Finite element potentials

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Abstract

We present an explicit and efficient way for constructing finite elements with assigned gradient, or curl, or divergence. Some simple notions of homology theory and graph theory applied to the finite element mesh are basic tools for devising the solution algorithms.

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1. Introduction

Determining the necessary and sufficient conditions for assuring that a vector field defined in a bounded and sufficiently smooth three-dimensional domain Ω is the gradient of a scalar potential or the curl of a vector potential is one of the most classical problem of vector analysis.

The answer is well-known, and shows an interesting interplay of differential calculus and topology (see, e.g., Cantarella et al. [3]):

- a vector field is the gradient of a scalar potential if and only if it is curl free and its line integral is vanishing on all the closed curves that give a basis of the first homology group of $\overline{\Omega}$;
- a vector field is the curl of a vector potential if and only if it is divergence free and its flux is vanishing across (all but one) the connected components of $\partial\Omega$.

Less interesting is the problem of finding a vector field with assigned divergence f: this problem is very simply solved by taking the gradient of the solution φ of the elliptic problem $\Delta\varphi = f$ in Ω, φ vanishing on the boundary $\partial\Omega$; no compatibility conditions on f are needed, no topological properties of Ω come into play.

However, a less clarified situation takes shape when, given a suitable *finite element* vector field, we want to furnish an explicit and efficient procedure for constructing its *finite element* scalar potential and vector potential. Note also that at this level the construction of a finite element vector field with an assigned divergence comes back on the table: in fact, the gradient of a (standard) finite element approximate solution of $\Delta\varphi = f$ has a distributional divergence which is not a function, and therefore this divergence cannot be equal to an assigned finite element.

The aim of this paper is to furnish a simple and efficient way for constructing finite elements with assigned gradient, or curl, or divergence. Clearly, in numerical computations this is important any time one has to reduce a given problem to an associated one with vanishing data.

It is worth noting that the computational cost of all the algorithms we propose depends linearly on the number of unknowns.

2. Notation and preliminary results

Let Ω be a bounded polyhedral domain of $\mathbb{R}^3$ with Lipschitz boundary and let $(\partial\Omega)_0, \ldots, (\partial\Omega)_p$ be the connected components of $\partial\Omega$. Consider a tetrahedral triangulation $\mathcal{T}_h = (V, E, F, T)$ of $\overline{\Omega}$. Here V is the set of vertices, E the set of edges, F the set of faces and T the set of tetrahedra in $\mathcal{T}_h$.

We consider the following spaces of finite elements (for a complete presentation, see Monk [7]). The space L_h of continuous piecewise-linear finite elements; its dimension is n_v, the number of vertices in $\mathcal{T}_h$. The space N_h of Nédélec edge elements of degree 1; its dimension is n_e, the number of edges in $\mathcal{T}_h$. The space RT_h of Raviart–Thomas finite elements of degree 1; its dimension is n_f, the number of faces in $\mathcal{T}_h$. The space PC_h of piecewise-constant elements; its dimension is n_t, the number of tetrahedra in $\mathcal{T}_h$.

It is well-known that $L_h \subset H^1(\Omega)$, $N_h \subset H(\mathrm{curl};\Omega)$, $RT_h \subset H(\mathrm{div};\Omega)$ and $PC_h \subset L^2(\Omega)$, where

$$\begin{aligned} H^1(\Omega) &= \{\phi \in L^2(\Omega) \mid \mathrm{grad}\,\phi \in (L^2(\Omega))^3\}\,, \\ H(\mathrm{curl};\Omega) &= \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\mathbf{v} \in (L^2(\Omega))^3\}\,, \\ H(\mathrm{div};\Omega) &= \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}\,. \end{aligned}$$

Moreover $\mathrm{grad}\, L_h \subset N_h$, $\mathrm{curl}\, N_h \subset RT_h$ and $\mathrm{div}\, RT_h \subset PC_h$.

Fix a total ordering $v_1, \ldots, v_{n_v}$ of the elements of V. This induces an orientation on the elements of E, F and T: if the end points of e_j are v_a and v_b for some $a, b \in \{1, \ldots, n_v\}$ with $a < b$, then the oriented edge e_j is denoted by $[v_a, v_b]$, and therefore the unit tangent vector of e_j is given by $\boldsymbol{\tau} = \frac{v_b - v_a}{|v_b - v_a|}$. On the other hand, if the face f has vertices v_a, v_b and v_c with $a < b < c$, the oriented face f is denoted by $[v_a, v_b, v_c]$ and its unit normal vector $\boldsymbol{\nu}$ is obtained by the right hand rule. Finally, if the tetrahedron t has vertices v_a, v_b, v_c and v_d with $a < b < c < d$, the oriented tetrahedron f is denoted by $[v_a, v_b, v_c, v_d]$

Let us consider a basis of L_h, $\{\Phi_{h,1}, \ldots, \Phi_{h,n_v}\}$, such that

$$\Phi_{h,i}(v_j) = \delta_{i,j}$$

for $1 \le i, j \le n_v$, a basis of N_h, $\{\mathbf{w}_{h,1}, \ldots \mathbf{w}_{h,n_e}\}$, such that

$$\int_{e_j} \mathbf{w}_{h,i} \cdot \boldsymbol{\tau} = \delta_{i,j}$$

for $1 \leq i, j \leq n_e$, a basis of RT_h, $\{\mathbf{r}_{h,1}, \dots \mathbf{r}_{h,n_f}\}$, such that

$$\int_{f_j} \mathbf{r}_{h,i} \cdot \boldsymbol{\nu} = \delta_{i,j}$$

for $1 \leq i, j \leq n_f$, and the basis of PC_h, $\{g_{h,1}, \dots g_{h,n_t}\}$, given by the characteristic functions of the tetrahedron t_i.

In the following we introduce some notions of homology theory (see, e.g., Munkres [8]). We start from the mesh $\mathcal{T}_h = (V, E, F, T)$ on $\overline{\Omega}$, having assigned the orientation to the edges and faces as explained before. The basic concept is that of chain: a 2-chain is a formal linear combination of oriented faces, a 1-chain is a formal linear combination of oriented edges, and a 0-chain is a formal linear combination of vertices, in all cases taking the coefficients in $\mathbb{Z}$. We denote by $C_k(\mathcal{T}_h, \mathbb{Z})$ the set of all the k-chains in $\mathcal{T}_h$, $k = 0, 1, 2$.

Now we can define the boundary operator $\partial_k : C_k(\mathcal{T}_h, \mathbb{Z}) \to C_{k-1}(\mathcal{T}_h, \mathbb{Z})$ for $k = 1, 2$. For the oriented face $f = [v_{a_0}, v_{a_1}, v_{a_2}]$ we have

$$\partial_2 f := [v_{a_1}, v_{a_2}] - [v_{a_0}, v_{a_2}] + [v_{a_0}, v_{a_1}] \, .$$

Analogously for the oriented edge $e = [v_a, v_b]$ we have

$$\partial_1 e := v_b - v_a .$$

We extend the definition of the boundary operator to chains by linearity.

A 1-chain c of $\mathcal{T}_h$ is a 1-cycle if $\partial_1 c = 0$, and is a 1-boundary if there exists a 2-chain C such that $\partial_2 C = c$. Notice that all 1- boundaries are 1-cycles but, in general, not all 1-cycles are 1-boundaries.

Let us denote by $Z_1(\mathcal{T}_h, \mathbb{Z})$ the set of 1-cycles, $Z_1(\mathcal{T}_h, \mathbb{Z}) := \ker(\partial_1)$, and $B_1(\mathcal{T}_h, \mathbb{Z})$ the set of 1-boundaries, $B_1(\mathcal{T}_h, \mathbb{Z}) := \mathrm{im}(\partial_2)$. Two 1-cycles c and c' are called homologous in $\mathcal{T}_h$ if $c - c'$ is a 1-boundary in $\mathcal{T}_h$. If c is homologous to the trivial 1-cycle (namely, it is a 1-boundary), then we say that c bounds in $\mathcal{T}_h$.

The first homology group of $\mathcal{T}_h$ consists of all homology classes of 1-cycles of $\mathcal{T}_h$, that is, it is the quotient group

$$\mathcal{H}_1(\mathcal{T}_h, \mathbb{Z}) = Z_1(\mathcal{T}_h, \mathbb{Z}) / B_1(\mathcal{T}_h, \mathbb{Z}) \, .$$

Let $\{\sigma_n\}_{n=1}^g$ be a set of 1-cycles in $\mathcal{T}_h$ such that the equivalence classes $\{[\sigma_n]\}_{n=1}^g$ are a basis of the homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$.

As told before, aim of this paper is to devise effective algorithms for the solution of the following problems:

- *Grad problem.* Given $\mathbf{F}_h \in N_h$ such that $\operatorname{curl} \mathbf{F}_h = \mathbf{0}$ and $\oint_{\sigma_n} \mathbf{F}_h \cdot d\mathbf{s} = 0$ for any 1-cycle σ_n, $n = 1, \dots, g$, find $\Psi_h \in L_h$ such that $\operatorname{grad} \Psi_h = \mathbf{F}_h$.
- *Curl problem.* Given $\mathbf{J}_h \in RT_h$ such that $\operatorname{div} \mathbf{J}_h = 0$ and $\int_{(\partial\Omega)_r} \mathbf{J}_h \cdot \mathbf{n} = 0$ for (all but one) the connected components $(\partial\Omega)_r$ of $\partial\Omega$, $r = 1, \dots, p$, find $\mathbf{Z}_h \in N_h$ such that $\operatorname{curl} \mathbf{Z}_h = \mathbf{J}_h$.

- *Div problem.* Given $f_h \in PC_h$ find $\mathbf{v}_h \in RT_h$ such that $\operatorname{div} \mathbf{v}_h = f_h$.

Notice that none of these problems has a unique solution. The solution of the grad problem is unique up to a constant, the solution of the curl problem is unique up to a gradient or a harmonic field belonging to $\mathcal{H}(m;\Omega)$, where

$$\mathcal{H}(m;\Omega) = \{\mathbf{w} \in (L^2(\Omega))^3 \mid \operatorname{curl} \mathbf{w} = \mathbf{0}, \operatorname{div}\, \mathbf{w} = 0, \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} ,$$

while the solution of the div problem is unique up to a curl or a harmonic field belonging to $\mathcal{H}(e;\Omega)$, where

$$\mathcal{H}(e;\Omega) = \{\mathbf{w} \in (L^2(\Omega))^3 \mid \operatorname{curl} \mathbf{w} = \mathbf{0}, \operatorname{div}\, \mathbf{w} = 0, \mathbf{w} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\} .$$

From the computational point of view, it is clear that more efficient algorithms can be proposed if one is able to find suitable formulations of these problems for which uniqueness holds.

3. Grad problem

As we already noted, the solution is not unique but to obtain uniqueness it is enough to fix the value of the solution in a vertex of V, for instance in v_1. So we consider the following problem: given $\mathbf{F}_h \in N_h$ such that $\operatorname{curl} \mathbf{F}_h = \mathbf{0}$ and $\oint_{\sigma_n} \mathbf{F}_h \cdot d\mathbf{s} = 0$ for any 1-cycle σ_n, $n = 1, \ldots, g$, find $\Psi_h \in L_h$ such that

$$\begin{aligned} &\operatorname{grad} \Psi_h = \mathbf{F}_h \quad \text{in } \Omega \\ &\Psi_h(v_1) = 0 \,. \end{aligned} \tag{1}$$

The set of vertices and edges (V, E) of the mesh $\mathcal{T}_h = (V, E, F, T)$ form a graph. A spanning tree is a maximal subgraph of (V, E) (maximal because it visits all vertices) without loops (this means that it is a tree). Let $\mathcal{S}_h = (V, L)$ be a spanning tree of the graph (V, E).

For each $v_i \in V$, $v_i \neq v_1$ let us denote by C_{v_i} the unique 1-chain in $\mathcal{S}_h$ such that $\partial_1 C_{v_i} = v_i - v_1$. We set

$$\kappa_i = \int_{C_{v_i}} \mathbf{F}_h \cdot \boldsymbol{\tau} \,.$$

It is not difficult to see that $\Psi_h := \sum_{i=1}^{n_v} \kappa_i \Phi_{h,i}$ is a solution of (1).

In fact, first of all we have $\Psi_h(v_1) = 0$. Then, given an oriented edge $e = [v_a, v_b] \in E$, we consider the 1-cycle $D_e = C_{v_a} + e - C_{v_b}$. Since $\operatorname{curl} \mathbf{F}_h = \mathbf{0}$ and $\oint_{\sigma_n} \mathbf{F}_h \cdot d\mathbf{s} = 0$ for any 1-cycles σ_n, $n = 1, \ldots, g$, from Helmholtz decomposition (see, e.g., Cantarella et al. [3])) it follows that $\mathbf{F}_h$ is a gradient, hence its line integral on each 1-cycle vanishes. Therefore we have

$$0 = \oint_{D_e} \mathbf{F}_h \cdot d\mathbf{s} = \Psi_h(v_a) + \int_e \mathbf{F}_h \cdot \boldsymbol{\tau} - \Psi_h(v_b) = \int_e \mathbf{F}_h \cdot \boldsymbol{\tau} - \int_e \operatorname{grad} \Psi_h \cdot \boldsymbol{\tau}$$

for all $e \in E$, hence $\operatorname{grad} \Psi_h = \mathbf{F}_h$.

In particular, on the spanning tree we have

$$\Psi_h(v_b) - \Psi_h(v_a) = \int_e \mathbf{F}_h \cdot \boldsymbol{\tau} \quad \forall\, e = [v_a, v_b] \in L\,. \tag{2}$$

Since $\Psi_h(v_1) = 0$, an easy elimination algorithm for the computation of Ψ_h can be implemented going down from the root v_1 along the spanning tree.

Precisely, it reads as follows. Let us denote by R the set of the vertices where the value of Ψ_h is already known and by P the set of $e \in L$ with exactly one vertex, $v'(e)$, not in R. Initially $R = \{v_1\}$ and $P = E(v_1) \cap L$, where $E(v) := \{e \in E : v \in e\}$, namely, the set of edges in L incident to the root.

Algorithm 1.

1. $R = \{v_1\}$, $P = E(v_1) \cap L$
2. *while* $R \neq V$
   (a) *pick* $e \in P$
   (b) *compute* $\Psi_h(v'(e))$ *from (2)*
   (c) *update* R*:* $R = R \cup \{v'(e)\}$
   (d) *update* P*:* $P = [P \cup (E(v'(e)) \cap L)] \setminus \{e\}$.

Clearly, this algorithm stops only when $R = V$.

4. Curl problem

Since the solution is not unique, we start adding some topological equations, that filter out harmonic fields belonging to $\mathcal{H}(m;\Omega)$. Therefore, we consider the problem: given $\mathbf{J}_h \in RT_h$ such that $\operatorname{div} \mathbf{J}_h = 0$ and $\int_{(\partial\Omega)_r} \mathbf{J}_h \cdot \mathbf{n} = 0$ for (all but one) the connected components $(\partial\Omega)_r$ of $\partial\Omega$, $r = 1, \ldots, p$, find $\mathbf{Z}_h \in N_h$ such that

$$\begin{array}{ll} \operatorname{curl} \mathbf{Z}_h = \mathbf{J}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \ldots, g\,, \end{array} \tag{3}$$

where $\kappa_1, \ldots, \kappa_g$ are real numbers.

The solution of this problem is not yet unique, as it is unique up to a gradient. More precisely, if $\mathbf{Z}_h \in N_h$ satisfies $\operatorname{curl} \mathbf{Z}_h = \mathbf{0}$ and $\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = 0$ for all $n = 1, \ldots, g$, we know that $\mathbf{Z}_h = \operatorname{grad} \phi_h$ with $\phi_h \in L_h$ (see Section 3, or Monk [7, Lemma 5.28]).

Therefore, a way for obtaining a unique solution is to consider the problem

$$\begin{array}{ll} \operatorname{curl} \mathbf{Z}_h = \mathbf{J}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \ldots, g \\ \int_\Omega \mathbf{Z}_h \cdot \operatorname{grad} \eta_h = 0 & \forall\, \eta_h \in L_h\,. \end{array} \tag{4}$$

Note that we have added $n_v - 1$ equations, as that is the dimension of $\operatorname{grad} L_h$.

However, the orthogonality condition is not the best suited for an effective implementation. Therefore, we prefer to resort to another way of filtering out the gradients, and we use a (well-known) method that dates back to Kirchhoff and to circuit theory (see also Alonso Rodríguez et al. [1]).

This way of selecting a unique solution reads as follows. Let $\mathcal{S}_h = (V, L)$ be a spanning tree of the graph (V, E); remember that the number of edges in the spanning tree is exactly $n_v - 1$. We claim that there exists a unique solution of the problem

$$\begin{array}{ll} \operatorname{curl} \mathbf{Z}_h = \mathbf{J}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \dots, g \\ \int_e \mathbf{Z}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e \in L\,. \end{array} \tag{5}$$

The uniqueness of the solution is easily proved. In fact, let us suppose that in (5) we have $\mathbf{J}_h = \mathbf{0}$ and $\kappa_n = 0$ for each $n = 1, \dots, g$. Then the first two conditions say that $\mathbf{Z}_h = \operatorname{grad} \phi_h$ with $\phi_h \in L_h$. Moreover, if $\int_e \operatorname{grad} \phi_h \cdot \boldsymbol{\tau} = 0$ for all $e \in L$ then $\phi_h(v) = \phi_h(v_1)$ for all $v \in V$, hence ϕ_h is constant and $\operatorname{grad} \phi_h = \mathbf{0}$.

Concerning the existence we will construct explicitly the solution. Since we are looking for a solution $\mathbf{Z}_h$ in N_h, we need to compute its degrees of freedom $\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau}$, for all $e \in E$. Initially we will distinguish two cases: either $\mathbf{J}_h = \mathbf{0}$ (loop fields), or $\mathbf{J}_h \neq \mathbf{0}$ with $\operatorname{div} \mathbf{J}_h = 0$ in Ω (and, moreover, imposing the condition $\mathbf{J}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$, that is somehow more restrictive than the necessary one, $\int_{(\partial\Omega)_r} \mathbf{J}_h \cdot \mathbf{n} = 0$ for $r = 1, \dots, p$). After these steps, it will be easy to solve the general problem.

4.1. Loop fields

By *loop fields* we mean irrotational vector fields $\mathbf{T}_0$ that cannot be expressed in Ω as the gradient of any single-valued scalar potential (therefore, there exists a loop in Ω such that the line integral of $\mathbf{T}_0$ on it is different from 0).

Let us now assume that we know a set of 1-cycles $\{\sigma_n\}_{n=1}^g \cup \{\widehat{\sigma}_n\}_{n=1}^g$ of $\partial\Omega$ such that: $\{[\sigma_n]\}_{n=1}^g \cup \{[\widehat{\sigma}_n]\}_{n=1}^g$ are a basis of the homology group of $\partial\Omega$; $\{[\sigma_n]\}_{n=1}^g$ (respectively, $\{[\widehat{\sigma}_n]\}_{n=1}^g$) is a basis of the homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$ (respectively, of the homology group $\mathcal{H}_1(\mathbb{R}^3 \setminus \Omega, \mathbb{Z})$). (For the construction of these 1-cycles, see Hiptmair and Ostrowski [6].)

We denote by $[\sigma_j]^+$ the homology class of σ_j in $\overline{\Omega}$ and by $[\widehat{\sigma}_j]^-$ the homology class of $\widehat{\sigma}_j$ in $\mathbb{R}^3 \setminus \Omega$. Here below we also introduce the 1-cycle $R^-\widehat{\sigma}_j$, a representative of $[\widehat{\sigma}_j]^-$ whose support is completely contained in $\mathbb{R}^3 \setminus \overline{\Omega}$. $R^-\widehat{\sigma}_j$ can be obtained by slightly "retracting" $\widehat{\sigma}_j$ inside $\mathbb{R}^3 \setminus \overline{\Omega}$.

The Biot–Savart law gives the magnetic field generated by a unitary density current concentrated along the cycle $R^-\widehat{\sigma}_j$ by means of the formula:

$$\widehat{\mathbf{H}}_j(\mathbf{x}) = \frac{1}{4\pi} \oint_{R^-\widehat{\sigma}_j} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}(\mathbf{y}) \quad , \quad \mathbf{x} \notin R^-\widehat{\sigma}_j\,.$$

Since the cycle $R^-\widehat{\sigma}_j$ is external to $\overline{\Omega}$, one has $\operatorname{curl} \widehat{\mathbf{H}}_j = \mathbf{0}$ in Ω. Moreover, on each cycle $\gamma \subset \overline{\Omega}$ that is linking the current passing in $R^-\widehat{\sigma}_j$ one finds $\oint_\gamma \widehat{\mathbf{H}}_j \cdot d\mathbf{s} \neq 0$, hence $\widehat{\mathbf{H}}_j$ is a loop field. (There are cycles γ with the required property: for instance, at least one of the generators σ_n of the first homology group of $\overline{\Omega}$.)

Clearly, the Nédélec interpolant $\Pi^{N_h}\widehat{\mathbf{H}}_j$ is a finite element loop field. For each $e \in E$, its degrees of freedom are given by

$$\widehat{q}_e = \frac{1}{4\pi}\int_e \left(\oint_{R^-\widehat{\sigma}_j} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}(\mathbf{y})\right) \cdot \boldsymbol{\tau}(\mathbf{x})$$

Consider now the spanning tree $\mathcal{S}_h = (V, L)$ and define the scalar function $\phi_h \in L_h$ in the all vertices of $\mathcal{T}_h$ as $\phi_h(v_1) = 0$ and

$$\phi_h(v_j) = \int_{C_{v_j}} \Pi^{N_h}\widehat{\mathbf{H}}_j \cdot \boldsymbol{\tau}$$

for all $j = 2, \ldots, n_v$.

Let us set $\mathbf{T}_{0,j} = \Pi^{N_h}\widehat{\mathbf{H}}_j - \operatorname{grad}\phi_h \in N_h$. Clearly it is a loop field and its degree of freedom on the oriented edge $e = [v_a, v_b]$ is given by

$$\begin{aligned}
\textstyle\int_e \mathbf{T}_{0,j} \cdot \boldsymbol{\tau} &= \textstyle\int_e \Pi^{N_h}\widehat{\mathbf{H}}_j \cdot \boldsymbol{\tau} - [\phi_h(v_b) - \phi_h(v_a)] \\
&= \textstyle\int_e \Pi^{N_h}\widehat{\mathbf{H}}_j \cdot \boldsymbol{\tau} - \int_{C_{v_b}} \Pi^{N_h}\widehat{\mathbf{H}}_j \cdot \boldsymbol{\tau} + \int_{C_{v_a}} \Pi^{N_h}\widehat{\mathbf{H}}_j \cdot \boldsymbol{\tau} \\
&= \textstyle\oint_{D_e} \Pi^{N_h}\widehat{\mathbf{H}}_j \cdot d\mathbf{s} = \oint_{D_e} \widehat{\mathbf{H}}_j \cdot d\mathbf{s}\,.
\end{aligned}$$

More explicitly, the degrees of freedom of $\mathbf{T}_{0,j}$ are given by

$$\int_e \mathbf{T}_{0,j} \cdot \boldsymbol{\tau} = \frac{1}{4\pi}\oint_{D_e} \left(\oint_{R^-\widehat{\sigma}_j} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}(\mathbf{y})\right) \cdot d\mathbf{s}(\mathbf{x}) \tag{6}$$

for all $e \in E$. In particular, if $e \in L$ it follows that D_e is trivial and therefore $\int_e \mathbf{T}_{0,j} \cdot \boldsymbol{\tau} = 0$.

Remark 1. *This explicit formula for the determination of the loop fields has been also obtained in Alonso Rodríguez et al. [1], following a different procedure based on the use of linking numbers. Let us recall that the linking number is an integer that, given two closed and disjoint curves in the three-dimensional space, represents the number of times that each curve winds around the other (see, e.g., Rolfsen [9, pp. 132–136]).*

The linking number is defined as a double line integral: given γ and γ', two 1-cycles in $\mathbb{R}^3$ with disjoint supports, the linking number of γ and γ' is

$$\begin{aligned}
\ell_{\mathcal{K}}(\gamma, \gamma') &:= \tfrac{1}{4\pi}\int_\gamma \int_{\gamma'} \tfrac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \cdot d\mathbf{s}(\mathbf{x}) \times d\mathbf{s}(\mathbf{y}) \\
&= \tfrac{1}{4\pi}\oint_\gamma \left(\oint_{\gamma'} \tfrac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}(\mathbf{y})\right) \cdot d\mathbf{s}(\mathbf{x})\,.
\end{aligned}$$

From this definition it is easy to check that the discrete loop field $\mathbf{T}_{0,j}$ is the unique solution of the following problem:

$$\begin{array}{ll}
\operatorname{curl}\mathbf{T}_{0,j} = \mathbf{0} & \text{in } \Omega \\
\oint_{\sigma_n} \mathbf{T}_{0,j} \cdot d\mathbf{s} = \ell_{\mathcal{K}}(\sigma_n, R^-\widehat{\sigma}_j) & \forall\, n = 1, \ldots, g \\
\int_e \mathbf{T}_{0,j} \cdot \boldsymbol{\tau} = 0 & \forall\, e \in L\,.
\end{array} \tag{7}$$

Moreover from (6) we have

$$\int_e \mathbf{T}_{0,j} \cdot \boldsymbol{\tau} = \ell_{\mathcal{K}}(D_e, R^{-}\widehat{\sigma}_j)$$

for all $e \in E$.

4.2. Source field

Given $\mathbf{J}$, the vector fields $\mathbf{H}_e$ satisfying $\operatorname{curl}\mathbf{H}_e = \mathbf{J}$ in Ω are often called *source fields* in the electromagnetic literature. It is well know that a source field can be obtained by means of the Biot–Savart formula. In the following we assume that $\mathbf{J}_h \in RT_h$, $\operatorname{div}\mathbf{J}_h = 0$ in Ω, and also the additional condition $\mathbf{J}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$. Defining

$$\mathbf{H}^{BS}(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega \mathbf{J}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\,,$$

one has $\operatorname{curl}\mathbf{H}^{BS} = \mathbf{J}_h$ in Ω (here the condition $\mathbf{J}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$ has played a role).

Theorem 1. *Assuming that* $\mathbf{J}_h \in RT_h$, $\operatorname{div}\mathbf{J}_h = 0$ *in* Ω *and* $\mathbf{J}_h \cdot \mathbf{n} = 0$ *on* $\partial\Omega$, *the solution* $\mathbf{H}_{e,h} \in N_h$ *of*

$$\begin{array}{ll} \operatorname{curl}\mathbf{H}_{e,h} = \mathbf{J}_h & \textit{in } \Omega \\ \oint_{\sigma_n} \mathbf{H}_{e,h} \cdot d\mathbf{s} = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s} & \forall\, n = 1,\ldots,g \\ \int_e \mathbf{H}_{e,h} \cdot \boldsymbol{\tau} = 0 & \forall\, e \in L \end{array} \tag{8}$$

satisfies

$$\int_e \mathbf{H}_{e,h} \cdot \boldsymbol{\tau} = \oint_{D_e} \mathbf{H}^{BS} \cdot d\mathbf{s} = \frac{1}{4\pi}\oint_{D_e}\left(\int_\Omega \mathbf{J}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\right) \cdot d\mathbf{s}(\mathbf{x})$$

for all $e \in E$.

Proof. Let us consider the Nédélec interpolant $\Pi^{N_h}\mathbf{H}^{BS}$. Notice that $\Pi^{N_h}\mathbf{H}^{BS}$ is well defined because $\mathbf{H}^{BS}$ belongs to $H^1(\Omega)$ and $\operatorname{curl}\mathbf{H}^{BS} = \mathbf{J}_h \in L^p(\Omega)$ for $p > 2$ (see, e.g., Amrouche et al. [2]). In fact $\mathbf{H}^{BS}$ is defined in $\mathbb{R}^3$ and $\mathbf{H}^{BS} = \operatorname{curl}\mathbf{A}^{BS}$ with $-\Delta\mathbf{A}^{BS} = \widetilde{\mathbf{J}}_h$ being $\widetilde{\mathbf{J}}_h$ the extension of $\mathbf{J}_h$ by $\mathbf{0}$ outside Ω. We have $\mathbf{A}^{BS} \in (H^2(\Omega))^3$, hence $\mathbf{H}^{BS} \in (H^1(\Omega))^3$. Clearly $\operatorname{curl}\Pi^{N_h}\mathbf{H}^{BS} = \Pi^{RT_h}(\operatorname{curl}\mathbf{H}^{BS}) = \mathbf{J}_h$ and $\oint_e \Pi^{N_h}\mathbf{H}^{BS} \cdot d\mathbf{s} = \oint_e \mathbf{H}^{BS} \cdot d\mathbf{s}$ for all $e \in E$, in particular, $\oint_{\sigma_n} \Pi^{N_h}\mathbf{H}^{BS} \cdot d\mathbf{s} = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s}$ for all $n = 1,\ldots,g$. However, for $e \in L$ we have $\int_e \Pi^{N_h}\mathbf{H}^{BS} \cdot \boldsymbol{\tau} \neq 0$. Therefore, to obtain the solution of (8) we need to correct the interpolant by a gradient. The procedure is similar to the one used for the loop fields.

Given the spanning tree L with root v_1, we define the scalar function $\psi_h \in L_h$ in all vertices of $\mathcal{T}_h$ as $\psi_h(v_1) = 0$ and

$$\psi_h(v_j) = \int_{C_{v_j}} \Pi^{N_h}\mathbf{H}^{BS} \cdot \boldsymbol{\tau}$$

for all $j = 2, \ldots, n_v$. Then for any $e = [v_a, v_b] \in E$ one finds

$$
\begin{aligned}
\int_e (\Pi^{N_h} \mathbf{H}^{BS} - \operatorname{grad} \psi_h) \cdot \boldsymbol{\tau} &= \int_e \Pi^{N_h} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - [\psi_h(v_b) - \psi_h(v_a)] \\
&= \int_e \Pi^{N_h} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \left[\int_{C_{v_b}} \Pi^{N_h} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \int_{C_{v_a}} \Pi^{N_h} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} \right] \\
&= \oint_{D_e} \Pi^{N_h} \mathbf{H}^{BS} \cdot d\mathbf{s} = \oint_{D_e} \mathbf{H}^{BS} \cdot d\mathbf{s} \,.
\end{aligned}
$$

In particular, if $e \in L$ it follows that D_e is trivial and thus

$$
\int_e (\Pi^{N_h} \mathbf{H}^{BS} - \operatorname{grad} \psi_h) \cdot \boldsymbol{\tau} = 0 \,,
$$

hence $\mathbf{H}_{e,h} = \Pi^{N_h} \mathbf{H}^{BS} - \operatorname{grad} \psi_h$. $\square$

4.3. Explicit formula for the solution of the general problem

Let us come back to the general problem: given $\mathbf{J}_h \in RT_h$ with $\operatorname{div} \mathbf{J}_h = 0$ in Ω and $\mathbf{J}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$, and $\kappa_n \in \mathbb{R}$, $n = 1, \ldots, g$, find $\mathbf{Z}_h \in N_h$ such that

$$
\begin{array}{ll}
\operatorname{curl} \mathbf{Z}_h = \mathbf{J}_h & \text{in } \Omega \\
\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \ldots, g \\
\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e \in L \,.
\end{array}
\tag{9}
$$

Let us introduce the matrix M with entries $m_{n,j} = \ell_{\mathcal{K}}(\sigma_n, R^- \widehat{\sigma}_j)$ and the vector $\boldsymbol{\beta}$ with components $\beta_n = \kappa_n - \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s}$. The $g \times g$ matrix M is non-singular (see Munkres [8, Sect. 71] and Seifert and Threlfall [10, point 47, p. 337]). Then consider the solution $\boldsymbol{\varrho}$ of the linear system $M\boldsymbol{\varrho} = \boldsymbol{\beta}$, and denote by ϱ_j its components. A straightforward check shows that the solution to (9) is given by

$$
\mathbf{Z}_h = \mathbf{H}_{e,h} + \sum_{j=1}^{g} \varrho_j \mathbf{T}_{0,j} \,. \tag{10}
$$

In fact, using (8) and (7) we see $\operatorname{curl} \mathbf{Z}_h = \mathbf{J}_h$ in Ω and $\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau} = 0$ for all $e \in L$. Moreover

$$
\begin{aligned}
\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} &= \oint_{\sigma_n} \mathbf{H}_{e,h} \cdot d\mathbf{s} + \textstyle\sum_{j=1}^{g} \varrho_j \oint_{\sigma_n} \mathbf{T}_{0,j} \cdot d\mathbf{s} \\
&= \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s} + \textstyle\sum_{j=1}^{g} \varrho_j \ell_{\mathcal{K}}(\sigma_n, R^- \widehat{\sigma}_j) \\
&= \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s} + (M\boldsymbol{\varrho})_n = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s} + \beta_n = \kappa_n \,.
\end{aligned}
$$

A direct application of the previous results gives

$$
\begin{aligned}
\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau} &= \int_e \mathbf{H}_{e,h} \cdot \boldsymbol{\tau} + \textstyle\sum_{j=1}^{g} \varrho_j \int_e \mathbf{T}_{0,j} \cdot \boldsymbol{\tau} \\
&= \frac{1}{4\pi} \oint_{D_e} \left(\int_\Omega \mathbf{J}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \, d\mathbf{y} \right) \cdot d\mathbf{s}(\mathbf{x}) \\
&\quad + \frac{1}{4\pi} \textstyle\sum_{j=1}^{g} \varrho_j \oint_{D_e} \left(\oint_{R^- \widehat{\sigma}_j} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \times d\mathbf{s}(\mathbf{y}) \right) \cdot d\mathbf{s}(\mathbf{x}) \,.
\end{aligned}
\tag{11}
$$

4.4. Elimination algorithm

Relations $(5)_1$ are in fact a linear system with n_f (number of faces in $\mathcal{T}_h$) equations and n_e (number of edges in $\mathcal{T}_h$) unknowns. For each face $f \in F$ we have

$$\int_f \mathbf{J}_h \cdot \boldsymbol{\nu} = \int_f \operatorname{curl} \mathbf{Z}_h \cdot \boldsymbol{\nu} = \oint_{\partial_2 f} \mathbf{Z}_h \cdot d\mathbf{s}\,. \tag{12}$$

so the equation corresponding to the face f has exactly three non-zero entries, corresponding to the three edges of $\partial_2 f$. In addition we have the equations

$$\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n \tag{13}$$

for $n = 1, \ldots, g$.

The complete system has also the equations $(5)_3$, that are simply saying that, for each edge in the spanning tree L, the associated degree of freedom is set to be equal to 0. Then, the other unknowns can be easily eliminated by the following algorithm, proposed, in a simplified form, by Webb and Forghani [11]:

Algorithm 2.

1. *$D = L$, $N = F$*
2. *while $D \neq E$*
   (a) *$n_D := card(D)$*
   (b) *for every $f \in N$*
      i. *if every edge of $\partial_2 f$ belongs to D, then $N = N \setminus \{f\}$*
      ii. *if exactly one edge e of $\partial_2 f$ does not belong to D*
         A. *compute $\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau}$ from (12)*
         B. *$D = D \cup \{e\}$*
   (c) *for $n = 1, \ldots, g$*
      i. *if exactly one edge e of σ_n does not belong to D*
         A. *compute $\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau}$ from (13)*
         B. *$D = D \cup \{e\}$*
   (d) *if $card(D) = n_D$ then STOP.*

If the spanning tree $\mathcal{S}_h$ is constructed in a suitable way, for instance a breadth-first spanning tree, the algorithm does start; in fact, in these cases, setting $D = L$, there exist faces in F with exactly one side not belonging to D. However the algorithm can terminate with $D \neq E$. A careful analysis of the termination properties of this algorithm can be found in Dłotko and Specogna [4]. They are strongly dependent on the choice of the spanning tree.

If the algorithm fails, the explicit formula for the computation of $\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau}$ for any $e \in E$ derived in the previous section can be used in the following way: if the elimination procedure stops without having determined all the degrees of freedom, pick one edge $e \in E \setminus D$, compute $\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau}$ using the explicit formula (11) and restart the algorithm.

The numerical experiments in Alonso Rodríguez et al. [1] suggest that, using a breadth-first spanning tree, the number of times this formula has to be used is very small.

5. Div problem

Also in this case the solution is not unique, thus we start adding some topological equations that filter out harmonic fields belonging to $\mathcal{H}(e;\Omega)$. Therefore, we consider the problem: given $f_h \in PC_h$, find $\mathbf{v}_h \in RT_h$ such that

$$\begin{array}{ll} \operatorname{div}\mathbf{v}_h = f_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r & \forall\, r = 1,\dots,p\,, \end{array} \tag{14}$$

where $c_1,\dots,c_p$ are real numbers.

The solution of this problem is not yet unique: in fact, it is unique up to a curl. To devise suitable additional conditions we need some preliminaries.

First of all, if $\mathbf{v}_h \in RT_h$ is such that $\operatorname{div}\mathbf{v}_h = 0$, $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all $r = 1,\dots,p$, then Helmholtz decomposition (see, e.g., Cantarella et al. [3]) says that $\mathbf{v}_h = \operatorname{curl}\mathbf{q}$, with $\operatorname{div}\mathbf{q} = 0$ in Ω and $\mathbf{q}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Hence $\mathbf{q} \in H^s(\Omega)$ with $s > 1/2$ (see Amrouche et al. [2]), and the Nédélec interpolant is well-defined (see Monk [7, Lemma 5.38]). Thus we have $\operatorname{curl}\Pi^{N_h}\mathbf{q} = \Pi^{RT_h}\operatorname{curl}\mathbf{q} = \mathbf{v}_h$, namely, $\mathbf{v}_h = \operatorname{curl}\mathbf{q}_h$ with $\mathbf{q}_h \in N_h$.

A way for obtaining a unique solution is therefore to consider the problem: find $\mathbf{v}_h \in RT_h$ such that

$$\begin{array}{ll} \operatorname{div}\mathbf{v}_h = f_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r & \forall\, r = 1,\dots,p \\ \int_\Omega \mathbf{v}_h \cdot \operatorname{curl}\mathbf{w}_h = 0 & \forall\, \mathbf{w}_h \in N_h\,. \end{array} \tag{15}$$

It is well-known that the dimension of the space $\operatorname{curl} N_h$ is equal to the number of the edges minus the dimension of the kernel in N_h of the curl operator. The kernel of the curl is given by gradients of finite elements in L_h and finite elements loop fields, hence its dimension is $n_v - 1 + g$; in conclusion, the dimension of $\operatorname{curl} N_h$ is $n_e - n_v + 1 - g$. Since by the Euler–Poincaré formula we have $n_v - n_e + n_f - n_t = 1 - g + p$, it follows that the dimension of $\operatorname{curl} N_h$ can be rewritten as $n_f - n_t - p$.

A first result is therefore that system (15) is a square linear system of n_f equations and unknowns. For arriving to an algorithm which can be implemented in a better way, we can think to add $n_f - n_t - p$ equations different from $(15)_3$.

To do that, let us consider the following dual graph: the dual vertices are $W = T \cup \Sigma$, where the elements of T are the tetrahedra of the mesh and the elements of Σ are the $p+1$ connected components of $\partial\Omega$; the set of dual edges is F, the set of the faces of the mesh.

An internal face connects two tetrahedra, while a boundary face connects a tetrahedron and a connected component of $\partial\Omega$. So the dual graph is given by $\mathcal{G}_h = (W, F)$. Since Ω is connected, $\mathcal{G}_h$ is a connected graph.

The number of dual vertices is equal to $n_t + p + 1$, hence a spanning tree $\mathcal{M}_h = (W, M)$ of $\mathcal{G}_h$ has $n_t + p$ dual edges (and consequently its cotree has

$n_f - n_t - p$ dual edges). Therefore the linear system

$$\begin{array}{ll} \operatorname{div} \mathbf{v}_h = f_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r & \forall\, r = 1, \ldots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = 0 & \forall\, f \notin M \end{array} \tag{16}$$

is again a square linear system of n_f equations and unknowns.

Now we show that it has a unique solution. The procedure is constructive, similar in some sense to the elimination procedure used for the grad problem but now going up, along the dual spanning tree, starting from the leaves. (Let us recall that the leaves of a spanning tree $\mathcal{M}_h = (W, M)$ are the vertices of W that have only one edge of M incident to them.)

Remembering that we have imposed $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ if $f \notin M$, we can reduce the problem to the faces $f \in M$, hence to the spanning tree $\mathcal{M}_h = (W, M)$.

Given $w \in W$ (a tetrahedron or a connected component of $\partial\Omega$), let us set $F(w) := \{f \in F \,:\, f \subset w\}$; the elements of this set are faces of the primal mesh, therefore dual edges in the dual mesh. The leaves of the spanning tree $\mathcal{M}_h = (W, M)$ are the vertices $w \in W$ such that $F(w) \cap M$ reduces to exactly one dual edge (namely, to a face).

If w is a leave of $\mathcal{M}_h$ and $f(w) \in F$ is the unique dual edge (face) in M incident to w, we can easily compute the degree of freedom corresponding to $f(w)$, as we know that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all $f \notin M$. In fact we have

$$\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu} = \begin{cases} \int_{\partial w} \mathbf{v}_h \cdot \boldsymbol{\nu} = \int_w f_h & \text{if } w \in T \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r & \text{if } w = (\partial\Omega)_r\,,\ r = 1, \ldots, p \\ \int_{(\partial\Omega)_0} \mathbf{v}_h \cdot \boldsymbol{\nu} = \int_\Omega f_h - \sum_{r=1}^p c_r & \text{if } w = (\partial\Omega)_0\,, \end{cases}$$

having used the divergence theorem in the first and third lines.

Hence it is clear that if $\mathbf{v}_h \in RT_h$ is such that $\operatorname{div} \mathbf{v}_h = 0$, $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all $r = 1, \ldots, p$, and $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all $f \notin M$ then $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all the faces $f(w)$ associated to the leaves $w \in \mathcal{M}_h$.

We can iterate this argument: if we remove from the spanning tree a leave and its corresponding incident edge, the remaining graph is still a tree. The edges of this new tree are the faces where the degree of freedom is still unknown. Repeating the previous procedure, we can easily compute the degrees of freedom corresponding to the faces incident to the leaves of this new tree, finding that they are vanishing. After a finite number of steps the remaining tree reduces to just on vertex, and we have obtained $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all $f \in F$. Therefore, since (16) is a square linear system, this proves that it has a unique solution.

We can also furnish an explicit way for computing the values of the degrees of freedom. In fact, at a step of the previous procedure let us call N the set of the vertices w and G the set of edges f of the reduced dual graph. Then G is the set of faces where the degree of freedom $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}$ is still unknown. If w is a leave of (N, G), there exists exactly one face $f(w) \in F(w)$ belonging to G.

Then

$$\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu} = A_w - \sum_{f \in F(w) \cap (F \setminus G)} \int_f \mathbf{v}_h \cdot \boldsymbol{\nu} \tag{17}$$

where

$$A_w = \begin{cases} \int_w f_h & \text{if } w \in T \\ c_r & \text{if } w = (\partial\Omega)_r \,,\ r = 1, \ldots, p \\ \int_\Omega f_h - \sum_{r=1}^{p} c_r & \text{if } w = (\partial\Omega)_0 \,. \end{cases}$$

This can be rephrased as an elimination algorithm for the computation of $\mathbf{v}_h$.

Algorithm 3.

1. $G = M$, $N = W$
2. *while* $G \neq \emptyset$
   (a) *pick a leave* w *of the tree* (N, G)
   (b) *compute* $\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu}$ *from (17)*
   (c) *update* G*:* $G = G \setminus \{f(w)\}$
   (d) *update* N*:* $N = N \setminus \{w\}$

Notice that at any step of the algorithm (N, G) is a tree, so while $G \neq \emptyset$ a leave w of the tree (N, G) always exists.

Remark 2. *It is worth noting that the set of vector functions* $\{\mathbf{W}_{0,s}\}_{s=1}^{p}$*, solutions to problem (16) with* $f_h = 0$ *and* $c_r = \delta_{r,s}$*,* $r = 1, \ldots, p$*, is a basis of the second de Rham cohomology group of* Ω*.*

6. Stability

This section is devoted to investigate how we can construct stable finite element potentials, namely, potentials whose natural norms can be estimated in terms of the norms of the data, uniformly with respect to the mesh size h.

Before starting, let us remark that, very often, the construction of finite element potentials is a preliminary step in the procedure aiming at solving a partial differential equation. In this respect, the solution u_h will be written as $u_h = U_h + W_h$, W_h being the finite element potential and U_h the solution of an auxiliary problem in which W_h contributes at the right hand side. In this situation, what is interesting is the stability of the solution u_h, and not that of W_h and U_h; in many cases, an unstable W_h produces an unstable U_h but a stable u_h (see, e.g., the solution of the magnetostatic problem in Alonso Rodríguez et al. [1], that satisfies

$$\|\mathbf{H}_h\|_{(L^2(\Omega))^3} \leq C_* \|\operatorname{curl} \mathbf{H}_h\|_{(L^2(\Omega))^3} = C_* \|\mathbf{J}_h\|_{(L^2(\Omega))^3}$$

for $C_* > 0$ independent of h).

Let us come now to the construction of stable potentials. As before, we consider the three cases: gradient, curl, divergence.

6.1. Grad problem

Having constructed the solution Ψ_h of problem (1), take the solution $c_{0,h} \in \mathbb{R}$ of

$$\int_\Omega c_{0,h} = \int_\Omega \Psi_h \,,$$

namely, $c_{0,h} = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \Psi_h$ (it is the $(L^2(\Omega))^3$-projection of Ψ_h over the space of constants).

Then define $\widehat{\Psi}_h = \Psi_h - c_{0,h}$. Since $\widehat{\Psi}_h$ satisfies $\int_\Omega \widehat{\Psi}_h = 0$, from Poincaré inequality we find

$$\|\widehat{\Psi}_h\|_{L^2(\Omega)} \leq K_G \|\operatorname{grad} \widehat{\Psi}_h\|_{(L^2(\Omega))^3} = C_G \|\mathbf{F}_h\|_{(L^2(\Omega))^3}$$

for a constant $K_G > 0$ independent of h.

6.2. Curl problem

Given the solution $\mathbf{Z}_h$ of problem (5) with $\kappa_n = 0$ for all $n = 1, \ldots, g$, let $\varphi_h \in L_h$, $d_h \in \mathbb{R}$ be the solution of the Neumann problem (in the saddle-point formulation)

$$\begin{aligned} &\textstyle\int_\Omega \operatorname{grad} \varphi_h \cdot \operatorname{grad} \eta_h + d_h \eta_h(v_1) = \int_\Omega \mathbf{Z}_h \cdot \operatorname{grad} \eta_h \quad \forall\, \eta_h \in L_h \\ &\varphi_h(v_1) = 0\,. \end{aligned} \tag{18}$$

Note at once that a solution to (18) satisfies $d_h = 0$. In fact, by choosing the piecewise-linear function η_h equal to the constant d_h, it follows that $\int_\Omega d_h^2 = 0$, hence $d_h = 0$. Therefore, a solution φ_h to (18) satisfies $\int_\Omega \operatorname{grad} \varphi_h \cdot \operatorname{grad} \eta_h = \int_\Omega \mathbf{Z}_h \cdot \operatorname{grad} \eta_h$ for all $\eta_h \in L_h$, hence $\operatorname{grad} \varphi_h$ is the $(L^2(\Omega))^3$-projection of $\mathbf{Z}_h$ over $\operatorname{grad} L_h$.

The unique solvability (18) is also easily verified: for $\mathbf{Z}_h = \mathbf{0}$, taking $\eta_h = \varphi_h$ gives at once $\operatorname{grad} \varphi_h = \mathbf{0}$, therefore $\varphi_h = \text{const}$. Then the condition $\varphi_h(v_1) = 0$ gives $\varphi_h = 0$.

Define now $\widehat{\mathbf{Z}}_h = \mathbf{Z}_h - \operatorname{grad} \varphi_h$. As we will see here below, this is a stable vector potential. Note that it satisfies $\widehat{\mathbf{Z}}_h \perp \operatorname{grad} L_h$ and $\oint_{\sigma_n} \widehat{\mathbf{Z}}_h \cdot d\mathbf{s} = 0$ for each $n = 1, \ldots, g$.

Proposition 1. *Let us assume that the family of triangulations $\mathcal{T}_h$ is regular. Then the vector potential $\widehat{\mathbf{Z}}_h$ satisfies*

$$\|\widehat{\mathbf{Z}}_h\|_{(L^2(\Omega))^3} \leq K_C \|\operatorname{curl} \widehat{\mathbf{Z}}_h\|_{(L^2(\Omega))^3} = K_C \|\mathbf{J}_h\|_{(L^2(\Omega))^3}\,,$$

for a constant $K_C > 0$ independent of h.

Proof. Clearly, it is enough to find a vector function $\mathbf{W}^{(h)}$ such that

$$\begin{aligned} &\|\mathbf{W}^{(h)}\|_{(L^2(\Omega))^3} \leq C_1 \|\operatorname{curl} \widehat{\mathbf{Z}}_h\|_{(L^2(\Omega))^3} \\ &\|\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}\|_{(L^2(\Omega))^3} \leq C_1 \|\operatorname{curl} \widehat{\mathbf{Z}}_h\|_{(L^2(\Omega))^3}\,, \end{aligned}$$

for a constant $C_1 > 0$ independent of h.

By adapting the proof of Theor. 4.7 in Hiptmair [5], we see that this vector function is the solution to

$$\begin{array}{ll} \operatorname{curl} \mathbf{W}^{(h)} = \operatorname{curl} \widehat{\mathbf{Z}}_h & \text{in } \Omega \\ \operatorname{div} \mathbf{W}^{(h)} = 0 & \text{in } \Omega \\ \mathbf{W}^{(h)} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{W}^{(h)} \cdot d\mathbf{s} = 0 & \forall\, n = 1, \dots, g\,. \end{array}$$

It satisfies $\|\mathbf{W}^{(h)}\|_{(H^s(\Omega))^3} \leq C_2 \|\operatorname{curl} \widehat{\mathbf{Z}}_h\|_{(L^2(\Omega))^3}$, for a suitable $s > \frac{1}{2}$. Moreover, one easily sees that $\mathbf{Q}_h = \widehat{\mathbf{Z}}_h - \Pi^{N_h} \mathbf{W}^{(h)}$ satisfies $\operatorname{curl} \mathbf{Q}_h = \mathbf{0}$ in Ω and $\oint_{\sigma_n} \mathbf{Q}_h \cdot d\mathbf{s} = 0$ for all $n = 1, \dots, g$, hence it belongs to $\operatorname{grad} L_h$, as proved in Section 3 (see also Monk [7, Lemma 5.28]). This permits to conclude that $\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}$ is orthogonal to $\widehat{\mathbf{Z}}_h - \Pi^{N_h} \mathbf{W}^{(h)}$ and therefore

$$\begin{aligned} \|\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}\|^2_{(L^2(\Omega))^3} &= \int_\Omega (\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}) \cdot (\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}) \\ &= \int_\Omega (\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}) \cdot (\Pi^{N_h} \mathbf{W}^{(h)} - \mathbf{W}^{(h)}) \\ &\leq \|\widehat{\mathbf{Z}}_h - \mathbf{W}^{(h)}\|_{(L^2(\Omega))^3} \|\Pi^{N_h} \mathbf{W}^{(h)} - \mathbf{W}^{(h)}\|_{(L^2(\Omega))^3}\,; \end{aligned}$$

thus the interpolation error estimate concludes the proof. □

6.3. Div problem

Given the solution $\mathbf{v}_h$ of problem (16) with $c_r = 0$ for all $r = 1, \dots, p$, let $\mathbf{q}_h \in N_h$, $\psi_h \in L_h$, $m_h \in \mathbb{R}$, $\beta_j \in \mathbb{R}$, $j = 1, \dots, g$, be the solution of

$$\begin{array}{ll} \int_\Omega \operatorname{curl} \mathbf{q}_h \cdot \operatorname{curl} \mathbf{p}_h + \int_\Omega \operatorname{grad} \psi_h \cdot \mathbf{p}_h & \\ \quad + \sum_{j=1}^g \beta_j \oint_{\sigma_j} \mathbf{p}_h \cdot d\mathbf{s} = \int_\Omega \mathbf{v}_h \cdot \operatorname{curl} \mathbf{p}_h & \forall\, \mathbf{p}_h \in N_h \\ \int_\Omega \mathbf{q}_h \cdot \operatorname{grad} \eta_h + m_h\, \eta_h(v_1) = 0 & \forall\, \eta_h \in L_h \\ \psi_h(v_1) = 0 & \\ \oint_{\sigma_n} \mathbf{q}_h \cdot d\mathbf{s} = 0 & \forall\, n = 1, \dots, g\,. \end{array} \tag{19}$$

Taking the piecewise-linear function η_h equal to the constant m_h, it is easily checked that a solution to (19) satisfies $m_h = 0$; moreover, choosing $\mathbf{p}_h = \operatorname{grad} \psi_h$ gives $\operatorname{grad} \psi_h = \mathbf{0}$ in Ω, thus $\psi_h = 0$. Finally, the choice $\mathbf{p}_h = \mathbf{T}_{0,n}$, the discrete loop field associated to σ_n, furnishes $\beta_n = 0$ for each $n = 1, \dots, g$. In conclusion, $\operatorname{curl} \mathbf{q}_h$ is the $(L^2(\Omega))^3$-projection of $\mathbf{v}_h$ over $\operatorname{curl} N_h$, $\mathbf{q}_h \perp \operatorname{grad} L_h$ and $\oint_{\sigma_n} \mathbf{q}_h \cdot d\mathbf{s} = 0$ for all $n = 1, \dots, g$.

The unique solvability (19) is also easily verified: when $\mathbf{v}_h = \mathbf{0}$, it follows $\operatorname{curl} \mathbf{q}_h = \mathbf{0}$, and therefore, as proved in Section 3 (see also Monk [7, Lemma 5.28]), $\mathbf{q}_h \in \operatorname{grad} L_h$ and finally, by the orthogonality property, $\mathbf{q}_h = \mathbf{0}$.

Define now $\widehat{\mathbf{v}}_h = \mathbf{v}_h - \operatorname{curl} \mathbf{q}_h$. As we will see here below, this is a stable potential. Note that it satisfies $\widehat{\mathbf{v}}_h \perp \operatorname{curl} N_h$ and $\int_{(\partial\Omega)_r} \widehat{\mathbf{v}}_h \cdot \boldsymbol{\nu} = 0$ for all $r = 1, \dots, p$.

Proposition 2. *Let us assume that the family of triangulations $\mathcal{T}_h$ is regular. Then the potential $\widehat{\mathbf{v}}_h$ satisfies*

$$\|\widehat{\mathbf{v}}_h\|_{(L^2(\Omega))^3} \leq K_D \| \operatorname{div} \widehat{\mathbf{v}}_h \|_{L^2(\Omega)} = K_D \|f_h\|_{L^2(\Omega)} \,,$$

for a constant $K_D > 0$ independent of h.

Proof. Clearly, it is enough to find a vector function $\mathbf{V}^{(h)}$ such that

$$\begin{aligned} &\|\mathbf{V}^{(h)}\|_{(L^2(\Omega))^3} \leq C_3 \| \operatorname{div} \widehat{\mathbf{v}}_h \|_{L^2(\Omega)} \\ &\|\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}\|_{(L^2(\Omega))^3} \leq C_3 \| \operatorname{div} \widehat{\mathbf{v}}_h \|_{L^2(\Omega)} \,, \end{aligned}$$

for a constant $C_3 > 0$ independent of h.

Adapting also in this case the proof of Theor. 4.7 in Hiptmair [5], we see that this vector function is the solution to

$$\begin{array}{ll} \operatorname{curl} \mathbf{V}^{(h)} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div} \mathbf{V}^{(h)} = \operatorname{div} \widehat{\mathbf{v}}_h & \text{in } \Omega \\ \mathbf{V}^{(h)} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{V}^{(h)} \cdot \boldsymbol{\nu} = 0 & \forall\ r = 1, \ldots, p \,. \end{array}$$

It satisfies $\|\mathbf{V}^{(h)}\|_{(H^s(\Omega))^3} \leq C_4 \| \operatorname{div} \widehat{\mathbf{v}}_h \|_{L^2(\Omega)}$, for a suitable $s > \frac{1}{2}$. Moreover, let us denote by Π^{RT_h} the Raviart–Thomas interpolation operator; then one easily sees that $\mathbf{R}_h = \widehat{\mathbf{v}}_h - \Pi^{RT_h} \mathbf{V}^{(h)}$ satisfies $\operatorname{div} \mathbf{R}_h = \mathbf{0}$ in Ω and $\int_{(\partial\Omega)_r} \mathbf{R}_h \cdot \boldsymbol{\nu} = 0$ for all $r = 1, \ldots, p$, hence it belongs to $\operatorname{curl} N_h$ (see, e.g., Alonso Rodríguez et al. [1]). This permits to conclude that $\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}$ is orthogonal to $\widehat{\mathbf{v}}_h - \Pi^{RT_h} \mathbf{V}^{(h)}$ and therefore

$$\begin{aligned} \|\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}\|^2_{(L^2(\Omega))^3} &= \textstyle\int_\Omega (\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}) \cdot (\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}) \\ &= \textstyle\int_\Omega (\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}) \cdot (\Pi^{RT_h} \mathbf{V}^{(h)} - \mathbf{V}^{(h)}) \\ &\leq \|\widehat{\mathbf{v}}_h - \mathbf{V}^{(h)}\|_{(L^2(\Omega))^3} \|\Pi^{RT_h} \mathbf{V}^{(h)} - \mathbf{V}^{(h)}\|_{(L^2(\Omega))^3} \,; \end{aligned}$$

thus the interpolation error estimate concludes the proof. □

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