Graphs, spanning trees and divergence-free finite elements in general topological domains

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Abstract

We construct sets of basis functions of the space of divergence-free finite elements of Raviart–Thomas type in a general topological domain. Two different methods are presented: one using a suitable selection of the curls of Nédélec finite elements, the other based on an efficient algebraic procedure. The first approach looks to be more useful for numerical approximation, as the basis functions have a localized support.

1. Introduction

The Hilbert space $H(\mathrm{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \,|\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$ furnishes a natural framework for the variational formulation of several elliptic problems. A couple of examples are the Darcy problem

$$\begin{cases} \mathbf{u} + K\nabla p = \mathbf{g} \\ \mathrm{div}\,\mathbf{u} = 0\,, \end{cases} \tag{1}$$

or the saddle point formulation of the second order problem $\mathrm{div}(A\nabla p) = f$, which is given by

$$\begin{cases} \mathbf{u} - A\nabla p = \mathbf{0} \\ \mathrm{div}\,\mathbf{u} = f\,. \end{cases} \tag{2}$$

Note that (2) can be expressed in the form (1) by finding an auxiliary unknown $\mathbf{u}_f$ such that $\mathrm{div}\,\mathbf{u}_f = f$, and then solving for $\mathbf{w} = \mathbf{u} - \mathbf{u}_f$. Let us focus on (1), and, for the sake of exposition, assume that the boundary condition is given by $p = 0$ on $\partial\Omega$.

An integration by parts leads to the standard mixed variational formulation

$$\begin{cases} \int_\Omega K^{-1}\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\,\mathrm{div}\,\mathbf{v} = \int_\Omega K^{-1}\mathbf{g}\cdot\mathbf{v} \\ \int_\Omega q\,\mathrm{div}\,\mathbf{u} = 0\,, \end{cases} \tag{3}$$

where $\mathbf{u}, \mathbf{v} \in H(\mathrm{div};\Omega)$ and $p, q \in L^2(\Omega)$.

However, it is worth noting that an even simpler variational formulation is given by

$$\int_\Omega K^{-1}\mathbf{u}\cdot\mathbf{v} = \int_\Omega K^{-1}\mathbf{g}\cdot\mathbf{v}\,, \tag{4}$$

where $\mathbf{u},\mathbf{v}\in H^0(\mathrm{div};\Omega) = \{\mathbf{v}\in H(\mathrm{div};\Omega)\,|\,\mathrm{div}\,\mathbf{v}=0 \text{ in } \Omega\}$.

The numerical approximation by the finite element method of problems like (3) is a well-known option (see, e.g., Boffi, Brezzi and Fortin [1]). Instead, the numerical approximation of problem (4) has not been frequently considered, as a conforming approximation of $H^0(\mathrm{div};\Omega)$ presents some difficulties, essentially related to the divergence-free constraint.

The delicate point is in fact the construction of a basis for the space of divergence-free finite elements; in the three-dimensional case, to our knowledge this has been done by Hecht [2], Dubois [3] and Scheichl [4] for a simply-connected domain with a non-connected boundary, and by Rapetti, Dubois and Bossavit [5] for a κ-fold torus; see also some related results by Gustafson and Hartman [6], [7] concerned with hydrodynamics problems.

Due to the importance of topological issues in many problems of mathematical physics (e.g., in fluid dynamics, or in electromagnetism) in this paper we present two simple and explicit constructions of a basis for divergence-free finite elements in a domain with an arbitrary topology. The first construction performs an accurate selection of the curls of the Nédélec finite elements, based on an algorithm proposed by Hiptmair and Ostrowski [8] (and extended by Alonso Rodríguez et al. [9]) which furnishes a suitable basis of the first homology group $\mathcal{H}_1(\overline{\Omega},\mathbb{Z})$ of $\overline{\Omega}$. The second construction is grounded on a more direct algebraic procedure, which however leads to basis functions with non-localized support.

Having available a set of finite element basis functions can furnish an efficient tool for the numerical approximation of (1) or (2), or else of other boundary value problems, for instance the curl–div system

$$\begin{cases} \mathrm{curl}\,\mathbf{u} = \mathbf{J} \\ \mathrm{div}\,\mathbf{u} = g \\ \mathbf{u}\times\boldsymbol{\nu} = \mathbf{a} \quad (\text{or} \quad \mathbf{u}\cdot\boldsymbol{\nu} = b)\,, \end{cases} \tag{5}$$

which, after having determined a vector field $\mathbf{u}^\star$ such that $\mathrm{div}\,\mathbf{u}^\star = g$, can be formulated in the space $H^0(\mathrm{div};\Omega)$. This will be the subject of a forthcoming paper.

2. Notation and preliminary results

Let Ω be a bounded polyhedral domain of $\mathbb{R}^3$ with Lipschitz boundary and let $(\partial\Omega)_0,\ldots,(\partial\Omega)_p$ be the connected components of $\partial\Omega$, $(\partial\Omega)_0$ being the external one. Consider a tetrahedral triangulation $\mathcal{T}_h = (V,E,F,T)$ of $\overline{\Omega}$. Here V is the set of vertices, E the set of edges, F the set of faces and T the set of tetrahedra of $\mathcal{T}_h$.

We consider the following spaces of finite elements (for a complete presentation, see Monk [10]). The space L_h of continuous piecewise-linear finite elements:

its dimension is n_v, the number of vertices in $\mathcal{T}_h$. The space N_h of Nédélec edge elements of degree 1: its dimension is n_e, the number of edges in $\mathcal{T}_h$. The space RT_h of Raviart-Thomas finite elements of degree 1; its dimension is n_f, the number of faces in $\mathcal{T}_h$. The space PC_h of piecewise-constant elements; its dimension is n_t, the number of tetrahedra in $\mathcal{T}_h$.

It is well-known that $L_h \subset H^1(\Omega) = \{\phi \in L^2(\Omega) \,|\, \operatorname{grad}\phi \in (L^2(\Omega))^3\}$, $N_h \subset H(\operatorname{curl};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \,|\, \operatorname{curl}\mathbf{v} \in (L^2(\Omega))^3\}$, $RT_h \subset H(\operatorname{div};\Omega)$ and $PC_h \subset L^2(\Omega)$. Moreover $\operatorname{grad} L_h \subset N_h$, $\operatorname{curl} N_h \subset RT_h$, and $\operatorname{div} RT_h \subset PC_h$. Let us consider a basis of N_h, $\{\mathbf{w}_{h,1}, \dots \mathbf{w}_{h,n_e}\}$, such that $\int_{e_j} \mathbf{w}_{h,i} \cdot \boldsymbol{\tau} = \delta_{i,j}$ for $1 \le i,j \le n_e$, and a basis of RT_h, $\{\mathbf{r}_{h,1}, \dots \mathbf{r}_{h,n_f}\}$, such that $\int_{f_k} \mathbf{r}_{h,l} \cdot \boldsymbol{\nu} = \delta_{l,k}$ for $1 \le k,l \le n_f$.

Fix a total ordering $v_1, \dots, v_{n_v}$ of the elements of V. This induces an orientation on the elements of E and F: if the end points of e_j are v_a and v_b for some $a, b \in \{1, \dots, n_v\}$ with $a < b$, then the oriented edge e_j will be denoted by $[v_a, v_b]$, with unit tangent vector $\boldsymbol{\tau} = \frac{v_b - v_a}{|v_b - v_a|}$; moreover, if the face f_k has vertices v_a, v_b and v_c with $a < b < c$, the oriented face f_k will be denoted by $[v_a, v_b, v_c]$ and its unit normal vector $\boldsymbol{\nu} = \frac{(v_b - v_a) \times (v_c - v_a)}{|(v_b - v_a) \times (v_c - v_a)|}$ is obtained by the right hand rule.

We finally need to introduce a set of 1-cycles in $\mathcal{T}_h$ that are representatives of a basis of the first homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$ (whose rank will be denoted by g): in other words, it is a maximal set of non-bounding 1-cycles in $\mathcal{T}_h$. An explicit and efficient construction of these 1-cycles is presented in Hiptmair and Ostrowski [8]; we denote them by $\{\sigma_n\}_{n=1}^g$. For a more detailed presentation of the homological concepts that are useful in the numerical approximation of PDEs, see, e.g., Bossavit [11], Hiptmair [12], Gross and Kotiuga [13]; see also Benedetti, Frigerio and Ghiloni [14], Alonso Rodríguez et al. [9], Alonso Rodríguez et al. [15].

3. Construction of a basis of $H^0(\operatorname{div};\Omega) \cap RT_h$

For constructing a basis of $H^0(\operatorname{div};\Omega) \cap RT_h$ we present two different procedures.

Let us start with some remarks. First of all, one clearly has $\operatorname{curl} N_h \subset H^0(\operatorname{div};\Omega) \cap RT_h$; however, taking the curl of a basis of N_h furnishes a set of vector fields that are not linearly independent, as there are functions in N_h that are curl-free (for instance, the gradients of the nodal elements L_h). Moreover, if $\partial\Omega$ is not connected, namely $p \ge 1$, then $\operatorname{curl} N_h \ne H^0(\operatorname{div};\Omega) \cap RT_h$. In fact, for each $s = 1, \dots, p$, the following problem

$$\begin{cases} \operatorname{div} \mathbf{v}_{h,s} = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_{h,s} \cdot \boldsymbol{\nu} = \delta_{s,r} & \forall\, r = 1, \dots, p \end{cases}$$

has a unique solution orthogonal to $\operatorname{curl} N_h$ (see Alonso Rodríguez and Valli [16]).

This description also shows that the dimension of $H^0(\operatorname{div};\Omega) \cap RT_h$ is equal to p plus the dimension of $\operatorname{curl} N_h$. It is well-known that the dimension of the space $\operatorname{curl} N_h$ is equal to $n_e - (n_v - 1) - g$ so the dimension of $H^0(\text{div};\Omega) \cap RT_h$ is $n_e - (n_v - 1) - g + p = n_f - n_t$, because $n_v - n_e + n_f - n_t = 1 - g + p$ by the Euler–Poincaré formula.

Having in mind this picture, the first procedure we present for the construction of a basis of $H^0(\text{div};\Omega) \cap RT_h$ is an extension of previous approaches, that are indeed restricted to special domains: Dubois [3] and Scheichl [4] assume that Ω is simply-connected (which is equivalent to require $g = 0$, see Benedetti, Frigerio and Ghiloni [14, Sect. 3.2])); Rapetti, Dubois and Bossavit [5] assume that Ω is a κ-fold torus, thus in particular $p = 0$. Our method consists in the determination of a suitable basis of $\operatorname{curl} N_h$, making use of some tools of graph theory, with the addition of p linearly independent functions in $H^0(\text{div};\Omega) \cap RT_h$ that are not in $\operatorname{curl} N_h$. (Let us recall that the use of graph theory in numerical fluid dynamics has been also proposed by Hecht [2], Gustafson and Hartman [6], [7].)

A second option is to follow an algebraic approach, that is related to what presented in Alotto and Perugia (see [17], [18]). This procedure is essentially grounded on the solution of the following problem: given $f_h \in PC_h$, find $\mathbf{v}_h \in RT_h$ such that $\operatorname{div} \mathbf{v}_h = f_h$. This problem is in fact a linear system of n_t equations and n_f unknowns and the matrix associated to this linear system is the transpose of the incidence matrix that for each tetrahedron returns its faces. Thus, the construction of a basis of $H^0(\text{div};\Omega) \cap RT_h$ coincides with the computation of a basis of the kernel of this matrix.

3.1. Using the curls

Recall that $\{\mathbf{w}_{h,j}\}_{j=1}^{n_e}$ is a basis of N_h. Then the set of functions $\{\operatorname{curl} \mathbf{w}_{h,j}\}_{j=1}^{n_e}$ generates a subspace of $H^0(\text{div};\Omega) \cap RT_h$ but these functions are not linearly independent, because there are functions in N_h that are curl-free.

The main idea is quite natural, but in the most general case it is somehow hidden behind some technical aspects. Therefore we prefer to start with a case that very often occurs in numerical computations (and we will return later to the general case): we assume to know a set of 1-cycles σ_n, $n = 1, \ldots, g$, in $\mathcal{T}_h$, representing a basis of $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$, and that these 1-cycles are mutually disjoint polygonal loops without self-intersection. For each $n = 1, \ldots, g$, select one edge $\epsilon_n^\star$ belonging to σ_n; the set $\sigma_n \setminus \{\epsilon_n^\star\}$ is therefore a tree, and we can find a spanning tree $S = (V, M)$ of the graph (V, E) such that all the edges of each $\sigma_n \setminus \{\epsilon_n^\star\}$ belong to this spanning tree, while the edges $\{\epsilon_n^\star\}_{n=1}^g$ belong to the cotree (namely, the set of edges not belonging to the tree). Let us also assume that we have numbered the edges in such a way that $\{\epsilon_n^\star\}_{n=1}^g$ are the first g edges and that the edges belonging to the tree are the last $(n_v - 1)$ edges.

The first set of basis functions is selected by choosing the curl of the Nédélec basis functions $\mathbf{w}_{h,l}$ corresponding to the edges belonging to the cotree, but different from the "closing" edges $\{\epsilon_n^\star\}_{n=1}^g = \{e_l\}_{l=1}^g$. We recall that some authors have given the name of "belted tree" to the graph $(V, M \cup \{\epsilon_n^\star\}_{n=1}^g)$ (that indeed is not a tree), see, e.g., Ren and Razek [19], Kettunen et al. [20], Rapetti et al. [5]: using this notation, we are selecting the curls of the Nédélec basis functions corresponding to the edges not belonging to the "belted tree".

Proposition 1. *The vector fields*

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=g+1}^{n_e-(n_v-1)} \subset H^0(\operatorname{div};\Omega) \cap RT_h$$

are linearly independent.

Proof. Suppose that we have

$$0 = \sum_{l=g+1}^{n_e-(n_v-1)} \alpha_l \operatorname{curl} \mathbf{w}_{h,l} = \operatorname{curl} \left(\sum_{l=g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} \right) .$$

For each $n = 1, \ldots, g$ we find

$$\oint_{\sigma_n} \left(\sum_{l=g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} \right) \cdot d\mathbf{s} = 0 \,,$$

as, by construction, σ_n is composed by the "closing" edge $\epsilon_n^\star = e_n$ and by edges belonging to the spanning tree. Thus we can conclude that $\sum_{l=g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l}$ is a gradient, say, $\operatorname{grad} \varphi_h$. Indeed, it is easily seen that φ_h is constant: in fact, for all $i = 2, \ldots, n_v$ we have

$$\varphi_h(v_i) - \varphi_h(v_1) = \int_{c_i} \operatorname{grad} \varphi_h \cdot \boldsymbol{\tau} = \sum_{l=g+1}^{n_e-(n_v-1)} \alpha_l \int_{c_i} \mathbf{w}_{h,l} \cdot \boldsymbol{\tau} \,,$$

where c_i denotes the unique path, composed by edges belonging to the spanning tree, connecting v_1 and v_i; hence $\int_{c_i} \mathbf{w}_{h,l} \cdot \boldsymbol{\tau} = 0$ for each $l = g+1, \ldots, n_e-(n_v-1)$ and $\varphi_h(v_i) - \varphi_h(v_1) = 0$. In conclusion $\varphi_h \equiv \varphi_h(v_1)$ and $\sum_{l=g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} = \operatorname{grad} \varphi_h = 0$. Since $\{\mathbf{w}_{h,j}\}_{j=1}^{n_e}$ is a basis of N_h, it follows $\alpha_l = 0$ for all $l = g+1, \ldots, n_e - (n_v - 1)$. □

Note that if the topology of Ω is trivial (namely, $g = p = 0$, and therefore Ω is homeomorphic to a cube, see Benedetti, Frigerio and Ghiloni [14, Theor. 3.2]) the procedure above reduces to the determination of the basis as the set $\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=1}^{n_e-(n_v-1)}$, the curls of the Nédélec basis functions associated to the edges of the cotree, and nothing else must be done: in fact, in this case the dimension of $H^0(\operatorname{div};\Omega) \cap RT_h$ is $n_e - (n_v - 1)$.

If $g \neq 0$ the result proved in Proposition 1 shows that one has to disregard also the curls associated to some edges belonging to the cotree, namely, to the "closing" edges $\{\epsilon_n^\star\}_{n=1}^g = \{e_l\}_{l=1}^g$. Instead, when $p \neq 0$ we have to add some other basis functions.

Suppose now that $p \geq 1$ and consider the following dual graph: the dual vertices are $W = T \cup \Sigma$, where the elements of T are the tetrahedra of the mesh and the elements of Σ are the $p + 1$ connected components of $\partial\Omega$. The set of dual arcs is F, the set of the faces of the mesh; an internal face connects two tetrahedra, while a boundary face connects a tetrahedron and a connected component of $\partial\Omega$. So the dual graph is given by (W, F).

The number of dual vertices is equal to $n_t + p + 1$, hence a spanning tree $\mathcal{S} = (W, \mathcal{M})$ of (W, F) has $n_t + p$ dual arcs (and consequently its cotree has $n_f - n_t - p$ dual arcs). Therefore the linear system

$$\begin{cases} \operatorname{div} \mathbf{v}_h = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r & \forall\, r = 1, \dots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu} = d_f & \forall\, f \notin \mathcal{M} \end{cases} \tag{6}$$

is a square linear system of n_f equations and unknowns. In Alonso Rodríguez and Valli [16] it has been shown that it is uniquely solvable, and it is also described how to construct the solution $\mathbf{v}_h$ in an efficient way.

For each $s = 1, \dots, p$, let us denote $\mathbf{v}_{h,s} \in RT_h$ the unique solution of

$$\begin{cases} \operatorname{div} \mathbf{v}_{h,s} = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_{h,s} \cdot \boldsymbol{\nu} = \delta_{s,r} & \forall\, r = 1, \dots, p \\ \int_f \mathbf{v}_{h,s} \cdot \boldsymbol{\nu} = 0 & \forall\, f \notin M\,. \end{cases} \tag{7}$$

Theorem 1. *Let $\{\mathbf{v}_{h,s}\}_{s=1}^p$ be the vector fields introduced in (7). The set*

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=g+1}^{n_e-(n_v-1)} \cup \{\mathbf{v}_{h,s}\}_{s=1}^p$$

is a basis of $H^0(\operatorname{div}; \Omega) \cap RT_h$.

Proof. Since the flux of a curl through a closed surface is vanishing, the flux through $(\partial\Omega)_r$ of a linear combination of these vector fields reduces to

$$\sum_{s=1}^p \alpha_s \int_{(\partial\Omega)_r} \mathbf{v}_{h,s} \cdot \boldsymbol{\nu} = \alpha_r\,.$$

Hence we have a set of linearly independent vector fields, and their number is $n_e - (n_v - 1) - g + p = n_f - n_t$, the dimension of $H^0(\operatorname{div}; \Omega) \cap RT_h$. □

3.1.1. The general case

From the theoretical point of view every bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, equipped with a triangulation $\mathcal{T}_h$, admits a "belted tree" and hence there always exist 1-cycles $\{\sigma_n\}_{n=1}^g$ in $\mathcal{T}_h$, representing a basis of $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$, with the additional properties that they are mutually disjoint polygonal loops without self-intersection. However, we do not know any algorithm able to compute them explicitly. On the other hand, as shown in Hiptmair and Ostrowski [8] (see also Alonso Rodríguez et al. [9]), it is possible to construct the 1-cycles $\{\sigma_n\}_{n=1}^g$ without the additional properties: each σ_n is a formal sum of edges in $\mathcal{T}_h$ with integer coefficients.

To make precise this argument, we must slightly modify the procedure used before. First, let us consider the graph given by the vertices and the edges of $\mathcal{T}_h$ on $\partial\Omega$. The number of connected components of this graph coincides with the number of connected components of $\partial\Omega$. For each $r = 0, 1, \dots, p$ let

$S^r_{\partial\Omega} = (V^r_{\partial\Omega}, M^r_{\partial\Omega})$ be a spanning tree of the corresponding connected component of the graph. Then consider the graph (V, E), given by all the vertices and edges of $\mathcal{T}_h$, and a spanning tree of this graph, $S = (V, M)$, such that $M^r_{\partial\Omega} \subset M$ for each $r = 0, 1, \dots, p$. Let us order the edges in such a way that the edge e_l belongs to the cotree for $l = 1, \dots, n_e - (n_v - 1)$ and the last $n_v - 1$ edges belong to the tree, namely, the edges $e_{n_e-(n_v-1)+i}$ belong to the tree for $i = 1, \dots, n_v - 1$. In particular, denote by e_q, $q = 1, \dots, 2g$, the set of edges of $\partial\Omega$, constructed by Hiptmair and Ostrowski [8], that have the following properties: they all belong to the cotree, and each one of them "closes" a 1-cycle γ_q that is a representative of a basis of the first homology group $\mathcal{H}_1(\partial\Omega, \mathbb{Z})$ (whose rank is indeed equal to $2g$). The difference with respect to the previous case is that, instead of the g edges $\epsilon^\star_n$ that are "closing" edges for σ_n, the representatives of a basis of the first homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$, now we know the $2g$ edges e_q that are "closing" edges for γ_q, the representatives of a basis of the first homology group $\mathcal{H}_1(\partial\Omega, \mathbb{Z})$.

Having clarified the situation, we recall that the 1-cycles σ_n can be expressed as the formal sum

$$\sigma_n = \sum_{q=1}^{2g} A_{n,q}\gamma_q = \sum_{q=1}^{2g} A_{n,q} e_q + \sum_{i=n_e-(n_v-1)+1}^{n_e} a_{n,i} e_i \, , \tag{8}$$

for suitable and explicitly computable integers $A_{n,q}$.

The idea of the method is now the following: first, consider the set

$$\{\mathrm{curl}\, \mathbf{w}_{h,l}\}_{l=2g+1}^{n_e-(n_v-1)} \, ,$$

that, as in Proposition 1, is easily shown to be composed by linearly independent vector fields. However, since the index l starts from $2g + 1$ and not from $g + 1$, for replacing Proposition 1 with another somehow equivalent statement we need to select other g independent functions. The procedure reads as follows.

Look for g functions $\mathbf{z}_{h,\lambda} \in RT_h$, $\lambda = 1, \dots, g$, of the form

$$\mathbf{z}_{h,\lambda} = \sum_{\upsilon=1}^{2g} c^{(\lambda)}_\upsilon \, \mathrm{curl}\, \mathbf{w}_{h,\upsilon} \, ,$$

where the linearly independent vectors $\mathbf{c}^{(\lambda)} \in \mathbb{R}^{2g}$ are chosen in such a way that

$$\oint_{\sigma_n} \left(\sum_{\upsilon=1}^{2g} c^{(\lambda)}_\upsilon \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} = 0$$

for $n = 1, \dots, g$. This can be done since σ_n is formed by the "closing" edges e_q, $q = 1, \dots, 2g$, and by edges belonging to the spanning tree, so that

$$\oint_{\sigma_n} \left(\sum_{\upsilon=1}^{2g} c^{(\lambda)}_\upsilon \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} \; = \sum_{q=1}^{2g} A_{n,q} \int_{e_q} \left(\sum_{\upsilon=1}^{2g} c^{(\lambda)}_\upsilon \mathbf{w}_{h,\upsilon} \right) \cdot \boldsymbol{\tau} = \sum_{q=1}^{2g} A_{n,q} c^{(\lambda)}_q \, ,$$

and the matrix $A \in \mathbb{Z}^{g\times 2g}$ with entries $A_{n,q}$ has rank g (see Hiptmair and Ostrowski [8], Alonso Rodríguez et al. [9, Sect. 6]). Thus we only have to determine a basis $\mathbf{c}^{(\lambda)} \in \mathbb{R}^{2g}$ of the kernel of A, $\lambda = 1, \dots, g$.

Since in all the cases interesting for applications the matrix A has a relatively small dimension (the genus g of $\overline{\Omega}$ is very often a small number, say, less than twenty), finding the vectors $\mathbf{c}^{(\lambda)}$ is an easy task. However, it is worth noting that, since the 1-cycles σ_n have been determined by means of the procedure proposed by Hiptmair and Ostrowski [8], a suitable choice of the vector fields $\mathbf{c}^{(\lambda)}$ is already available.

In fact, let us denote by $2g_r$, $r = 0, 1, \dots, p$, the rank of the first homology group $\mathcal{H}_1((\partial\Omega)_r, \mathbb{Z})$; clearly we have $\sum_{r=0}^{p} 2g_r = 2g$. Acting on each connected component $(\partial\Omega)_r$ of the boundary $\partial\Omega$ and proceeding as in Hiptmair and Ostrowski [8] we construct the 1-cycles $\gamma_t^{(r)}$, $t = 1, \dots, 2g_r$, that are representatives of a basis of the first homology group $\mathcal{H}_1((\partial\Omega)_r, \mathbb{Z})$ (and that, all together, are representatives of a basis of the first homology group $\mathcal{H}_1(\partial\Omega, \mathbb{Z})$). We also determine an edge $e_t^{(r)}$, that is the only edge of $\gamma_t^{(r)}$ belonging to the cotree and that "closes" the 1-cycle $\gamma_t^{(r)}$. Let us order these "closing" edges as $\{e_t^{(0)}\}_{t=1}^{2g_0}$, $\{e_t^{(1)}\}_{t=1}^{2g_1}$, $\dots$, $\{e_t^{(p)}\}_{t=1}^{2g_p}$, and similarly the 1-cycles $\{\sigma_n\}_{n=1}^{g}$ as $\{\sigma_m^{(0)}\}_{m=1}^{g_0}$, $\{\sigma_m^{(1)}\}_{m=1}^{g_1}$, $\dots$, $\{\sigma_m^{(p)}\}_{m=1}^{g_p}$. For each $m = 1, \dots, g_r$ we can write

$$\sigma_m^{(r)} = \sum_{t=1}^{2g_r} A_{m,t}^{(r)} \gamma_t^{(r)} ,$$

for suitable integer coefficients $A_{m,t}^{(r)}$. The $(g \times 2g)$-matrix A in (8) is the block matrix having $A^{(r)}$ as diagonal blocks.

Set $Q_0 = 0$ and $Q_r = \sum_{s=0}^{r-1} g_s$, $r = 1, \dots, p$. As before, for each $r = 0, 1, \dots, p$ we want to select a set of vector fields $\mathbf{z}_{h,\omega}^{(r)} \in RT_h$, $\omega = 1, \dots, g_r$, of the form

$$\mathbf{z}_{h,\omega}^{(r)} = \sum_{\rho=1}^{2g_r} c_\rho^{(r,\omega)} \operatorname{curl} \mathbf{w}_{h,\rho+2Q_r} , \tag{9}$$

where the linearly independent vectors $\mathbf{c}^{(r,\omega)} \in \mathbb{R}^{2g_r}$ are chosen in such a way that

$$\oint_{\sigma_m^{(r)}} \mathbf{z}_{h,\omega}^{(r)} \cdot d\mathbf{s} = \oint_{\sigma_m^{(r)}} \left(\sum_{\rho=1}^{2g_r} c_\rho^{(r,\omega)} \mathbf{w}_{h,\rho+2Q_r} \right) \cdot d\mathbf{s} = 0 \tag{10}$$

for each $m = 1, \dots, g_r$.

It is now useful to introduce some notation. Denote by $\ell_{\mathcal{K}}(\gamma, \gamma')$ the linking number between two disjoint 1-cycles γ and γ' and by $G^{(r)} = (G_{\rho,t}^{(r)}) \in \mathbb{Z}^{2g_r \times 2g_r}$ the matrix with entries $G_{\rho,t}^{(r)} = \ell_{\mathcal{K}}(\gamma_\rho^{(r)}, R^+\gamma_t^{(r)})$, where $R^+\gamma_t^{(r)}$ is a 1-cycle homologous to $\gamma_t^{(r)}$ and completely contained in Ω (therefore not intersecting $\gamma_\rho^{(r)}$). The rank of the matrix $G^{(r)}$ turns out to be equal to g_r (see Alonso Rodríguez et al. [9] for a more detailed presentation of these tools and arguments). We can prove:

Lemma 1. *For each* $r = 0, 1, \ldots, p$ *a set of linearly independent vectors* $\mathbf{c}^{(r,m)} \in \mathbb{Z}^{2g_r}$, $m = 1, \ldots, g_r$, *to be used in (9) for obtaining (10), is given by* g_0 *independent columns of the matrix* $G^{(0)}$ *(if* $r = 0$*) and by* g_r *independent rows of the matrix* $G^{(r)}$ *(if* $r = 1, \ldots, p$*).*

Proof. Since $\sigma_m^{(r)} = \sum_{t=1}^{2g_r} A_{m,t}^{(r)} \gamma_t^{(r)} = \sum_{t=1}^{2g_r} A_{m,t}^{(r)} e_t^{(r)} + \sum_{i=n_e-(n_v-1)+1}^{n_e} a_{m,i}^{(r)} e_i$ and for $\rho \in \{1, \ldots, 2g_r\}$ the index $\rho + 2Q_r$ denotes edges belonging to the cotree, we can rewrite (10) as

$$\begin{aligned} 0 &= \oint_{\sigma_m^{(r)}} \left(\sum_{\rho=1}^{2g_r} c_\rho^{(r,m)} \mathbf{w}_{h,\rho+2Q_r} \right) \cdot d\mathbf{s} \\ &= \sum_{t=1}^{2g_r} A_{m,t}^{(r)} \left(\sum_{\rho=1}^{2g_r} c_\rho^{(r,m)} \oint_{e_t^{(r)}} \mathbf{w}_{h,\rho+2Q_r} \cdot d\mathbf{s} \right) = \sum_{t=1}^{2g_r} A_{m,t}^{(r)} c_t^{(r,m)} \end{aligned}$$

for each $m = 1, \ldots, g_r$, since $\oint_{e_t^{(r)}} \mathbf{w}_{h,\rho+2Q_r} \cdot d\mathbf{s} = \delta_{t,\rho}$. From the results in Alonso Rodríguez et al. [9, Sect. 6] (see also Hiptmair and Ostrowski [8, Sect. 4]) we know that

$$\begin{aligned} &\sum_{t=1}^{2g_0} (G^{(0)})_{\rho,t}^T A_{m,t}^{(0)} = 0 \quad , \quad \forall\, m = 1, \ldots, g_0, \rho = 1, \ldots, 2g_0 \\ &\sum_{t=1}^{2g_r} G_{\rho,t}^{(r)} A_{m,t}^{(r)} = 0 \quad , \quad \forall\, m = 1, \ldots, g_r, \rho = 1, \ldots, 2g_r \ (r = 1, \ldots, p) \, . \end{aligned}$$

Therefore, in order to complete the proof, it is sufficient to define the vectors $\mathbf{c}^{(0,m)}$ as g_0 linearly independent rows of $(G^{(0)})^T$ (namely, g_0 independent columns of $G^{(0)}$), and the vectors $\mathbf{c}^{(r,m)}$, $r = 1, \ldots, p$, as g_r linearly independent rows of $G^{(r)}$. □

Note that the square matrices $G^{(r)}$ have dimension $2g_r$, a very small number in all the cases interesting for applications. Therefore determining g_r independent rows or columns of $G^{(r)}$ is quite cheap.

We are now in a position to conclude, taking into account the block structure of the problem.

Proposition 2. *Let* $\mathbf{z}_{h,\lambda}$, $\lambda = 1, \ldots, g$, *be the vector fields of the form* $\mathbf{z}_{h,\lambda} = \sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \operatorname{curl} \mathbf{w}_{h,\upsilon}$, *with* $\mathbf{c}^{(\lambda)} \in \mathbb{Z}^{2g}$ *given by* $\mathbf{c}^{(\lambda)} = (\mathbf{c}^{(0,\lambda)}, \mathbf{0}, \ldots, \mathbf{0})$ *for* $\lambda = 1, \ldots, g_0$, $\mathbf{c}^{(\lambda)} = (\mathbf{0}, \mathbf{c}^{(1,\lambda-Q_1)}, \mathbf{0}, \ldots, \mathbf{0})$ *for* $\lambda = Q_1 + 1, \ldots, Q_1 + g_1$, $\ldots$, $\mathbf{c}^{(\lambda)} = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{c}^{(p,\lambda-Q_p)})$ *for* $\lambda = Q_p + 1, \ldots, Q_p + g_p = g$, *the vectors* $\mathbf{c}^{(r,m)} \in \mathbb{Z}^{2g_r}$ *being determined in Lemma 1. The vector fields*

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=2g+1}^{n_e-(n_v-1)} \cup \{\mathbf{z}_{h,\lambda}\}_{\lambda=1}^{g} \subset H^0(\operatorname{div}; \Omega) \cap RT_h$$

are linearly independent.

Proof. Suppose that we have

$$\begin{aligned} 0 &= \sum_{l=2g+1}^{n_e-(n_v-1)} \alpha_l \operatorname{curl} \mathbf{w}_{h,l} + \sum_{\lambda=1}^{g} \beta_\lambda \mathbf{z}_{h,\lambda} \\ &= \operatorname{curl} \left(\sum_{l=2g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} + \sum_{\lambda=1}^{g} \beta_\lambda \sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) . \end{aligned}$$

The 1-cycle σ_n is formed by the "closing" edges e_q, $q = 1, \ldots, 2g$, and by edges belonging to the spanning tree $S = (V, M)$, thus for each $n = 1 \ldots, g$ we have

$$\oint_{\sigma_n} \left(\sum_{l=2g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} \right) \cdot d\mathbf{s} = 0 \,.$$

Moreover, since the 1-cycle σ_n is equal to a 1-cycle $\sigma_m^{(r)}$ for a suitable $r \in \{0, 1, \ldots, p\}$ and $m \in \{1, \ldots, g_r\}$, we have

$$\begin{aligned} \oint_{\sigma_m^{(r)}} \left(\sum_{\lambda=1}^{g} \beta_\lambda \sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} &= \sum_{\lambda=1}^{g} \beta_\lambda \oint_{\sigma_m^{(r)}} \left(\sum_{\rho=1}^{2g_r} c_{\rho+2Q_r}^{(\lambda)} \mathbf{w}_{h,\rho+2Q_r} \right) \cdot d\mathbf{s} \\ &= \sum_{\omega=1}^{g_r} \beta_{\omega+Q_r} \oint_{\sigma_m^{(r)}} \left(\sum_{\rho=1}^{2g_r} c_\rho^{(r,\omega)} \mathbf{w}_{h,\rho+2Q_r} \right) \cdot d\mathbf{s} = 0 \,, \end{aligned}$$

having used (10).

As in Proposition 1, we conclude that

$$\begin{aligned} \mathbf{0} &= \sum_{l=2g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} + \sum_{\lambda=1}^{g} \beta_\lambda \sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \\ &= \sum_{l=2g+1}^{n_e-(n_v-1)} \alpha_l \mathbf{w}_{h,l} + \sum_{\upsilon=1}^{2g} \left(\sum_{\lambda=1}^{g} \beta_\lambda c_\upsilon^{(\lambda)} \right) \mathbf{w}_{h,\upsilon} \,. \end{aligned}$$

Since $\mathbf{w}_{h,j}$, $j = 1, \ldots, n_e$, are linearly independent, we find $\alpha_l = 0$ for $l = 2g+1, \ldots, n_e - (n_v - 1)$ and $\sum_{\lambda=1}^{g} \beta_\lambda c_\upsilon^{(\lambda)} = 0$ for $\upsilon = 1, \ldots, 2g$. The vectors $\mathbf{c}^{(\lambda)}$, $\lambda = 1, \ldots, g$, are also linearly independent, hence we have $\beta_\lambda = 0$ for $\lambda = 1, \ldots, g$. □

The final theorem now reads:

Theorem 2. *Let $\{\mathbf{z}_{h,\lambda}\}_{\lambda=1}^{g}$ be the vector fields introduced in Proposition 2 and $\{\mathbf{v}_{h,s}\}_{s=1}^{p}$ the vector fields introduced in (7). The set*

$$\{\mathrm{curl}\, \mathbf{w}_{h,l}\}_{l=2g+1}^{n_e-(n_v-1)} \cup \{\mathbf{z}_{h,\lambda}\}_{\lambda=1}^{g} \cup \{\mathbf{v}_{h,s}\}_{s=1}^{p}$$

is a basis of $H^0(\mathrm{div}; \Omega) \cap RT_h$.

The proof is the same than that of Theorem 1.

Remark 1. *Let us mention that the numerical approximation of the solution to problem (4) in a domain homeomorphic to a cube (namely, $g = p = 0$) has been considered in Hiptmair and Hoppe [21], using as divergence-free finite elements the curls of all the Nédélec elements, without eliminating those associated to some edges. In this way the resulting algebraic system is expressed by a* *symmetric and positive semidefinite singular matrix. However, a suitable multi-level algorithm is shown to be an efficient way for determining the approximate solution.*

Clearly, being able to select appropriate edges leads to an algebraic problem that is associated to a symmetric and positive definite non-singular matrix of smaller size ($n_e - n_v + 1$ rows instead of n_e rows), for which iterative algorithms and suitable preconditioners are easier to devise.

3.2. The algebraic approach

For describing the algebraic approach, let us start with some definitions. The degrees of freedom z_j of a Raviart–Thomas finite element $\mathbf{z}_h = \sum_{j=1}^{n_f} z_j \mathbf{r}_{h,j}$ are the fluxes across the faces of the triangulation, having chosen the orientation of the unit normal vector $\boldsymbol{\nu}$ as indicated before. Let t_i be a tetrahedron and f_j a face of $\mathcal{T}_h$. We define the integer $o_{t_i}(f_j)$ by 0 if f_j is not a face of t_i, by 1 if f_j is a face of t_i for which the chosen orientation coincides with the external to t_i, by -1 if f_j is a face of t_i for which the chosen orientation is opposite to the external to t_i.

Since the divergence of $\mathbf{z}_h \in RT_h$ is piecewise-constant, the condition $\operatorname{div} \mathbf{z}_h = 0$ can be written as $\int_{t_i} \operatorname{div} \mathbf{z}_h = 0$ for all $i = 1, \ldots, n_t$. Therefore, we have $\operatorname{div} \mathbf{z}_h = 0$ if and only if $\int_{\partial t_i} \mathbf{z}_h \cdot \boldsymbol{\nu} = 0$ for all $i = 1, \ldots, n_t$, namely, if and only if

$$0 = \int_{\partial t_i} \mathbf{z}_h \cdot \boldsymbol{\nu} = \sum_{j=1}^{n_f} o_{t_i}(f_j) z_j \quad \forall\, i = 1, \ldots, n_t \,. \tag{11}$$

Note that the sum in (11) reduces indeed to only four terms.

Let us consider the following dual graph (it is a slight modification of the dual graph used in Section 3.1, and this choice makes easier the following arguments): the nodes are the tetrahedra of $\mathcal{T}_h$ plus an additional node, t_0, representing $\mathbb{R}^3 \setminus \Omega$; the arcs are the faces of $\mathcal{T}_h$. The faces in Ω connect two tetrahedra while a face on $\partial\Omega$ connects the tetrahedron containing that face with the additional node t_0. Let us denote by $\widetilde{W}$ the set $T \cup \{t_0\}$; a spanning tree $\widetilde{\mathcal{S}} = (\widetilde{W}, \widetilde{\mathcal{M}})$ of this graph $(\widetilde{W}, F)$ contains $n_t + 1 - 1 = n_t$ arcs while the cotree contains $n_f - n_t$ arcs.

The main point now is to note that, if $\mathbf{z}_h \in RT_h$ and $\operatorname{div} \mathbf{z}_h = 0$, the degrees of freedom corresponding to a face in the spanning tree can be expressed in terms of the degrees of freedom corresponding to faces in the cotree (this fact has been previously called "tree–cotree condensation", see, e.g., Alotto and Perugia [17], [18]).

The approach adopted in Alonso Rodríguez and Valli [16] clearly illustrates this assertion, and reads in this way. The leaves of the spanning tree are either a tetrahedron t_i with exactly one face in the spanning tree or the node t_0 representing $\mathbb{R}^3 \setminus \Omega$ (if just one face of $F_{\partial\Omega}$ belongs to the spanning tree). Consider a leave $t_i \neq t_0$: using (11), the degree of freedom corresponding to the unique face of t_i belonging to the tree can be computed in terms of the degrees of freedom associated to the three other faces of t_i in the cotree. Now we can eliminate from the spanning tree the leaves and the arcs arriving to the leaves, and repeat the procedure with the new tree (it is not longer a spanning tree of the original graph, but this is not relevant).

The overall procedure can be formally expressed as a linear map from the $n_f - n_t$ unknowns associated to the cotree of $\widetilde{\mathcal{S}}$ in the dual graph to the n_t unknowns associated to the spanning tree $\widetilde{\mathcal{S}}$. An explicit construction of the matrix expressing this procedure is given in Alotto and Perugia [17], [18], and can be described as follows.

First of all, note that each face $f_k \in \widetilde{\mathcal{M}}$ splits the spanning tree $\widetilde{\mathcal{S}}$ into two connected components. One of them, denoted by $\widetilde{\mathcal{S}}(f_k) = (\widetilde{\mathcal{W}}(f_k), \widetilde{\mathcal{M}}(f_k))$, does not contain t_0. Let us indicate by $\mathcal{V}_k$ the set given by the union of the tetrahedra belonging to $\widetilde{\mathcal{W}}(f_k)$, that is, $\mathcal{V}_k = \cup_{t_i \in \widetilde{\mathcal{W}}(f_k)} t_i \subset \overline{\Omega}$. Since f_k is joining the two connected components of the spanning tree, it follows that $f_k \subset \partial\mathcal{V}_k$; moreover, f_k is the unique face in $\partial\mathcal{V}_k$ belonging to $\widetilde{\mathcal{M}}$.

Let us identify $\mathcal{V}_k$ with the vector $\mathbf{V}^{(k)} \in \mathbb{N}^{n_t}$ with coefficients $v_i^{(k)} = 1$ if $t_i \in \widetilde{\mathcal{W}}(f_k)$ and $v_i^{(k)} = 0$ otherwise, and, as usual, each $\mathbf{z}_h \in RT_h$ with the vector $\mathbf{Z}$ of its coefficients z_j in the canonical base, $j = 1, \ldots, n_f$. Let us denote by B the incidence matrix, namely, the $(n_f \times n_t)$-matrix that for each tetrahedron returns its faces; it is worth noting that B is the transpose of the matrix D expressing the divergence operator.

We are now in a position to conclude: if $\mathbf{z}_h \in RT_h$ satisfies $\operatorname{div} \mathbf{z}_h = 0$, the divergence theorem ensures that $\int_{\partial\mathcal{V}_k} \mathbf{z}_h \cdot \boldsymbol{\nu} = 0$, therefore for each k such that $f_k \in \widetilde{\mathcal{M}}$ we have found

$$B\mathbf{V}^{(k)} \cdot \mathbf{Z} = 0\,. \tag{12}$$

Since the unique face in $\partial\mathcal{V}_k$ belonging to $\widetilde{\mathcal{M}}$ is f_k, from (12) we can write z_k in terms of the degrees of freedom associated to the remaining faces in $\partial\mathcal{V}_k$ that belong to the cotree. In other words, having ordered the faces in such a way that the face f_l belongs to the cotree for $l = 1, \ldots, n_f - n_t$ and the face $f_{n_f - n_t + i}$ belongs to the tree for $i = 1, \ldots, n_t$, we have seen that, if $\mathbf{z}_h = \sum_{j=1}^{n_f} z_j \mathbf{r}_{h,j}$ satisfies $\operatorname{div} \mathbf{z}_h = 0$, for each $i = 1, \ldots, n_t$ it holds

$$z_{n_f - n_t + i} - \sum_{l=1}^{n_f - n_t} m_{i,l} z_l = 0\,,$$

where the coefficients $m_{i,l}$ take the values -1, 0 or 1.

This statement can be made more precise:

Proposition 3. *Let* $\mathbf{z}_h = \sum_{j=1}^{n_f} z_j \mathbf{r}_{h,j}$ *be an element of* RT_h *and define by* $\mathbf{Z}$ *the vector with coefficients* z_j*,* $j = 1, \ldots, n_f$*. Then we have* $\operatorname{div} \mathbf{z}_h = 0$ *in* Ω *if and only if*

$$\begin{bmatrix} -M & I_{n_t} \end{bmatrix} \mathbf{Z} = \mathbf{0}\,, \tag{13}$$

where the matrix M *has entries* $m_{i,l}$*,* $i = 1, \ldots, n_t$*,* $l = 1, \ldots, n_f - n_t$*, and* I_{n_t} *denotes the identity of dimension* n_t*.*

Proof. We have already shown that if $\operatorname{div} \mathbf{z}_h = 0$ then (13) is satisfied.

Thus it remains to prove that (13) implies $\operatorname{div} \mathbf{z}_h = 0$. Since, after reordering, (13) is equivalent to (12), namely, to $\int_{\partial \mathcal{V}_k} \mathbf{z}_h \cdot \boldsymbol{\nu} = 0$ for each k such that the face f_k belongs to the spanning tree $\widetilde{\mathcal{M}}$, we have to prove that if $\int_{\partial \mathcal{V}_k} \mathbf{z}_h \cdot \boldsymbol{\nu} = 0$ for each k then $\int_t \operatorname{div} \mathbf{z}_h = 0$ for all $t \in T$.

Without loosing generality, we can assume that the additional node t_0, representing $\mathbb{R}^3 \setminus \Omega$, is the root of the spanning tree. First of all, it is clear that for a leave $\hat{t} \neq t_0$ one has $\hat{t} = \mathcal{V}_{\hat{k}}$, being $f_{\hat{k}}$ the only face in $\widetilde{\mathcal{M}}$ incident to the leave $\hat{t}$. Consequently, $\int_{\hat{t}} \operatorname{div} \mathbf{z}_h = 0$, as $\int_{\hat{t}} \operatorname{div} \mathbf{z}_h = \int_{\partial \hat{t}} \mathbf{z}_h \cdot \boldsymbol{\nu} = \int_{\partial \mathcal{V}_{\hat{k}}} \mathbf{z}_h \cdot \boldsymbol{\nu}$.

For a tetrahedron $t \in T$ let us define the distance $d(t, t_0)$ as the number of the faces that connect t with t_0 along the path of $\widetilde{\mathcal{M}}$; moreover, set $\mu = \max_{t \in T} d(t, t_0)$. We have just proved that $\int_t \operatorname{div} \mathbf{z}_h = 0$ for all $t \in T$ such that $d(t, t_0) = \mu$ (in fact, $d(t, t_0) = \mu$ says that t is a leave).

We use now an induction procedure. Supposing that the result is true for all tetrahedra t with $d(t, t_0) \geq m + 1$ for some m with $1 \leq m \leq \mu - 1$, we show that it is true for all tetrahedra at distance m. Consider $\hat{t}$ with $d(\hat{t}, t_0) = m$ and take the unique t^* with $d(t^*, t_0) = m - 1$ and $t^* \cap \hat{t} = f_{\hat{k}} \in \widetilde{\mathcal{M}}$. Let us set $\widetilde{\mathcal{W}}_-(f_{\hat{k}}) = \widetilde{\mathcal{W}}(f_{\hat{k}}) \setminus \{\hat{t}\}$; we clearly have $\mathcal{V}_{\hat{k}} = \hat{t} \cup \left(\cup_{t_i \in \widetilde{\mathcal{W}}_-(f_{\hat{k}})} t_i \right)$; moreover, notice that for all $t_i \in \widetilde{\mathcal{W}}_-(f_{\hat{k}})$ one has with $d(t_i, t_0) \geq m + 1$. Therefore we have

$$0 = \int_{\partial \mathcal{V}_{\hat{k}}} \mathbf{z}_h \cdot \boldsymbol{\nu} = \int_{\hat{t}} \operatorname{div} \mathbf{z}_h + \sum_{t_i \in \widetilde{\mathcal{W}}_-(f_{\hat{k}})} \int_{t_i} \operatorname{div} \mathbf{z}_h = \int_{\hat{t}} \operatorname{div} \mathbf{z}_h \,,$$

the last equality being true by the induction assumption. □

This proposition shows that a basis of $H^0(\operatorname{div}; \Omega) \cap RT_h$ can be derived from a basis of the kernel of $\begin{bmatrix} -M & I_{n_t} \end{bmatrix}$.

Theorem 3. *A basis of $H^0(\operatorname{div}; \Omega) \cap RT_h$ is given by the set of functions $\{\sum_{j=1}^{n_f} X_{j,l} \mathbf{r}_{h,j}\}_{l=1}^{n_f - n_t}$, where $X = \begin{bmatrix} I_{n_f - n_t} \\ M \end{bmatrix}$ and $I_{n_f - n_t}$ denotes the identity of dimension $n_f - n_t$.*

Proof. These vector fields are linearly independent. In fact, since $\{\mathbf{r}_{h,j}\}_{j=1}^{n_f}$ is a basis of the space RT_h, from

$$0 = \sum_{l=1}^{n_f - n_t} \alpha_l \left(\sum_{j=1}^{n_f} X_{j,l} \mathbf{r}_{h,j} \right) = \sum_{j=1}^{n_f} \left(\sum_{l=1}^{n_f - n_t} X_{j,l} \alpha_l \right) \mathbf{r}_{h,j} \,,$$

it follows that $\sum_{l=1}^{n_f - n_t} X_{j,l} \alpha_l = 0$ for each $j = 1, \ldots, n_f$. This means that the vector with entries α_l belongs to the kernel of $X = \begin{bmatrix} I_{n_f - n_t} \\ M \end{bmatrix}$, and the kernel of X is clearly trivial.

Moreover, from Proposition 3 it follows at once that a function $\sum_{j=1}^{n_f} X_{j,l} \mathbf{r}_{h,j}$ belongs to $H^0(\operatorname{div}; \Omega) \cap RT_h$ for each $l = 1, \ldots, n_f - n_t$.

The proof ends recalling that the dimension of the finite element space $H^0(\mathrm{div};\Omega) \cap RT_h$ is $n_f - n_t$. □

Remark 2. *It is worth noting that the mass matrix associated to the basis functions determined in Theorems 1 or 2 is a sparse matrix, except for very few rows and columns. In fact, the support of the basis functions* $\mathrm{curl}\, \mathbf{w}_{h,l}$ *and* $\mathbf{z}_{h,\lambda}$ *is localized; only the functions* $\mathbf{v}_{h,s}$ *have a non-localized support, but these functions are only* p*, and in all the cases interesting for applications* $p+1$ *(the number of connected components of the boundary* $\partial\Omega$*) is a small number.*

Instead, the mass matrix associated to the basis functions determined in Theorem 3 is not sparse. This probably explains why in the literature it is often asserted that the finite element basis functions of $H^0(\mathrm{div};\Omega)$ *have a non-localized support. We have seen that this is not always the case, and depends on which basis functions have been constructed.*

Remark 3. *An alternative construction of a basis of* $H^0(\mathrm{div};\Omega) \cap RT_h$ *can be done solving* $(n_f - n_t)$ *times problem (6) (for instance, with the method presented in Alonso Rodríguez and Valli [16]), each time with a right hand side having only one non-vanishing value among all* $\{c_r\}$*,* $r = 1, \ldots, p$*, and* $\{d_f\}$*,* $f \notin \mathcal{M}$*.*

Clearly, this procedure would furnish basis functions with non-localized support; moreover, its computational cost is much higher than that of the other two proposed approaches.

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