

PEANO ON DEFINITION OF SURFACE AREA

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On the occasion of the 150th anniversary of the birth of Giuseppe Peano

ABSTRACT. In this paper we investigate the evolution of the concept of area in PEANO's works, taking into account the main role played by GRASSMANN's geometric-vector calculus and PEANO's theory on derivative of measures. *Geometric* (1887) and *bi-vectorial* (1888) PEANO's approaches to surface area mark the development of this topic during the first half of the last century. In the sequel we will present some significative contributions on surface area that are inspired and/or closely related to PEANO's definition.

1. INTRODUCTION

In 1882 PEANO at the age of 24 discovers that the definition of area of a surface presented by SERRET in his *Course d'Analyse* [45, (1868) vol. 2, p. 296] was not correct. According to SERRET's proposal, the area of a surface should be given by the limit of the area of the inscribed polyhedral surfaces, but this definition cannot be applied even to a cylindrical surface. In fact PEANO observes that in this case it is possible to choose a suitable sequence of inscribed polyhedral surfaces whose areas converge to infinity (see [37, (1890)], [39, (1902) p. 300-301]).

GENOCCHI, PEANO's teacher, dampens the enthusiasm of the young mathematician, by communicating him that a similar counterexample was already been discovered by SCHWARZ. In fact, in a letter of May 26, 1882, GENOCCHI writes SCHWARZ [44, (1890) vol. 2, p. 369]:

C'est précisément Mr. Peano, qui m'amène à vous parler d'un autre sujet. Devant aborder la quadrature des surfaces courbes, il s'est aperçu que la définition d'une aire courbe donnée par Serret n'était pas bonne, et m'a expliqué les raisons qui ne lui permettaient pas de l'adopter. Alors je l'ai informé du jugement que vous en aviez porté dans plusieurs de vos lettres (20 et 26 décembre 1880, 8 janvier 1881), ce qui l'a beaucoup intéressé.

GENOCCHI and SCHWARZ¹ were conscious of the problem and of the lack of a "correct" definition of surface area, suitable to handle at least the area of elementary figures. In 1882 GENOCCHI writes another letter to SCHWARZ and invites him to propose an alternative definition, but SCHWARZ declines and stresses the difficulties:

Vous avez voulu que je donne la rectification de la définition incomplète; mais ce n'est pas facile. On peut rectifier cette définition *de plusieurs*

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¹ SCHWARZ communicates his counterexample also to CASORATI and to BELTRAMI (1880); see the correspondence between CASORATI and PEANO in GABBA [15, (1957)].

manières et il me semble qu'il suffit de donner expressément une seule possibilité qui convient avec la définition donnée par Sturm.

STURM's definition [46, (1877) vol. 1, p. 427] is the following:

On appelle aire d'une surface courbe, terminée à un contour quelconque, la limite vers laquelle tend l'aire d'une surface polyédrique composée de faces planes, qui en diminuant toutes indéfiniment, tendent à devenir tangentes à la surface considérée. On suppose d'ailleurs que le contour qui termine la surface polyédrique se rapproche indéfiniment de celui qui termine la surface courbe.

We can think that the drawbacks communicated by SCHWARZ to GENOCCHI were also due to the lack of a choice criterion between the several possible definitions of surface area. In any case a good definition of area should at least be compatible with the Lagrange formula of area of a Cartesian surface ²:

$$(1.1) \quad \iint_D \sqrt{1 + |\nabla f(x, y)|^2} \, dx dy$$

for C^1 functions $f : D \rightarrow \mathbb{R}$ (D being any rectangular subset of \mathbb{R}^2).

PEANO's definition of area in *Applicazioni geometriche del calcolo infinitesimale* [35, (1887) p. 164] overcomes the drawbacks of SERRET's approach, yielding the Lagrange formula (1.1). PEANO's proposal is deeply influenced by GRASSMANN's geometric-vector calculus in affine spaces, that gives a mathematical formalization of geometrical and physical concepts (vectors, pair of vectors, moment and so on) and allows also to take into account properties related to orientation, without using drawings or tricky and intuitive constructions. It is not surprising that PEANO's definition via GRASSMANN's calculus is suitable to handle oriented integrals and, consequently, to prove main results (such as Stokes theorem and Green formula), and to develop formulae leading to the integration of 2-forms of CARTAN ³.

Besides PEANO's proposal, in the literature several definitions of surface area have been given ⁴: nowadays the most famous and commonly accepted as definitive are grounded on Hausdorff measures.

The aim of the present paper is the investigation of the evolution and use of the concept of area in PEANO's works, taking into account the main role played by GRASSMANN's geometric-vector calculus and PEANO's theory on derivative of measures. PEANO's approach to surface area marks the development of this topic during the first half of the last century. In the sequel we will present contributions concerning surface area that are inspired and/or closely related to PEANO's definition.

² Nowadays we know that the Lagrange formula is sufficient to define the area of a C^1 -submanifold, but the extension of the formula (1.1) from a rectangle D to a more general 2-dimensional set is not trivial and hides some pitfalls.

³ In 1899 CARTAN [7] introduced the calculus of differential forms

Ce calcul présente aussi de nombreuses analogies avec le calcul de Grassmann; il est d'ailleurs identique au calcul géométrique dont se sert M. Burali-Forti dans un Livre récent (*Introduction à la Géométrie différentielle, suivant la méthode de Grassmann*, Gauthier-Villars, 1898).

BURALI-FORTI was one of the prominent scholars of PEANO. Together with MARCOLONGO he developed and applied GRASSMANN's vector calculus to geometry, mechanics and physics.

⁴ For a detailed presentation of the several possible definitions of surface area see CESARI [12, (1954)] and FEDERER [13, (1969)]

PEANO's definition of measure of surfaces is grounded on elementary formulae of area of planar polygons (see Theorems 2.1, 2.2 and their proofs). The surprising absence of results concerning area of planar polygons in several modern encyclopedic books (see for example ALEXANDROV [1, (2005)] and BERGER [5, (1977)]) motivates us to try to trace the history of such formulae that, as we shall see, are deeply connected with statics and can be found in their final form in the works by MÖBIUS and BELLAVITIS. The generalization of the formula of area from planar to non-planar polygons and to closed curves allowed PEANO to specify and to evaluate area of surface at an infinitesimal level (see Section 4).

The paper is organized as follows. In Section 2 the main definitions and results on Grassmann's geometric-vector calculus are presented in a modern fashion, according to GRECO, PAGANI [20, (2010)]. In Section 3 the historical development of the formulae of area of polygons and volume of polyhedra is investigated. Section 4 is devoted to the description of PEANO's definition of area. In Section 5 we recall PEANO's bi-vector integral formula and other ways of associating a *number* to a given oriented closed curve. In Section 6 we will list main propositions and theorems about area given by PEANO in his works. In Section 7 we recall some significant mathematical contributions inspired and/or closely related to PEANO's definition. In particular we present the re-formulations of PEANO's and GEÖCZE's area (due to several mathematicians) in order to make them coincident with LEBESGUE's area.

This article concerns some historical aspects. From a methodological point of view, we are focussed on primary sources, that is on mathematical facts and not on the elaborations or interpretations of these fact by other Scholars of history of mathematics.

2. GRASSMANN-PEANO GEOMETRIC-VECTOR CALCULUS ON THREE DIMENSIONAL AFFINE SPACES

We present here the Grassmann-Peano geometric-vector calculus, as described in GRECO, PAGANI [20, (2010)]. The aim of this section is to understand the mathematical basis used by PEANO in the construction of his notion of area. Such a formalism will be useful not only to clarify the genesis of the vectorial formulae for area of polygons and volume of polyhedra, but also to understand the deep connection between some concepts of statics (points, applied forces, momenta, Poincaré pairs and so on) and of geometry (geometric forms of first, second and n -degree, namely, points, vectors, bi-points, tri-points, quadri-points, and so on).

PEANO is one of the first mathematician who presents GRASSMANN's work [17, (1844)], [18, (1862)], [16] to the mathematical community. Actually he rebuilds GRASSMANN's calculus using a original "functional" approach that relies only on the assignment of a volume form on a given affine space (see GRECO, PAGANI [20] for a detailed presentation of this subject).

For convenience of the reader we choose to rebuild here *Grassmann graded exterior algebra on an affine space* (Grassmann affine algebra, for short), using an approach based on the usual notion of graded exterior algebra on a vector space.

The starting point for the construction of Grassmann affine algebra on the ordinary 3-dimensional affine space is the introduction of a 4-dimensional *Möbius space*, i.e., a couple (\mathbb{W}, ω) , where \mathbb{W} is 4-dimensional vector space and $\omega : \mathbb{W} \rightarrow \mathbb{R}$ is a non-vanishing linear form, called *mass*. Given the Möbius space (\mathbb{W}, ω) , let us consider the subspace $\mathbb{V} := \{w \in \mathbb{W} : \omega(w) = 0\}$ of \mathbb{W} and the subset

$\mathbb{P} := \{w \in \mathbb{W} : \omega(w) = 1\}$. Elements of \mathbb{V} and \mathbb{P} will be called ω -vectors and ω -points of (\mathbb{W}, ω) , respectively.

The affine space $(\mathbb{P}, \mathbb{V}, -)$ (where $-$ stands for the difference between elements of \mathbb{W}) may be identified with the 3-dimensional Euclidean space. Therefore, the form ω allows a non ambiguous selection of the ω -vectors and the ω -points from the elements of the Möbius space \mathbb{W} . The elements of \mathbb{W} with $\omega(x) \neq 0$ are called *weighted ω -points*.

Let us consider the graded exterior algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V})$ on the vector spaces \mathbb{W} and \mathbb{V} respectively. We have explicitly

$$\mathbb{G}(\mathbb{W}) = \Lambda^0(\mathbb{W}) \oplus \Lambda^1(\mathbb{W}) \oplus \Lambda^2(\mathbb{W}) \oplus \Lambda^3(\mathbb{W}) \oplus \Lambda^4(\mathbb{W})$$

where $\Lambda^0(\mathbb{W}) := \mathbb{R}$ and $\Lambda^k(\mathbb{W})$, $k = 1, \dots, 4$, is the vector space generated by the products of k vectors of \mathbb{W} . The elements of $\Lambda^k(\mathbb{W})$ are called *geometric forms of degree k* . Since \mathbb{W} is the vector space generated by ω -points, it is worth observing that $\Lambda^k(\mathbb{W})$ is generated by the products of k ω -points. In a similar way we have

$$\mathbb{G}(\mathbb{V}) = \Lambda^0(\mathbb{V}) \oplus \Lambda^1(\mathbb{V}) \oplus \Lambda^2(\mathbb{V}) \oplus \Lambda^3(\mathbb{V})$$

where $\Lambda^0(\mathbb{V}) := \mathbb{R}$ and $\Lambda^k(\mathbb{V})$, $k = 1, 2, 3$, are linear combinations of products of k ω -vectors. The elements of $\Lambda^k(\mathbb{V})$ are called *geometric vector forms of degree k* .⁵

The linear form $\omega : \mathbb{W} \rightarrow \mathbb{R}$ can be extended to a linear map from the whole $\mathbb{G}(\mathbb{W})$ to $\mathbb{G}(\mathbb{V})$ by means of the following relations

$$\omega(1) = 0$$

$$\omega(P_0) = 1$$

$$\omega(P_0 P_1) = P_1 - P_0$$

$$\omega(P_0 P_1 P_2) = (P_1 - P_0)(P_2 - P_0)$$

$$\omega(P_0 P_1 P_2 P_3) = (P_1 - P_0)(P_2 - P_0)(P_3 - P_0),$$

for every $P_0, P_1, P_2, P_3 \in \mathbb{P}$.⁶ Actually $\Lambda^k(\mathbb{V})$ is a vector subspace of $\Lambda^k(\mathbb{W})$ and the linear map ω connects the graded algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V})$ in the following way:

$$(2.3) \quad \omega(\Lambda^k(\mathbb{W})) = \Lambda^{k-1}(\mathbb{V}) = \text{Ker}(\omega|_{\Lambda^{k-1}(\mathbb{W})}) \quad k = 0, \dots, 4.$$

Restrictions of ω to $\Lambda^k(\mathbb{W})$, denoted by ω_k , are called $(k-1)$ -vector-masses because, by the first equality of formula (2.3), ω transforms a k -degree geometric form into geometric vector forms of $(k-1)$ -degree. The second equality says that a k -vector-mass is null on geometric vector forms of degree k ; in particular $\omega \circ \omega = 0$ and the *reduction formula* holds:

$$(2.4) \quad x = P\omega(x) + \omega(Px), \quad \forall P \in \mathbb{P}, x \in \Lambda^k(\mathbb{W}), k = 0, 1, 2, 3, 4.$$

⁵ The product in $\mathbb{G}(\mathbb{W})$ and in $\mathbb{G}(\mathbb{V})$ will be denoted as juxtaposition of symbols. Recall that the algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V})$ are anticommutative, i.e. $xy = (-1)^{rs}yx$, for any $x \in \Lambda^r(\mathbb{W}), y \in \Lambda^s(\mathbb{W})$. Clearly, $\mathbb{G}(\mathbb{V})$ is a sub-algebra of $\mathbb{G}(\mathbb{W})$; moreover, due to anti-commutativity of the product and to the dimension of the space \mathbb{W} and \mathbb{V} we have that $\Lambda^k(\mathbb{W}) = 0$ (for $k > 4$) and $\Lambda^k(\mathbb{V}) = 0$ (for $k > 3$).

⁶ The extension of the linear form ω on the whole graded algebra $\mathbb{G}(\mathbb{W})$ can be uniquely determined by the conditions

$$(2.1) \quad \omega : \Lambda^k(\mathbb{W}) \rightarrow \Lambda^{k-1}(\mathbb{V}) \quad (\text{we assume } \Lambda^{-1}(\mathbb{V}) := \{0\});$$

$$(2.2) \quad \omega(xy) = \omega(x)y + (-1)^{\deg(x)}x\omega(y) \quad (\text{graded Leibnitz rule}).$$

A quadri-point $ABCD$ (with $A, B, C, D \in \mathbb{P}$ regarded as the four vertices of a tetrahedron) suggests the construction of particular basis of $\Lambda^r(\mathbb{W})$, whenever they are not co-planar (i.e. the vectors $B - A, C - A, D - A$ are linearly independent). The four vertices A, B, C, D are a basis of $\Lambda^1(\mathbb{W})$, the six bi-points AB, AC, AD, BC, BD, CD , corresponding to the six edges of the tetrahedron, are a basis of $\Lambda^2(\mathbb{W})$, the four tri-points ABC, ACD, ABD, BCD , corresponding to the four faces of the tetrahedron, are a basis of $\Lambda^3(\mathbb{W})$, and the quadri-point $ABCD$ is a basis of $\Lambda^4(\mathbb{W})$.

If A', B', C', D' are the four vertices of another tetrahedron, then

$$(2.5) \quad A'B'C'D' = \frac{\det(B' - A', C' - A', D' - A')}{\det(B - A, C - A, D - A)} ABCD,$$

where $(B' - A', C' - A', D' - A')$ is the 3×3 matrix whose columns are the coordinates of the vectors $B' - A', C' - A', D' - A'$ along the basis $B - A, C - A, D - A$. Equality (2.5) enlightens the geometrical interpretation of a quadri-point in terms of an oriented volume. The equality between two elements $x, y \in \Lambda^r(\mathbb{W})$ can be expressed by means the following condition:

$$(2.6) \quad x = y \quad \Leftrightarrow \quad xz = yz \quad \forall z \in \Lambda^{4-r}(\mathbb{W})$$

Several elements of the graded exterior algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V})$ admit interesting geometrical and mechanical interpretation.

Let $A, B, C \in \mathbb{P}$. The bi-point AB can be seen as the applied vector $A(B - A)$ (for instance, as a “force” $B - A$ “applied” in A), and the tri-point ABC can be represented by a triangle or by an applied bi-vector $A(B - A)(C - A)$ (applied in A). Elements of $\Lambda^2(\mathbb{V})$, i.e. the bi-vectors, can be seen as Poincot couples or as oriented boundary of triangles, and the elements of $\Lambda^3(\mathbb{V})$, i.e. tri-vectors, as oriented surfaces of tetrahedrons.

Besides mechanical interpretations, a system of applied forces can be represented by an element of $\Lambda^2(\mathbb{W})$, more precisely as a sum of bi-points. The equivalence between two systems of applied forces $\{A_i B_i\}_{i=1, \dots, n}$ and $\{C_j D_j\}_{j=1, \dots, m}$ can be expressed as the equality between the corresponding elements $\sum_i A_i B_i$ and $\sum_j C_j D_j$ of $\Lambda^2(\mathbb{W})$. Indeed by means of equation (2.5), given a bi-point PQ , the product $A_i B_i PQ$ can be recognized as the axial moment of the force $A_i B_i$ with respect to the axis passing through P and Q . As a consequence the equality (2.6) between elements of $\Lambda^2(\mathbb{W})$ reduces the equivalence between two systems of applied forces to the equality of their axial moments with respect to every axis. As a particular case, a system of forces with vanishing resultant (a Poincot couple), can be represented by an element of $\Lambda^2(\mathbb{V})$. It is interesting to note that POINOT’S theorem concerning the sum of Poincot couples emerges naturally from the structure of vector space of $\Lambda^2(\mathbb{V})$.

Pursuing the analogy with statics, the image of the operator ω acting on $\Lambda^2(\mathbb{W})$ represents the resultant of a system of forces (a special case of the 1-vector-mass introduced above). The reduction formula (2.4) can be directly translated into the reduction formula for a system of forces: given an arbitrary point P , a system of forces x is equivalent to a system formed by the resultant $\omega(x)$ applied in P , and by the Poincot couple $\omega(Px)$.

The formalism presented so far, allows a direct proof of the following results:

Theorem 2.1 (Area of a plane polygon⁷). *For any planar polygon with consecutive vertices A_1, \dots, A_n , the sum*

$$(2.7) \quad \sum_{i=1}^n PA_i A_{i+1}$$

(with $A_{n+1} = A_1$) does not depend on the choice of the point P , with P belonging to the plane of the vertices.

The vector space of third degree forms in a plane is 1-dimensional; a base for this space is provided by an arbitrary triangle RST with non collinear vertices. Then $PA_i A_{i+1} = a RST$, where a is the oriented area of $PA_i A_{i+1}$ with respect to RST ⁸. Therefore formula (2.7) gives the sum of the oriented areas of triangles $PA_i A_{i+1}$, termed by PEANO as the *area bounded by the oriented closed polygonal line $A_1, \dots, A_n, A_{n+1} = A_1$* . As observed by PEANO, this area coincides with the usual measure of area if the polygonal line is convex or, more generally, is not interlaced.

Theorem 2.2 (Area of a non-planar polygon⁹). *For any closed polygonal line (not necessarily planar) there exists a triangle such that the area of any projection of the polygonal line on an arbitrary plane is equal to the area of the projection of the triangle.*¹⁰

Theorem 2.3 (Volume of an oriented polyhedron¹¹). *Let us consider a closed oriented polyhedral surface made of triangular faces $A_i B_i C_i$, $i = 1 \dots, n$. The sum of the oriented volumes*

$$(2.8) \quad \sum_{i=1}^n PA_i B_i C_i$$

of the tetrahedra $PA_i B_i C_i$ does not depend on the choice of the vertex P .

For convenience of the reader, we give the proofs of the previous theorems accordingly to Section 2 on Grassmann-Peano geometric-vector calculus.

Proof of theorem 2.1. Denote by π the plane of the vertices A_1, \dots, A_n . Let us consider the element $x \in \Lambda^2(\mathbb{W})$ given by $x = \sum_{i=1}^n A_i A_{i+1}$. Since $\omega_2(x) = \sum_{i=1}^n (A_{i+1} - A_i) = A_{n+1} - A_1 = 0$, we have $x \in \ker(\omega_2) = \Lambda^2(\mathbb{V})$; therefore, x is a bi-vector and, hence, there exist three points X, Y, Z in the plane π such that $x = (Y - X)(Z - X)$. Given a generic point P in the plane π , we have $\omega_4(PXYZ) = (X - P)(Y - P)(Z - P) = 0$ (the three vectors are linearly dependent); therefore, by the reduction formula (2.4), we have

$$XYZ = P\omega_3(XYZ) + \omega_4(PXYZ) = P\omega_3(XYZ) = Px = \sum_{i=1}^n PA_i A_{i+1}$$

and the conclusion follows. \square

⁷ See PEANO's *Applicazioni geometriche del calcolo infinitesimale* [35, (1887) p. 21], *Calcolo geometrico* [36, (1888) p. 59], *Lezioni di analisi infinitesimale* [38, (1893) vol. II, p. 32].

⁸ In other words, $|a|$ is the ratio between the areas of the two triangles; the sign of a is positive if the triangles have the same orientation.

⁹ See PEANO's *Calcolo geometrico* [36, (1888) p. 137].

¹⁰ In this case the polygon is said "equipollent" to the triangle.

¹¹ See PEANO's *Applicazioni geometriche del calcolo infinitesimale* [35, (1887) p. 26-27], *Calcolo geometrico* [36, (1888) p. 66], *Lezioni di analisi infinitesimale* [38, (1893) vol. II, p. 35].

Proof of theorem 2.2. Let us consider a (non planar) polygons with vertex A_1, \dots, A_n , with $A_{n+1} = A_1$. By reasoning as in the proof of Theorem 2.1, the element $x \in \Lambda^2(\mathbb{W})$ given by $x = \sum_{i=1}^n A_i A_{i+1}$ can be represented by a bi-vector and there exist three points X, Y, Z such that $x = (Y - X)(Z - X)$. Given a generic plane γ and a generic point A , let us consider a unitary vector u orthogonal to the plane; consequently Au represents a “vector applied” applied at the point A . Relation (2.6) implies that $(Y - X)(Z - X)Au = \sum_{i=1}^n A_i A_{i+1} Au$. By formula (2.5) the element $(Y - X)(Z - X)Au \in \Lambda^4(\mathbb{W})$ represents the volume of the tetrahedron with unitary height and area of basis equal to the area of the projection of the triangle XYZ on the plane γ , and the conclusion follows. \square

Proof of theorem 2.3. The closedness of the polyhedral surface made of triangular faces $A_i B_i C_i$ amount to the condition $\omega_3(\sum_{i=1}^n A_i B_i C_i) = 0$, which implies that the element $\sum_{i=1}^n A_i B_i C_i \in \Lambda^3(\mathbb{W})$ is a tri-vector. This implies that there exist four points X, Y, Z, U , such that $\sum_{i=1}^n A_i B_i C_i = (Y - X)(Z - X)(U - X)$. For any point P , the reduction formula (2.4) gives

$$\begin{aligned} XYZU &= P\omega_4(XYZU) + \omega_5(PXYZU) = P\omega_4(XYZU) \\ (2.9) \quad &= P(Y - X)(Z - X)(U - X) = \sum_{i=1}^n P A_i B_i C_i, \end{aligned}$$

and the conclusion follows. ¹² \square

3. MÖBIUS AND BELLAVITIS ON THE AREA OF POLYGONS AND VOLUME OF POLYHEDRA

At the beginning of the 19th century an increasing interest is devoted to the study of polygons and polyhedra. This interest is paved by the researches by LEGENDRE and POINSOT, who follow the way traced by EUCLID, KEPLER, DESCARTES and EULER. LEGENDRE in 1794 gives a proof of the famous EULER’s formula (1750) for polyhedra:

$$(3.1) \quad V - E + F = 2,$$

where V, E and F denote the number of vertices, edges and faces, respectively. On the other hand, POINSOT [40], according to the “Géométrie de situation” of LEIBNITZ [24], in 1810 started the classification of polygons and polyhedra, discovering some new “star polyhedra”. In 1813 CAUCHY [8] gave a new proof of EULER’s formula (3.1) showing that there are no star polyhedra different from those described by POINSOT. Moreover, urged by LEGENDRE, CAUCHY gave the famous rigidity theorem for convex polyhedra, as he said in *Sur les polygones et les polyèdres* [9, p. 87]:

[...] chercher la démonstration du théorème renfermé dans la définition 9, placée à la tête du onzième Livres *Elements d’Euclide*, savoir que deux polyèdres convexes sont égaux lorsqu’ils sont compris sous un même nombre de faces égales chacune à chacune.

¹² The independence of the sum of volumes (2.9) has been proved starting from the equality $\omega_3(\sum_{i=1}^n A_i B_i C_i) = 0$. It is worth noting that the converse is still valid; in other words the equality $\omega_3(\sum_{i=1}^n A_i B_i C_i) = 0$ holds if and only if “ $v \sum_{i=1}^n A_i B_i C_i = 0$ for every $v \in \mathbb{V}$ ”.

One of the first book devoted to polyhedra was written by DESCARTES [14], but many other authors devote their efforts to the study of this topic.

An evidence of the importance which was given to polygons and polyhedra in the 19th century is the *Gran Prix* “Perfectionner dans quelque point important la théorie géométrique des polyèdres” organized in 1858 by the Accademy of Sciences of Paris. Indeed the Accademy decided to assign a prize only in presence of a significative and revolutionary contribution to the theory of polyhedra. Several important scientists participate, including MÖBIUS. As other participants, MÖBIUS’s goal was to provide a complete classification of polyhedra, but very soon he discovered that this is really an arduous task and decided to change his aims, proposing an innovative work concerning the concept of orientation. Despite of this, the Accademy does not judge any contribution sufficiently important and does not assign the prize to any participant.

Among several results present in the mathematical literature, we restrict ourselves to analyze in details the works of MÖBIUS and BELLAVITIS, due to their influence on PEANO. The formula of area of polygons (2.7) can be found for the first time in MÖBIUS’s *Barycentrische Calcul* [33, (1827) p.201], where it appears in a remark, as an application of the analogous formula for triangles and as a direct geometrical consequence of the notion of barycentric coordinates. BELLAVITIS presents the formula in *Teoremi generali per determinare le aree dei poligoni e i volumi di poliedri* [2, (1834)] as a “trivial consequence” of a well known properties due to POINSOT. BELLAVITIS says indeed:

[La formula dell’area] esprime la proprietà di un sistema di forze aventi la risultante nulla di produrre una stessa coppia (*couple*), in qualunque punto comune tutte esse si trasportino. ¹³

Later MÖBIUS himself deduced the formula (2.7) of previous Section, as an application of statics, in his book *Der Statik* [31, (1837) p.61-64]. In our opinion this correlation with statics, where the couples of consecutive vertices (= bi-points) of a closed polygonal line are interpreted as forces with vanishing resultant, is important from a historical point of view and may be emphasized into the following:

Metatheorem 3.1. *The following two propositions are equivalent:*

- (3.2) *Formula of area (2.7) for planar polygons holds.*
- (3.3) *Any system of planar forces with vanishing resultant is equivalent to a couple.*

According to BELLAVITIS also the formula of volume of polyhedra (2.8) can be seen as a consequence of the static theorem of POINSOT: “the sum of couples is a couple”. Later, references to formulae (2.7) and (2.8) can be found in BELLAVITIS’s *Metodo delle equipollenze* [4, (1838) pp.95-97] and *Sposizione del metodo delle equipollenze* [3, (1854)].

Concerning MÖBIUS, both formulae (2.7) and (2.8) can be found in his article appeared in 1865 *Über die Bestimmung des Inhaltes eines Polyhedres* [32, pp. 486, 494].

The methods of proof of BELLAVITIS and MÖBIUS are quite different. BELLAVITIS is one of the first mathematicians developing vector calculus, and he uses it

¹³ [[In the formula of area] one can see the property satisfied by a system of applied forces with vanishing resultant to be equivalent to a couple, independently of the common point in which the forces are translated.]

in most of his proofs. Moreover the deep connection between statics and geometry is strongly emphasized. It is also worthwhile to note that BELLAVITIS applies the duality relation between polygons and polyhedra, then it is not surprising that both formulae (2.7) and (2.8) appear in the same article. In the work by MÖBIUS emerges the revolutionary concept of orientation. MÖBIUS is conscious that orientability of polyhedra is an important condition for the validity of the formula of volume, and it cannot be ignored, as well as he was aware of the existence of non oriented polyhedra. In BELLAVITIS's work the necessity of orientability is not transparent, and he handles only with polyhedra which are dual of polygons that are oriented by construction.

We did not find any trace (but we cannot exclude it) concerning area of non-planar polygons either in MÖBIUS or in BELLAVITIS even if a statement similar to Metatheorem 3.1 is still valid for non planar polygons and non planar forces:

Metatheorem 3.2. *The following two propositions are equivalent:*

(3.4) *For any closed polygonal line (not necessarily planar) there exists a triangle such that the area of any projection of the polygonal line on an arbitrary plane is equal to the area of the projection of the triangle.*

(3.5) *Any system of forces with vanishing resultant is equivalent to a couple.*

4. PEANO'S DEFINITIONS OF AREA

In PEANO's works we recognize two definitions of area of a non-planar surface, the first one, referred as *geometric*, is based on the notion of Jordan-Peano area of planar sets; the second one, referred as *bi-vectorial*, is based on the notion of bi-vector associated to a closed curve bounding a pieces of surface.

In *Applicazioni Geometriche del Calcolo Infinitesimale* [35, p. 164] PEANO introduces his geometric definition of area in the following terms:

AREE DI SUPERFICIE NON PIANE. Abbiasi una superficie qualunque. Proiettandola ortogonalmente sopra un piano avremo una figura piana; supporremo che questo abbia un'area propria, e che la superficie data si possa decomporre in parti che godano della stessa proprietà. Si scomponga la superficie data in parti e, dopo averle trasportate comunque nello spazio, si proiettino queste ortogonalmente su d'uno stesso piano. La somma delle aree di queste proiezioni sarà un'area piana, variabile col variare del modo di divisione della superficie e del modo con cui si dispongono queste parti. Il limite superiore dei valori di quest'area piana si dirà l'*area* della superficie data. Si deduce immediatamente dalla definizione che l'area di una superficie qualunque è maggiore della sua proiezione ortogonale su d'un piano qualunque.¹⁴

¹⁴ [AREA OF NON PLANAR SURFACES. Let us consider an arbitrary surface. Performing an orthogonal projection on a plane, we get a plane figure; we assume that this figure have an "area propria" (i.e., it is Peano-Jordan measurable) and that the given surface can be decomposed into parts having the same property.

Let us decompose the given surface into pieces and, after carrying these pieces arbitrarily in the space, let us project all these pieces on the same plane. The sum of the areas of these projections is a planar area, depending on the decomposition of the surface and on the way its pieces are located. The supremum of the values of these planar areas will be defined as the *area* of the surface.

Paraphrasing the content of the paper *Sulla definizione dell'area di una superficie* [37, (1890), p. 56], we may have the following bi-vectorial definition of area, that may help the reader in comparing the two definitions of area given by PEANO:

Let us consider an arbitrary surface delimited by a closed oriented curve. Performing an orthogonal projection on a plane, we get a plane figure delimited by a closed oriented curve; we assume that to the latter there corresponds a bi-vector which magnitude gives the planar area of the figure, and that the given surface can be decomposed into pieces having the same property.

Let us decompose the given surface into pieces and, after carrying these pieces arbitrarily in the space, let us project all these pieces on the same plane. The sum of magnitudes of the closed oriented curves of these projections depends on the decomposition of the surface and on the way its pieces are located. The supremum of these sums will be defined as the *bi-vector area* of the surface.

In 1890 PEANO in *Sulla definizione dell'area di una superficie* [37] examines historically various definitions of area and restates his definition. He starts by presenting the definitions of length of a convex planar arc and the area of a convex surface, given by ARCHIMEDES, as the limit of inscribed and circumscribed polygons and, respectively, as the limit of inscribed and circumscribed convex polyhedral surfaces. PEANO, aware of the fact that ARCHIMEDE's proposal is suitable enough to define the area of a cylindrical surface, tries to propose a definition of area preserving the analogy between length of arcs and area of surfaces present in ARCHIMEDE's work. In the case of non planar curves a good definition of length can be obtained by considering only the inscribed polygons, but in the case of surfaces, PEANO observes that ARCHIMEDE's definition cannot be applied to non convex ones. PEANO's aim is to extend Archimede's definition in order to handle more general surfaces, such as the concave ones.

Later, PEANO criticizes the definitions of area present in the literature, including Serret's definition, and explaining that

L'errore principale commesso da SERRET sta nel ritenere che il piano passante per tre punti di una superficie abbia per limite il piano tangente alla medesima. ¹⁵

He criticizes also LAGRANGE's definition:

Il risultato è ottenuto da LAGRANGE per mezzo di un'asserzione non esatta. ¹⁶

He also criticizes HARNACH's modification of SERRET's definition, saying that, even if the faces of the polyhedron considered by HARNACH tend to the tangent planes, HARNACH's definition fails even in the case of a cartesian surface of equation $z = f(x, y)$. PEANO also recalls that the non correctness of SERRET's definition has already been noted by SCHWARZ. The definition proposed by HERMITE as a consequence of SCHWARZ's remark, even if considered sufficiently "rigorous" by PEANO,

It follows immediately from this definition that the area of an arbitrary surface is greater than its orthogonal projection on an arbitrary plane.]

¹⁵ [The main mistake of SERRET is his belief that the plane passing through three points of a surface tends to the tangent plane.]

¹⁶ [The result has been obtained by LAGRANGE by means of a not exact statement.]

is not completely satisfactory, because depends on the choice of the coordinate system.

Finally PEANO observes that any difficulty can be overcome by using the concept of *oriented* area, attributed by him to CHELINI, MÖBIUS, BELLAVITIS, GRASSMANN and HAMILTON¹⁷. The bi-vectorial definition of non-planar surface area of PEANO is based on the concept of GRASSMANN's *bi-vector*: PEANO extends the equipollence between closed polygonal lines and triangles (see Theorem 2.2). Thus closed lines are represented by bi-vectors:

Data una linea chiusa (non piana) l , si può sempre determinare una linea piana chiusa o bivettore l' , in guisa che, proiettando le due linee l l' su d'un piano arbitrario, con raggi paralleli di direzione arbitraria, le aree [con segno] limitate dalle loro proiezioni risultino sempre uguali.¹⁸

The logical evidence of this proposition is not trivial for a modern reader¹⁹.

By presenting the mathematical instruments for the proof, we observe what PEANO says in order to understand the necessary mathematical background.

questa proposizione è conseguenza immediata della somma, o composizione, dei bivettori [poiché tale somma è essa stessa un bivettore,] quando la linea l è poligonale.²⁰

The trivialness of this part is a consequence of Theorem 2.2 of Section 2.

¹⁷ It is interesting to note as CHELINI, MÖBIUS, BELLAVITIS and GRASSMANN in their work refer directly to POINSOT.

¹⁸ [[Given a closed (not planar) line l , it is always possible to determine a closed planar line or bi-vector l' in such a way that by projecting both lines on an arbitrary plane, with parallel rays along an arbitrary direction, the [signed] areas defined by their projections coincide.]

¹⁹ In a letter written by PEANO to CASORATI (26 October 1889), PEANO presents his note, later published in 1990 in *Rendiconti dell'Accademia dei Lincei*:

Questi contorni chiusi sono analoghi ai *segmenti* o *vettori*, cui corrispondono per dualità; si possono identificare con le *coppie* della meccanica. Siccome secondo Grassmann sono i prodotti di due vettori, si possono chiamare *bivettori*. Dica si grandezza di un bivettore C l'area, in valore assoluto, del triangolo T di cui si parla nel precedente teorema. Se proiettando C su una terna di piani ortogonali si ottengono le aree a, b, c allora la grandezza di C vale $\sqrt{a^2 + b^2 + c^2}$. I bivettori si possono sommare, o comporre, analogamente ai vettori, e precisamente come le coppie di forze. Se una porzione di superficie si scompone in parti, il bivettore (o contorno) di quella superficie è la somma dei bivettori delle sue parti come se un arco di linea si scompone in parti, il vettore (corda) dell'arco è uguale alla somma (risultante) dei vettori delle sue parti.

[[These closed contours are analogous, by duality, to *segments* or *vectors*; they can be identified with the couples of mechanics. According to Grassmann they are products of two vectors and can be called *bi-vectors*. Let us call the magnitude of a bi-vector C the area, in absolute value, of the triangle T described in the previous Theorem. If, by projecting C on a tern of orthogonal planes, one obtains the areas a, b, c , then the size of C is given by $\sqrt{a^2 + b^2 + c^2}$. The bi-vectors can be added, or composed, in an analogous way as the vectors, and more precisely as the couples of forces. If a part of a surface is decomposed into pieces, the bi-vector (or contour) of that surface is the sum of the bi-vectors of its pieces, as in the case of an arc of a line is decomposed into pieces, the vector (cord) of the arc is the sum (resultant) of the vectors of its pieces.]

²⁰ [[This proposition is a direct consequence of the sum, or composition, of bi-vectors [since such a sum is a bi-vector,] when the line l is polygonal.]

Il solito passaggio al limite permette di dimostrarla quando la l è una linea curva, descritta da un punto avente sempre derivata finita, ed anche in altri casi.²¹

Concerning this part, the approximation of a line by means of polygons provides the direct way to transfer properties of closed polygons to closed continuous curves. It is worthwhile to note that limits of polygons and triangles are included in the topological concepts introduced by PEANO concerning geometric forms (see Section 2). The condition of finite derivative, besides guaranteeing the continuity of the curve, assures that any projection of the closed line is the boundary of a set which is measurable in the sense of Jordan-Peano.

Moreover, PEANO underlines that area must be thought as “oriented”:

Le aree si devono considerare tenendo in debito conto i segni.²²

This part underlines the fact that the orientation of closed lines has always to be taken into account and this element becomes fundamental in the case of self-intersecting lines.

Thanks to the notion of equipollence between closed lines, PEANO observes that:

se si proietta ortogonalmente una linea chiusa (non [necessariamente] piana) l su un piano variabile, il massimo dell’area limitata dalla proiezione di l vale la grandezza del bivettore [associato a] l ; e questo massimo avviene quando il piano su cui si proietta ha la giacitura [del bivettore associato a] l .²³

In 1890 PEANO presents a new and more clear formulation of its definition of area:

L’area di una porzione di superficie è il limite superiore della somma delle grandezze dei bivettori delle sue parti.²⁴

More pragmatically, this quotation suggests the following re-statement of “bi-vectorial definition of area”:

Given an arbitrary non planar surface, we consider a decomposition into pieces. For each of these pieces we consider its oriented boundary and the magnitude of the corresponding bi-vector. The supremum, with respect to all decompositions of the surface, of the sums of the magnitudes of the bi-vectors of the pieces of the decomposition, will be defined as the bi-vector area of the surface.

With this formulation PEANO provides the fundamental property leading to the formula of area (1.1):

La giacitura del bivettore di una porzione infinitesima di superficie è quella del piano tangente; il rapporto fra la sua grandezza e l’area di quella porzione è l’unità.²⁵

²¹ [The usual limiting procedure allows one to prove this fact when l is described by a point having finite derivative, and also in other cases.]

²² [Areas must be considered by taking their sign into account.]

²³ [If one projects orthogonally a (not planar) closed line l on a variable plane, the maximum of the area delimited by the projection of l is equal to the size of the bi-vector l . This maximum is achieved by projecting on a plane on which l lies.]

²⁴ [The area of a portion of surface is the upper limit of the sum of its parts.]

²⁵ [The bi-vector corresponding to an infinitesimal part of the surface lies on the tangent plane; the ratio between its size and the area of that part is equal to 1.]

In this way PEANO shows the complete analogy between length of arcs and area of surfaces: in fact PEANO observes that the direction of the vector with endpoints on an infinitesimal arc coincides with the tangent, and the rate between their lengths is equal to one.²⁶ Commenting on this fact, we may say that PEANO's definition grasps the essence of the measure of area at the infinitesimal level.

The idea of projection on planes and the selection of the projection which maximizes the area is present also in CARATHÉODORY's work of 1914. His ideas are further developed by HAUSDORFF, who extends CARATHÉODORY's results in the case of HAUSDORFF measures with integer exponent. Nowadays the most famous measure is the HAUSDORFF measure, which allows one to define the measure of rather general sets by including also the concept of dimension. One of the first results proved by HAUSDORFF is the LAGRANGE formula of area (1.1).

5. ORIENTED CLOSED CURVES AND BI-VECTORS

For convenience of the reader we outline in a formal way how to associate a bi-vector to a closed oriented curve accordingly with Theorem 2.2.

In *Calcolo geometrico* [36, (1888)] PEANO provides a formula to valuate the bi-vector associated to a closed curve. Let $A: [t_0, t_1] \rightarrow \mathbb{P}_3$ be a C^1 closed curve. The bi-vector associated to A is given by²⁷

$$(5.1) \quad \int_{t_0}^{t_1} A(t)A'(t)dt .$$

In the case of a closed planar curve A , the area of the triangle $X \int_{t_0}^{t_1} A(t)A'(t)dt$ does not depend on the point $X \in \mathbb{P}_3$ belonging to the plane of the curve; such area is called by PEANO "area delimited by the closed planar curve A ".

PEANO gives an example of bi-vector associated to the non-planar closed curve $A: [-r, 2\pi + r + h2\pi] \rightarrow \mathbb{P}_3$ formed by a cylindrical helix of radius r and pitch $h2\pi$ and three rectilinear pieces, two horizontal and one vertical, according to the definition:

$$(5.2) \quad A(t) := \begin{cases} O + (r + t)\mathbf{i} & \text{for } -r \leq t \leq 0 \\ O + r \cos t\mathbf{i} + r \sin t\mathbf{j} + ht\mathbf{k} & \text{for } 0 \leq t \leq 2\pi \\ O + (2\pi + r - t)\mathbf{i} + h2\pi\mathbf{k} & \text{for } 2\pi \leq t \leq 2\pi + r \\ O + (h2\pi + 2\pi + r - t)\mathbf{k} & \text{for } 2\pi + r \leq t \leq 2\pi + r + h2\pi \end{cases}$$

where O is a point of \mathbb{P}_3 and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthogonal base of \mathbb{V} . A straightforward calculations gives for the integral (5.1) the value $2\pi r^2 \mathbf{ij}$, where \mathbf{ij} denotes the bi-vector product of \mathbf{i} and \mathbf{j} . This coincides with the value of the bi-vector corresponding to the orthogonal projection of the curve A on the plane \mathbf{i}, \mathbf{j} .

In previous Section we have outlined several properties related to bi-vectors associated to closed curves. Now we formulate these properties in terms of the following propositions, leaving the proofs to the reader.

²⁶ Besides these properties of areas, PEANO gives an estimate of the difference between the lengths of an arc and its cord and between the area of a surface and its bi-vector.

²⁷ Recall that \mathbb{P}_3 denotes the set of points according to Grassmann-Peano vector calculus (see Section 2) and that $A'(t)$ denotes the derivative of A at t .

Proposition 5.1. *Let $\gamma: [0, 1] \rightarrow \mathbb{P}_3$ be a continuous closed curve lying on a plane π . There exists a unique bi-vector (denoted by α_γ) associated to γ such that for any point O in π*

$$(5.3) \quad O\alpha_\gamma = \lim_{\{t_i\}} \sum_{i=0}^{m-1} O\gamma(t_i)\gamma(t_{i+1})$$

where the limit is evaluated on the subdivisions $0 = t_0 < \dots < t_i < t_{i+1} < \dots < t_m = 1$ of the interval $[0, 1]$ for $\max\{t_{i+1} - t_i : i = 0, \dots, m-1\} \rightarrow 0$.

Theorem 5.2. *Let $\mu: [0, 1] \rightarrow \mathbb{P}_3$ be a continuous closed (non necessarily planar) curve. There exists a unique bi-vector associated to μ such that, for every plane π and for every parallel projection on π , the bi-vector α_{μ^*} associated to the curve μ^* , projection of the curve μ on the plane π , is equal to the projection of the bi-vector on π .*

Theorem 5.3. *Let $\sigma: [0, 1] \times [0, 1] \rightarrow \mathbb{P}_3$ be a C^1 surface. For any $x, y \in (0, 1)$ let's consider the infinitesimal square $Q_\varepsilon = [x, x+\varepsilon] \times [y, y+\varepsilon]$, its counterclockwise oriented boundary ∂^+Q_ε , and the infinitesimal element of surface $\sigma(Q_\varepsilon)$. Then the ratio between the magnitude of the bi-vector $\alpha_{\sigma(\partial^+Q_\varepsilon)}$, associated to the closed curve ∂^+Q_ε , and the PEANO's area of $\sigma(Q_\varepsilon)$ tends to 1 when ε tends to 0.*

6. USE OF THE CONCEPT OF AREA BY PEANO

In this section we analyze the use of the concept of area by PEANO into the following works: *Applicazioni geometriche* [35, (1887)], *Calcolo geometrico* [36, (1888)], *Lezioni di analisi infinitesimale* [38, (1893)] and *Formulario mathematico* (1895-1908).

PEANO, by means of the notions of inner and outer measures on Euclidean spaces of dimension 1, 2, 3, that have been introduced by him in *Sull'integrabilit  delle funzioni* [34, (1883)], refounds in *Applicazioni geometriche* [35, (1887)] the notion of Riemann integral and extends it to abstract measures. The development of the theory of measure is based on a solid topological and logical ground and on a deep knowledge of set theory.

PEANO in *Applicazioni geometriche* and later JORDAN in *Cours d'Analyse* [22, (1893)] develop the well known concepts of classical measure theory: measurability, change of variables, fundamental theorems of calculus.

The mathematical tools employed by PEANO were really innovative both on geometrical and topological level. PEANO used extensively the geometric vector calculus introduced by GRASSMANN (see Section 2). A revolutionary tool is the notion of differentiation of distributive set functions, that suggests to regard area of a non-planar surface as a distributive set function and to compare it, at the infinitesimal level, with the area of a planar set. In this context the evaluation of the area of a non-planar surface is reduced to the integration of a numerical function obtained by differentiation of the area of a non-planar surface with respect to the area of planar sets. ²⁸

²⁸ As observed in our paper *Peano on derivative of measures: strict derivative of distributive set functions* [19, (2010)], differentiation of distributive set functions gives a mathematical implementation of the *massa-density paradigm* (mass and volume are distributive set functions and the density is obtained by differentiating mass with respect to volume).

In this rich mathematical context PEANO gives his first definition of area of non-planar surfaces (see first quotation of Section 4) and derives general formulae for planar and non-planar surfaces.

(6.1) *Formula for planar area* (see [35, (1887, Th. 47, p.237)], [38, (1893, Vol. 2, §394 p.224-225)]). Let $A, B : [t_0, t_1] \rightarrow \mathbb{R}^2$ be two C^1 functions, such that the segments $A(t)B(t)$ and $A(t')B(t')$ have empty intersection for any $t, t' \in [t_0, t_1], t \neq t'$. The set spanned by the segment $A(t)B(t)$, with $t \in [t_0, t_1]$, namely the set $\cup_{t \in [t_0, t_1]} A(t)B(t)$ has an area u given by the formula

$$u = \frac{1}{2} \int_{t_0}^{t_1} (B(t) - A(t)) \cdot \left(\frac{dA(t)}{dt} + \frac{dB(t)}{dt} \right) dt$$

where, following PEANO's terminology, $(B(t) - A(t)) \cdot \left(\frac{dA(t)}{dt} + \frac{dB(t)}{dt} \right)$ denotes the magnitude of the bi-vector given by the product of the vectors $(B(t) - A(t))$ and $\left(\frac{dA(t)}{dt} + \frac{dB(t)}{dt} \right)$.²⁹

(6.2) *Formula for non-planar area* (see [35, (1887, Th. 49, p.243)], [38, (1893, Vol. 2, §396 p.229-232)]). Let $P : D \rightarrow \mathbb{R}^3$ be a C^1 function over $D := \{(u, v) \in \mathbb{R}^2 : a < u < b, \theta_0(u) < v < \theta_1(u)\}$ where θ_0 and θ_1 are continuous functions defined on the interval $[a, b]$. The surface formed by points $P(u, v)$, with $(u, v) \in D$, has an area S given by the formula

$$(6.3) \quad S = \int_a^b du \int_{\theta_0(u)}^{\theta_1(u)} \omega(u, v) dv$$

where $\omega(u, v)$ is the magnitude of the bi-vector product of the vectors $\frac{\partial P}{\partial u}$ and $\frac{\partial P}{\partial v}$.³⁰

PEANO uses formulae (6.1) and (6.2) to obtain classical formulae for elementary surfaces (planar and non-planar). Moreover from (6.1) he derives in ([35, (1887), p.242]) and in ([38, (1893, Vol. 2, §394 p.225-226)]) formulae that have been recovered one century later by MAMIKON A. MNATSKANYAN in his paper *On the area of the region on a developable surface* [30, (1981)].

Particular instances of formula (6.1), considered by PEANO, are the following:

- (6.4) The point A moves along a straight line and the angle of the segment AB with that line is constant;
- (6.5) The point A is fixed;
- (6.6) The segment AB is tangent at the point A to the curve described by A ;
- (6.7) The segment AB is of constant length and normal to the curve described by its midpoint.

²⁹ In modern language, the magnitude of this bi-vector is the norm of the vector product $(B(t) - A(t)) \wedge \left(\frac{dA(t)}{dt} + \frac{dB(t)}{dt} \right)$. Therefore the formula (6.1) becomes

$$u = \frac{1}{2} \int_{t_0}^{t_1} \left\| (B(t) - A(t)) \wedge \left(\frac{dA(t)}{dt} + \frac{dB(t)}{dt} \right) \right\| dt .$$

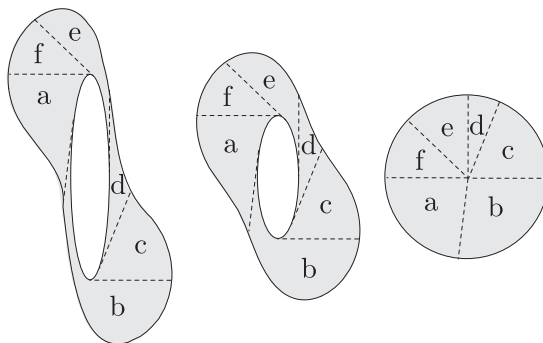
³⁰ In modern language, the magnitude of this bi-vector is the norm of the vector product $\frac{\partial P}{\partial u}(u, v) \wedge \frac{\partial P}{\partial v}(u, v)$. Therefore the formula (6.2) becomes

$$S = \int_a^b du \int_{\theta_0(u)}^{\theta_1(u)} \left\| \frac{\partial P}{\partial u}(u, v) \wedge \frac{\partial P}{\partial v}(u, v) \right\| dv .$$

In the case (6.6), formula (6.1) becomes

$$u = \frac{1}{2} \int_{t_0}^{t_1} \left| \det \begin{pmatrix} v_1(t) & v_2(t) \\ v_1'(t) & v_2'(t) \end{pmatrix} \right| dt,$$

where $v_1(t), v_2(t)$ are the components of $B(t) - A(t)$ and $t \in [t_0, t_1]$. It is clear from this formula, that the area depends only on the differences of the points $B(t) - A(t)$ and not on the particular positions of the points $A(t), B(t)$. As a consequence of this PEANO derives the content of what is nowadays stated as MAMIKON's Theorem: *the area of a tangent sweep of a curve is equal to the area of its corresponding tangent cluster*. The three figures have the same area, because they are swept by the same tangent vector to the inner ellipsis (or point). The areas marked by the same letter have the same area as well.



MAMIKON's theorem has numerous applications, as it enables one to obtain area of complicated figures almost without calculation, by reducing the problem to the calculus of area of simple figures; see, for examples, MAMIKON A. MNATSKANYAN and APOSTOL in [27],[25], [26].³¹

Finally the formula (6.3), already obtained by PEANO from his geometric definition of area of surfaces, is proved by him also using his bi-vectorial definition. This coincidence is valid in the case of C^1 surfaces, but it does not hold for arbitrary surfaces.

Concerning the area, in the five editions of *Formulario mathematico*, in addition to some properties outlined above, we find: another definition of area [39, (1902) p. 300], due to BORCHARDT, and the well-known counter-example to the definition of SERRET on area [39, (1902) p. 300-301].

In *Formulario mathematico* PEANO adopts BORCHARDT's area [6, (1854) p. 369]³², defined for every set S of points in \mathbb{R}^3 of null volume by the following formula:

$$(6.8) \quad \lim_{h \rightarrow 0^+} \frac{\text{Volum}\{x \in \mathbb{R}^3 : \text{dist}(x, S) < h\}}{2h}$$

The counter-example to SERRET's definition is based on the construction of a polyhedral surface $\mathcal{S}_{m,n}$, with m, n positive integers, inscribed into a cylinder of

³¹ In [28, (2009)] APOSTOL and MNATSKANYAN, using Mamikon theorem, prove the property of ROBERVAL: "The area of a cycloidal sector is three times the area described by the generating disk along its motion". This property was proved by PEANO [38, (1893) Vol. 2, §395 p.226-228] using (6.1).

³² BORCHARDT's area, usually called Minkowski area, was rediscovered by MINKOWSKI [29, (1901)] 47 years later.

height 1 and radius 1, formed by mn triangles with the following vertices:

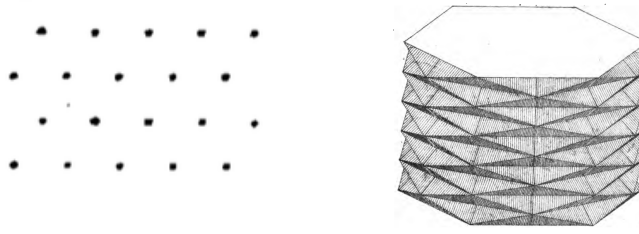
$$\left(\cos\left[\frac{2\pi r}{m}\right], \sin\left[\frac{2\pi r}{m}\right], \frac{s}{n}\right), \left(\cos\left[\frac{2\pi[r+1]}{m}\right], \sin\left[\frac{2\pi[r+1]}{m}\right], \frac{s}{n}\right), \left(\cos\left[\frac{\pi[2r+1]}{m}\right], \sin\left[\frac{\pi[2r+1]}{m}\right], \frac{s+1}{n}\right)$$

and by mn triangles with the following vertices:

$$\left(\cos\left[\frac{2\pi r}{m}\right], \sin\left[\frac{2\pi r}{m}\right], \frac{s}{n}\right), \left(\cos\left[\frac{\pi[2r-1]}{m}\right], \sin\left[\frac{\pi[2r-1]}{m}\right], \frac{s+1}{n}\right), \left(\cos\left[\frac{\pi[2r+1]}{m}\right], \sin\left[\frac{\pi[2r+1]}{m}\right], \frac{s+1}{n}\right)$$

with $r = 0, 1, \dots, m - 1$ and $s = 0, 1, \dots, n - 1$.

The following pictures show the positions of vertices of triangles in the plane development of the cylindrical surface (as appears in PEANO [39, (1902) p. 300-301], with $m = 5, n = 3$) and the shape of the polyhedral surface $\mathcal{S}_{m,n}$ (as appears in HERMITE [21, (1883) p. 36] with $m = 6, n = 10$).



A straightforward calculations gives the area $a_{m,n}$ of the polyhedral surface $\mathcal{S}_{m,n}$:

$$(6.9) \quad a_{m,n} = 2m \sin\left(\frac{\pi}{m}\right) \sqrt{1 + 4n^2 \sin^4 \frac{\pi}{2m}} \quad .$$

Clearly

$$(6.10) \quad \lim_{m \rightarrow \infty} a_{m,m} = 2\pi, \quad \lim_{m \rightarrow \infty} a_{m,m^2} = 2\pi \sqrt{1 + \frac{\pi^4}{4}}, \quad \lim_{m \rightarrow \infty} a_{m,m^3} = +\infty$$

Consequently the limit of the area of the polyhedra $\mathcal{S}_{m,n}$ for $m, n \rightarrow \infty$ does not exist.

7. ON THE INFLUENCE OF PEANO ON DEFINITION OF AREA

With LEBESGUE's Thesis *Intégrale, Longueur, Aire* [23, (1902)], PEANO's definition of area acquires notoriety. LEBESGUE is acquainted with the bi-vectorial definition of area given by PEANO in 1890, but ignores the original definition of 1887 and any other contribution of this Author (with the exception of the PEANO's curve). As a consequence of this, it is not surprising that, in almost all contributions on the definition of area, references to the other PEANO's works on area and, in particular, to the books *Applicazioni geometriche* [35, (1887)] and *Calcolo geometrico* [36, (1888)], are absent.

LEBESGUE's area of a parameterized surface is defined by him as the lower limit of the area of the polyhedral surfaces that approximate uniformly the surface.

In the mathematical literature, we find definitions of area that implement PEANO's inequality, namely the "area of surface is greater or equal to the area of its orthogonal projection on an arbitrary plane". Different implementations correspond to the different way of defining the "area of the orthogonal projection on a plane".

Other definitions of area implement the PEANO's bi-vectorial inequality, namely that the "area of a surface bounded by a closed oriented contour is greater or equal to the magnitude of the bi-vector associated with the contour itself". In this case,

the implementations correspond to the different ways to associate a number to a given oriented closed curve.

After SCHWARZ and PEANO, as observed by RADÓ in [42, (1956) p. 513], “many definitions [of surface area] have been proposed, and an enormous amount of efforts have been expended in the study of . . . various concepts of surface area”. For this reason we are forced to present only some contributions. Interested readers may find detailed historical and mathematical facts in CESARI’s *Surface area* [12, (1954)] and RADÓ’s *Length and area* [43, (1948)].

In addition to the one given by PEANO, remarkable definitions are the LEBESGUE’s and GEÖCZE’s area. The original definitions of PEANO and GEÖCZE provide an evaluation of area that is greater than or equal to LEBESGUE’s area. Observe that PEANO’s and GEÖCZE’s area relies on the evaluation of the area of the orthogonal projection on planes of pieces of the given surface. Therefore many authors have proposed different ways to define the area of a plane surface, in order to make PEANO’s and GEÖCZE’s area coincident with LEBESGUE’s area for a wider class of continuous parametric surfaces (see RADÓ [41, (1928)] and CECCONI [10, (1950)], [11, (1951)]).

CESARI [12, (1956)] reformulates the definitions given by PEANO and GEÖCZE in a suitable way in order “to preserve” elementary area of polyhedral surfaces and, above all, lower semicontinuity. CESARI states the following theorem:

Theorem 7.1. *For every continuous surface S we have $\mathcal{L}(S) = \mathcal{V}(S) = \mathcal{P}(S)$, where $\mathcal{L}(S)$, $\mathcal{V}(S)$ and $\mathcal{P}(S)$ denote Lebesgue area, Geöcze area and Peano area.*

More precisely, PEANO’s and GEÖCZE’s definitions are reformulated by CESARI in terms of *topological index* of planar closed curves. This index is denoted with $O(P, \gamma)$ by CESARI, where γ is a closed planar curve, and $P = (x, y)$ is a point of the plane π of γ . It is worth observing that, as for the bi-vector associated to a closed planar curve, the integral $\int_{\pi} |O(P, \gamma)| dx dy$ (denoted in the following by $v(\gamma, \pi)$) is interpreted, as “area of the planar surface delimited by γ ” (see CESARI [42, (1956) p. 104]).

Now, let S be a parametric surface in \mathbb{R}^3 , parameterized by a continuous $\varphi: A \rightarrow S$ (i.e. $S = \varphi(A)$), where A is an *admissible* set⁽³³⁾. Given a plane α and a curve γ in A , let denote with $\gamma^{*\alpha}$ the orthogonal projection on α of the image γ^* of γ under the parameterization φ .

The reformulation $\mathcal{P}(S)$ of PEANO’s area of the surface S , given by CESARI (see [12, (1956) p. 137]), is the following:

$$(7.1) \quad \mathcal{P}(S) := \sup_{\{\gamma_i\}_i} \sum_i \sup_{\alpha} v(\gamma_i^{*\alpha}, \alpha)$$

where $\{\gamma_i\}_i$ runs over all finite families of simple closed polygonal curves in A delimiting non-overlapping regions and α runs over all planes in \mathbb{R}^3 .

Concerning GEÖCZE’s area, let us consider the coordinate planes α_{xy} , α_{yz} and α_{zx} in the Euclidean space. The reformulation $\mathcal{V}(S)$ of GEÖCZE’s area of the surface S , given by CESARI (see [12, (1956) p. 117]), is the following:

$$(7.2) \quad \mathcal{V}(S) := \sup_{\{\gamma_i\}_i} \sum_i \sqrt{[v(\gamma_i^{*\alpha_{xy}}, \alpha_{xy})]^2 + [(v(\gamma_i^{*\alpha_{yz}}, \alpha_{yz}))]^2 + [(v(\gamma_i^{*\alpha_{zx}}, \alpha_{zx}))]^2}$$

³³ Among the admissible sets (see CESARI [12, (1956) p. 27]), we mention: planar sets delimited by a Jordan simple curve or finite union of such sets, and open sets.

A great deal of research has been dedicated to find an axiomatic characterization of a notion of surface area, namely to the problem of establishing properties characterizing univocally the notion of area. CECCONI in [11, (1951)] gives the following properties characterizing LEBESGUE's area (and, consequently, Peano area (7.1) and Geöcze area (7.2)):

Theorem 7.2. *Let Φ be a functional defined over all continuous parametric surfaces S on 2-cells. Then Φ coincides with Lebesgue area if the following properties are satisfied:*

(7.3) Φ is lower semi-continuous;

(7.4) Φ coincides with usual elementary area for polyhedral surfaces;

(7.5) Φ is super-additive⁽³⁴⁾;

(7.6) Φ satisfies Peano inequality⁽³⁵⁾.

In the proof of this Theorem, given by CECCONI, a crucial step consists in the inequality $\mathcal{P}(S) \leq \Phi(S) \leq \mathcal{L}(S)$ that, together with the equality $\mathcal{P}(S) = \mathcal{L}(S)$ (see Theorem 7.1), leads to the expected coincidence $\Phi(S) = \mathcal{L}(S)$.

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³⁴ Namely, for every subdivision of the surface S in parts, the area of S is greater than the sum of the areas of the parts.

³⁵ Namely, for every S , and for every plane α , one has $\Phi(S) \geq \text{mis}(\{P \in \alpha : O(P; C_\alpha) \neq 0\})$, where C_α is the projection of the contour of S on α , and $O(P; C_\alpha)$ is the topological index of P with respect to the curve C_α defined above.

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