THE TANGENCY OF A C^1 SMOOTH SUBMANIFOLD WITH RESPECT TO A NON-INVOLUTIVE C^1 DISTRIBUTION HAS NO SUPERDENSITY POINTS

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Abstract. Consider a C^1 smooth n-dimensional submanifold M of $\mathbb{R}^{n+m}$ and a C^1 distribution $\mathcal{D}$ of rank n on $\mathbb{R}^{n+m}$. Let $\tau(M,D)$ denote the set of all points $z \in M$ such that $D(z)$ is tangent to M at z. We prove that if $\mathcal{D}$ is not involutive at every point of M then $\tau(M,D)$ has no superdensity points.

1. Introduction

Let us consider a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^{n+m}$, that is a map $\mathcal{D}$ assigning an n-dimensional vector subspace $\mathcal{D}(z)$ of $\mathbb{R}^{n+m}$ to each point $z \in U$ and satisfying the following property: If $z \in U$ then there exists a family $\{X_1^{(z)}, \dots, X_n^{(z)}\}$ of vector fields of class C^1 in a neighbourhood $V^{(z)} \subset U$ of z such that $\{X_1^{(z)}(z'), \dots, X_n^{(z)}(z')\}$ is a basis of $\mathcal{D}(z')$ for all $z' \in V^{(z)}$. Recall that the distribution $\mathcal{D}$ is said to be involutive at $z \in U$ if $[X_i^{(z)}, X_j^{(z)}](z) \in \mathcal{D}(z)$ for all $i, j \in \{1, \dots, n\}$.

Also recall that the distribution $\mathcal{D}$ can be described through the formalism of differential forms, compare [5, Section 3.2] and [13, Section 2.11]. According to this approach, if $z \in U$ then there exists a family of m linearly independent differential 1-forms of class C^1 in $V^{(z)}$, that is

$$\theta^{(j)} = \sum_{i=1}^{n+m} a_i^{(j)} dz_i \qquad (j = 1, \dots, m)$$

with $a_i^{(j)} \in C^1(V^{(z)})$, such that (for $z' \in V^{(z)}$)

$$\mathcal{D}(z') = \ker(\theta_{z'}^{(1)}) \cap \dots \cap \ker(\theta_{z'}^{(m)}) = \left[\operatorname{span}\{a^{(1)}(z'), \dots, a^{(m)}(z')\}\right]^{\perp}$$

where

$$a^{(j)} := (a_1^{(j)}, \cdots, a_{n+m}^{(j)})^t.$$

Observe that, for all $j = 1, \dots, m$ and $i = 1, \dots, n$, the function

$$f_{ji}^{(z)} : V^{(z)} \to \mathbb{R}, \quad z' \mapsto \theta_{z'}^{(j)}(X_i^{(z)}(z'))$$

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is identically zero (in that $X_i^{(z)}(z') \in \mathcal{D}(z')$). Hence, by a well-known formula (compare [5, Theorem 2.3] or [13, Proposition 2.6.6]) one has

$$\begin{aligned}(d\theta^{(j)})_{z'}(X_i^{(z)}(z'), X_k^{(z)}(z')) &= X_i^{(z)}(z')(\theta^{(j)}(X_k^{(z)})) - X_k^{(z)}(z')(\theta^{(j)}(X_i^{(z)})) \\ &\quad - \theta^{(j)}_{z'}([X_i^{(z)}, X_k^{(z)}](z')) \\ &= X_i^{(z)}(z')(f_{jk}^{(z)}) - X_k^{(z)}(z')(f_{ji}^{(z)}) - \theta^{(j)}_{z'}([X_i^{(z)}, X_k^{(z)}](z')) \\ &= -\theta^{(j)}_{z'}([X_i^{(z)}, X_k^{(z)}](z'))\end{aligned}$$

whenever $z' \in V^{(z)}$, for all $j = 1, \ldots, m$ and $i, k = 1, \ldots, n$. It follows that $\mathcal{D}$ is involutive at $z' \in V^{(z)}$ if and only if

$$(d\theta^{(j)})_{z'}|_{\mathcal{D}(z')\times\mathcal{D}(z')} = 0 \qquad (j = 1, \ldots, m)$$

for all $j = 1, \ldots, m$.

If $\mathcal{N}_{\mathcal{D}}$ denotes the open set of all points $z \in U$ such that $\mathcal{D}$ is not involutive at z, then the classical Frobenius theorem establishes the following fact, compare [13, Section 2.11]: The open set $\mathcal{N}_{\mathcal{D}}$ is empty if and only if for all $z_0 \in U$ there exists an n-dimensional regular embedded C^1 submanifold M of U such that $z_0 \in M$ and $\tau(M, \mathcal{D}) = M$, where $\tau(M, \mathcal{D})$ stands for the tangency set

$$\{z \in M \, : \, T_z M = \mathcal{D}(z)\}$$

of M with respect to $\mathcal{D}$ (see [3]).

In general, if no restriction about involutivity is assumed on $\mathcal{D}$, the Frobenius theorem cannot be applied and it is natural to look at the number

$$\rho_{C^\sigma}(\mathcal{D}) := \sup\{\dim_H(\tau(M, \mathcal{D})) \, : \, M \in \mathcal{M}^n_{C^\sigma}(U)\}$$

where $\dim_H$ denotes the Hausdorff dimension and $\mathcal{M}^n_{C^\sigma}(U)$ is the family of n-dimensional submanifolds of U of class C^σ. We note that $\rho_{C^\sigma}(\mathcal{D})$ increases as the regularity σ "decreases", e.g., $\rho_{C^2}(\mathcal{D}) \le \rho_{C^1}(\mathcal{D})$, but such an obvious observation is too vague to be useful. Far more significant is the upper bound for $\rho_{C^2}(\mathcal{D})$ provided in [3, Theorem 1.3] (see also [8] for an alternative proof, based on the implicit function theorem). An interesting example is provided by the horizontal subbundle $H\mathbb{H}^k$ of the tangent bundle $T\mathbb{H}^k$ to the Heisenberg group $\mathbb{H}^k$, that is the distribution of rank $2k$ on $\mathbb{R}^{2k+1}$ defined as

$$H\mathbb{H}^k(z) := \left[\text{span}\left\{-\sum_{i=1}^{k} z_{k+i}\, e_i + \sum_{i=k+1}^{2k} z_{i-k}\, e_i - e_{2k+1}\right\}\right]^{\perp} \qquad (z \in \mathbb{R}^{2k+1})$$

where $e_1, \ldots, e_{2k+1}$ denotes the standard basis of $\mathbb{R}^{2k+1}$. In this case one can assume $V^{(z)} = \mathbb{R}^{2k+1}$ for all $z \in \mathbb{R}^{2k+1}$ and

$$\theta^{(1)}_z := -\sum_{i=1}^{k} z_{k+i}\, dz_i + \sum_{i=k+1}^{2k} z_{i-k}\, dz_i - dz_{2k+1} \qquad (z \in \mathbb{R}^{2k+1})$$

so that

$$(d\theta^{(1)})_z = 2\sum_{i=1}^{k} dz_i \wedge dz_{k+i} \qquad (z \in \mathbb{R}^{2k+1}).$$

Since

$$e_1 - z_{k+1}e_{2k+1},\, e_{k+1} + z_1 e_{2k+1} \in H\mathbb{H}^k(z)$$

and

$$(d\theta^{(1)})_z(e_1 - z_{k+1}e_{2k+1}, e_{k+1} + z_1 e_{2k+1}) = 2dz_1 \wedge dz_{k+1}(e_1, e_{k+1}) = 2$$

for all $z \in \mathbb{R}^{2k+1}$, one has $\mathcal{N}_{H\mathbb{H}^k} = \mathbb{R}^{2k+1}$. The result [3, Theorem 1.3] is used in [3, Example 6.5] to prove the following estimate for the Hausdorff dimension of the characteristic set $C(M) = \tau(M, H\mathbb{H}^k)$ of a codimension 1 submanifold M of class C^2 in $\mathbb{H}^k$

$$\dim_H(C(M)) = \dim_H(\tau(M, H\mathbb{H}^k)) \le k, \text{ i.e., } \rho_{C^2}(H\mathbb{H}^k) \le k$$

which is actually an earlier result by Balogh [2, Theorem 1.2]. Another interesting application of [3, Theorem 1.3] is [3, Theorem 4.5], which generalizes the Derridj's theorem [9, Theorem 1] about the size of tangencies in the context of Hörmander distributions. Further related work on stratified groups can be found in [11, 12].

The size of the tangency set $\tau(M, \mathcal{D})$ for $M \in \mathcal{M}^n_{C^1}(U)$ has been investigated in [3] in the case when $z = (z_1, \dots, z_{n+m}) \mapsto \mathcal{D}(z)$ is translation-invariant with respect to the last m variables, i.e., $\mathcal{D}(z)$ does not depend on $(z_{n+1}, \dots, z_{n+m})$. In this special situation, including $H\mathbb{H}^k$, it turns out that the following facts hold [3, Proposition 8.2]:

- $\rho_{C^{1,1}}(\mathcal{D}) = n$;
- For all $z_0 \in U$ (in particular for all $z_0 \in \mathcal{N}_{\mathcal{D}}$) there exists $M_0 \in \cap_{\alpha \in (0,1)} \mathcal{M}^n_{C^{1,\alpha}}(U)$ such that $z_0 \in M_0$ and $\mathcal{H}^n(\tau(M_0, \mathcal{D})) > 0$, hence $M_0 \in \mathcal{M}^n_{C^1}(U)$ and $\rho_{C^1}(\mathcal{D}) = \dim_H(\tau(M_0, \mathcal{D})) = n$.

In particular, one has $\rho_{C^{1,1}}(H\mathbb{H}^k) = 2k$ and for all $z_0 \in \mathbb{R}^{2k+1}$ there exists a surface $S_0 \in \cap_{\alpha \in (0,1)} \mathcal{M}^{2k}_{C^{1,\alpha}}(\mathbb{R}^{2k+1})$ such that $z_0 \in S_0$ and $\mathcal{H}^{2k}(\tau(S_0, H\mathbb{H}^k)) > 0$.

So it can happen to bump into tangency sets $\tau(M, \mathcal{D})$ of positive $\mathcal{H}^n$ measure (even with $M \subset \mathcal{N}_{\mathcal{D}}$) and indeed we are inclined to think that the second fact above can be proved for any C^1 distribution of rank n, at least in the following weaker form: For every C^1 distribution $\mathcal{D}$ of rank n such that $\mathcal{N}_{\mathcal{D}} \neq \emptyset$ and for all $z_0 \in \mathcal{N}_{\mathcal{D}}$, there exists $M_0 \in \mathcal{M}^n_{C^1}(\mathcal{N}_{\mathcal{D}})$ such that $z_0 \in M_0$ and $\mathcal{H}^n(\tau(M_0, \mathcal{D})) > 0$.

In this paper we shall prove that a tangency set can never be too dense at the points of $\mathcal{N}_{\mathcal{D}}$, even when it has positive measure. To understand exactly what this means, we recall first of all that $z_0 \in M$, with $M \in \mathcal{M}^n_{C^1}(U)$, is said to be a superdensity point of $H \subset M$ if

$$\mathcal{H}^n(B_M(z_0, r) \setminus H) = o(r^{n+1}) \qquad (\text{as } r \to 0+)$$

where $B_M(z_0, r) \subset M$ denotes the metric ball of radius r centered at z_0, compare Section 2 below. We are finally able to state precisely our main result and its corollaries (where it is assumed, without loss of generality, that the differential 1-forms $\theta^{(j)}$ describing $\mathcal{D}$ are defined in U):

Theorem 1.1. *Let M be an n-dimensional regularly embedded C^1 submanifold of U and assume that one between conditions (I) and (II) below is satisfied:*

(I) *Let $z_0 \in M$ be a superdensity point of $\tau(M, \mathcal{D})$, i.e.,*
$$\mathcal{H}^n(B_M(z_0, r) \setminus \tau(M, \mathcal{D})) = o(r^{n+1}) \qquad (\text{as } r \to 0+);$$
(II) *Let $z_0 \in M$ be an ordinary point of density of $\tau(M, \mathcal{D})$, i.e.,*
$$\mathcal{H}^n(B_M(z_0, r) \setminus \tau(M, \mathcal{D})) = o(r^n) \qquad (\text{as } r \to 0+)$$
and let M be locally of class C^2 at z_0.

Then $z_0 \in \tau(M, \mathcal{D})$ and $\mathcal{D}$ is involutive at z_0, i.e.,
$$(d\theta^{(j)})_{z_0}|_{T_{z_0}M \times T_{z_0}M} = (d\theta^{(j)})_{z_0}|_{\mathcal{D}(z_0) \times \mathcal{D}(z_0)} = 0$$
for all $j = 1, \dots, m$.

The following corollary follows trivially from Theorem 1.1.

Corollary 1.1. *Let M be an n-dimensional regularly embedded C^1 submanifold of U and let z_0 be a point of density of $\tau(M, \mathcal{D})$ (hence $z_0 \in \tau(M, \mathcal{D})$, by Remark 5.1). Moreover assume $\mathcal{D}$ is not involutive at z_0, namely there is $j_0 \in \{1, \dots, m\}$ such that*
$$(d\theta^{(j_0)})_{z_0}|_{T_{z_0}M \times T_{z_0}M} = (d\theta^{(j_0)})_{z_0}|_{\mathcal{D}(z_0) \times \mathcal{D}(z_0)} \neq 0.$$
Then

(1) *z_0 is not a superdensity point of $\tau(M, \mathcal{D})$;*
(2) *The submanifold M is not locally of class C^2 at z_0.*

Observe that the first statement of Corollary 1.1 can be rephrased as follows.

Theorem 1.2. *If M is an n-dimensional regularly embedded C^1 submanifold of U, then the set $\tau(M \cap \mathcal{N}_\mathcal{D}, \mathcal{D})$ has no superdensity points. In particular, if $M \subset \mathcal{N}_\mathcal{D}$ then $\tau(M, \mathcal{D})$ has no superdensity points.*

In particular, as a consequence of Theorem 1.2 and the remarks above, it turns out that $\tau(M, H\mathbb{H}^k)$ has no superdensity points, whatever $M \in \mathcal{M}^{2k}_{C^1}(\mathbb{R}^{2k+1})$, even if one has $\dim_H \tau(M, H\mathbb{H}^k) = 2k$.

Finally, let us spend a few words on how the paper is organized. In Section 2 we introduce the notation. In Section 3 we have collected some expected results, including the density-preserving property of C^1 embeddings and the description of the tangent space to an arbitrary set at a point of density. Section 4 is actually the main one, where we prove the hardest part of Theorem 1.1 in the case when M is the image of an injective immersion of class C^1. Eventually, Section 5 provides the proof of Theorem 1.1 in its whole generality.

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2. General notation

The standard basis of $\mathbb{R}^{n+m}$ is denoted by $e_1, \dots, e_{n+m}$. We will often have to deal with maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and with their graphs. Due to this fact and other technical reasons, we distinguish the notation relative to the first n coordinates from the notation relative to the last m coordinates: they are denoted by $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, respectively. We may write $\mathbb{R}^n_x$ in place of $\mathbb{R}^n$ and $\mathbb{R}^m_y$ in place of $\mathbb{R}^m$. As one expects, the dual basis of $e_1, \dots, e_{n+m}$ is indicated with

$$dx_1, \dots, dx_n, dy_1, \dots, dy_m.$$

Also we need the trivial isomorphism $J : \mathbb{R}^n \times \mathbb{R}^m \to (\mathbb{R}^n \times \mathbb{R}^m)^*$ mapping every e_i to its corresponding member in the dual basis, i.e.,

$$J(e_i) = \begin{cases} dx_i & \text{if } i = 1, \dots, n \\ dy_{i-n} & \text{if } i = n+1, \dots, n+m. \end{cases}$$

The open ball in $\mathbb{R}^n$ centered at p with radius $r > 0$ will be denoted by $B(p, r)$. Let M be an n-dimensional regularly embedded C^1 submanifold of $\mathbb{R}^{n+m}$ and $z_0 \in M$. Let δ_M be the distance defined on each connected component of M by taking the infimum of the length over the joining paths (compare [4, Section 1.6]) and, for $r > 0$, let us define

$$B_M(z_0, r) := \{z \in M^{(z_0)} : \delta_M(z, z_0) < r\}$$

where $M^{(z_0)}$ is the connected component of M containing z_0. The tangent space to M at z_0 is denoted by $T_{z_0}M$. According to [3], we also define the tangent space $\mathrm{Tan}_{p_0}(E)$ to an arbitrary set $E \subset \mathbb{R}^h$ at $p_0 \in \overline{E}$ as the vector space spanned by

$$\mathrm{Dir}_{p_0}(E) := \left\{ u \in \mathbb{S}^{h-1} : u = \lim_{i\to\infty} \frac{p_i - p_0}{|p_i - p_0|} \text{ for some } \{p_i\}_{i=1}^{\infty} \subset E \setminus \{p_0\} \text{ with } p_i \to p_0 \right\}.$$

Let us now recall a way to define a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^n_x \times \mathbb{R}^m_y$ through the formalism of differential forms, compare [5, Section 3.2] and [13, Section 2.11]. Consider a family of m linearly independent differential 1-forms of class C^1 in U, that is

$$\theta^{(j)} = \sum_{i=1}^{n} a_i^{(j)} dx_i + \sum_{h=1}^{m} a_{n+h}^{(j)} dy_h \qquad (j = 1, \dots, m)$$

with $a_i^{(j)} \in C^1(U)$. Then, for $z \in U$, let $\mathcal{D}(z)$ be the set of vectors in $\mathbb{R}^{n+m}$ solving the so-called Pfaffian system of equations

$$\theta_z^{(j)} = 0 \qquad (j = 1, \dots, m)$$

namely

$$\begin{aligned}\mathcal{D}(z) &:= \ker(\theta_z^{(1)}) \cap \dots \cap \ker(\theta_z^{(m)}) \\ &= \left[\operatorname{span}\{J^{-1}(\theta_z^{(1)})\}\right]^\perp \cap \dots \cap \left[\operatorname{span}\{J^{-1}(\theta_z^{(m)})\}\right]^\perp \\ &= \left[\operatorname{span}\{J^{-1}(\theta_z^{(1)}), \dots, J^{-1}(\theta_z^{(m)})\}\right]^\perp\end{aligned}$$

i.e.

$$\mathcal{D}(z) = \left[\operatorname{span}\{a^{(1)}(z), \dots, a^{(m)}(z)\}\right]^\perp \tag{2.1}$$

where

$$a^{(j)} := (a_1^{(j)}, \cdots, a_{n+m}^{(j)})^t.$$

Consider an open set $A \subset \mathbb{R}^n$, $\varphi \in C^1(A, \mathbb{R}^{n+m})$ and let ω be a differential form of class C^1 in an open subset of $\mathbb{R}^{n+m}$ including $\varphi(A)$. Then $(\varphi^*\omega)_a$ is the pullback of ω under φ at $a \in A$.

The Lebesgue outer measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff outer measure on $\mathbb{R}^{n+m}$ will be denoted by $\mathcal{L}^n$ and $\mathcal{H}^n$, respectively. A point $p \in \mathbb{R}^n$ is said to be a superdensity point of $E \subset \mathbb{R}^n$ if $\mathcal{L}^n(B(p,r) \setminus E) = o(r^{n+1})$ as $r \to 0+$. The set of all these points p is denoted by $E^{(n+1)}$. One has the following result which generalizes the Schwarz theorem about the equality of cross derivatives, compare [6, 7].

Theorem 2.1. *Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $f \in C^1(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. If define $K := \{p \in \Omega : Df(p) = F(p)\}$ and if $p_0 \in \Omega \cap K^{(n+1)}$ then one has $p_0 \in K$ and $(DF(p_0))^t = DF(p_0)$.*

The property stated in Theorem 2.1 is false, in general, if one simply assumes that p_0 is a point of density of K, compare [6, Remark 2.2] where [1, Theorem 1] is used to provide a counterexample. A way to get the symmetry of DF at a point of density of K in Ω is to assume that $f \in C^2(\Omega)$, compare Corollary 3.2 below.

3. Some preliminaries

3.1. **A result from linear algebra.** Define

$$X := \mathbb{R}^n_x \times \{0_{\mathbb{R}^m_y}\}, \quad Y := \{0_{\mathbb{R}^n_x}\} \times \mathbb{R}^m_y$$

and consider the standard projections $\pi_X : \mathbb{R}^n_x \times \mathbb{R}^n_y \to X$, $\pi_Y : \mathbb{R}^n_x \times \mathbb{R}^n_y \to Y$ defined as follows

$$\pi_X(x_1, \dots, x_n, y_1, \dots, y_m) := (x_1, \dots, x_n, 0, \dots, 0)$$

and

$$\pi_Y(x_1,\dots,x_n,y_1,\dots,y_m) := (0,\dots,0,y_1,\dots,y_m).$$

Recall that $e_1,\dots,e_n$ is a basis of X. The following result holds.

Proposition 3.1. *Let Z be an n-dimensional vector subspace of $\mathbb{R}^n_x \times \mathbb{R}^m_y$ such that $\pi_X(Z) = X$. Then $\pi_Y(Z^\perp) = Y$.*

Proof. We first prove that

$$Z^\perp \cap X = \{0\}. \tag{3.1}$$

For $i = 1,\dots,n$, let $v_i \in Z$ be such that $\pi_X(v_i) = e_i$. Hence

$$\varepsilon_i := e_i - v_i \in Y \qquad (i = 1,\dots,n).$$

Thus, if $u \in Z^\perp \cap X = Z^\perp \cap Y^\perp$, one has

$$u \cdot e_i = u \cdot v_i + u \cdot \varepsilon_i = 0$$

for all $i = 1,\dots,n$. This yields $u = 0$ and concludes the proof of (3.1).

Now consider a basis $w_1,\dots,w_m$ of $Z^\perp$ and let $c_1,\dots,c_m \in \mathbb{R}$ be such that $c_1\pi_Y(w_1) + \ldots + c_m\pi_Y(w_m) = 0$. Then $\pi_Y(c_1w_1 + \ldots + c_mw_m) = 0$, i.e., $c_1w_1 + \ldots + c_mw_m \in X$. From (3.1) it follows that $c_1w_1 + \ldots + c_mw_m = 0$, hence the constants c_i have to be zero. This proves that $\pi_Y(w_1),\dots,\pi_Y(w_m)$ is a basis of Y. □

3.2. C^1 **embeddings preserve density-degree.** Let V and Ω be open subsets of $\mathbb{R}^n$, let $\Phi : V \to \Omega$ be a diffeomorphism of class C^1 and let $a_0 \in V$. Since $D\Phi$ is continuous in V, we can find a closed proper ball $B \subset V$ centered at a_0 and a positive constant C such that

$$|\Phi(a) - \Phi(a_0)| \le C|a - a_0|, \text{ for all } a \in B. \tag{3.2}$$

Analogously, possibly taking a larger C, we can find a closed proper ball $B' \subset \Omega$ centered at $\Phi(a_0)$ such that

$$|\Phi^{-1}(x) - a_0| \le C|x - \Phi(a_0)|, \text{ for all } x \in B'. \tag{3.3}$$

The inequalities (3.2) and (3.3) prove that

$$\Phi(B(a_0, r/C)) \subset B(\Phi(a_0), r) \subset \Phi(B(a_0, Cr)) \tag{3.4}$$

provided r is small enough.

Now let A be an open subset of $\mathbb{R}^n$ and consider an injective immersion of class C^1

$$\varphi : A \to \mathbb{R}^{n+m}$$

so that $S := \varphi(A)$ is an n-dimensional regularly embedded C^1 submanifold of $\mathbb{R}^{n+m}$. Then one has the following result which extends (3.4).

Proposition 3.2. *If $a_0 \in A$ then there exists $C > 1$ such that $B(a_0, Cr) \subset A$ and*

$$\varphi(B(a_0, r/C)) \subset B_S(\varphi(a_0), r) \subset \varphi(B(a_0, Cr))$$

provided r is small enough.

Proof. Consider the field of inner products $A \ni a \mapsto g(a) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$g(a)[u, v] := d\varphi_a(u) \cdot d\varphi_a(v), \quad u, v \in \mathbb{R}^n$$

and recall [10, 3.2.46]. In particular:

- There exist orthonormal base vectors $\varepsilon_1, \dots, \varepsilon_n$ of $\mathbb{R}^n$ (i.e., with respect to the canonical Euclidean scalar product) such that
$$d\varphi_{a_0}(\varepsilon_i) \cdot d\varphi_{a_0}(\varepsilon_j) = 0, \text{ whenever } i \neq j;$$
- If $L : \mathbb{R}^n \to \mathbb{R}^n$ is the linear operator defined by
$$L(\varepsilon_i) := |d\varphi_{a_0}(\varepsilon_i)|\, \varepsilon_i \qquad (i = 1, \dots, n)$$
then
$$\frac{1}{2}|L(a - a_0)| \leq \delta_S(\varphi(a), \varphi(a_0)) \leq 2\, |L(a - a_0)| \tag{3.5}$$
for all $a \in B(a_0, r)$, provided r is small enough.

Since L is nonsingular, one has

$$\nu := \min_{u \in \mathbb{S}^{n-1}} |Lu| > 0$$

hence (by (3.5))

$$\frac{\nu}{2}|a - a_0| \leq \delta_S(\varphi(a), \varphi(a_0)) \leq 2\|L\|\, |a - a_0|$$

for all $a \in B(a_0, r)$, provided r is small enough. The conclusion follows by taking $C > 0$ satisfying $C \geq 2\|L\|$ and $1/C \leq \nu/2$. □

Proposition 3.3. *Let M be an n-dimensional regularly embedded C^1 submanifold of $\mathbb{R}^{n+m}$ and let $\varphi : A \to \mathbb{R}^{n+m}$, where $A \subset \mathbb{R}^n$ is open, be an injective immersion of class C^1 such that $\varphi(A) \subset M$. Moreover let E be a subset of A and let $a_0 \in A$. Then (for $k > 0$)*

$$\mathcal{L}^n(B(a_0, r) \setminus E) = o(r^k) \qquad (\text{as } r \to 0+)$$

if and only if

$$\mathcal{H}^n(B_M(\varphi(a_0), r) \setminus \varphi(E)) = o(r^k) \qquad (\text{as } r \to 0+).$$

Proof. Let $z_0 := \varphi(a_0)$. Then from Proposition 3.2 we obtain

$$B_M(z_0, r) \setminus \varphi(E) \subset \varphi(B(a_0, Cr)) \setminus \varphi(E) = \varphi(B(a_0, Cr) \setminus E)$$

and

$$B_M(z_0, r) \setminus \varphi(E) \supset \varphi(B(a_0, r/C)) \setminus \varphi(E) = \varphi(B(a_0, r/C) \setminus E)$$

provided r is small enough. Hence, by the area formula [10, Theorem 3.2.3]

$$\int_{B(a_0,r/C)\setminus E} J_n\varphi \, d\mathcal{L}^n \leq \mathcal{H}^n(B_M(z_0,r)\setminus\varphi(E)) \leq \int_{B(a_0,Cr)\setminus E} J_n\varphi \, d\mathcal{L}^n \tag{3.6}$$

provided r is small enough, where $J_n\varphi$ denotes the n-dimensional Jacobian of φ, i.e.,

$$J_n\varphi(a) = \sqrt{\det[(D\varphi(a))^t(D\varphi(a))]}, \quad a \in A.$$

Since φ is an (injective) immersion of class C^1, there exists $\nu \geq 1$ such that

$$\frac{1}{\nu} \leq J_n\varphi \leq \nu$$

in a neighborhood of a_0. Then (3.6) implies

$$\frac{1}{\nu}\,\mathcal{L}^n(B(a_0,r/C)\setminus E) \leq \mathcal{H}^n(B_M(z_0,r)\setminus\varphi(E)) \leq \nu\,\mathcal{L}^n(B(a_0,Cr)\setminus E)$$

provided r is small enough. Hence the conclusion follows immediately. □

3.3. **The tangent space to an arbitrary set at a point of density.**

Proposition 3.4. *Let p_0 be a point of density of $D \subset \mathbb{R}^n$, i.e., $p_0 \in \mathbb{R}^n$ and $\mathcal{L}^n(B(p_0,r)\setminus D) = o(r^n)$ as $r \to 0+$. Then $\mathrm{Dir}_{p_0}(D) = \mathbb{S}^{n-1}$, hence $\mathrm{Tan}_{p_0}(D) = \mathbb{R}^n$.*

Proof. Let $u \in \mathbb{S}^{n-1}$ and consider the open semi-cones

$$V_i := \{p \in \mathbb{R}^n \,:\, (p-p_0)\cdot u > (1-1/i)|p-p_0|\} \qquad (i = 1,2,\dots).$$

We can prove that

$$B(p_0,1/i)\cap V_i \cap D \neq \emptyset \tag{3.7}$$

for all positive integer i, by reductio ad absurdum. Indeed, if this property does not hold then a positive integer i_0 has to exist such that $B(p_0,1/i_0)\cap V_{i_0}\cap D = \emptyset$, hence $B(p_0,r)\cap V_{i_0}\cap D = \emptyset$ for all $r \in (0,1/i_0]$. Thus

$$B(p_0,r)\cap V_{i_0} = B(p_0,r)\cap V_{i_0}\cap D^c \subset B(p_0,r)\cap D^c$$

for all $r \in (0,1/i_0]$, which yields (since p_0 is a point of density of D) the following absurd identity

$$\lim_{r\to 0+}\frac{\mathcal{L}^n(B(p_0,r)\cap V_{i_0})}{r^n} = 0.$$

Now, from (3.7) it follows that for each i there exists $p_i \in V_i \cap D$ such that $0 < |p_i - p_0| < 1/i$. It follows that $p_i \to p_0$ (as $i \to +\infty$) and

$$\left|\frac{p_i-p_0}{|p_i-p_0|} - u\right|^2 = 2 - 2\,\frac{(p_i-p_0)\cdot u}{|p_i-p_0|} \leq 2 - 2\left(1-\frac{1}{i}\right) = \frac{2}{i}$$

hence

$$\frac{p_i-p_0}{|p_i-p_0|} \to u \qquad (\text{as } i \to +\infty).$$

The conclusion follows from the arbitrariness of $u \in \mathbb{S}^{n-1}$. □

Corollary 3.1. *Let M be an n-dimensional regularly embedded C^1 submanifold of $\mathbb{R}^{n+m}$. If z_0 is a point of density of $R \subset M$, i.e., $z_0 \in M$ and $\mathcal{H}^n(B_M(z_0, r) \setminus R) = o(r^n)$ as $r \to 0+$, then $\mathrm{Tan}_{z_0}(R) = T_{z_0}M$.*

Proof. By assumption, there exist an open set $A \subset \mathbb{R}^n$ and an injective immersion $\varphi : A \to \mathbb{R}^{n+m}$ of class C^1 such that $z_0 \in \varphi(A) \subset M$. Let $D := \varphi^{-1}(R)$ and observe that

$$\varphi(D) = R \cap \varphi(A). \tag{3.8}$$

Since $\varphi(A)$ is a relatively open subset of M containing z_0, the identity (3.8) implies

$$B_M(z_0, r) \setminus \varphi(D) = B_M(z_0, r) \setminus R$$

provided r is small enough. Then

$$\mathcal{H}^n(B_M(z_0, r) \setminus \varphi(D)) = o(r^n) \qquad (\text{as } r \to 0+)$$

by assumption. From Proposition 3.3, with $E := D$ and $k := n$, we obtain that $a_0 := \varphi^{-1}(z_0) \in A$ is a point of density of D. Hence $\mathrm{Tan}_{a_0}(D) = \mathbb{R}^n$, by Proposition 3.4. Finally

$$\mathrm{Tan}_{z_0}(R) = \mathrm{Tan}_{z_0}(\varphi(D)) = d\varphi_{a_0}(\mathrm{Tan}_{a_0}(D)) = d\varphi_{a_0}(\mathbb{R}^n) = T_{z_0}M$$

by [3, Proposition 2.2]. □

From Proposition 3.4 we get also the following simple result related to Theorem 2.1.

Corollary 3.2. *Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $f \in C^2(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. If $p_0 \in \Omega$ is a point of density of $K := \{p \in \Omega : Df(p) = F(p)\}$, then one has $p_0 \in K$ and $DF(p_0)$ coincides with the Hessian of f at p_0. In particular $(DF(p_0))^t = DF(p_0)$.*

Proof. One has $p_0 \in K$ in that K is closed relatively to Ω and $p_0 \in \Omega$.

Let u_k be the k-th element of the standard basis of $\mathbb{R}^n$. Then, by Proposition 3.4, there are $p_1, p_2, \ldots \in K \setminus \{p_0\}$ such that

$$\frac{p_i - p_0}{|p_i - p_0|} \to u_k \qquad (\text{as } i \to +\infty). \tag{3.9}$$

By the differentiability of F_h and $D_h f$ at p_0, one has also

$$F_h(p_i) - F_h(p_0) - DF_h(p_0) \cdot (p_i - p_0) = o(|p_i - p_0|)$$

and

$$D_h f(p_i) - D_h f(p_0) - D(D_h f)(p_0) \cdot (p_i - p_0) = o(|p_i - p_0|)$$

as $i \to +\infty$. Since $F_h(p_i) = D_h f(p_i)$ and $F_h(p_0) = D_h f(p_0)$, we obtain

$$DF_h(p_0) \cdot (p_i - p_0) = D(D_h f)(p_0) \cdot (p_i - p_0) + o(|p_i - p_0|)$$

that is

$$[DF_h(p_0) - D(D_h f)(p_0)] \cdot \frac{p_i - p_0}{|p_i - p_0|} = \frac{o(|p_i - p_0|)}{|p_i - p_0|}$$

as $i \to +\infty$. The conclusion follows from (3.9). □

Remark 3.1. From Corollary 3.2 it follows immediately the following property: If $f \in C^2(\Omega)$, ω is a differential 1-form of class C^1 in Ω (where Ω is an open subset of $\mathbb{R}^n$) and $p_0 \in \Omega$ is a point of density of $K := \{p \in \Omega \,:\, df_p = \omega_p\}$, then one has $(d\omega)_{p_0} = 0$.

4. The main preliminary result

Theorem 4.1. *Let U' be an open subset of $\mathbb{R}^n_x \times \mathbb{R}^m_y$ and let ω be a differential 1-form of the type*

$$\omega = \sum_{i=1}^{n} \beta_i dx_i - dy_j$$

with $\beta_i \in C^1(U')$, for all $i = 1, \dots, n$. Moreover, let Ω be an open subset of $\mathbb{R}^n_x$ and let $f \in C^1(\Omega, \mathbb{R}^m_y)$ be such that $\psi(x) := (x, f(x)) \in U'$ for all $x \in \Omega$. If define

$$H := \{x \in \Omega \,:\, (\psi^*\omega)_x = 0\}$$

then one has

$$[\psi^*(d\omega)]_{x_0} = 0$$

for all $x_0 \in \Omega \cap H^{(n+1)}$.

Proof. Let $x \in \Omega$. Then:

- One has

$$[\psi^*(\omega)]_x(e_h) = \omega_{(x,f(x))}\left(e_h + \sum_{p=1}^{m} D_h f_p(x) e_{n+p}\right) = \beta_h(x, f(x)) - D_h f_j(x)$$

  for all $h = 1, \dots, n$, hence

$$H = \left\{x \in \Omega \,:\, Df_j(x) = \Big(\beta_1(x, f(x)), \dots, \beta_n(x, f(x))\Big)^t\right\};$$

  It follows that

$$D_h[\beta_k(x, f(x))]_{x=x_0} = D_k[\beta_h(x, f(x))]_{x=x_0} \tag{4.1}$$

  for all $x_0 \in \Omega \cap H^{(n+1)}$, by Theorem 2.1.

- If $\langle \cdot\,;\cdot\rangle$ denotes the standard inner product in $\Lambda_2(\mathbb{R}^n_x \times \mathbb{R}^m_y)$, one has

$$\begin{aligned}
[\psi^*(d\omega)]_x(e_h, e_k) &= (d\omega)_{(x,f(x))}\left(e_h + \sum_{p=1}^m D_h f_p(x) e_{n+p}\,,\; e_k + \sum_{q=1}^m D_k f_q(x) e_{n+q}\right)\\
&= \Big\langle \sum_{i=1}^n \left(\sum_{r=1}^n (D_{x_r}\beta_i)(x,f(x))e_r + \sum_{s=1}^m (D_{y_s}\beta_i)(x,f(x))e_{n+s}\right)\wedge e_i\,;\\
&\qquad \left(e_h + \sum_{p=1}^m D_h f_p(x)e_{n+p}\right)\wedge\left(e_k + \sum_{q=1}^m D_k f_q(x)e_{n+q}\right)\Big\rangle\\
&= \Big\langle \sum_{i,r=1}^n (D_{x_r}\beta_i)(x,f(x))e_r\wedge e_i + \sum_{i=1}^n\sum_{s=1}^m (D_{y_s}\beta_i)(x,f(x))e_{n+s}\wedge e_i\,;\\
&\qquad e_h\wedge e_k + \sum_{q=1}^m D_k f_q(x) e_h\wedge e_{n+q} + \sum_{p=1}^m D_h f_p(x) e_{n+p}\wedge e_k\Big\rangle\\
&= (D_{x_h}\beta_k)(x,f(x)) - (D_{x_k}\beta_h)(x,f(x))\\
&\qquad -\sum_{q=1}^m (D_{y_q}\beta_h)(x,f(x))D_k f_q(x) + \sum_{q=1}^m (D_{y_q}\beta_k)(x,f(x))D_h f_q(x)
\end{aligned}$$

  namely

$$[\psi^*(d\omega)]_x(e_h, e_k) = D_h[\beta_k(x,f(x))] - D_k[\beta_h(x,f(x))]. \tag{4.2}$$

  for all $h,k = 1,\ldots,n$.

The conclusion follows from (4.1) and (4.2). □

Now consider an open subset U of $\mathbb{R}^n_x \times \mathbb{R}^m_y$ and a family of m linearly independent differential 1-forms of class C^1 in U, that is

$$\theta^{(j)} = \sum_{i=1}^n a_i^{(j)} dx_i + \sum_{h=1}^m a_{n+h}^{(j)} dy_h \qquad (j = 1,\ldots,m)$$

with $a_i^{(j)} \in C^1(U)$. Then let $\mathcal{D}$ be the C^1 distribution of rank n on U determined by the Pfaffian system of equations

$$\theta^{(j)} = 0 \qquad (j = 1,\ldots,m)$$

compare Section 2.

Define the vector fields ($j = 1,\ldots,m$)

$$a^{(j)} := (a_1^{(j)},\ldots,a_{n+m}^{(j)})^t, \qquad a_Y^{(j)} := (a_{n+1}^{(j)},\ldots,a_{n+m}^{(j)})^t$$

and the matrix fields

$$M := \begin{pmatrix} a_1^{(1)} & \cdots & a_1^{(m)}\\ \vdots & \ddots & \vdots\\ a_n^{(1)} & \cdots & a_n^{(m)}\end{pmatrix}, \quad N := (a_Y^{(1)}|\cdots|a_Y^{(m)}) = \begin{pmatrix} a_{n+1}^{(1)} & \cdots & a_{n+1}^{(m)}\\ \vdots & \ddots & \vdots\\ a_{n+m}^{(1)} & \cdots & a_{n+m}^{(m)}\end{pmatrix}$$

$$P := (a^{(1)}|\cdots|a^{(m)}) = \begin{pmatrix} a_1^{(1)} & \cdots & a_1^{(m)} \\ \vdots & \ddots & \vdots \\ a_{n+m}^{(1)} & \cdots & a_{n+m}^{(m)} \end{pmatrix}.$$

Moreover let A be an open subset of $\mathbb{R}^n$ and let

$$\varphi : A \to U$$

be an injective immersion of class C^1. Then $S := \varphi(A)$ is an n-dimensional C^1 submanifold of $\mathbb{R}^n_x \times \mathbb{R}^m_y$.

The following very simple fact holds.

Proposition 4.1. *Let*

$$K := \Big\{a \in A \,:\, (\varphi^*\theta^{(j)})_a = 0 \text{ for all } j = 1,\dots,m\Big\}. \tag{4.3}$$

Then

$$\varphi(K) = \tau(S,\mathcal{D}) = \{z \in S \,:\, T_zS = \mathcal{D}(z)\}.$$

Proof. Recall that, for all $a \in A$, one has

$$(\varphi^*\theta^{(j)})_a = \theta^{(j)}_{\varphi(a)} \circ d\varphi_a, \quad d\varphi_a(\mathbb{R}^n) = T_{\varphi(a)}S.$$

Hence $a \in K$ if and only if

$$T_{\varphi(a)}S \subset \mathcal{D}(\varphi(a)), \text{ i.e., } T_{\varphi(a)}S = \mathcal{D}(\varphi(a)).$$

The conclusion follows trivially. □

Theorem 4.2 (main lemma)**.** *Let $a_0 \in A \cap K^{(n+1)}$, where K is the set defined in* (4.3). *Then $a_0 \in K$ and*

$$[\varphi^*(d\theta^{(j)})]_{a_0} = 0 \quad \Big(i.e.\ (d\theta^{(j)})_{\varphi(a_0)}|_{T_{\varphi(a_0)}S\times T_{\varphi(a_0)}S} = 0\Big)$$

for all $j = 1,\dots,m$.

Proof. One has $a_0 \in K$ in that K is closed relatively to A and $a_0 \in A$. Without loss of generality, we can assume that

$$\det\Big[D(\varphi_1,\dots,\varphi_n)^t(a_0)\Big] \neq 0$$

hence there exists an open neighborhood V of $a_0 \in \mathbb{R}^n$ such that

$$\Phi := (\varphi_1,\dots,\varphi_n)^t|_V : V \to \Omega := \operatorname{Im}\Big((\varphi_1,\dots,\varphi_n)^t|_V\Big)$$

is a diffeomorphism of class C^1. It follows that $\varphi(V)$ is the graph of the C^1-function

$$f := (\varphi_{n+1},\dots,\varphi_{n+m})^t \circ \Phi^{-1} : \Omega \to \mathbb{R}^m_y.$$

One obviously has $a_0 \in K$, hence

$$\mathcal{D}(\varphi(a_0)) = T_{\varphi(a_0)}S$$

by Proposition 4.1. It follows that (with the notation of Section 3.1)

$$\pi_X(\mathcal{D}(\varphi(a_0))) = X = \mathbb{R}^n_x \times \{0_{\mathbb{R}^m_y}\}.$$

Then (by replacing V with a smaller neighborhood of a_0, if need be) we can assume that

$$\pi_X(\mathcal{D}(\varphi(a))) = X = \mathbb{R}^n_x \times \{0_{\mathbb{R}^m_y}\}$$

for all $a \in V$. Thus

$$\pi_Y\Big(\mathrm{span}\{a^{(1)}(\varphi(a)), \dots, a^{(m)}(\varphi(a))\}\Big) = Y = \{0_{\mathbb{R}^n_x}\} \times \mathbb{R}^m_y$$

for all $a \in V$, by Proposition 3.1 and (2.1), which is equivalent to

$$\mathrm{span}\{a^{(1)}_Y(\varphi(a)), \dots, a^{(m)}_Y(\varphi(a))\} = \mathbb{R}^m_y$$

for all $a \in V$. This proves that the matrix $N(\varphi(a))$ is invertible, for all $a \in V$. Thus there must be an open subset U' of U such that $\varphi(a_0) \in U'$ and $N(z) : \mathbb{R}^m \to \mathbb{R}^m$ is invertible for all $z \in U'$.

Now let $z \in U'$. Then one has

$$P(z) \times [-N(z)^{-1}] = \begin{pmatrix} M(z) \\ N(z) \end{pmatrix} \times [-N(z)^{-1}] = \begin{pmatrix} M(z) \times [-N(z)^{-1}] \\ -I_m \end{pmatrix} \tag{4.4}$$

where I_m is the identity matrix of size m. If $\alpha_{hj}(z)$ and $\beta_{ij}(z)$ denote the (h, j)-entry of $-N(z)^{-1}$ and the (i, j)-entry of $M(z) \times [-N(z)^{-1}]$, respectively, then (4.4) shows that

$$\sum_{h=1}^{m} \alpha_{hj}(z)\theta^{(h)}_z = \sum_{i=1}^{n} \beta_{ij}(z)dx_i - dy_j \qquad (j = 1, \dots, m). \tag{4.5}$$

Observe that all the functions $z \mapsto \alpha_{hj}(z)$ and $z \mapsto \beta_{ij}(z)$ are in $C^1(U')$. Hence, in particular, the differential 1-forms

$$\omega^{(j)} := \sum_{i=1}^{n} \beta_{ij} dx_i - dy_j \qquad (j = 1, \dots, m)$$

are of class C^1 in U'.

Now on, without loss of generality, we can suppose that $\varphi(V) \subset U'$. Consider the map $\psi : \Omega \to \mathbb{R}^n_x \times \mathbb{R}^m_y$ of class C^1 defined as

$$\psi(x) := (x, f(x)), \quad x \in \Omega$$

so that $\psi \circ \Phi = \varphi|_V$. By recalling (4.5), we find

$$\Phi^*(\psi^*\omega^{(j)}) = (\varphi|_V)^*\omega^{(j)} = \sum_{h=1}^{m} (\alpha_{hj} \circ \varphi|_V)\,(\varphi|_V)^*\theta^{(h)} \quad (j = 1, \dots, m). \tag{4.6}$$

It follows that

$$[\Phi^*(\psi^*\omega^{(j)})]_a = 0, \text{ for all } a \in K \cap V \qquad (j = 1, \dots, m) \tag{4.7}$$

namely

$$(\psi^*\omega^{(j)})_x = 0, \text{ for all } x \in \Phi(K) \cap \Omega \qquad (j = 1, \dots, m)$$

which is equivalent to

$$\Phi(K)\cap\Omega\subset\bigcap_{j=1}^{m}H_j,\ \text{with } H_j:=\{x\in\Omega\,|\,(\psi^*\omega^{(j)})_x=0\}. \tag{4.8}$$

Moreover, since $a_0\in K^{(n+1)}\cap V$ and recalling (3.4), one has

$$\Phi(a_0)\in\Phi(K)^{(n+1)}\cap\Omega=[\Phi(K)\cap\Omega]^{(n+1)}. \tag{4.9}$$

From (4.8) and (4.9) we obtain

$$\Phi(a_0)\in\left(\bigcap_{j=1}^{m}H_j\right)^{(n+1)}\subset\bigcap_{j=1}^{m}H_j^{(n+1)}$$

hence

$$[\psi^*(d\omega^{(j)})]_{\Phi(a_0)}=0\qquad(j=1,\dots,m)$$

by Theorem 4.1. Then

$$[(\varphi|_V)^*(d\omega^{(j)})]_{a_0}=0\qquad(j=1,\dots,m). \tag{4.10}$$

Since $a_0\in K\cap V$ one has also

$$[(\varphi|_V)^*\omega^{(j)}]_{a_0}=0\qquad(j=1,\dots,m) \tag{4.11}$$

by (4.7).

Now, for $z\in U'$, let $\nu_{ij}(z)$ be the (i,j)-entry of $-N(z)$ and observe that the functions $z\mapsto\nu_{ij}(z)$ belong to $C^1(U')$. From (4.4) or (4.5) we obtain

$$\theta^{(j)}|_{U'}=\sum_{h=1}^{m}\nu_{hj}\omega^{(h)}\qquad(j=1,\dots,m).$$

Thus

$$d(\theta^{(j)}|_{U'})=\sum_{h=1}^{m}d\nu_{hj}\wedge\omega^{(h)}+\sum_{h=1}^{m}\nu_{hj}d\omega^{(h)}\qquad(j=1,\dots,m)$$

hence

$$[\varphi^*(d\theta^{(j)})]_{a_0}=\sum_{h=1}^{m}[(\varphi|_V)^*d\nu_{hj}]_{a_0}\wedge[(\varphi|_V)^*\omega^{(h)}]_{a_0}+\sum_{h=1}^{m}\nu_{hj}(\varphi(a_0))[(\varphi|_V)^*d\omega^{(h)}]_{a_0}$$

for all $j=1,\dots,m$. The conclusion follows at once from (4.10) and (4.11). □

Remark 4.1. From (4.6) it follows that $a\in K\cap V$ if and only if

$$(\psi^*\omega^{(j)})_{\Phi(a)}=0\qquad(j=1,\dots,m).$$

Moreover, for all $x \in \Omega$, one has

$$\begin{aligned}(\psi^*\omega^{(j)})_x &= \sum_{i=1}^n \beta_{ij}(\psi(x))(\psi^*(dx_i))_x - (\psi^*(dy_j))_x \\ &= \sum_{i=1}^n \beta_{ij}(x, f(x))dx_i - (df_j)_x \\ &= \sum_{i=1}^n \left[\beta_{ij}(x, f(x)) - D_i f_j(x)\right] dx_i.\end{aligned}$$

Then

$$\Phi(K \cap V) = \Big\{x \in \Omega \,:\, Df(x) = \beta^t(x, f(x))\Big\}$$

where β denotes the matrix whose (i, j)-entry is β_{ij}, i.e., $\beta := -M \times N^{-1}$. In particular, in the special case when $\mathcal{D}$ is translation-invariant along $\mathbb{R}^m_y$, namely if the coefficients $a_i^{(j)}$ do not depend on y (hence also the β_{ij} do not depend on y), then the problem of determining a local integral surface of $\mathcal{D}$ is reduced to the problem of determining a mapping with locally prescribed Jacobian matrix. This fact and [2, Theorem 4.1] have been used in [3] to prove the following result: *If $\mathcal{D}$ is translation-invariant along $\mathbb{R}^m_y$ and $z_0 \in U$, then there exists an n-dimensional regularly embedded submanifold M_0 of U of class $\cap_{\alpha\in(0,1)} C^{1,\alpha}$ such that*

$$z_0 \in M_0, \quad 0 < \mathcal{H}^n(M_0) < +\infty, \quad \mathcal{H}^n(\tau(M_0, \mathcal{D})) > 0. \tag{4.12}$$

Compare [3, Proposition 8.2]. We observe that if one is interested only in an n-dimensional regularly embedded submanifold of U of class C^1 satisfying (4.12), this can be obtained also from [1, Theorem 1]. The following example for $n = 2k$ and $m = 1$ has been considered in [2] in connection with characteristic sets, in the context of the Heisenberg group:

$$U := \mathbb{R}^{2k+1}, \quad \theta^{(1)}_{(x,y)} := -\sum_{i=1}^k x_{k+i}\, dx_i + \sum_{i=k+1}^{2k} x_{i-k}\, dx_i - dy_1. \tag{4.13}$$

In this case the matrix field N has a single entry identically equal to -1. Then $N(z)$ is obviously invertible for all $z \in \mathbb{R}^{2k+1}$, so we can choose $U' = \mathbb{R}^{2k+1}$ and a trivial computation shows that $\omega^{(1)} = \theta^{(1)}$.

Remark 4.2. In the special case when φ is of class C^2, even under the weaker assumption that $a_0 \in A$ is merely a point of density of K, we can provide the following two much simpler proofs of Theorem 4.2:

- *(First proof)* Observe that the differential 1-forms $\varphi^*\theta^{(j)}$ are of class C^1 in A and that

$$K = \bigcap_{j=1}^m K_j$$

where $K_j := \{a \in A \,:\, (\varphi^*\theta^{(j)})_a = 0\}$. Since a_0 is a point of density of K, then a_0 is a point of density of K_j for all $j = 1, \dots, m$. Recalling the property in Remark

3.1 (with $\Omega := A$, $f := 0$, $\omega := \varphi^*\theta^{(j)}$) we conclude that

$$[\varphi^*(d\theta^{(j)})]_{a_0} = [d(\varphi^*\theta^{(j)})]_{a_0} = 0$$

for all $j = 1, \dots, m$.

- *(Second proof)* Since a_0 is a point of density of K then $\mathrm{Tan}_{a_0}(K) = \mathbb{R}^n$, by Proposition 3.4. Hence

$$\begin{aligned}\mathrm{Tan}_{\varphi(a_0)}(\tau(S,\mathcal{D})) &= \mathrm{Tan}_{\varphi(a_0)}(\varphi(K)) = d\varphi_{a_0}(\mathrm{Tan}_{a_0}(K)) = d\varphi_{a_0}(\mathbb{R}^n)\\ &= T_{\varphi(a_0)}S\end{aligned}$$

by Proposition 4.1 and [3, Proposition 2.2]. The conclusion follows from [3, Lemma 3.2].

5. The proof of Theorem 1.1

As in Section 4, let $\mathcal{D}$ be a C^1 distribution of rank n on an open set $U \subset \mathbb{R}^n_x \times \mathbb{R}^m$ determined by a Pfaffian system of equations

$$\theta^{(j)} = 0 \qquad (j = 1, \dots, m).$$

Remark 5.1. Let M be an n-dimensional regularly embedded C^1 submanifold of U and let $z_0 \in M$ be a point of density of $\tau(M,\mathcal{D})$, i.e., $\mathcal{H}^n(B_M(z_0,r) \setminus \tau(M,\mathcal{D})) = o(r^n)$ as $r \to 0+$. Since $\tau(M,\mathcal{D})$ is closed relatively to M, then one has $z_0 \in \tau(M,\mathcal{D})$.

We can finally prove our main result, namely Theorem 1.1, as a corollary of Theorem 4.2 and Proposition 3.3. We recall its statement, for the reader's convenience.

Theorem. *Let M be an n-dimensional regularly embedded C^1 submanifold of U and assume that one between conditions (I) and (II) below is satisfied:*

(I) *Let $z_0 \in M$ be a superdensity point of $\tau(M,\mathcal{D})$, i.e.,*

$$\mathcal{H}^n(B_M(z_0,r) \setminus \tau(M,\mathcal{D})) = o(r^{n+1}) \qquad (\text{as } r \to 0+);$$

(II) *Let $z_0 \in M$ be an ordinary point of density of $\tau(M,\mathcal{D})$, i.e.,*

$$\mathcal{H}^n(B_M(z_0,r) \setminus \tau(M,\mathcal{D})) = o(r^n) \qquad (\text{as } r \to 0+)$$

and let M be locally of class C^2 at z_0.

Then $z_0 \in \tau(M,\mathcal{D})$ and $\mathcal{D}$ is involutive at z_0, i.e.,

$$(d\theta^{(j)})_{z_0}|_{T_{z_0}M \times T_{z_0}M} = (d\theta^{(j)})_{z_0}|_{\mathcal{D}(z_0)\times\mathcal{D}(z_0)} = 0 \tag{5.1}$$

for all $j = 1, \dots, m$.

Proof. **If (I) is satisfied.** One has $z_0 \in \tau(M,\mathcal{D})$, by Remark 5.1. By assumption, there exist an open set $A \subset \mathbb{R}^n$ and an injective immersion $\varphi : A \to U$ of class C^1 which parametrizes M around z_0, that is $z_0 \in \varphi(A) \subset M$. Let K be the set defined in (4.3) and observe that

$$\varphi(K) = \{z \in \varphi(A) \,:\, T_z M = \mathcal{D}(z)\} = \tau(M,\mathcal{D}) \cap \varphi(A) \tag{5.2}$$

by Proposition 4.1. Since $\varphi(A)$ is a relatively open subset of M containing z_0, the identity (5.2) yields

$$B_M(z_0,r) \setminus \varphi(K) = B_M(z_0,r) \setminus \tau(M,\mathcal{D})$$

provided r is small enough. Then

$$\mathcal{H}^n(B_M(z_0,r) \setminus \varphi(K)) = o(r^{n+1}) \qquad (\text{as } r \to 0+) \tag{5.3}$$

by assumption. From Proposition 3.3, with $E := K$ and $k = n+1$, we obtain $a_0 := \varphi^{-1}(z_0) \in K^{(n+1)}$. Hence

$$[\varphi^*(d\theta^{(j)})]_{a_0} = 0, \text{ for all } j = 1,\dots,m \tag{5.4}$$

by Theorem 4.2. Since $T_{z_0}M = \mathrm{Im}(d\varphi_{a_0})$, the identity (5.4) is equivalent to (5.1).

If (II) is satisfied. The argument above continues to work, with some obvious changes. First of all φ can be assumed to be of class C^2. In place of (5.3) we obtain

$$\mathcal{H}^n(B_M(z_0,r) \setminus \varphi(K)) = o(r^n) \qquad (\text{as } r \to 0+)$$

hence $a_0 := \varphi^{-1}(z_0)$ turns out to be a point of density of K (by Proposition 3.3, with $E := K$ and $k = n$). Then (5.4) follows by recalling Remark 4.2. □

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