 Preliminary results

We consider the space $L^2(\Omega)$ with its corresponding norm $\|\cdot\|_{0,\Omega}$. For convenience, we denote $\|\cdot\|_{0,\Omega}$ the norm of $L^2(\Omega)^3$, too. Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable functions with compact support in Ω and $\mathcal{D}(\overline{\Omega}) := \{\phi_{|\Omega} : \phi \in \mathcal{D}(\mathbb{R}^3)\}$.

Let us introduce the Hilbert spaces

$$
\begin{aligned}
&H^1(\Omega) := \left\{ v \in L^2(\Omega) \,:\, \nabla v \in L^2(\Omega)^3 \right\} \,, \\
&H^1_0(\Omega) := \left\{ v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma \right\} \,, \\
&H(\mathrm{div};\Omega) := \left\{ \boldsymbol{v} \in L^2(\Omega)^3 \,:\, \mathrm{div}\boldsymbol{v} \in L^2(\Omega) \right\} \,, \\
&H(\mathrm{div}^0;\Omega) := \{ \boldsymbol{v} \in H(\mathrm{div};\Omega) \,:\, \mathrm{div}\boldsymbol{v} \,=\, 0 \text{ in } \Omega \} \,, \\
&H_0(\mathrm{div};\Omega) := \{ \boldsymbol{v} \in H(\mathrm{div};\Omega) \,:\, \boldsymbol{v}\cdot\boldsymbol{n} \,=\, 0 \text{ on } \Gamma \} \,, \\
&H_0(\mathrm{div}^0;\Omega) := H(\mathrm{div}^0;\Omega) \cap H_0(\mathrm{div};\Omega) \,, \\
&H(\mathbf{curl};\Omega) := \left\{ \boldsymbol{v} \in L^2(\Omega)^3 \,:\, \mathbf{curl}\boldsymbol{v} \in L^2(\Omega)^3 \right\} \,, \\
&H(\mathbf{curl}^0;\Omega) := \{ \boldsymbol{v} \in H(\mathbf{curl};\Omega) \,:\, \mathbf{curl}\boldsymbol{v} \,=\, \mathbf{0} \text{ in } \Omega \} \,, \\
&H_0(\mathbf{curl};\Omega) := \{ \boldsymbol{v} \in H(\mathbf{curl};\Omega) \,:\, \boldsymbol{v}\times\boldsymbol{n} \,=\, \mathbf{0} \text{ on } \Gamma \} \,.
\end{aligned}
$$

The spaces $H(\mathrm{div};\Omega)$ and $H(\mathbf{curl};\Omega)$ are respectively endowed with the norms defined by

$$
\|\boldsymbol{v}\|^2_{\mathrm{div},\Omega} \,:=\, \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\boldsymbol{v}\|^2_{0,\Omega} \quad \text{and} \quad \|\boldsymbol{v}\|^2_{\mathbf{curl},\Omega} \,:=\, \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathbf{curl}\boldsymbol{v}\|^2_{0,\Omega} \,.
$$

We recall the classical Helmholtz decomposition (cf. [11, Theorem I.2.7]):

$$
L^2(\Omega)^3 = H_0(\mathrm{div}^0;\Omega) \overset{\perp}{\oplus} \nabla H^1(\Omega). \tag{3}
$$

Let $H^{1/2}(\Gamma)$ be the space of traces on Γ of functions in $H^1(\Omega)$, with dual space $H^{-1/2}(\Gamma)$. For the ease of notation, the duality pairing will be simply denotes by the integral on Γ; this simplification will be also adopted for other duality pairings, for instance the duality pairing between the space of tangential traces of $H(\mathbf{curl};\Omega)$ and the space of tangential components of $H(\mathbf{curl};\Omega)$. Notice that the conditions $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{v}\times\boldsymbol{n} = \mathbf{0}$ on Γ must be understood in the sense of $H^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)^3$, respectively.

We will also consider the Hilbertian Sobolev space $H^s(\Omega)$ $(0 < s < 1)$ with norm $\|\cdot\|_{s,\Omega}$, which is well-known to satisfy

$$
H^1(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^2(\Omega) \,,
$$

both inclusions being compact (see, for instance, [11, Sect. I.1.1]), and the space

$$
H^s(\mathbf{curl};\Omega) \,:=\, \{\boldsymbol{v} \in H^s(\Omega)^3 \,:\, \mathbf{curl}\boldsymbol{v} \in H^s(\Omega)^3\}.
$$

Let us remark that, according to [2, Theorem 2.9 and Proposition 3.7] there exists $s > 1/2$ such that

$$
H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div};\Omega) \hookrightarrow H^s(\Omega)^3, \tag{4}
$$

the inclusion being continuous.

Having fixed a unit normal vector $\boldsymbol{n}_j$ on each Σ_j, we denote its two faces by Σ_j^+ and Σ_j^-, with $\boldsymbol{n}_j$ being the 'outer' normal to $\partial\Omega^0$ on Σ_j^+. For any $\psi \in H^1(\Omega^0)$, we denote by $[\![\psi]\!]_{\Sigma_j} := \psi|_{\Sigma_j^+} - \psi|_{\Sigma_j^-}$ the jump of ψ across Σ_j along $\boldsymbol{n}_j$. The tangent unit vector $\boldsymbol{t}_j$ is oriented counterclockwise with respect to Σ_j^+. A similar notation is used on each Σ'_j.

We have the following Green's identity (proved in [2, Lemma 3.10]).

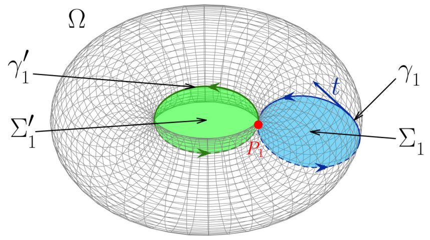

Fig. 1 Toroidal domain. Σ_1 and Σ_1' represent the 'cutting' surfaces of Ω and Ω', respectively.

Lemma 1 *For all* $\boldsymbol{v} \in H_0(\mathrm{div};\Omega)$, $\boldsymbol{v}\cdot\boldsymbol{n}_j|_{\Sigma_j} \in H^{1/2}(\Sigma_j)'$, $1 \leq j \leq g$, *and the following Green's formulas hold true:*

$$\int_{\Sigma_j} \boldsymbol{v}\cdot\boldsymbol{n}_j [\![\psi]\!]_{\Sigma_j} = \int_{\Omega\setminus\Sigma_j} \boldsymbol{v}\cdot\nabla\psi + \int_{\Omega\setminus\Sigma_j} (\mathrm{div}\boldsymbol{v})\,\psi \quad \forall\psi \in H^1(\Omega\setminus\Sigma_j).$$

Note that, in general, the functions $\psi \in H^1(\Omega^0)$ do not admit an extension to the whole Ω that lies in the space $H^1(\Omega)$. However, any extension of $\nabla\psi$ obviously belongs to $L^2(\Omega)^3$. We denote such an extension $\widetilde{\nabla}\psi$.

Let us consider the space of the so-called harmonic Neumann fields:

$$\boldsymbol{\mathcal{K}}_T := H(\mathbf{curl}^0;\Omega) \cap H_0(\mathrm{div}^0;\Omega)\,. \tag{5}$$

This is a finite-dimensional space, its dimension being equal to the first Betti number g of Ω, namely, the number of the 'cutting' surfaces. The following lemma gives a representation of the basis $\boldsymbol{\rho}_j$ of the space $\boldsymbol{\mathcal{K}}_T$.

Lemma 2 *A basis of the space* $\boldsymbol{\mathcal{K}}_T$ *is given by* $\{\boldsymbol{\rho}_j\}_{j=1}^g$ *where* $\boldsymbol{\rho}_j := \widetilde{\nabla}\phi_j$, $j = 1,\ldots,g$, *and* $\phi_j \in H^1(\Omega\setminus\Sigma_j)/\mathbb{R}$ *is the unique solution of*

$$\begin{aligned} \Delta\phi_j &= 0 \quad \text{in} \quad \Omega\setminus\Sigma_j\,,\\ \partial_{\boldsymbol{n}}\phi_j &= 0 \quad \text{on} \quad \partial\Omega\,,\\ [\![\partial_{\boldsymbol{n}}\phi_j]\!]_{\Sigma_j} &= 0\,,\\ [\![\phi_j]\!]_{\Sigma_j} &= 1\,. \end{aligned}$$

Proof The result follows by noticing that, for $j = 1,\ldots,g$, ϕ_j is the unique solution of the following problem: find $w \in H^1(\Omega\setminus\Sigma_j)/\mathbb{R}$ such that $[\![w]\!]_{\Sigma_j} = 1$ and

$$\int_\Omega \nabla w\cdot\nabla v = 0 \quad \forall v \in H^1(\Omega).$$

For details see, for instance, [10, Lemma 1.3]. □

A similar construction can be done also for the harmonic Neumann vector fields $\boldsymbol{\rho}_j'$ defined in Ω' with tangential component equal zero on $\partial\Omega' = \partial B \cup \Gamma$, where we recall that B is an open ball containing $\overline{\Omega}$ and $\Omega' := B\setminus\overline{\Omega}$.

The following result will be helpful in the subsequent analysis.

Lemma 3 *There holds*

$$H(\mathbf{curl}^0;\Omega) = \boldsymbol{\mathcal{K}}_T \overset{\perp}{\oplus} \nabla H^1(\Omega).$$

Proof As consequence of the Helmholtz decomposition (3), the fact that $\nabla H^1(\Omega) \subset H(\mathbf{curl}^0;\Omega)$ and the definition (5) of $\boldsymbol{\mathcal{K}}_T$, we obtain that

$$H(\mathbf{curl}^0;\Omega) = H(\mathbf{curl}^0;\Omega) \cap [H_0(\mathrm{div}^0;\Omega) \overset{\perp}{\oplus} \nabla H^1(\Omega)] = \boldsymbol{\mathcal{K}}_T \overset{\perp}{\oplus} \nabla H^1(\Omega)$$

which concludes the proof. □

As we already remarked, we need to give a precise meaning to the additional conditions $\oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i = 0$ and $\oint_{\gamma'_j} \boldsymbol{u} \cdot \boldsymbol{t}'_j = 0$. For functions $\boldsymbol{u} \in H(\mathbf{curl};\Omega)$ with $\mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ this can be done as follows: since $\mathbf{curl}\boldsymbol{u} \in H_0(\mathrm{div}^0;\Omega)$, as an analogy to Stokes Theorem, by using Lemma 1 we can write, for each $1 \le i \le g_1$,

$$\oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i := \int_{\Sigma_i} \mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{n}_i = \int_{\Omega \setminus \Sigma_i} \mathbf{curl}\boldsymbol{u} \cdot \nabla\psi \,,$$

with $\psi \in H^1(\Omega \setminus \Sigma_i)$, $[\![\psi]\!]_{\Sigma_i} = 1$. In particular, we can take as ψ the function ϕ_i introduced in Lemma 2, and we find

$$\int_{\Omega \setminus \Sigma_i} \mathbf{curl}\boldsymbol{u} \cdot \nabla\phi_i = \int_{\Omega} \mathbf{curl}\boldsymbol{u} \cdot \widetilde{\nabla}\phi_i = \int_{\Omega} \mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{\rho}_i \,,$$

an alternative definition of $\oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i$. Moreover, since $\mathbf{curl}\boldsymbol{\rho}_i = \mathbf{0}$ in Ω, by integration by parts we have

$$\int_{\Omega} \mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{\rho}_i = \int_{\Gamma} \boldsymbol{n} \times \boldsymbol{u} \cdot \boldsymbol{\rho}_i \,;$$

in conclusion, for a vector field satisfying $\mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ we can also write

$$\oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i = \int_{\Gamma} \boldsymbol{n} \times \boldsymbol{u} \cdot \boldsymbol{\rho}_i = \int_{\Omega} \mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{\rho}_i \,. \tag{6}$$

Finally, we clearly have that

$$\left| \oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i \right| \le C \|\boldsymbol{u}\|_{\mathbf{curl},\Omega} \,, \quad \text{for } 1 \le i \le g_1.$$

We proceed similarly for $\oint_{\gamma'_j} \boldsymbol{u} \cdot \boldsymbol{t}'_j$ for $g_1 + 1 \le j \le g$. Thanks to [17, Theorem 3.34] (see, alternatively, [5, Theorem 1], [6, Theorem 7.1]) we know that there exists a bounded extension $\widetilde{\boldsymbol{u}}$ of $\boldsymbol{u}$ to $\mathbb{R}^3$, with support contained in B, and such that

$$\|\widetilde{\boldsymbol{u}}\|_{\mathbf{curl};B} \le C \|\boldsymbol{u}\|_{\mathbf{curl};\Omega}.$$

By using Lemma 1 again and the previous inequality, we have that for $g_1 + 1 \le j \le g$

$$\oint_{\gamma'_j} \boldsymbol{u} \cdot \boldsymbol{t}'_j := \int_{\Sigma'_j} \mathbf{curl}\widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}_j = \int_{\Omega' \setminus \Sigma'_j} \mathbf{curl}\widetilde{\boldsymbol{u}} \cdot \nabla\psi'$$

with $\psi' \in H^1(\Omega' \setminus \Sigma'_j)$, $[[\psi']]_{\Sigma'_j} = 1$. As before, we can also write $\oint_{\gamma'_j} \boldsymbol{u} \cdot \boldsymbol{t}'_j = \int_{\Omega'} \mathbf{curl}\widetilde{\boldsymbol{u}} \cdot \boldsymbol{\rho}'_j$ or

$$\oint_{\gamma'_j} \boldsymbol{u} \cdot \boldsymbol{t}'_j = -\int_{\Gamma} \boldsymbol{n} \times \boldsymbol{u} \cdot \boldsymbol{\rho}'_j \tag{7}$$

(the sign due to the fact that $\boldsymbol{n}$ on Γ is the inward normal with respect to Ω'). We also have

$$\left| \oint_{\gamma'_j} \boldsymbol{u} \cdot \boldsymbol{t}'_j \right| \leq C \|\widetilde{\boldsymbol{u}}\|_{\mathbf{curl},B} \leq C \|\boldsymbol{u}\|_{\mathbf{curl},\Omega}, \quad \text{for } g_1 + 1 \leq j \leq g.$$

Remark 2 It is straightforward to check that, when $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ and $g_1 = g$, the conditions $\oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = 0$ for $1 \leq i \leq g_1 = g$ are equivalent to $\mathbf{curl}\, \boldsymbol{v} \perp \boldsymbol{\mathcal{K}}_T$. In fact, by (6) we have

$$\oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = \int_{\Omega} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{\rho}_i \,.$$

On the other hand, we have also seen that, using the Stokes theorem, there holds $\oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = \int_{\Sigma_i} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}_i$, therefore the vanishing of the line integrals on γ_i can also be expressed as the zero-flux conditions

$$\int_{\Sigma_i} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}_i = 0, \quad 1 \leq i \leq g\,.$$

The former point of view is adopted in [19], [20], [21]; the latter in [14], [25], [15].

3 Mixed variational formulation, well-posedness and spectral analysis.

Let us start by recalling the following result, that is easily obtained by integration by parts:

Lemma 4 *For each $\xi \in H^1(\Omega)$ and $\boldsymbol{\omega} \in H(\mathbf{curl};\Omega)$ or $\boldsymbol{\omega} \in H(\mathbf{curl};\Omega')$ there holds*

$$\int_{\Gamma} \nabla_{\boldsymbol{t}} \xi \cdot (\boldsymbol{\omega} \times \boldsymbol{n}) = -\int_{\Gamma} \xi\, \mathbf{curl} \boldsymbol{\omega} \cdot \boldsymbol{n}\,,$$

where $\nabla_t \xi = \boldsymbol{n} \times \nabla \xi \times \boldsymbol{n}$. Hence, if $\mathbf{curl}\, \boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ on Γ, it follows

$$\int_{\Gamma} \nabla_{\boldsymbol{t}} \xi \cdot (\boldsymbol{\omega} \times \boldsymbol{n}) = 0\,.$$

Proof If $\boldsymbol{\omega} \in H(\mathbf{curl};\Omega)$ then

$$\int_{\Gamma} \nabla_{\boldsymbol{t}} \xi \cdot (\boldsymbol{\omega} \times \boldsymbol{n}) = -\int_{\Omega} \nabla \xi \cdot \mathbf{curl} \boldsymbol{\omega} = -\int_{\Gamma} \xi\, \mathbf{curl} \boldsymbol{\omega} \cdot \boldsymbol{n}\,.$$

On the other hand if $\boldsymbol{\omega} \in H(\mathbf{curl};\Omega')$, let $\widetilde{\xi} \in H^1_0(B)$ be an extension of ξ; then

$$\int_{\Gamma} \nabla_{\boldsymbol{t}} \xi \cdot (\boldsymbol{\omega} \times \boldsymbol{n}) = \int_{\Omega'} \nabla \widetilde{\xi} \cdot \mathbf{curl} \boldsymbol{\omega} = -\int_{\Gamma} \xi\, \mathbf{curl} \boldsymbol{\omega} \cdot \boldsymbol{n}\,,$$

the minus sign coming from the fact that $\boldsymbol{n}$ is the unit normal vector outer to Ω. □

We also recall the following result (see [12], Lemma 6.11 and Lemma 6.12) that will be helpful in the sequel, and for which we furnish a complete proof under our assumptions.

Lemma 5 *There holds*

$$\int_\Gamma (\boldsymbol{\rho}_j \times \boldsymbol{n})\cdot \boldsymbol{\rho}_k = 0 \ , \ \int_\Gamma (\boldsymbol{\rho}_j \times \boldsymbol{n})\cdot \boldsymbol{\rho}'_k = \delta_{j,k} = -\int_\Gamma (\boldsymbol{\rho}'_k \times \boldsymbol{n})\cdot \boldsymbol{\rho}_j \ , \ \int_\Gamma (\boldsymbol{\rho}'_j \times \boldsymbol{n})\cdot \boldsymbol{\rho}'_k = 0$$

for $1 \leq j,k \leq g$.

Proof The first equation follows easily by integration by parts:

$$\int_\Gamma (\boldsymbol{\rho}_j \times \boldsymbol{n}) \cdot \boldsymbol{\rho}_k = \int_\Omega (\boldsymbol{\rho}_j \cdot \mathbf{curl}\boldsymbol{\rho}_k - \mathbf{curl}\boldsymbol{\rho}_j \cdot \boldsymbol{\rho}_k) = 0\,.$$

The proof of the third equation is similar to that of the first one but working in Ω' instead of Ω.

Concerning the second equation, for any $j = 1,\dots,g$ let us consider a function $\psi_j \in H^1(\Omega \setminus \Sigma_j)$, such that $[\![\psi_j]\!]_{\Sigma_j} = 1$ and $\widetilde{\nabla}\psi_j$ is regular enough to give a meaning to $\oint_{\gamma'_k} \widetilde{\nabla}\psi_j \cdot \boldsymbol{t}'_k$ for each $k = 1,\dots,g$ in the classical sense. Then, by proceeding as done to obtain (7), we have

$$\int_\Gamma (\widetilde{\nabla}\psi_j \times \boldsymbol{n}) \cdot \boldsymbol{\rho}'_k = \oint_{\gamma'_k} \widetilde{\nabla}\psi_j \cdot \boldsymbol{t}'_k = \delta_{j,k}$$

for $1 \leq k \leq g$, the last equality stemming from the fact that $[\![\psi_j]\!]_{\Sigma_j} = 1$ and $[\![\psi_j]\!]_{\Sigma_k} = 0$, $k \neq j$.

However the function $\boldsymbol{\rho}_j = \widetilde{\nabla}\phi_j$ is not regular enough on Γ to define the line integral in a classical sense (see its definition in Lemma 2). Since $\boldsymbol{\rho}_j$ is regular in Ω, one could replace γ'_k with an internal cycle in the same homology class; however, a proof of the result is possible even without modifying γ'_k.

Now we consider the curl-free vector field $\mathbf{T}_j = \widetilde{\nabla}\psi^\star_j$, where $\psi^\star_j \in H^1(\Omega \setminus \Sigma_j)$ is a smooth function in $\Omega \setminus \Sigma_j$ having $[\![\psi^\star_j]\!]_{\Sigma_j} = 1$ (and $[\![\psi^\star_j]\!]_{\Sigma_k} = 0$ for $k \neq j$, being $\psi^\star_j$ in $H^1(\Omega \setminus \Sigma_j)$) (see [15, Lemma 3.12] for an example in a toroidal domain). Clearly, for $\mathbf{T}_j$ the line integrals $\oint_{\gamma'_k} \mathbf{T}_j \cdot \boldsymbol{t}'_k$ are well-defined and take the values $\delta_{j,k}$. For these vector fields, that are regular, we can also write $\oint_{\gamma'_k} \mathbf{T}_j \cdot \boldsymbol{t}'_k = \int_\Gamma (\mathbf{T}_j \times \boldsymbol{n}) \cdot \boldsymbol{\rho}'_k$, thus $\int_\Gamma (\mathbf{T}_j \times \boldsymbol{n}) \cdot \boldsymbol{\rho}'_k = \delta_{j,k}$.

We define $\varphi_j \in H^1(\Omega) \setminus \mathbb{R}$ such that $\int_\Omega \nabla\varphi_j \cdot \nabla\phi = \int_\Omega \mathbf{T}_j \cdot \nabla\phi$ for all $\phi \in H^1(\Omega) \setminus \mathbb{R}$. Thus, from the proof of Lemma 2, it follows that $\boldsymbol{\rho}_j = \mathbf{T}_j - \nabla\varphi_j$. Moreover, by using Lemma 4 we obtain

$$\int_\Gamma (\boldsymbol{\rho}_j \times \boldsymbol{n}) \cdot \boldsymbol{\rho}'_k = \int_\Gamma [(\mathbf{T}_j - \nabla\varphi_j) \times \boldsymbol{n}] \cdot \boldsymbol{\rho}'_k = \int_\Gamma (\mathbf{T}_j \times \boldsymbol{n}) \cdot \boldsymbol{\rho}'_k = \delta_{j,k}\,.$$

The result $\int_\Gamma (\boldsymbol{\rho}'_k \times \boldsymbol{n}) \cdot \boldsymbol{\rho}_j = -\delta_{j,k}$ follows at once by exchanging the order of the factors. □

Let us define some function spaces that we will use to give a convenient variational formulation of Problem 1:

$$\boldsymbol{\mathcal{X}} := \{\boldsymbol{v} \in H(\mathbf{curl};\Omega) \,:\, \mathbf{curl}\boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}\,,$$

$$\boldsymbol{\mathcal{X}}_0 := \{\boldsymbol{v} \in \boldsymbol{\mathcal{X}} \,:\, \oint_{\gamma_i} \boldsymbol{v}\cdot\boldsymbol{t}_i = 0 \text{ for } i = 1,\dots,g_1\}\,,$$

$$\boldsymbol{\mathcal{X}}_\star := \{\boldsymbol{v} \in \boldsymbol{\mathcal{X}} \,:\, \oint_{\gamma_j'} \boldsymbol{v}\cdot\boldsymbol{t}_j' = 0 \text{ for } j = g_1+1,\dots,g\}\,,$$

$$\boldsymbol{\mathcal{Z}} := \boldsymbol{\mathcal{X}}_0 \cap \boldsymbol{\mathcal{X}}_\star\,,$$

$$\boldsymbol{\mathcal{H}} := \nabla H^1(\Omega) \oplus \operatorname{span}\{\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_{g_1}\}\,,$$

where $0 \le g_1 \le g$.

Lemma 6 *The following equality holds true*

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{X}}_\star \cap H(\mathbf{curl}^0;\Omega) = \boldsymbol{\mathcal{Z}} \cap H(\mathbf{curl}^0;\Omega)\,.$$

Proof Clearly $\boldsymbol{\mathcal{H}} \subset H(\mathbf{curl}^0;\Omega)$, and Lemmas 4 and 5 assure that $\boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{X}}_\star$. Conversely, from Lemma 3, a vector field $\boldsymbol{v} \in H(\mathbf{curl}^0;\Omega)$ can be written as $\boldsymbol{v} = \nabla\psi + \sum_{l=1}^{g} \alpha_l \boldsymbol{\rho}_l$, with $\oint_{\gamma_l'} \boldsymbol{v}\cdot\boldsymbol{t}_l' = \alpha_l$; therefore, as a consequence of the constraints $\oint_{\gamma_j'} \boldsymbol{v}\cdot\boldsymbol{t}_j' = 0$ for $j = g_1+1,\dots,g$, a vector field $\boldsymbol{v} \in \boldsymbol{\mathcal{X}}_\star \cap H(\mathbf{curl}^0;\Omega)$ can be written as $\boldsymbol{v} = \nabla\psi + \sum_{i=1}^{g_1} \alpha_i \boldsymbol{\rho}_i$.

It remains to show that $H(\mathbf{curl}^0;\Omega) \subset \boldsymbol{\mathcal{X}}_0$, and this is clear as, by definition (6), $\oint_{\gamma_i} \boldsymbol{v}\cdot\boldsymbol{t}_i = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\rho}_i$, $i = 1,\dots,g$, and the last integral vanishes for $\boldsymbol{v} \in H(\mathbf{curl}^0;\Omega)$. (In homological language, we could say that the result is clear since the cycles γ_i are bounding in $\overline{\Omega}$.) □

Now we introduce the following commuting property, that will be the basis for the spectral characterization of the problem, and that also will lead us to the choice of the space $\boldsymbol{\mathcal{Z}}$ as suitable space for a viable variational formulation.

Lemma 7 *For all* $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$,

$$\int_\Omega (\mathbf{curl}\boldsymbol{w}\cdot\overline{\boldsymbol{v}} - \boldsymbol{w}\cdot\mathbf{curl}\overline{\boldsymbol{v}}) = 0.$$

Proof Let $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$. We know that

$$\int_\Omega \mathbf{curl}\boldsymbol{w}\cdot\overline{\boldsymbol{v}} - \int_\Omega \boldsymbol{w}\cdot\mathbf{curl}\overline{\boldsymbol{v}} = -\int_\Gamma (\boldsymbol{w}\times\boldsymbol{n})\cdot\overline{\boldsymbol{v}}\,.$$

We recall from [5], [12] that $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}$ can be written on Γ as

$$\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n} = \nabla_{\boldsymbol{t}}\pi + \sum_{k=1}^{g} \zeta_k \boldsymbol{\rho}_{k,\boldsymbol{t}} + \sum_{l=1}^{g} \eta_l \boldsymbol{\rho}_{l,\boldsymbol{t}}'\,.$$

$$\boldsymbol{n}\times\boldsymbol{w}\times\boldsymbol{n} = \nabla_{\boldsymbol{t}}\vartheta + \sum_{j=1}^{g} \delta_j \boldsymbol{\rho}_{j,\boldsymbol{t}} + \sum_{i=1}^{g} \varepsilon_i \boldsymbol{\rho}_{i,\boldsymbol{t}}'\,,$$

where we are using the notation $\mathbf{z}_{\boldsymbol{t}} = \boldsymbol{n} \times \mathbf{z} \times \boldsymbol{n}$ for all $\mathbf{z} \in H(\mathbf{curl}; \Omega)$ or $\mathbf{z} \in H(\mathbf{curl}; \Omega')$. From Lemma 4 we have, for $1 \le k \le g$,

$$\int_\Gamma (\nabla_{\boldsymbol{t}} \vartheta \times \boldsymbol{n}) \cdot \nabla_{\boldsymbol{t}} \overline{\pi} = 0\,, \quad \int_\Gamma (\nabla_{\boldsymbol{t}} \vartheta \times \boldsymbol{n}) \cdot \boldsymbol{\rho}_k = 0\,, \quad \int_\Gamma (\nabla_{\boldsymbol{t}} \vartheta \times \boldsymbol{n}) \cdot \boldsymbol{\rho}'_k = 0\,,$$

and the same holds for $\nabla_{\boldsymbol{t}} \pi$. Therefore, from Lemma 5

$$\int_\Gamma (\boldsymbol{w} \times \boldsymbol{n}) \cdot \overline{\boldsymbol{v}} = \sum_{j=1}^{g} \delta_j \overline{\eta_j} - \sum_{i=1}^{g} \varepsilon_i \overline{\zeta_i}\,.$$

Moreover, given that $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$, from (7), Lemma 4 and Lemma 5 it follows

$$0 = \oint_{\gamma'_k} \boldsymbol{v} \cdot \boldsymbol{t}'_{\boldsymbol{k}} = -\int_\Gamma \boldsymbol{n} \times \boldsymbol{v} \cdot \boldsymbol{\rho}'_k = \zeta_k \qquad \text{for } k = g_1 + 1, \dots, g$$

and, analogously, $\delta_j = 0$ for $j = g_1 + 1, \dots, g$. Using (6) instead of (7) we obtain

$$0 = \oint_{\gamma_l} \boldsymbol{v} \cdot \boldsymbol{t}_{\boldsymbol{l}} = \int_\Gamma \boldsymbol{n} \times \boldsymbol{v} \cdot \boldsymbol{\rho}_l = \eta_l \qquad \text{for } l = 1, \dots, g_1\,.$$

and $\varepsilon_i = 0$ for $i = 1, \dots, g_1$. Thus we have $\int_\Gamma (\boldsymbol{w} \times \boldsymbol{n}) \cdot \overline{\boldsymbol{v}} = 0$, which concludes the proof. $\square$

Now, we introduce a mixed formulation of Problem 1.
Problem 2. Find $\lambda \in \mathbb{C}$ and $(\boldsymbol{u}, \boldsymbol{q}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$, $\boldsymbol{u} \neq \mathbf{0}$, such that

$$\int_\Omega \mathbf{curl}\boldsymbol{u} \cdot \mathbf{curl}\overline{\boldsymbol{v}} + \int_\Omega \boldsymbol{q} \cdot \overline{\boldsymbol{v}} = \lambda \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\overline{\boldsymbol{v}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}\,, \tag{8a}$$

$$\int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{p}} = 0 \quad \forall\, \boldsymbol{p} \in \boldsymbol{\mathcal{H}}. \tag{8b}$$

Remark 3 It is worthy to note that an eigenvalue in Problem 2 must be different from 0 and real. In fact, taking $\boldsymbol{v} = \boldsymbol{q} \in \boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{Z}}$ in (8a) it follows $\boldsymbol{q} = \mathbf{0}$. Hence, if we suppose $\lambda = 0$, we obtain $\mathbf{curl}\boldsymbol{u} = \mathbf{0}$ in Ω, and consequently $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}$ by Lemma 6. Choosing $\boldsymbol{p} = \boldsymbol{u}$ in (8b) it follows $\boldsymbol{u} = \mathbf{0}$, and we conclude that $\lambda = 0$ is not admissible. Moreover, as a consequence of Lemma 7 it is standard to prove that $\lambda \in \mathbb{R}$.

The following result establishes the equivalence between this variational formulation and Problem 1.

Lemma 8 *If $(\lambda, \boldsymbol{u})$, $\lambda \neq 0$, is a solution to Problem 1, then $(\lambda, \boldsymbol{u}, \mathbf{0})$ is a solution to Problem 2. If $(\lambda, \boldsymbol{u}, \boldsymbol{q})$ is a solution to Problem 2, then $\boldsymbol{q} = \mathbf{0}$ and $(\lambda, \boldsymbol{u})$ is a solution to Problem 1.*

Proof If $(\lambda, \boldsymbol{u})$, $\lambda \neq 0$, is a solution to Problem 1, then clearly (8a) is satisfied with $\boldsymbol{q} = \mathbf{0}$. Since $\boldsymbol{u} \in H_0(\mathrm{div}^0; \Omega)$, it follows that $\boldsymbol{u}$ is orthogonal to the gradients. Therefore, recalling that $\boldsymbol{p} \in \boldsymbol{\mathcal{H}}$ can be written as $\boldsymbol{p} = \nabla\psi + \sum_{i=1}^{g_1} \alpha_i \boldsymbol{\rho}_i$, we have only to prove that $\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\rho}_i = 0$ for $i = 1, \dots, g_1$. Due to the definition $\oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i = \int_\Omega \mathbf{curl}\boldsymbol{u} \cdot \boldsymbol{\rho}_i$, the condition $\oint_{\gamma_i} \boldsymbol{u} \cdot \boldsymbol{t}_i = 0$ can be interpreted as $\lambda \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\rho}_i = 0$, and the thesis follows as $\lambda \neq 0$.

If $(\lambda, \boldsymbol{u}, \boldsymbol{q})$ is a solution to Problem 2, we have already seen in Remark 3 that $\boldsymbol{q} = \boldsymbol{0}$. Moreover, taking in (8b) $\boldsymbol{p} \in \nabla H^1(\Omega) \subset \boldsymbol{\mathcal{H}}$ it follows $\operatorname{div}\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ.

Therefore, we only need to prove that $\mathbf{curl}\boldsymbol{u} = \lambda\boldsymbol{u}$ in Ω. Choosing $\boldsymbol{v} \in \mathcal{D}(\Omega)^3$ and integrating by parts (8a) we find $\mathbf{curl}(\mathbf{curl}\boldsymbol{u} - \lambda\boldsymbol{u}) = \boldsymbol{0}$ in Ω. Repeating the previous computation, but now for $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$, we obtain

$$\int_\Gamma (\mathbf{curl}\boldsymbol{u} - \lambda\boldsymbol{u}) \cdot \boldsymbol{n} \times \overline{\boldsymbol{v}} = 0\,.$$

Since $\mathbf{curl}\boldsymbol{u} - \lambda\boldsymbol{u}$ is curl-free, thanks to Lemma 3 we can write $\mathbf{curl}\boldsymbol{u} - \lambda\boldsymbol{u} = \nabla\varphi + \sum_{s=1}^{g} \beta_s \boldsymbol{\rho}_s$. Moreover, for $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$ and following the arguments in the proof of Lemma 7, on Γ we have

$$\boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{n} = \nabla_{\boldsymbol{t}}\pi + \sum_{k=1}^{g_1} \zeta_k \boldsymbol{\rho}_{\boldsymbol{t},k} + \sum_{l=g_1+1}^{g} \eta_l \boldsymbol{\rho}'_{\boldsymbol{t},l}\,.$$

Thus

$$\begin{aligned} 0 &= \int_\Gamma (\mathbf{curl}\boldsymbol{u} - \lambda\boldsymbol{u}) \cdot \boldsymbol{n} \times \overline{\boldsymbol{v}} \\ &= \int_\Gamma \Big(\nabla_t\varphi + \sum_{s=1}^{g} \beta_s \boldsymbol{\rho}_s\Big) \cdot \boldsymbol{n} \times \left(\nabla_{\boldsymbol{t}}\overline{\pi} + \sum_{k=1}^{g_1} \overline{\zeta_k} \boldsymbol{\rho}_{\boldsymbol{t},k} + \sum_{l=g_1+1}^{g} \overline{\eta_l} \boldsymbol{\rho}'_{\boldsymbol{t},l}\right) \\ &= \sum_{l=g_1+1}^{g} \beta_l \overline{\eta_l}\,. \end{aligned}$$

Since η_l are arbitrary, it follows that $\beta_l = 0$ for $l = g_1 + 1, \ldots, g$. As a consequence, we can write $\mathbf{curl}\boldsymbol{u} - \lambda\boldsymbol{u} = \nabla\varphi + \sum_{i=1}^{g_1} \beta_i \boldsymbol{\rho}_i \in \boldsymbol{\mathcal{H}}$.

Therefore by (8b) $\boldsymbol{u}$ is orthogonal to $\overline{\lambda\boldsymbol{u} - \mathbf{curl}\boldsymbol{u}}$ and we have

$$0 = \int_\Omega \lambda\boldsymbol{u} \cdot (\overline{\lambda\boldsymbol{u} - \mathbf{curl}\boldsymbol{u}}) = \int_\Omega (\lambda\boldsymbol{u} - \mathbf{curl}\boldsymbol{u}) \cdot (\overline{\lambda\boldsymbol{u} - \mathbf{curl}\boldsymbol{u}}) + \int_\Omega \mathbf{curl}\boldsymbol{u} \cdot (\overline{\lambda\boldsymbol{u} - \mathbf{curl}\boldsymbol{u}})\,;$$

the last integral vanishes due to (8a), and we have proved that $\mathbf{curl}\boldsymbol{u} = \lambda\boldsymbol{u}$ in Ω. □

In order to obtain a spectral characterization of problem (8) we introduce the following solution operator:

$$\begin{aligned} \boldsymbol{T} : \boldsymbol{\mathcal{Z}} &\longrightarrow \boldsymbol{\mathcal{Z}}\,, \\ \boldsymbol{f} &\longmapsto \boldsymbol{T}\boldsymbol{f} := \boldsymbol{w}\,, \end{aligned}$$

with $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$ such that there exist $\boldsymbol{q} \in \boldsymbol{\mathcal{H}}$ satisfying

$$\int_\Omega \mathbf{curl}\boldsymbol{w} \cdot \mathbf{curl}\overline{\boldsymbol{v}} + \int_\Omega \boldsymbol{q} \cdot \overline{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{v}} \tag{9a}$$

$$\int_\Omega \boldsymbol{w} \cdot \overline{\boldsymbol{p}} = 0 \tag{9b}$$

for all $(\boldsymbol{v}, \boldsymbol{p}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$.

The well-posedness of this problem is a consequence of the following lemmas.

Lemma 9 *In* $\boldsymbol{\mathcal{Y}} := \boldsymbol{\mathcal{X}} \cap H_0(\mathrm{div};\Omega)$ *the seminorm*

$$|||w||| := \left\{ \|\mathbf{curl}\boldsymbol{w}\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{w}\|_{0,\Omega}^2 + \sum_{i=1}^{g_1} \left| \int_\Omega \boldsymbol{w}\cdot\boldsymbol{\rho}_i \right|^2 + \sum_{j=g_1+1}^{g} \left| \oint_{\gamma'_j} \boldsymbol{w}\cdot\boldsymbol{t}'_j \right|^2 \right\}^{1/2}$$

is equivalent to the norm

$$\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{Y}}} := \{\|\boldsymbol{w}\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{w}\|_{0,\Omega}^2 + \|\mathbf{curl}\boldsymbol{w}\|_{0,\Omega}^2\}^{1/2}.$$

Proof Since $|\int_\Omega \boldsymbol{w}\cdot\boldsymbol{\rho}_i| \le C_1\|\boldsymbol{w}\|_{0,\Omega}$, for $i = 1,\dots,g_1$, and $\left|\oint_{\gamma'_j} \boldsymbol{w}\cdot\boldsymbol{t}'_j\right| \le C_2\|\boldsymbol{w}\|_{\mathbf{curl};\Omega}$, for $j = g_1+1,\dots,g$, it is clear that $|||w|||^2 \le C\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{Y}}}^2$.

The other inequality will be proved by contradiction. We suppose that for all $C > 0$, there exists $\boldsymbol{v}_C \in \boldsymbol{\mathcal{Y}}$ such that $\|\boldsymbol{v}_C\|_{\boldsymbol{\mathcal{Y}}} > C\,|||\boldsymbol{v}_C|||$. In particular, for all $n \in \mathbb{N}$, there exists $\boldsymbol{v}_n \in \boldsymbol{\mathcal{Y}}$ such that $\|\boldsymbol{v}_n\|_{\boldsymbol{\mathcal{Y}}} > n\,|||\boldsymbol{v}_n|||$. Let $\boldsymbol{u}_n := \boldsymbol{v}_n/\|\boldsymbol{v}_n\|_{\boldsymbol{\mathcal{Y}}}$. It follows that $\|\boldsymbol{u}_n\|_{\boldsymbol{\mathcal{Y}}} = 1$ and

$$\|\mathbf{curl}\boldsymbol{u}_n\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{u}_n\|_{0,\Omega}^2 + \sum_{i=1}^{g_1} \left| \int_\Omega \boldsymbol{u}_n\cdot\boldsymbol{\rho}_i \right|^2 + \sum_{j=g_1+1}^{g} \left| \oint_{\gamma'_j} \boldsymbol{u}_n\cdot\boldsymbol{t}'_j \right|^2 < \frac{1}{n^2}, \quad \forall n \in \mathbb{N}. \tag{10}$$

We know from (4) that $\boldsymbol{\mathcal{Y}}$ is compactly included in $L^2(\Omega)^3$; hence, since the sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\boldsymbol{\mathcal{Y}}$, there exists a (not relabeled) subsequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{Y}}$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^2(\Omega)^3$. Thus from (10) we obtain that

$$\begin{aligned}\|\boldsymbol{u}_n - \boldsymbol{u}_m\|_{\boldsymbol{\mathcal{Y}}}^2 \le C\,\{&\|\boldsymbol{u}_n - \boldsymbol{u}_m\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{u}_n\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{u}_m\|_{0,\Omega}^2 \\ &+ \|\mathbf{curl}\boldsymbol{u}_n\|_{0,\Omega}^2 + \|\mathbf{curl}\boldsymbol{u}_m\|_{0,\Omega}^2\} \xrightarrow{n,m} 0.\end{aligned}$$

Then, $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $\boldsymbol{\mathcal{Y}}$ which implies that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{\mathcal{Y}}$ with $\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{Y}}} = 1$. Notice that from (10) we obtain that $\mathrm{div}\boldsymbol{u} = 0$ and $\mathbf{curl}\boldsymbol{u} = \mathbf{0}$ in Ω. So, $\boldsymbol{u} \in \boldsymbol{\mathcal{K}}_T$, say, $\boldsymbol{u} = \sum_{k=1}^{g} \alpha_k\boldsymbol{\rho}_k$. In addition, from (10) we also obtain that $\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\rho}_i = 0$ for $i = 1,\dots,g_1$, and $\oint_{\gamma'_j} \boldsymbol{u}\cdot\boldsymbol{t}'_j = 0$ for $j = g_1+1,\dots,g$. In particular, by (7) and Lemma 5, for $j = g_1+1,\dots,g$ we have

$$0 = \oint_{\gamma'_j} \boldsymbol{u}\cdot\boldsymbol{t}'_j = \int_\Gamma \boldsymbol{u}\times\boldsymbol{n}\cdot\boldsymbol{\rho}'_j = \sum_{k=1}^{g} \alpha_k \int_\Gamma \boldsymbol{\rho}_k\times\boldsymbol{n}\cdot\boldsymbol{\rho}'_j = \alpha_j\,.$$

Then we also have, for $i = 1,\dots,g_1$,

$$0 = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{\rho}_i = \sum_{k=1}^{g_1} \alpha_k \int_\Omega \boldsymbol{\rho}_k\cdot\boldsymbol{\rho}_i\,;$$

this implies $\alpha_k = 0$ for $k = 1,\dots,g_1$, as the $g_1\times g_1$ matrix with entries $\int_\Omega \boldsymbol{\rho}_k\cdot\boldsymbol{\rho}_i$ is symmetric and positive definite, due to the fact that $\boldsymbol{\rho}_i$ are linearly independent. In conclusion, we have found $\boldsymbol{u} = \mathbf{0}$ in Ω and a contradiction is produced. □

Lemma 10 (ellipticity in the kernel) *There exists* $\alpha > 0$ *such that*

$$\int_\Omega |\mathbf{curl}\boldsymbol{v}|^2 \ge \alpha\|\boldsymbol{v}\|_{\mathbf{curl},\Omega}^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}},$$

being

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{H}}^{\perp_{\boldsymbol{\mathcal{Z}}}} = \left\{ \boldsymbol{v} \in \boldsymbol{\mathcal{Z}} : \int_\Omega \boldsymbol{v}\cdot\overline{\boldsymbol{q}} = 0, \quad \forall \boldsymbol{q} \in \boldsymbol{\mathcal{H}} \right\}.$$

Proof It is easy to check that $\boldsymbol{\mathcal{V}}$ can be characterized as

$$\boldsymbol{\mathcal{V}} = \left\{ \boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \cap H_0(\mathrm{div}^0; \Omega) \,:\, \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\rho}_i = 0, \ \text{for} \ i = 1, \ldots, g_1 \right\}.$$

Then the ellipticity in the kernel $\boldsymbol{\mathcal{V}}$ follows from the fact that $\|\boldsymbol{v}\|_{\mathbf{curl},\Omega} \leq C\|\mathbf{curl}\boldsymbol{v}\|_{0,\Omega}$ for all $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$ (see Lemma 9). □

Lemma 11 (inf–sup condition) *There exists $\beta > 0$ such that*

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{\mathcal{Z}}\\ \boldsymbol{v}\neq\mathbf{0}}} \frac{\left|\int_\Omega \boldsymbol{v} \cdot \overline{\boldsymbol{p}}\right|}{\|\boldsymbol{v}\|_{\mathbf{curl},\Omega}} \geq \beta \|\boldsymbol{p}\|_{0,\Omega}, \qquad \forall \boldsymbol{p} \in \boldsymbol{\mathcal{H}}.$$

Proof The inf–sup condition follows by taking $\boldsymbol{v} = \boldsymbol{p} \in \boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{Z}}$. □

By virtue of Lemmas 10 and 11, problem (9) is a well-posed problem, as the Babuška–Brezzi conditions for saddle-point problems are satisfied. Moreover, clearly, $\boldsymbol{T}\boldsymbol{u} = \mu\boldsymbol{u}$, with $\mu \neq 0$, if and only if $(\lambda, \boldsymbol{u}, \mathbf{0})$ is a solution of Problem 2, with $\lambda = 1/\mu$. Thus, we focus on characterizing the spectrum of $\boldsymbol{T}$. To this end, we start by introducing the following result.

Lemma 12 *The operator $\boldsymbol{T}$ is continuous and satisfies $\boldsymbol{T}(\boldsymbol{\mathcal{Z}}) \subset \boldsymbol{\mathcal{V}}$. Moreover, there exists $s > 1/2$ and $C > 0$ such that, for all $\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$, $\boldsymbol{w} = \boldsymbol{T}\boldsymbol{f} \in H^s(\mathbf{curl}; \Omega)$ and*

$$\|\boldsymbol{w}\|_{s,\Omega} + \|\mathbf{curl}\boldsymbol{w}\|_{s,\Omega} \leq C\|\boldsymbol{f}\|_{\mathbf{curl};\Omega}. \tag{11}$$

Consequently, $\boldsymbol{T}$ is compact in $\boldsymbol{\mathcal{Z}}$. In addition, there holds $(\mathbf{curl}\boldsymbol{w} - \boldsymbol{f}) \in \boldsymbol{\mathcal{H}}$.

Proof Let $\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$ then $\boldsymbol{w} = \boldsymbol{T}\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$ and from (9b), $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}$. The same arguments used in the proof of Lemma 8 apply to the problem defining $\boldsymbol{T}$ and thus we can prove that $\boldsymbol{q} = \mathbf{0}$ and $\mathbf{curl}(\mathbf{curl}\boldsymbol{w} - \boldsymbol{f}) = \mathbf{0}$ in Ω. Hence, $\mathbf{curl}\boldsymbol{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\mathrm{div}^0; \Omega)$. Thanks to (4), there exists $s > 1/2$ such that $H(\mathbf{curl}; \Omega) \cap H_0(\mathrm{div}^0; \Omega)$ is continuously embedded in $H^s(\Omega)^3$. So, $\boldsymbol{w} \in H^s(\mathbf{curl}; \Omega)$ and the estimate (11) holds true. The compactness of the inclusion $H^s(\mathbf{curl}; \Omega) \cap \boldsymbol{\mathcal{Z}} \hookrightarrow \boldsymbol{\mathcal{Z}}$ is a consequence of the fact that the inclusion $H^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Finally, we have to prove that $(\mathbf{curl}\boldsymbol{w} - \boldsymbol{f}) \in \boldsymbol{\mathcal{H}}$; this result is obtained by proceeding similarly to Lemma 8. □

Lemma 13 *The operator $\boldsymbol{T} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ is self-adjoint.*

Proof Given $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{\mathcal{Z}}$, let $\boldsymbol{w} := \boldsymbol{T}\boldsymbol{f}$ and $\boldsymbol{v} := \boldsymbol{T}\boldsymbol{g}$. From Lemma 12, we know that $\mathbf{curl}\boldsymbol{w} - \boldsymbol{f}$ and $\mathbf{curl}\boldsymbol{v} - \boldsymbol{g}$ belong to $\boldsymbol{\mathcal{H}}$. Using that $\boldsymbol{w}$, $\boldsymbol{v}$ satisfy the second equation of the problem defining $\boldsymbol{T}$ (cf. (9b)) and Lemma 7, we have that

$$\begin{aligned}
\int_\Omega \boldsymbol{T}\boldsymbol{f} \cdot \overline{\boldsymbol{g}} &= \int_\Omega \boldsymbol{w} \cdot \overline{\boldsymbol{g}} + \int_\Omega \boldsymbol{w} \cdot (\mathbf{curl}\overline{\boldsymbol{v}} - \overline{\boldsymbol{g}}) \\
&= \int_\Omega \boldsymbol{w} \cdot \mathbf{curl}\overline{\boldsymbol{v}} \\
&= \int_\Omega \mathbf{curl}\boldsymbol{w} \cdot \overline{\boldsymbol{v}} \\
&= \int_\Omega (\boldsymbol{f} - \mathbf{curl}\boldsymbol{w}) \cdot \overline{\boldsymbol{v}} + \int_\Omega \mathbf{curl}\boldsymbol{w} \cdot \overline{\boldsymbol{v}} \\
&= \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{T}\boldsymbol{g}}.
\end{aligned}$$

On the other hand, from (9a) and Lemma 7 again, we obtain that

$$\int_{\Omega} \mathbf{curl}(\boldsymbol{T}\boldsymbol{f}) \cdot \mathbf{curl}\overline{\boldsymbol{g}} = \int_{\Omega} \boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{g}} = \int_{\Omega} \mathbf{curl}\boldsymbol{f} \cdot \overline{\boldsymbol{g}} = \int_{\Omega} \mathbf{curl}\boldsymbol{f} \cdot \mathbf{curl}(\overline{\boldsymbol{T}\boldsymbol{g}}),$$

and we conclude the proof. □

Now, we are in a position to establish a spectral characterization of $\boldsymbol{T}$.

Theorem 1 *The spectrum of $\boldsymbol{T}$ decomposes as follows: sp($\boldsymbol{T}$)=$\{\mu_n\}_{n\in\mathbb{N}} \cup \{0\}$. Moreover,*

- $\mu_0 = 0$ *is an infinite-multiplicity eigenvalue and its associated eigenspace is $\boldsymbol{\mathcal{H}}$;*
- $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of finite-multiplicity eigenvalues (repeated according to their respective multiplicities) which converges to 0 and there exists a Hilbertian basis of associated eigenfunctions $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ of $\boldsymbol{\mathcal{V}}$ (i.e., such that $\boldsymbol{T}\boldsymbol{u}_n = \mu_n \boldsymbol{u}_n$, $n \in \mathbb{N}$).*

Proof Since $\boldsymbol{T} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ is compact, self-adjoint and $\boldsymbol{T}(\boldsymbol{\mathcal{Z}}) \subset \boldsymbol{\mathcal{V}}$, we only need to prove that Ker$\boldsymbol{T}$=$\boldsymbol{\mathcal{H}}$. We have

$$\mathrm{Ker}\,\boldsymbol{T} := \left\{ \boldsymbol{f} \in \boldsymbol{\mathcal{Z}} : \int_{\Omega} \boldsymbol{f} \cdot \mathbf{curl}\boldsymbol{v} = 0 \; \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \right\} \subset \boldsymbol{\mathcal{H}}$$

because $\mathcal{D}(\Omega) \subset \boldsymbol{\mathcal{Z}}$ and, by Lemma 6, $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{Z}} \cap H(\mathbf{curl}^0; \Omega)$. Conversely, if $\boldsymbol{f} \in \boldsymbol{\mathcal{H}}$, we have $\mathbf{curl}\boldsymbol{f} = \boldsymbol{0}$ in Ω and, thanks to Lemma 7, for all $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$ it holds $\int_{\Omega} \boldsymbol{f} \cdot \mathbf{curl}\boldsymbol{v} = \int_{\Omega} \mathbf{curl}\boldsymbol{f} \cdot \boldsymbol{v} = 0$, hence $\boldsymbol{f} \in \mathrm{Ker}\,\boldsymbol{T}$. □

4 Finite element approximation

In this section we introduce and study a Galerkin approximation of Problem 2 and prove convergence and error estimates for the computed eigenvalues and eigenfunctions. To that end, we assume that Ω is a polyhedron and we choose the "cutting" surfaces Σ_i, $i = 1, \cdots, g_1$, also polyhedral. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of tetrahedral partitions of $\overline{\Omega}$. We denote by E_h the set of all the edges of $\mathcal{T}_h$. It is not restrictive to assume that there exist sets $\mathsf{E}_h^{\gamma_i} \subset \mathsf{E}_h$, for $i = 1, \ldots, g_1$, and $\mathsf{E}_h^{\gamma'_j} \subset \mathsf{E}_h$, for $j = g_1 + 1, \ldots, g$ such that

$$\gamma_i = \bigcup_{e \in \mathsf{E}_h^{\gamma_i}} e\,, \;\text{ for } i = 1, \ldots, g_1, \quad \text{and} \quad \gamma'_j = \bigcup_{e \in \mathsf{E}_h^{\gamma'_j}} e\,, \;\text{ for } j = g_1 + 1, \ldots, g. \tag{12}$$

The mesh parameter h denotes the maximum diameter of all the tetrahedra $T \in \mathcal{T}_h$. For any $T \in \mathcal{T}_h$ and $k \geq 1$, let $\mathcal{N}^k(T) := \mathbb{P}_{k-1}(T)^3 \oplus \{\boldsymbol{p} \in \widetilde{\mathbb{P}}_k(T)^3 : \boldsymbol{p}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0\}$, where $\mathbb{P}_k$ is the set of polynomials of degree not greater than k and $\widetilde{\mathbb{P}}_k$ is the subset of homogeneous polynomials of degree k. The corresponding global space to approximate $H(\mathbf{curl}; \Omega)$ is the well-known Nédélec space defined as follows:

$$\mathcal{N}_h^k := \{\boldsymbol{v}_h \in H(\mathbf{curl}; \Omega) \,:\, \boldsymbol{v}_h \in \mathcal{N}^k(T)\,, \forall T \in \mathcal{T}_h\}.$$

Whence, the natural approximation space for $\boldsymbol{\mathcal{Z}}$ is

$$\begin{aligned}\boldsymbol{\mathcal{Z}}_h &:= \boldsymbol{\mathcal{Z}} \cap \mathcal{N}_h^k \\ &= \left\{ \boldsymbol{v}_h \in \mathcal{N}_h^k \, : \, \mathbf{curl} \boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \quad \oint_{\gamma_i} \boldsymbol{v}_h \cdot \boldsymbol{t}_i = 0 \,, \text{ for } i = 1, \dots, g_1 \right. \\ &\qquad \left. \text{and } \oint_{\gamma_j'} \boldsymbol{v}_h \cdot \boldsymbol{t}_j' = 0 \,, \text{ for } j = g_1 + 1, \dots, g \right\}.\end{aligned}$$

Let $\boldsymbol{I}_h^{\mathcal{R}}$ be the divergence-conforming Raviart–Thomas interpolant (see, for instance, [17, Sect. 5.4] for its definition and properties). This interpolant is well-defined for functions in $H^s(\Omega)^3$ with $s > 1/2$ ([17, Lemma 5.15]). In addition, we denote by $\boldsymbol{I}_h^{\mathcal{N}}$ the curl-conforming Nédélec interpolant. We refer to [17, Sect. 5.5] for its precise definition and the properties that we will use in the sequel. This interpolant is well-defined for function in $H^s(\mathbf{curl};\Omega)$ provided $s > 1/2$, so that $\boldsymbol{I}_h^{\mathcal{N}} : H^s(\mathbf{curl};\Omega) \to \mathcal{N}_h^k$ is a bounded linear operator. Moreover, as it is shown in the following lemma, the Nédélec interpolant of functions from $\boldsymbol{\mathcal{Z}}$ remains in this space.

Lemma 14 *For all $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \cap H^s(\mathbf{curl};\Omega)$ with $s > 1/2$, $\boldsymbol{I}_h^{\mathcal{N}} \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}_h$.*

Proof Let $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \cap H^s(\mathbf{curl};\Omega)$ with $s > 1/2$. Then, according to the definition of $\boldsymbol{\mathcal{Z}}$, $\mathbf{curl}\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, $\oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = 0$, $i = 1, \dots, g_1$ and $\oint_{\gamma_j'} \boldsymbol{v} \cdot \boldsymbol{t}_j' = 0$, $j = g_1 + 1, \dots, g$. Therefore,

$$\mathbf{curl}(\boldsymbol{I}_h^{\mathcal{N}} \boldsymbol{v}) \cdot \boldsymbol{n} = (\boldsymbol{I}_h^{\mathcal{R}} \mathbf{curl} \boldsymbol{v}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma,$$

where the first equality follows from [17, Lemma 5.40] and the second one from the fact that the Raviart–Thomas interpolant preserves vanishing normal components on the faces of the tetrahedra of the mesh (which follows from the definition of this interpolant). On the other hand, thanks to (12) and [17, Sect. 5.5], we obtain

$$\oint_{\gamma_i} \boldsymbol{I}_h^{\mathcal{N}} \boldsymbol{v} \cdot \boldsymbol{t}_i = \oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = 0, \qquad \text{for } i = 1, \dots, g_1$$

and, analogously,

$$\oint_{\gamma_j'} \boldsymbol{I}_h^{\mathcal{N}} \boldsymbol{v} \cdot \boldsymbol{t}_j' = \oint_{\gamma_j'} \boldsymbol{v} \cdot \boldsymbol{t}_j' = 0, \qquad \text{for } j = g_1 + 1, \dots, g.$$

Thus, $\boldsymbol{I}_h^{\mathcal{N}} \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}_h$. □

To discretize the Lagrange multiplier $\boldsymbol{q} \in \boldsymbol{\mathcal{H}}$ we use the finite element space

$$\boldsymbol{\mathcal{H}}_h := \boldsymbol{\mathcal{Z}}_h \cap H(\mathbf{curl}^0;\Omega).$$

Note that

$$\boldsymbol{\mathcal{H}}_h = \left\{ \boldsymbol{v}_h \in \mathcal{N}_h^k \, : \, \mathbf{curl} \boldsymbol{v}_h = \mathbf{0} \text{ in } \Omega, \oint_{\gamma_j'} \boldsymbol{v}_h \cdot \boldsymbol{t}_j' = 0 \,, \text{ for } j = g_1 + 1, \dots, g \right\},$$

since, by definition (6), $\oint_{\gamma_i} \boldsymbol{v}_h \cdot \boldsymbol{t}_i = \int_\Omega \mathbf{curl}\, \boldsymbol{v}_h \cdot \boldsymbol{\rho}_i$, $i = 1, \dots, g$, and the last integral vanishes as $\mathbf{curl}\, \boldsymbol{v}_h = \mathbf{0}$ in Ω (as above, we could say that the result is clear since the cycles γ_i are bounding in $\overline{\Omega}$).

Now, we are in position to introduce a finite element discretization of Problem 2.

Problem 3. Find $\lambda_h \in \mathbb{C}$ and $(\boldsymbol{u}_h, \boldsymbol{q}_h) \in \boldsymbol{\mathcal{Z}}_h \times \boldsymbol{\mathcal{H}}_h$, $\boldsymbol{u}_h \neq \boldsymbol{0}$, such that

$$\int_\Omega \mathbf{curl}\boldsymbol{u}_h \cdot \mathbf{curl}\overline{\boldsymbol{v}_h} + \int_\Omega \boldsymbol{q}_h \cdot \overline{\boldsymbol{v}_h} = \lambda_h \int_\Omega \boldsymbol{u}_h \cdot \mathbf{curl}\overline{\boldsymbol{v}_h}\,,$$
$$\int_\Omega \boldsymbol{u}_h \cdot \overline{\boldsymbol{p}_h} = 0$$

for all $(\boldsymbol{v}_h, \boldsymbol{p}_h) \in \boldsymbol{\mathcal{Z}}_h \times \boldsymbol{\mathcal{H}}_h$.

Notice that, also in this case, for any solution $(\lambda_h, \boldsymbol{u}_h, \varphi_h)$ of Problem 3, $\boldsymbol{q}_h$ vanishes. In fact, since $\boldsymbol{\mathcal{H}}_h \subset \boldsymbol{\mathcal{Z}}_h$, this result follows by taking $\boldsymbol{v}_h = \boldsymbol{q}_h$. Moreover, one easily obtains that $\lambda_h \neq 0$ and $\lambda_h \in \mathbb{R}$.

As for the continuous problem, we consider the corresponding discrete solution operator:

$$\begin{aligned} \boldsymbol{T}_h \,:\, \boldsymbol{\mathcal{Z}} &\longrightarrow \boldsymbol{\mathcal{Z}}\,, \\ \boldsymbol{f} &\longmapsto \boldsymbol{T}_h\boldsymbol{f} := \boldsymbol{w}_h\,, \end{aligned}$$

with $\boldsymbol{w}_h \in \boldsymbol{\mathcal{Z}}_h$ such that there exist $\boldsymbol{q}_h \in \boldsymbol{\mathcal{H}}_h$ satisfying

$$\begin{aligned} \int_\Omega \mathbf{curl}\boldsymbol{w}_h \cdot \mathbf{curl}\overline{\boldsymbol{v}_h} + \int_\Omega \boldsymbol{q}_h \cdot \overline{\boldsymbol{v}_h} &= \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{v}_h}\,, \\ \int_\Omega \boldsymbol{w}_h \cdot \overline{\boldsymbol{p}_h} &= 0 \end{aligned} \tag{13}$$

for all $(\boldsymbol{v}_h, \boldsymbol{p}_h) \in \boldsymbol{\mathcal{Z}}_h \times \boldsymbol{\mathcal{H}}_h$. In what follows, we check that the Babuška–Brezzi conditions are satisfied for problem (13).

Lemma 15 (discrete ellipticity in the kernel) *There exists $\alpha > 0$, independent of h, such that*

$$\int_\Omega |\mathbf{curl}\boldsymbol{v}_h|^2 \geq \alpha \|\boldsymbol{v}_h\|^2_{\mathbf{curl},\Omega} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h,$$

where

$$\boldsymbol{\mathcal{V}}_h := \left\{ \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h \,:\, \int_\Omega \boldsymbol{v}_h \cdot \overline{\boldsymbol{p}_h} = 0 \quad \forall \boldsymbol{p}_h \in \boldsymbol{\mathcal{H}}_h \right\}.$$

Proof We define the following operator:

$$\begin{aligned} \mathcal{R} \,:\, \boldsymbol{\mathcal{V}}_h &\longrightarrow H(\mathbf{curl};\Omega)\,, \\ \boldsymbol{v}_h &\longmapsto \mathcal{R}\boldsymbol{v}_h := \boldsymbol{v}_h - \boldsymbol{\Phi}_{\boldsymbol{v}_h}\,, \end{aligned}$$

with $\boldsymbol{\Phi}_{\boldsymbol{v}_h} \in \boldsymbol{\mathcal{H}}$ is such that

$$\int_\Omega \boldsymbol{\Phi}_{\boldsymbol{v}_h} \cdot \overline{\boldsymbol{p}} = \int_\Omega \boldsymbol{v}_h \cdot \overline{\boldsymbol{p}} \qquad \forall \boldsymbol{p} \in \boldsymbol{\mathcal{H}}\,.$$

From the inclusions $\boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{\mathcal{Z}}_h \subset \boldsymbol{\mathcal{Z}}$ we have at once $\mathcal{R}\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}$, and from the definition of $\boldsymbol{\Phi}_{\boldsymbol{v}_h}$ it follows $\mathcal{R}\boldsymbol{v}_h \perp \boldsymbol{\mathcal{H}}$. Consequently, $\mathcal{R}\boldsymbol{v}_h \in \boldsymbol{\mathcal{V}} \subset H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div}^0;\Omega) \subset H^s(\Omega)^3$ for $s > 1/2$ (see relation (4)). In addition, $\mathbf{curl}(\mathcal{R}\boldsymbol{v}_h) = \mathbf{curl}\boldsymbol{v}_h \in \mathbf{curl}(\mathcal{N}_h^k)$. Hence, thanks to [17, Theorem 5.41], we have that $\boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)$ is well-defined, $\mathbf{curl}\,\boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h) = \mathbf{curl}\boldsymbol{v}_h$, and

$$\|\mathcal{R}\boldsymbol{v}_h - \boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega} \leq C\left\{h^s\|\mathcal{R}\boldsymbol{v}_h\|_{s,\Omega} + h\|\mathbf{curl}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega}\right\}. \tag{14}$$

Notice that from relation (4), Lemma 9 and the fact that $\mathcal{R}\boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}$, we obtain

$$\|\mathcal{R}\boldsymbol{v}_h\|_{s,\Omega} \le C\|\mathcal{R}\boldsymbol{v}_h\|_{\boldsymbol{\mathcal{Y}}} \le \widetilde{C}\|\mathbf{curl}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega}. \tag{15}$$

Employing the previous result and (14), we have

$$\|\mathcal{R}\boldsymbol{v}_h - \boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega} \le C(h^s + h)\|\mathbf{curl}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega}. \tag{16}$$

The Nédélec interpolant $\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{\Phi}_{\boldsymbol{v}_h}$ is defined by $\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{\Phi}_{\boldsymbol{v}_h} = \boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{v}_h - \boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h) = \boldsymbol{v}_h - \boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)$ and, since $\mathbf{curl}\,\boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h) = \mathbf{curl}\boldsymbol{v}_h$, it follows that $\mathbf{curl}(\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{\Phi}_{\boldsymbol{v}_h}) = \mathbf{0}$ in Ω. Furthermore, $\oint_{\gamma_j'} \boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{\Phi}_{\boldsymbol{v}_h}\cdot\boldsymbol{t}_j' = \oint_{\gamma_j'} \boldsymbol{\Phi}_{\boldsymbol{v}_h}\cdot\boldsymbol{t}_j' = 0$ for $j = g_1+1, \ldots, g$, as $\boldsymbol{\Phi}_{\boldsymbol{v}_h} \in \boldsymbol{\mathcal{H}}$. In conclusion, $\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{\Phi}_{\boldsymbol{v}_h} \in \boldsymbol{\mathcal{H}}_h$.

Employing this result and the fact that $\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{v}_h = \boldsymbol{v}_h$, we obtain

$$\begin{aligned}\|\boldsymbol{v}_h\|_{0,\Omega}^2 &= \int_\Omega \boldsymbol{v}_h \cdot \overline{\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{v}_h} = \int_\Omega \boldsymbol{v}_h \cdot \left(\overline{\boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)} + \overline{\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{\Phi}_{\boldsymbol{v}_h}}\right) \\ &= \int_\Omega \boldsymbol{v}_h \cdot \overline{\boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)},\end{aligned}$$

as $\boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h$. Using inequalities (16) and (15), and the fact that $\mathbf{curl}(\mathcal{R}\boldsymbol{v}_h) = \mathbf{curl}\boldsymbol{v}_h$ in Ω, we obtain

$$\begin{aligned}\|\boldsymbol{v}_h\|_{0,\Omega} &\le \|\boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega} \le \|\mathcal{R}\boldsymbol{v}_h - \boldsymbol{I}_h^{\mathcal{N}}(\mathcal{R}\boldsymbol{v}_h)\|_{0,\Omega} + \|\mathcal{R}\boldsymbol{v}_h\|_{0,\Omega} \\ &\le C(h^s + h)\|\mathbf{curl}\boldsymbol{v}_h\|_{0,\Omega} + \widetilde{C}\|\mathbf{curl}\boldsymbol{v}_h\|_{0,\Omega} \\ &\le \widehat{C}\|\mathbf{curl}\boldsymbol{v}_h\|_{0,\Omega},\end{aligned}$$

which concludes the proof. □

Lemma 16 (discrete inf-sup condition) *There exists $\beta > 0$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h \\ \boldsymbol{v}_h \neq \mathbf{0}}} \frac{\left|\int_\Omega \boldsymbol{v}_h \cdot \overline{\boldsymbol{p}_h}\right|}{\|\boldsymbol{v}_h\|_{\mathbf{curl};\Omega}} \ge \beta\,\|\boldsymbol{p}_h\|_{0,\Omega}\,, \qquad \forall\, \boldsymbol{p}_h \in \boldsymbol{\mathcal{H}}_h.$$

Proof As in the continuous case, the inf–sup condition is easily checked by taking $\boldsymbol{v}_h = \boldsymbol{p}_h \in \boldsymbol{\mathcal{H}}_h \subset \boldsymbol{\mathcal{Z}}_h$. □

Clearly, as a consequence of Lemma 15 and Lemma 16, $\boldsymbol{T}_h$ is a well-defined bounded linear operator. Moreover, $\boldsymbol{T}_h\boldsymbol{u}_h = \mu_h\boldsymbol{u}_h$, with $\mu_h \neq 0$, if and only if $(\lambda_h, \boldsymbol{u}_h, \mathbf{0})$ is a solution of Problem 3, with $\lambda_h = 1/\mu_h$.

In order to prove that the eigenvalues and eigenfunctions of Problem 2 are well-approximated by those of Problem 3, we use the classical theory for compact operators from [3]. To this end we will prove that $\boldsymbol{T}_h$ converges in norm to $\boldsymbol{T}$. In fact, from Lemma 12, we know that $\boldsymbol{T}\boldsymbol{f} \in H^s(\mathbf{curl};\Omega)$ for a suitable $s > \frac{1}{2}$ and the following result is easy to prove :

Lemma 17 *There exists $C > 0$, independent of h, such that for all $\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f}\|_{\mathbf{curl};\Omega} \le Ch^{\min\{s,k\}}\|\boldsymbol{f}\|_{\mathbf{curl};\Omega}\,.$$

Proof Given $\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$, let $\boldsymbol{w} := \boldsymbol{T}\boldsymbol{f}$ and $\boldsymbol{w}_h := \boldsymbol{T}_h\boldsymbol{f}$. The following Céa-type estimate follows from the definitions of $\boldsymbol{T}$ and $\boldsymbol{T}_h$ and the fact that the Lagrange multipliers vanish:

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathbf{curl};\Omega} \le C \inf_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h} \|\boldsymbol{w} - \boldsymbol{v}_h\|_{\mathbf{curl};\Omega}. \tag{17}$$

By using [17, Theorem 5.41] and Lemma 12, we obtain

$$\|\boldsymbol{w} - \boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{w}\|_{\mathbf{curl};\Omega} \le Ch^{\min\{s,k\}}\left(\|\boldsymbol{w}\|_{s,\Omega} + \|\mathbf{curl}\boldsymbol{w}\|_{s,\Omega}\right) \le Ch^{\min\{s,k\}}\|\boldsymbol{f}\|_{\mathbf{curl};\Omega}. \tag{18}$$

Therefore, from (17) and (18) we conclude the proof. □

As a consequence of the previous lemma, according to [3], we have that the eigenvalues and eigenspaces of $\boldsymbol{T}$ are well-approximated by those of $\boldsymbol{T}_h$. More precisely, let λ be an eigenvalue of Problem 2 with multiplicity m and $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{Z}}$ the corresponding eigenspace; then, there exist exactly m eigenvalues $\lambda_h^{(1)}, \ldots, \lambda_h^{(m)}$ of Problem 3 (repeated according to their respective multiplicities) which converge to λ as $h \to 0$. Furthermore, let $\boldsymbol{\mathcal{E}}_h$ be the direct sum of the eigenspaces corresponding to $\lambda_h^{(1)}, ..., \lambda_h^{(m)}$ and let us introduce the so-called gap between the continuous and discrete eigenspaces, given by

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) := \max\{\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h), \delta(\boldsymbol{\mathcal{E}}_h, \boldsymbol{\mathcal{E}})\},$$

with $\delta(M, N) := \sup_{\substack{x \in M \\ \|x\|=1}} \operatorname{dist}(x, N)$; then, it follows that $\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \to 0$ as h goes to zero. Finally, the following estimates hold true:

Theorem 2 *Let $r > 0$ be such that $\boldsymbol{\mathcal{E}} \subset H^r(\mathbf{curl}; \Omega)$. There exist constants C_1, $C_2 > 0$, independents of h, such that, for small h,*

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \le C_1 h^{\min\{r,k\}}, \tag{19}$$

and

$$|\lambda - \lambda_h^{(i)}| \le C_2 h^{2\min\{r,k\}}, \quad i = 1, ..., m. \tag{20}$$

Proof First note that, by proceeding as in the proof of Lemma 17, we have that, for all $\boldsymbol{f} \in \boldsymbol{\mathcal{E}}$,

$$\begin{aligned} \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f}\|_{\mathbf{curl};\Omega} &\le Ch^{\min\{r,k\}}\left(\|\boldsymbol{T}\boldsymbol{f}\|_{r,\Omega} + \|\mathbf{curl}\boldsymbol{T}\boldsymbol{f}\|_{r,\Omega}\right) \\ &\le Ch^{\min\{r,k\}} \sup_{\boldsymbol{g} \in \boldsymbol{\mathcal{E}}} \frac{\|\boldsymbol{T}\boldsymbol{g}\|_{r,\Omega} + \|\mathbf{curl}\boldsymbol{T}\boldsymbol{g}\|_{r,\Omega}}{\|\boldsymbol{g}\|_{\mathbf{curl};\Omega}} \|\boldsymbol{f}\|_{\mathbf{curl};\Omega} \\ &\le C'h^{\min\{r,k\}}\|\boldsymbol{f}\|_{\mathbf{curl};\Omega}, \end{aligned} \tag{21}$$

where we have used the fact that $\boldsymbol{\mathcal{E}}$ is finite dimensional for the last inequality. Therefore (19) follows from (21) and [3, Chap. II, Theor. 7.1].

To prove (20) we will resort to [3, Chap. II, Theor. 7.3]. To this end, let $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{V}}$ be two eigenfunctions. Then, in particular, $\boldsymbol{g}$ satisfies

$$\mathbf{curl}\boldsymbol{g} = \lambda\boldsymbol{g} \text{ in } \Omega. \tag{22}$$

If we define $\boldsymbol{u} := \boldsymbol{T}\boldsymbol{f}$ and $\boldsymbol{u}_h := \boldsymbol{T}_h\boldsymbol{f}$, then the following identities are satisfied

$$\int_\Omega \mathbf{curl}\boldsymbol{u} \cdot \mathbf{curl}\overline{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{v}} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$$

$$\int_\Omega \mathbf{curl}\boldsymbol{u}_h \cdot \mathbf{curl}\overline{\boldsymbol{v}_h} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{v}_h} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h\,.$$

Subtracting both identities, we obtain

$$\int_\Omega \mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \mathbf{curl}\overline{\boldsymbol{v}_h} = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h. \tag{23}$$

In addition, let $\boldsymbol{w} := \boldsymbol{T}\boldsymbol{g}$ and $\boldsymbol{w}_h := \boldsymbol{T}_h\boldsymbol{g}$. Then we have analogous identities for $\boldsymbol{g}$ and, in particular,

$$\int_\Omega \mathbf{curl}\boldsymbol{w} \cdot \mathbf{curl}\overline{\boldsymbol{v}} = \int_\Omega \boldsymbol{g} \cdot \mathbf{curl}\overline{\boldsymbol{v}} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}. \tag{24}$$

Thus, thanks to (22), the fact that $\boldsymbol{u}$, $\boldsymbol{u}_h$, $\boldsymbol{g} \in \boldsymbol{\mathcal{Z}}$, Lemma 7, (23) and (24), we obtain

$$\begin{aligned}
\int_\Omega (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f} \cdot \overline{\boldsymbol{g}} &= \lambda^{-1} \int_\Omega (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \mathbf{curl}\overline{\boldsymbol{g}} \\
&= \lambda^{-1} \int_\Omega \mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \overline{\boldsymbol{g}} \\
&= \lambda^{-1} \int_\Omega \mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \mathbf{curl}(\overline{\boldsymbol{w}} - \overline{\boldsymbol{w}_h}) \\
&= \lambda^{-1} \int_\Omega \mathbf{curl}((\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f}) \cdot \mathbf{curl}((\boldsymbol{T} - \boldsymbol{T}_h)\overline{\boldsymbol{g}})\,.
\end{aligned}$$

Using Cauchy–Schwarz inequality and (21) we get

$$\begin{aligned}
\left| \int_\Omega (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f} \cdot \overline{\boldsymbol{g}} \right| &\le \lambda^{-1} \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f}\|_{\mathbf{curl};\Omega} \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{g}\|_{\mathbf{curl};\Omega} \\
&\le C\lambda^{-1} h^{2\min\{r,k\}} \|\boldsymbol{f}\|_{\mathbf{curl};\Omega} \|\boldsymbol{g}\|_{\mathbf{curl};\Omega}.
\end{aligned} \tag{25}$$

On the other hand, thanks to (23), Cauchy–Schwarz inequality, (21), Lemma 12 and [17, Theorem 5.41], we obtain

$$\begin{aligned}
\left| \int_\Omega \mathbf{curl}(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{g}} \right| &= \left| \int_\Omega \mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \mathbf{curl}\overline{\boldsymbol{g}} \right| \\
&= \left| \int_\Omega \mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \mathbf{curl}(\overline{\boldsymbol{g}} - \overline{\boldsymbol{I}_h^{\mathcal{N}}\boldsymbol{g}}) \right| \\
&\le C h^{\min\{r,k\}} \|\boldsymbol{f}\|_{\mathbf{curl};\Omega}\, h^{\min\{r,k\}} (\|\boldsymbol{g}\|_{r;\Omega} + \|\mathbf{curl}\boldsymbol{g}\|_{r,\Omega}) \\
&\le C' h^{2\min\{r,k\}} \|\boldsymbol{f}\|_{\mathbf{curl};\Omega} \|\boldsymbol{g}\|_{\mathbf{curl};\Omega}\,.
\end{aligned} \tag{26}$$

Thus, from (25) and (26) we conclude that

$$\sup_{\boldsymbol{f},\boldsymbol{g} \in \boldsymbol{\mathcal{E}}} \frac{\left| \int_\Omega (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f} \cdot \overline{\boldsymbol{g}} + \int_\Omega \mathbf{curl}(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{f} \cdot \mathbf{curl}\overline{\boldsymbol{g}} \right|}{\|\boldsymbol{f}\|_{\mathbf{curl};\Omega} \|\boldsymbol{g}\|_{\mathbf{curl};\Omega}} \le C h^{2\min\{r,k\}}\ .$$

Estimate (20) follows from [3, Chap. II, Theor. 7.3] and the fact that $\boldsymbol{T}$ is self adjoint. □

Remark 4 The theorem above always holds for $r = s$ as in Lemma 12. However, if $\boldsymbol{\mathcal{E}} \subset H^r(\mathbf{curl}; \Omega)$ with $r > s$, the theorem yields estimates in terms of r.

4.1 Finite element implementation

In this section we will describe how to impose in $\mathcal{N}_h^k$ the constraints defining $\boldsymbol{\mathcal{Z}}_h$ and $\boldsymbol{\mathcal{H}}_h$. To begin with, let us denote by $\mathcal{T}_h^\Gamma$ the triangulation induced on Γ by $\mathcal{T}_h$. We recall that E_h is the set of edges of $\mathcal{T}_h$ and denote by V_h and F_h the set of vertices and faces of $\mathcal{T}_h$. We also denote by V_h^Γ, E_h^Γ and F_h^Γ the set of vertices, edges and faces of $\mathcal{T}_h^\Gamma$.

Our first aim is to construct a basis of $\boldsymbol{\mathcal{Z}}_h$. Let $\{\boldsymbol{w}_{h,m}\}_{m=1}^M$ be a basis of $\mathcal{N}_h^k$; we assume that $\{\boldsymbol{w}_{h,m}\}_{m=1}^{M'} (M < M')$ is a basis of $\mathcal{N}_h^k \cap H_0(\mathbf{curl}; \Omega)$. We denote by $\mathcal{L}_h^k$ the space of Lagrange finite elements of degree k and by $\{\varphi_{h,l}\}_{l=1}^L$ a basis of $\mathcal{L}_h^k$. We also choose these basis functions so that the first L' of them correspond to a basis of $\mathcal{L}_h^k \cap H_0^1(\Omega)$.

Proceeding as in [16, Sect. 4] it can be proved that the set $\{\boldsymbol{w}_{h,m}\}_{m=1}^{M'} \cup \{\nabla\varphi_{h,l}\}_{l=L'+1}^{L-1}$ is a basis of $\widehat{\boldsymbol{\mathcal{X}}} \cap \mathcal{N}_h^k$, where

$$\widehat{\boldsymbol{\mathcal{X}}} = \left\{ \boldsymbol{v} \in \boldsymbol{\mathcal{X}} \,:\, \oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = \oint_{\gamma'_j} \boldsymbol{v} \cdot \boldsymbol{t}'_j = 0 \text{ for } 1 \leq i,j \leq g \right\}.$$

The idea is to complete this set to a basis of $\boldsymbol{\mathcal{Z}}_h$ by adding g linearly independent functions in span$\{\boldsymbol{w}_{h,m}\,,\, m = M'+1, \ldots, M\}$ not belonging to $\widehat{\boldsymbol{\mathcal{X}}}$.

We consider the curves $\gamma_i^+ := \partial\Sigma_i^+$, $\gamma_i^- := \partial\Sigma_i^-$, ${\gamma'_j}^+ := \partial{\Sigma'_j}^+$ and ${\gamma'_j}^- := \partial{\Sigma'_j}^-$. Then, for $i = 1, \ldots, g_1$ we denote by $\xi_{h,i} \in \mathcal{C}(\Gamma \setminus \gamma_i)$ the function such that $\xi_{h,i}|_f \in \mathbb{P}_1(f)\ \forall f \in \mathsf{F}_h^\Gamma$, $\xi_{h,i}|_{\gamma_i^+} = 1$, $\xi_{h,i}|_{\gamma_i^-} = 0$ and $\xi_{h,i}(P) = 0$ if $P \in \mathsf{V}_h^\Gamma \setminus \gamma_i$. Analogously for $j = g_1+1, \ldots, g$ we denote by $\xi'_{h,j} \in \mathcal{C}(\Gamma \setminus \gamma'_j)$ the function such that $\xi'_{h,j}|_f \in \mathbb{P}_1(f)\ \forall f \in \mathsf{F}_h^\Gamma$, $\xi'_{h,j}|_{{\gamma'_j}^+} = 1$, $\xi'_{h,j}|_{{\gamma'_j}^-} = 0$ and $\xi'_{h,j}(P) = 0$ if $P \in \mathsf{V}_h^\Gamma \setminus \gamma'_j$.

Let $\{e_n\}_{n=1}^{M_1}$ be the set of oriented edges E_h. Again we assume that $\{e_m\}_{m=M'_1+1}^{M_1}$ are the edges in E_h^Γ. Since we are considering oriented edges we will use also the following notation: $e_m = [P(e_m), Q(e_m)]$, where $P(e_m)$ and $Q(e_m)$ denote the initial and final vertices of e_m, respectively. We denote by $\boldsymbol{t}_{e_m}$ the unit tangent vector pointing from $P(e_m)$ towards $Q(e_m)$. Moreover we denote by $\mathring{e}_m := e_m \setminus \{P(e_m), Q(e_m)\}$.

Let $\{\boldsymbol{w}_{h,m}^{(1)}\}_{m=1}^{M_1}$ be the canonical basis of $\mathcal{N}_h^1$, namely, $\int_{e_n} \boldsymbol{w}_{h,m}^{(1)} \cdot \boldsymbol{t}_{e_n} = \delta_{n,m}$. For each edge $e_m \in \mathsf{E}_h^\Gamma$, we define

$$c_m(\xi_{h,i}) := \begin{cases} \displaystyle\lim_{\substack{s \to Q(e_m) \\ s \in \mathring{e}_m}} \xi_{h,i}(s) - \lim_{\substack{s \to P(e_m) \\ s \in \mathring{e}_m}} \xi_{h,i}(s), & \text{if } \mathring{e}_m \subset \Gamma \setminus \gamma_i\,, \\ 0, & \text{if } e_m \subset \gamma_i\,, \end{cases} \tag{27}$$

for $i = 1, ..., g_1$ and

$$c'_m(\xi'_{h,j}) := \begin{cases} \lim\limits_{\substack{s \to Q(e_m) \\ s \in \mathring{e}_m}} \xi'_{h,j}(s) - \lim\limits_{\substack{s \to P(e_m) \\ s \in \mathring{e}_m}} \xi'_{h,j}(s), & \text{if } \mathring{e}_m \subset \Gamma \setminus \gamma'_j\,, \\ 0, & \text{if } e_m \subset \gamma'_j\,, \end{cases}$$

for $j = g_1 + 1, ..., g$. Now we define the following functions:

$$\widehat{\boldsymbol{w}}_{h,i} := \sum_{m=M'_1+1}^{M_1} c_m(\xi_{h,i}) \boldsymbol{w}^{(1)}_{h,m}\,,$$

for $i = 1, ..., g_1$ and

$$\widehat{\boldsymbol{w}}'_{h,j} := \sum_{m=M'_1+1}^{M_1} c'_m(\xi'_{h,j}) \boldsymbol{w}^{(1)}_{h,m}\,,$$

for $j = g_1 + 1, ..., g$.

It is readily seen that the g functions $\{\widehat{\boldsymbol{w}}_{h,i}\}_{i=1}^{g_1} \cup \{\widehat{\boldsymbol{w}}'_{h,j}\}_{j=g_1+1}^{g}$ belong to $\boldsymbol{\mathcal{Z}}_h$. In fact, for all $T \subset \Gamma$ there holds $\int_T \mathbf{curl}\widehat{\boldsymbol{w}}_{h,i} \cdot \mathbf{n} = \oint_{\partial T} \widehat{\boldsymbol{w}}_{h,i} \cdot \boldsymbol{t}_{\partial T} = 0$, the latter being a consequence of (27). Analogously, there holds $\int_T \mathbf{curl}\widehat{\boldsymbol{w}}'_{h,j} \cdot \mathbf{n} = 0$. Moreover, for $1 \le k, l \le g$, one has

$$\begin{aligned} &\oint_{\gamma_k} \widehat{\boldsymbol{w}}_{h,i} \cdot \boldsymbol{t}_k = 0\,, \quad \oint_{\gamma'_l} \widehat{\boldsymbol{w}}_{h,i} \cdot \boldsymbol{t}'_l = \delta_{i,l} \quad && \text{for } 1 \le i \le g_1\,, \\ &\oint_{\gamma_k} \widehat{\boldsymbol{w}}'_{h,j} \cdot \boldsymbol{t}_k = \delta_{j,k}\,, \oint_{\gamma'_l} \widehat{\boldsymbol{w}}'_{h,j} \cdot \boldsymbol{t}'_l = 0 \quad && \text{for } g_1 + 1 \le j \le g\,. \end{aligned}$$

Hence, the g functions $\{\widehat{\boldsymbol{w}}_{h,i}\}_{i=1}^{g_1} \cup \{\widehat{\boldsymbol{w}}'_{h,j}\}_{j=g_1+1}^{g}$ do not belong to $\widehat{\boldsymbol{\mathcal{X}}}$, and are linearly independent. Finally for $\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h$ we see that

$$\boldsymbol{v}_h - \sum_{i=1}^{g_1} \left(\oint_{\gamma'_i} \boldsymbol{v}_h \cdot \boldsymbol{t}'_i \right) \widehat{\boldsymbol{w}}_{h,i} - \sum_{k=g_1+1}^{g} \left(\oint_{\gamma_j} \boldsymbol{v}_h \cdot \boldsymbol{t}_j \right) \widehat{\boldsymbol{w}}'_{h,j} \in \widehat{\boldsymbol{\mathcal{X}}} \cap \mathcal{N}^k_h\,.$$

So, we have proved the following result.

Proposition 1 $\{\boldsymbol{w}_{h,m}\}_{m=1}^{M'} \cup \{\nabla\varphi_{h,l}\}_{l=L'+1}^{L-1} \cup \{\widehat{\boldsymbol{w}}_{h,i}\}_{i=1}^{g_1} \cup \{\widehat{\boldsymbol{w}}'_{h,j}\}_{j=g_1+1}^{g}$ *is a basis of* $\boldsymbol{\mathcal{Z}}_h$.

For the lowest-order elements, the constraints $\mathbf{curl}\boldsymbol{v}_h \cdot \boldsymbol{n} = 0$ on Γ, $\oint_{\gamma_i} \boldsymbol{v} \cdot \boldsymbol{t}_i = 0\,,\ i = 1, \dots, g_1$ and $\oint_{\gamma'_j} \boldsymbol{v} \cdot \boldsymbol{t}'_j = 0\,,\ j = g_1 + 1, \dots, g$, in the definition of $\boldsymbol{\mathcal{Z}}_h$ can be imposed by means of a static condensation from the matrix of the classical Nédélec elements. To this end, we follow a similar approach to the one used in [15]. In what follows, by simplicity, we assume that the boundary Γ is connected (otherwise, the same procedure should be repeated for each of its connected components).

We denote as above $\{e_1, \dots, e_M\}$ the set of all edges in $\mathcal{T}_h$ and $\{\boldsymbol{w}_{h,m}\}_{m=1}^{M}$ the associated nodal basis of $\mathcal{N}^1_h$. Then, for any $\boldsymbol{u}_h \in \mathcal{N}^1_h$,

$$\boldsymbol{u}_h = \sum_{m=1}^{M} \alpha_m \boldsymbol{w}_{h,m}$$

where $\alpha_m := \int_{e_m} \boldsymbol{u}_h \cdot \boldsymbol{t}_{e_m}$, $m = 1, \ldots, M$. We recall that the edges lying on Γ are the last ones : $e_{M'+1}, \ldots, e_M$. According to Proposition 1, for $\boldsymbol{u}_h \in \boldsymbol{\mathcal{Z}}_h$, there exist numbers $\alpha'_1, \ldots, \alpha'_{M'}$, $\beta_1, \ldots, \beta_{K-1}$, $\widehat{\beta}_1, \ldots, \widehat{\beta}_{g_1}$ and $\widehat{\beta}'_{g_1+1}, \ldots, \widehat{\beta}'_g$ such that

$$\boldsymbol{u}_h = \sum_{m=1}^{M'} \alpha'_m \boldsymbol{w}_{h,m} + \sum_{l=1}^{K-1} \beta_l \nabla \varphi_{h,l} + \sum_{i=1}^{g_1} \widehat{\beta}_i \widehat{\boldsymbol{w}}_{h,i} + \sum_{j=g_1+1}^{g} \widehat{\beta}'_j \widehat{\boldsymbol{w}}'_{h,j} \,.$$

Then, from the definition of α_m and the above relation, we obtain that there exists a matrix $\mathbf{K} \in \mathbb{R}^{M \times (M'+K-1+g)}$ such that $\boldsymbol{\alpha} = \mathbf{K}\widehat{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_M)^t$, $\widehat{\boldsymbol{\alpha}} = (\alpha'_1, \ldots, \alpha'_{M'}, \beta_1, \ldots, \beta_{K-1}, \widehat{\beta}_1, \ldots, \widehat{\beta}_{g_1}, \widehat{\beta}'_{g_1+1}, \ldots, \widehat{\beta}'_g)^t$ and

$$\mathbf{K} = \left[\mathbf{D} \,\middle|\, \mathbf{E} \,\middle|\, \mathbf{F} \,\middle|\, \mathbf{F}' \right] ,$$

with

$$\mathbf{D}_{m,n} := \int_{e_m} \boldsymbol{w}_{h,n} \cdot \boldsymbol{t}_{e_m} = \delta_{m,n} \,; \quad \mathbf{E}_{m,l} := \int_{e_m} \nabla \varphi_{h,l} \cdot \boldsymbol{t}_{e_m} = \begin{cases} \pm 1 & \text{if } P_l \in e_m, \\ 0 & \text{elsewhere,} \end{cases}$$

where $P_l \in \mathsf{V}_h$ is such that $\varphi_{h,l}(P_l) = 1$;

$$\mathbf{F}_{m,i} := \int_{e_m} \widehat{\boldsymbol{w}}_{h,i} \cdot \boldsymbol{t}_{e_m} = \begin{cases} 1 \;\; \text{if } e_m \cap \gamma_i^+ = \{P\}, \\ 0 \text{ elsewhere,} \end{cases}$$

for $i = 1, ..., g_1$ and

$$\mathbf{F}'_{m,j} := \int_{e_m} \widehat{\boldsymbol{w}}'_{h,j} \cdot \boldsymbol{t}_{e_m} = \begin{cases} 1 \;\; \text{if } e_m \cap \gamma_j'^{+} = \{P\}, \\ 0 \text{ elsewhere,} \end{cases}$$

for $j = g_1 + 1, ..., g$.

On the other hand, the space $\boldsymbol{\mathcal{H}}_h$ is generated by the linear combinations of the gradients of the Lagrange nodal basis functions and g_1 curl-free Nédélec elements $\mathbf{T}_i$, $i = 1, \ldots, g_1$, satisfying for $j = 1, \ldots, g$

$$\oint_{\gamma'_j} \mathbf{T}_i \cdot \boldsymbol{t}'_j = \delta_{i,j} \,.$$

The construction of $\mathbf{T}_i$ can be done as in [15], considering a Lagrange nodal function $\phi_i^\star \in H^1(\Omega \setminus \Sigma_i)$ having $[\![\phi_i^\star]\!]_{\Sigma_i} = 1$ and taking $\mathbf{T}_i = \widetilde{\nabla} \phi_i^\star$. This is what we have used to construct a basis of the space $\boldsymbol{\mathcal{H}}_h$. An alternative can be found in [1, Theorem 3].

After assembling the matrices corresponding to Problem 3, under the previous considerations, we obtain the following algebraic generalized eigenvalue problem:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^t \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{q} \end{pmatrix} = \lambda_h \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{q} \end{pmatrix} ,$$

where $\boldsymbol{u}$ and $\boldsymbol{q}$ are the components in the above given basis of $\boldsymbol{u}_h$ and $\boldsymbol{q}_h$, respectively. We observe that both matrices are real symmetric, but none is positive definite. Thanks to the fact that $\boldsymbol{q} = \mathbf{0}$ and adapting the same argument used in [15], we obtain that the above problem is equivalent to

$$(\mathbf{A} + \mathbf{B}^t\mathbf{B})\, \boldsymbol{u} = \lambda_h \, \mathbf{C}\, \boldsymbol{u}$$

with a real symmetric and positive definite left-hand side matrix. This allows us to conclude that Problem 3 is well posed.

5 Numerical experiments

In this section we present some numerical examples which confirm the theoretical results proved in Section 4. We have developed a MATLAB code based on lowest-order Nédélec elements ($k = 1$) to solve Problem 3.

5.1 Test 1: Domain with first Betti number $g = 1$ and $g_1 = 1$

In order to validate the numerical implementation and to check the performance and convergence properties of the scheme, we consider a problem with a known analytical solution. When the domain Ω is an annular cylindrical domain of finite z-length, i.e., a toroidal domain of rectangular cross section $0.005 \leq R \leq 1$ and $-1/2 \leq z \leq 1/2$, the least positive eigenvalue is $\lambda = 1.73457\pi = 5.449312$ and corresponds to the solution to Problem 2 with $g = g_1 = 1$. Moreover, λ is an eigenvalue of multiplicity two (see [18]).

We have used meshes $\mathcal{T}_h$ with different levels of refinement; we identify each mesh by the corresponding number N_h of tetrahedra. We have compared the computed eigenvalues $\lambda_{h,1}$ and $\lambda_{h,2}$ with the analytical eigenvalue λ. Table 1 shows the results obtained. The table also includes an estimate of the order of convergence and the extrapolated more accurate approximation λ_{ex} for each eigenvalue.

Table 1 Annular cylindrical geometry. Computed eigenvalues, experimental rates of convergence, extrapolated eigenvalues, exact eigenvalues.

N_h	10286	18993	38304	60758	order	λ_{ex}	λ
$\lambda_{h,1}$	5.623953	5.562451	5.517461	5.499519	2.12	5.451849	5.449312
$\lambda_{h,2}$	5.625268	5.563706	5.518249	5.499739	2.12	5.449423	5.449312

It can be seen from the previous table that the obtained results show an estimated order of convergence close to the theoretical one.

5.2 Test 2: Domain with first Betti number $g = 1$ and $g_1 = 0$

As a second numerical test, we have applied our MATLAB code to compute the smallest positive eigenvalues in a toroidal domain as that shown in Figure 2, with $r_1 = 1$ and $r_2 = 0.5$. Given that the first Betti number of Ω is 1, we can solve Problem 1 by imposing either the condition on γ_1 or γ_1'. Here we focus in the condition on γ_1', namely, case $g = 1$ and $g_1 = 0$. Notice that the case $g = 1$ and $g_1 = 1$ has been already studied in [15]. In both cases, to the best of the authors knowledge, no analytical solution is available.

Similarly as in [15], for each computed eigenvalue we have estimated the order of convergence and a more accurate value by means of a least square fitting of the model $\lambda_{h,k} \approx \lambda_{\mathsf{ex}} + Ch^{\alpha}$ with $h = N_h^{-1/3}$. As in the previous example, N_h denotes the corresponding number of tetrahedra. In Table 2 we summarize the convergence history of the five smallest eigenvalues. We can see that the rate of convergence predicted by Theorem 2 is attained in all the cases. In addition, we observe that the first eigenvalue converge to an eigenvalue of the continuous problem of multiplicity

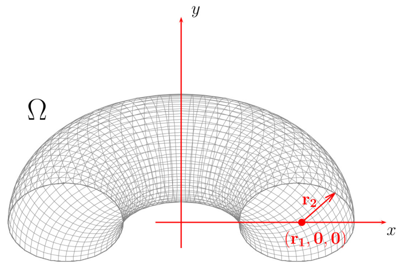

Fig. 2 Test 2. Half of the toroidal domain for the numerical test.

N_h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$	$\lambda_{h,5}$
15554	5.06643	6.63317	6.63631	6.70975	6.71578
24442	5.00805	6.49316	6.49505	6.55949	6.56238
33901	4.98580	6.43767	6.44054	6.50477	6.50574
47002	4.97127	6.40315	6.40724	6.46507	6.46619
65720	4.95832	6.37204	6.37570	6.43240	6.43321
80889	4.94705	6.34597	6.34966	6.40506	6.40614
114433	4.93518	6.31961	6.32088	6.37689	6.37744
129187	4.93139	6.31095	6.31176	6.36688	6.36760
147576	4.92779	6.30321	6.30384	6.35839	6.35889
195745	4.91944	6.28180	6.28230	6.33597	6.33606
211162	4.91801	6.27859	6.27886	6.33242	6.33279
247239	4.91511	6.27175	6.27241	6.32557	6.32565
λ_{ex}	4.89561	6.23065	6.22763	6.27980	6.28107
order	2.22	2.31	2.25	2.28	2.31

Table 2 Test 2. Smallest positive eigenvalues computed on different meshes.

one. The next two, to a double-multiplicity eigenvalue of the continuous problem and the last two, to another eigenvalue of multiplicity two. In Figure 3 we display a log-log plot of the computed errors for the eigenvalue λ_1 versus the number of tetrahedra. We observe once more that the quadratic order of convergence is attained.

We have compared the results obtained by the authors in [15] for the case $g = 1$ and $g_1 = 1$ (see Table 2). It can be seen that, except for $\lambda_{h,1}$, we obtain similar eigenvalues. This interesting similitude, for which we do not have an explanation, could be a subject of study. Note that the first eigenvalue does not appear if one considers the case $g_1 = 1$ (see the numerical experiments presented in [15]). We recall that, as already remarked the smallest eigenvalue is the most interesting from the physical point of view, as the associated eigenfunction realizes the minimum of the magnetic energy with fixed helicity.

Figure 8 shows the eigenfunction corresponding to the smallest positive eigenvalue $\lambda_{h,1}$.

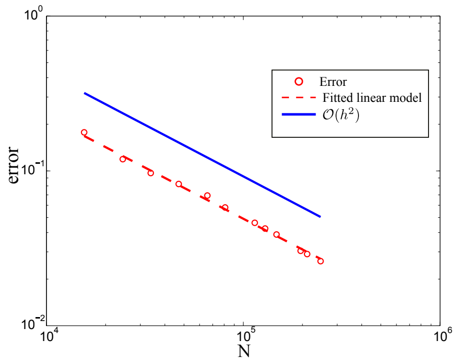

Fig. 3 Test 2. Error curve for the smallest positive eigenvalue: loglog plot of the computed error $|\lambda_{h,1} - \lambda_{\mathsf{ex}}|$ versus the number of tetrahedra N_h.

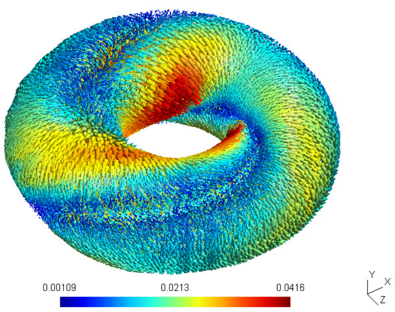

Fig. 4 Test 2. Beltrami field corresponding to the smallest positive eigenvalue.

5.3 Test 3: Domain with first Betti number $g = 2$ and different values of g_1

Although theory has been constructed by using finite elements on tetrahedra, it is easy to check that all the previous results hold true for hexahedral meshes with edges parallel to the coordinate axes. In fact, it is enough to observe that all the analysis done in Section 4 is does not change if the Lemmas 5.15, 5.40 and Theorem 5.41 in [17] are replaced by the Theorems 6.3, 6.7 and 6.6 in [17], respectively. Thus, we have implemented a MATLAB code based on lowest–order Nédélec elements on hexahedra ($k = 1$) to solve Problem 3. We have chosen a toroidal domain with two handles as that shown in Figure 5. As in the previous tests, we have used several regular meshes and we identify each mesh by the corresponding number N_h of hexahedra. Given that $g = 2$, on each mesh we have solved Problem 3 in three different cases: $g_1 = 0$, $g_1 = 1$ and $g_1 = 2$. Since for the three cases we do not have an analytical solution of the problem, we have estimated the order of convergence by means of least-squares fitting like in Test 2. Tables 3, 4 and 5 show the seven, six and five smallest positive eigenvalues computed in the same meshes, respectively. For each eigenvalue, the tables also include the extrapolated

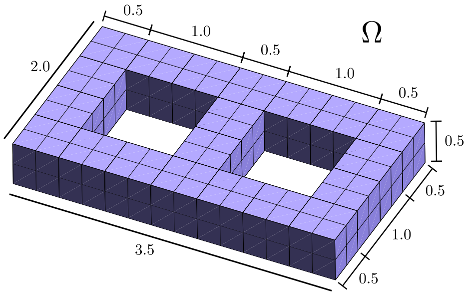

Fig. 5 Test 3. Toroidal domain for the numerical test.

N_h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$	$\lambda_{h,5}$	$\lambda_{h,6}$	$\lambda_{h,7}$
1280	9.3452	9.4559	11.4706	11.5594	12.2086	12.2378	12.4137
4320	8.9464	9.0467	10.7835	10.8426	11.4064	11.4227	11.5675
10240	8.8152	8.9123	10.5607	10.6107	11.1490	11.1617	11.2969
20000	8.7558	8.8514	10.4599	10.5059	11.0332	11.0444	11.1754
34560	8.7238	8.8187	10.4055	10.4494	10.9710	10.9814	11.1102
$\lambda_{\mathtt{ex}}$	8.6578	8.7543	10.2968	10.3417	10.8496	10.8578	10.9883
order	2.13	2.16	2.16	2.19	2.19	2.19	2.22

Table 3 Test 3. Smallest positive eigenvalues computed on different meshes for the problem corresponding to the case $g = 2$, $g_1 = 0$.

more accurate approximation $\lambda_{\mathtt{ex}}$ and the estimated order of convergence obtained with this fitting. The optimal order of convergence predicted by Theorem 2 are obtained in the three tables although the geometry contains some reentrant corners. In Figure 6 we display a log-log plot of the computed errors for the three smallest eigenvalues corresponding to the case $g = 2$, $g_1 = 0$ versus the number of hexahedra. We observe again that the quadratic order of convergence is achieved.

Additionally, if we compare the three tables we notice that there exists a correlation between the last five eigenvalues in Tables 3, 4 and 5. If we compare the extrapolated eigenvalues of each table, we observe that there are cases in which they are exactly the same in the three tables (see $\lambda_{\mathtt{ex}}$ corresponding to $\lambda_{h,4}$ or $\lambda_{h,7}$ in Table 3) and there are cases in which they seem to converge to the same values in the three tables, but we do not observe the same extrapolated eigenvalues (see $\lambda_{\mathtt{ex}}$ corresponding to $\lambda_{h,3}$, $\lambda_{h,5}$ and $\lambda_{h,6}$ in Table 3). We have observed numerically that, in the cases in which the eigenvalues are almost exactly the same, the associated eigenfunctions have the four possible circulations very close to zero (around 10^{-5} in the finest mesh). In addition we have compared the eigenfunctions associated to the eigenvalues in the three tables. We have observed that the eigenfunctions associated to $\lambda_{h,3}$ in Table 3, $\lambda_{h,2}$ in Table 4 and $\lambda_{h,1}$ in Table 5 have a similar behaviour. The same holds true for the eigenfunction associated to $\lambda_{h,4}$ in Table 3, $\lambda_{h,3}$ in Table 4 and $\lambda_{h,2}$ in Table 5, and so on up to the last column of the tables. In Figure 7 we show four eigenfunctions corresponding to the four smaller eigenvalues in Table 3 and in Figure 8 the eigenfunction associated to the smallest eigenvalue in Table 4.

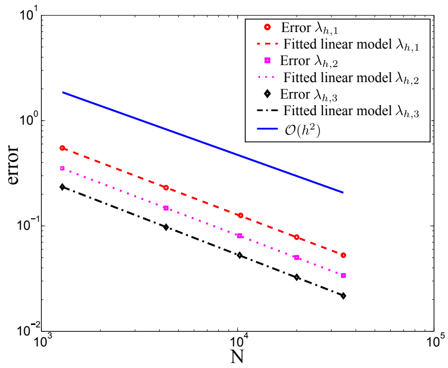

Fig. 6 Test 3. Error curve for the three smallest positive eigenvalue for the problem corresponding to the case $g = 2$, $g_1 = 0$: loglog plot of the computed error $|\lambda_{h,1} - \lambda_{\mathtt{ex}}|$ versus the number of tetrahedra N_h.

N_h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$	$\lambda_{h,5}$	$\lambda_{h,6}$
1280	9.4121	11.4653	11.5594	12.2060	12.2337	12.4137
4320	9.0070	10.7792	10.8426	11.4033	11.4207	11.5675
10240	8.8739	10.5568	10.6107	11.1456	11.1603	11.2969
20000	8.8136	10.4561	10.5059	11.0297	11.0432	11.1754
34560	8.7812	10.4018	10.4494	10.9675	10.9803	11.1102
$\lambda_{\mathtt{ex}}$	8.7140	10.2933	10.3417	10.8459	10.8571	10.9883
order	2.13	2.16	2.19	2.19	2.19	2.22

Table 4 Test 3. Smallest positive eigenvalues computed on different meshes for the problem corresponding to the case $g = 2$, $g_1 = 1$.

N_h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$	$\lambda_{h,5}$
1280	11.4617	11.5594	12.2025	12.2307	12.4137
4320	10.7764	10.8426	11.3989	11.4190	11.5675
10240	10.5541	10.6107	11.1409	11.1589	11.2969
20000	10.4536	10.5059	11.0247	11.0419	11.1754
34560	10.3993	10.4494	10.9623	10.9791	11.1102
$\lambda_{\mathtt{ex}}$	10.2910	10.3417	10.8407	10.8561	10.9883
order	2.16	2.19	2.19	2.19	2.22

Table 5 Test 3. Smallest positive eigenvalues computed on different meshes for the problem corresponding to the case $g = g_1 = 2$.

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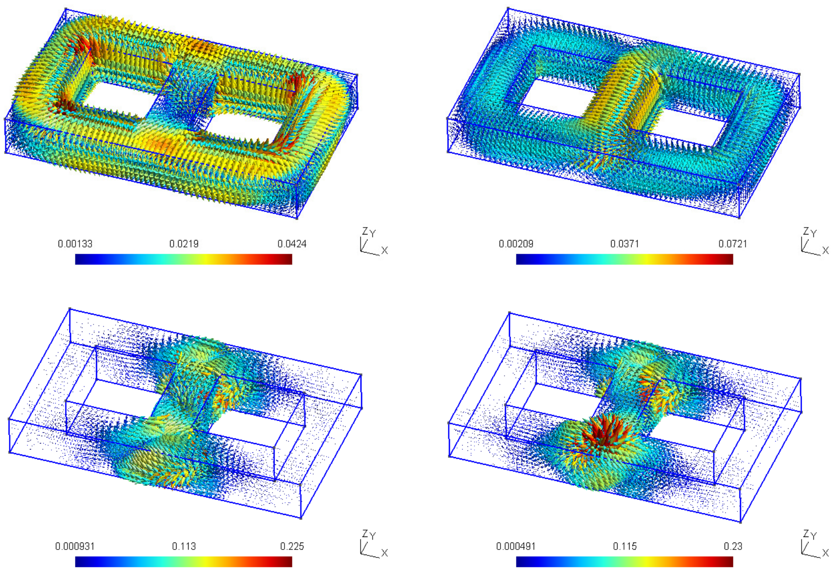

Fig. 7 Test 3. Left to right, top to bottom. Eigenfunctions corresponding to the eigenvalues $\lambda_{h,1} = 8.8152$, $\lambda_{h,2} = 8.9123$, $\lambda_{h,3} = 10.5607$, $\lambda_{h,4} = 10.6107$ for the case $g = 2$, $g_1 = 0$ (mesh with $N_h = 10240$).

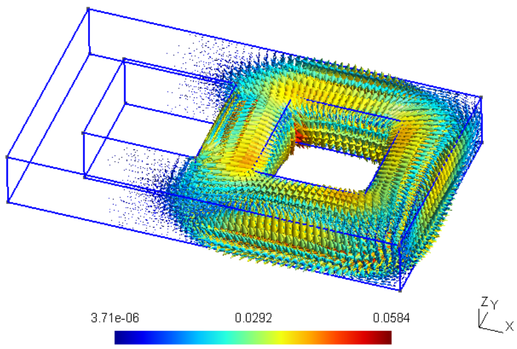

Fig. 8 Test 3. Eigenfunction corresponding to the eigenvalue $\lambda_{h,1} = 8.8739$ for the case $g = 2$, $g_1 = 1$ (mesh with $N_h = 10240$).

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