

# GROUPS THAT HAVE THE SAME HOLOMORPH AS A FINITE PERFECT GROUP

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ABSTRACT. We describe the groups that have the same holomorph as a finite perfect group. Our results are complete for centerless groups.

When the center is non-trivial, some questions remain open. The peculiarities of the general case are illustrated by a couple of examples that might be of independent interest.

## 1. INTRODUCTION

We are concerned with the question, when do two groups have the same holomorph? Recall that the *holomorph* of a group  $G$  is the natural semidirect product  $\text{Aut}(G)G$  of  $G$  by its automorphism group  $\text{Aut}(G)$ . To put this problem in proper context, recall that if  $\rho : G \rightarrow S(G)$  is the right regular representation of  $G$ , where  $S(G)$  is the group of permutations on the set  $G$ , then  $N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G)$  is isomorphic to the holomorph of  $G$ . We will also refer to  $N_{S(G)}(\rho(G))$  as the holomorph of  $G$ , and write it as  $\text{Hol}(G)$ . More generally, if  $N \leq S(G)$  is a regular subgroup, then  $N_{S(G)}(N)$  is isomorphic to the holomorph of  $N$ . We therefore begin to make the above question more precise by asking for which regular subgroup  $N$  of  $S(G)$  one has  $N_{S(G)}(N) = \text{Hol}(G)$ .

W.H. Mills has noted [17] that such an  $N$  need not be isomorphic to  $G$  (see Example 3.1, but also Example 1.2 below, and the comment following it). In this paper, we will be interested in determining the following set, and some naturally related ones

**Definition 1.1.**

$$\mathcal{H}(G) = \{N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G)\}.$$

G.A. Miller has shown [16] that the so-called *multiple holomorph* of  $G$

$$\text{NHol}(G) = N_{S(G)}(\text{Hol}(G))$$

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*Date:* 13 October 2017, 11:45 CEST — Version 9.08.

*2010 Mathematics Subject Classification.* 20B35 20D45 20D40.

*Key words and phrases.* holomorph, multiple holomorph, regular subgroups, finite perfect groups, central products, automorphisms.

The authors are members of INdAM—GNSAGA. The first author gratefully acknowledges support from the Department of Mathematics of the University of Trento.

acts transitively on  $\mathcal{H}(G)$ , and thus the group

$$T(G) = \text{NHol}(G)/\text{Hol}(G)$$

acts regularly on  $\mathcal{H}(G)$ .

Recently T. Kohl has described [15] the set  $\mathcal{H}(G)$  and the group  $T(G)$  for  $G$  dihedral or generalized quaternion. In [5], we have redone, via a commutative ring connection, the work of Mills [17], which determined  $\mathcal{H}(G)$  and  $T(G)$  for  $G$  a finitely generated abelian group.

In this paper we consider the case when  $G$  is a finite, *perfect* group, that is,  $G$  equals its derived subgroup  $G'$ .

If  $G$  has also trivial center, then one can show that if  $N \trianglelefteq \text{Hol}(G)$  is a regular subgroup, then  $N \in \mathcal{H}(G)$  (in particular,  $N \cong G$ ). The elements of  $\mathcal{H}(G)$  can be described in terms of a Krull-Remak-Schmidt decomposition of  $G$  as a group with  $\text{Aut}(G)$  as group of operators, that is, in terms of the unique decomposition of  $G$  as a direct product of non-trivial characteristic subgroups that are indecomposable as the direct product of characteristic subgroups. The group  $T(G)$  turns out to be an elementary abelian 2-group.

If  $G$  has non-trivial center, the regular subgroups  $N$  such that  $N \trianglelefteq \text{Hol}(G)$  can still be described in terms of the decomposition of  $G$  as the central product of non-trivial, perfect, characteristic subgroups, that are indecomposable as a central product of characteristic subgroups.

However, these  $N$  need not be isomorphic to  $G$  (see Example 1.2 below, and the comment following it), and the structure of  $T(G)$  in this case is not clear to us at the moment. The difficulties here are illustrated by the following examples, which might be of independent interest.

**Example 1.2.** There is a group  $G$  which is the central product of two characteristic subgroups  $Q_1, Q_2$ , such that  $G$  is not isomorphic to the group  $(G, \circ)$  obtained from  $G$  by replacing  $Q_1$  with its opposite.

Recall that the opposite of a group  $Q$  is the group obtained by exchanging the order of factors in the product of  $Q$ .

We will see in Section 7 that the group  $(G, \circ)$  is isomorphic to a regular subgroup  $N$  of  $S(G)$  such that  $N_{S(G)}(N) = \text{Hol}(G)$ . Therefore in our context the latter condition does not imply  $N \cong G$ .

**Example 1.3.** There is a group  $G$  which is the central product of three characteristic subgroups  $Q_1, Q_2, Q_3$  such that  $Q_1$  and  $Q_2$  are not characteristic in the group obtained from  $G$  by replacing  $Q_1$  with its opposite.

As we will see in Section 7, this example shows that if  $N \trianglelefteq \text{Hol}(G)$  is a regular subgroup, for  $G$  perfect, we may well have that  $N_{S(G)}(N)$  properly contains  $\text{Hol}(G)$ . However, if  $\text{Aut}(G)$  and  $\text{Aut}(N)$  have the same order, then  $N_{S(G)}(N) = \text{Hol}(G)$  (see Lemma 2.6(2)).

Example 1.2 and 1.3 are given in Subsection 7.2 as Proposition 7.10 and Proposition 7.9.

The plan of the paper is the following. Sections 2 and 3 introduce the holomorph and the multiple holomorph. In these sections, and in the following ones, we have chosen to repeat some elementary and well-known arguments, when we have deemed them handy for later usage. In Sections 4 and 5 we give a description of the regular subgroups  $N$  of  $\text{Hol}(G)$ , and of those that are normal in  $\text{Hol}(G)$ , in terms of a certain map  $\gamma : G \rightarrow \text{Aut}(G)$ . This leads to a group operation  $\circ$  on  $G$  such that  $N$  is isomorphic to  $(G, \circ)$ . In Section 6 we show that the values of  $\gamma$  on commutators are inner automorphisms, and this leads us to consider perfect groups.

In Section 7 we study the case of finite, perfect groups. We first obtain a description of the normal subgroups of  $\text{Hol}(G)$  that are regular, in terms of certain central product decompositions of  $G$  (Theorem 7.5). We then discuss separately, as explained above, the centerless case, where we can give a full picture, and the general case, where some questions remain open. Section 8 deals with a representation-theoretic method that is critical for the construction of the examples.

We note, as in [15], that this work is related to the enumeration of Hopf-Galois structures on separable field extensions, as C. Greither and B. Pareigis have shown [10] that these structures can be described through the regular subgroups of a suitable symmetric group, which are normalized by a given regular subgroup; this connection is exploited in the work of L. Childs [8], N.P. Byott [3], and Byott and Childs [4].

Our discussion of (normal) regular subgroups touches also on the subject of skew braces [13], see Remark 5.3.

We are very grateful to Robert Guralnick for several useful conversations. We are indebted to Derek Holt for kindly explaining to us in careful detail the example and the construction method which led to Proposition 7.11.

## 2. THE HOLOMORPH OF A GROUP

**Notation 2.1.** We write permutations as exponents, and denote compositions of maps by juxtaposition. We compose maps left-to-right.

The holomorph of a group  $G$  is the natural semidirect product

$$\text{Aut}(G)G$$

of  $G$  by its automorphism group  $\text{Aut}(G)$ . Let  $S(G)$  be the group of permutations on the set  $G$ . Consider the right and the left regular representations of  $G$ :

$$\left\{ \begin{array}{l} \rho : G \rightarrow S(G) \\ g \mapsto (x \mapsto xg) \end{array} \right. \quad \left\{ \begin{array}{l} \lambda : G \rightarrow S(G) \\ g \mapsto (x \mapsto gx) \end{array} \right.$$

**Notation 2.2.** We denote the inversion map  $g \mapsto g^{-1}$  on a group by  $\text{inv}$ .

**Definition 2.3.** The *opposite* of the group  $G$  is the group obtained by exchanging the order of factors in the product of  $G$ .

The map  $\text{inv}$  is an isomorphism between a group  $G$  and its opposite; compare with Proposition 2.4(4) below.

The following well-known fact should be compared with Lemma 2.4.2 of the paper [10] in which Greither and Pareigis set up the connection, already mentioned in the Introduction, between Hopf Galois extensions and regular subgroups of symmetric groups.

**Proposition 2.4.**

- (1)  $C_{S(G)}(\rho(G)) = \lambda(G)$  and  $C_{S(G)}(\lambda(G)) = \rho(G)$ .
- (2) The stabilizer of 1 in  $N_{S(G)}(\rho(G))$  is  $\text{Aut}(G)$ .
- (3) We have

$$N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G) = \text{Aut}(G)\lambda(G) = N_{S(G)}(\lambda(G)),$$

and this group is isomorphic to the holomorph  $\text{Aut}(G)G$  of  $G$ .

- (4) Inversion on  $G$  normalizes  $N_{S(G)}(\rho(G))$ , centralizes  $\text{Aut}(G)$ , and conjugates  $\rho(G)$  to  $\lambda(G)$ , that is

$$\rho(G)^{\text{inv}} = \lambda(G).$$

**Notation 2.5.** We write  $\text{Hol}(G) = N_{S(G)}(\rho(G))$ .

We will refer to either of the isomorphic groups  $N_{S(G)}(\rho(G))$  and  $\text{Aut}(G)G$  as the holomorph of  $G$ .

We now record another well-known fact.

**Lemma 2.6.**

- (1) Let  $\Omega$  be a set, and  $G$  a regular subgroup of  $S(\Omega)$ . Then there is an isomorphism  $S(\Omega) \rightarrow S(G)$  that sends  $G$  to  $\rho(G)$ , and thus  $N_{S(\Omega)}(G)$  to  $\text{Hol}(G)$ .
- (2) If  $N \leq S(G)$  is a regular subgroup, then  $N_{S(G)}(N)$  is isomorphic to the holomorph of  $N$ .

It is because of Lemma 2.6(1) that we have done without a set  $\Omega$ , and started directly with  $S(G)$  and its regular subgroup  $\rho(G)$ .

*Proof.* We only treat (2), for further reference.

Consider for such a regular subgroup  $N$  the bijection

$$\begin{aligned} \varphi : N &\rightarrow G \\ n &\mapsto 1^n. \end{aligned}$$

Then

$$\begin{aligned} \psi : S(G) &\rightarrow S(N) \\ \sigma &\mapsto \varphi\sigma\varphi^{-1} \end{aligned}$$

(recall that we compose left-to-right) is an isomorphism, which maps  $N$  onto  $\rho(N)$ , as for  $x, n \in N$  we have

$$x^{\varphi n \varphi^{-1}} = (1^{xn})^{\varphi^{-1}} = xn,$$

that is,

$$\varphi n \varphi^{-1} = \rho(n).$$

In particular,  $\psi(N_{S(G)}(N)) = N_{S(N)}(\rho(N)) = \text{Hol}(N)$ .  $\square$

### 3. GROUPS WITH THE SAME HOLOMORPH

In view of Lemma 2.6(2), one may inquire, what are the regular subgroups  $N \leq S(G)$  for which

$$(1) \quad \text{Hol}(N) \cong N_{S(G)}(N) = N_{S(G)}(\rho(G)) = \text{Hol}(G).$$

W.H. Mills has noted in [17] that if (1) holds, then  $G$  and  $N$  need not be isomorphic.

**Example 3.1.** The dihedral and the generalized quaternion groups of order  $2^n$ , for  $n \geq 4$ , have the same normalizer in  $S_{2^n}$ , in suitable regular representations. [14, 3.10], [15, 2.1].

When we restrict our attention to the regular subgroups  $N$  of  $S(G)$  for which  $N_{S(G)}(N) = \text{Hol}(G)$  and  $N \cong G$ , we can appeal to a result of G.A. Miller [16]. Miller found a characterization of these subgroups in terms of the *multiple holomorph* of  $G$

$$\text{NHol}(G) = N_{S(G)}(\text{Hol}(G)).$$

Consider the set

$$\mathcal{H}(G) = \{N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G)\}.$$

Using the well-known fact that two regular subgroups of  $S(G)$  are isomorphic if and only if they are conjugate in  $S(G)$ , Miller showed that the group  $\text{NHol}(G)$  acts transitively on  $\mathcal{H}(G)$  by conjugation. (See Lemma 4.2 in the next Section for a comment on this.) Clearly the stabilizer in  $\text{NHol}(G)$  of any element  $N \in \mathcal{H}(G)$  is  $N_{S(G)}(N) = \text{Hol}(G)$ . We obtain

**Theorem 3.2.** *The group*

$$T(G) = \text{NHol}(G) / \text{Hol}(G)$$

*acts regularly on  $\mathcal{H}(G)$  by conjugation.*

## 4. REGULAR SUBGROUPS OF THE HOLOMORPH

This section adapts to the nonabelian case the results of [6, Theorem 1] and [9, Proposition 2].

Let  $G$  be a finite group, and  $N \leq \text{Hol}(G)$  a regular subgroup. Since  $N$  is regular, for each  $g \in G$  there is a unique element  $\nu(g) \in N$ , such that  $1^{\nu(g)} = g$ . Now  $1^{\nu(g)\rho(g)^{-1}} = 1$ , so that  $\nu(g)\rho(g)^{-1} \in \text{Aut}(G)$  by Proposition 2.4(2). Therefore for  $g \in G$  we can write uniquely

$$(2) \quad \nu(g) = \gamma(g)\rho(g),$$

for a suitable map  $\gamma : G \rightarrow \text{Aut}(G)$ . We have

$$(3) \quad \nu(g)\nu(h) = \gamma(g)\rho(g)\gamma(h)\rho(h) = \gamma(g)\gamma(h)\rho(g^{\gamma(h)}h).$$

Since  $N$  is a subgroup of  $S(G)$ ,  $\gamma(g)\gamma(h) \in \text{Aut}(G)$ , and the expression (2) is unique, we have

$$\gamma(g)\gamma(h)\rho(g^{\gamma(h)}h) = \gamma(g^{\gamma(h)}h)\rho(g^{\gamma(h)}h),$$

from which we obtain

$$(4) \quad \gamma(g)\gamma(h) = \gamma(g^{\gamma(h)}h).$$

It is now immediate to obtain

**Theorem 4.1.** *Let  $G$  be a finite group. The following data are equivalent.*

- (1) *A regular subgroup  $N \leq \text{Hol}(G)$ .*
- (2) *A map  $\gamma : G \rightarrow \text{Aut}(G)$  such that*

$$(5) \quad \gamma(g)\gamma(h) = \gamma(g^{\gamma(h)}h).$$

*Moreover, under these assumptions*

- (a) *the assignment*

$$g \circ h = g^{\gamma(h)}h.$$

*for  $g, h \in G$ , defines a group structure  $(G, \circ)$  with the same unity as that of  $G$ .*

- (b) *There is an isomorphism  $\nu : (G, \circ) \rightarrow N$ .*
- (c) *For  $g, h \in G$ , one has*

$$g^{\nu(h)} = g \circ h.$$

*Proof.* Concerning the last statements, (5) implies that  $\circ$  is associative. Then for each  $h \in G$  one has that  $1 \circ h = 1^{\gamma(h)}h = h$ , as  $\gamma(h) \in \text{Aut}(G)$ , and that  $(h^{-1})^{\gamma(h)^{-1}}$  is a left inverse of  $h$  with respect to  $\circ$ . The bijection  $\nu$  introduced above is a homomorphism  $(G, \circ) \rightarrow N$  by (3) and (4). Finally,

$$g^{\nu(h)} = g^{\gamma(h)}h = g \circ h.$$

□

Note, for later usage, that (5) can be rephrased, setting  $k = g^{\gamma(h)}$ , as

$$(6) \quad \gamma(kh) = \gamma(k^{\gamma(h)^{-1}})\gamma(h).$$

We record the following Lemma, which will be useful later. We use the setup of Theorem 4.1.

**Lemma 4.2.** *Suppose  $N \in \mathcal{H}(G)$ , and let  $\vartheta \in \text{NHol}(G)$  such that  $\rho(G)^\vartheta = N$  and  $1^\vartheta = 1$ . Then*

$$\vartheta : G \rightarrow (G, \circ)$$

*is an isomorphism.*

*Conversely, an isomorphism  $\vartheta : G \rightarrow (G, \circ)$  conjugates  $\rho(G)$  to  $N$ .*

*Proof.* Note first that given any  $\vartheta \in \text{NHol}(G)$  such that  $\rho(G)^\vartheta = N$ , we can modify  $\vartheta$  by a suitable  $\rho(g)$ , and assume  $1^\vartheta = 1$ .

Suppose for  $y \in G$  one has  $\rho(y)^\vartheta = \nu(y^\sigma)$ , for some  $\sigma \in S(G)$ . Thus  $\rho(y)\vartheta = \vartheta\nu(y^\sigma)$ , so that for  $x, y \in G$  one has

$$(xy)^\vartheta = x^{\rho(y)\vartheta} = x^{\vartheta\nu(y^\sigma)} = x^{\vartheta\gamma(y^\sigma)}y^\sigma = x^\vartheta \circ y^\sigma.$$

Setting  $x = 1$  we see that  $\vartheta = \sigma$ , and thus

$$(xy)^\vartheta = x^\vartheta \circ y^\vartheta.$$

For the converse, if the last equation holds then

$$x^{\rho(y)\vartheta} = (x^{\vartheta^{-1}}y)^\vartheta = x \circ y^\vartheta = x^\nu(y^\vartheta).$$

□

## 5. NORMAL REGULAR SUBGROUPS OF THE HOLOMORPH

In this section, we adapt to the nonabelian case the results of [5, Theorem 3.1].

Consider the sets

$$\mathcal{I}(G) = \{N \leq S(G) : N \text{ is regular, } N_{S(G)}(N) = \text{Hol}(G)\}$$

and

$$\mathcal{J}(G) = \{N \leq S(G) : N \text{ is regular, } N \trianglelefteq \text{Hol}(G)\}.$$

Clearly we have

$$(7) \quad \mathcal{H}(G) \subseteq \mathcal{I}(G) \subseteq \mathcal{J}(G).$$

If  $N \trianglelefteq \text{Hol}(G)$ , then  $\text{Hol}(G) \leq N_{S(G)}(N)$ . However the latter may well be properly bigger than the former, as shown by the following simple example.

**Example 5.1.** Let  $G = \langle (1234) \rangle \leq S_4$ . Then  $N_{S_4}(G)$  has order 8, but its regular subgroup  $N = \langle (13)(24), (14)(23) \rangle$  is normal in the whole  $S_4$ .

Moreover, even when  $\text{Hol}(G) = N_{S(G)}(N)$ , Example 3.1 shows that  $G$  and  $N$  are not necessarily isomorphic. Therefore all inclusions in (7) may well be proper.

We will now give a characterization of the elements of  $\mathcal{J}(G)$  in terms of the description of Theorem 4.1. Suppose  $N \in \mathcal{J}(G)$ . To ensure that  $N \trianglelefteq \text{Hol}(G)$ , it is enough to make sure that  $N$  is normalized by  $\text{Aut}(G)$ , as if this holds, then the normalizer of  $N$  contains  $\text{Aut}(G)N$ , which is contained in  $\text{Hol}(G)$ , and has the same order as  $\text{Hol}(G)$ , as the regular subgroup  $N$  intersects  $\text{Aut}(G)$  trivially.

In order for  $\text{Aut}(G)$  to normalize  $N$ , we must have that for all  $\beta \in \text{Aut}(G)$  and  $g \in G$ , the conjugate  $\nu(g)^\beta$  of  $\nu(g)$  by  $\beta$  in  $S(G)$  lies in  $N$ . Since

$$\nu(g)^\beta = (\gamma(g)\rho(g))^\beta = \gamma(g)^\beta\rho(g)^\beta = \gamma(g)^\beta\rho(g^\beta)$$

and  $\gamma(g)^\beta \in \text{Aut}(G)$ , uniqueness of (2) implies

$$\gamma(g)^\beta\rho(g^\beta) = \gamma(g^\beta)\rho(g^\beta),$$

so that

$$(8) \quad \gamma(g^\beta) = \gamma(g)^\beta$$

for  $g \in G$  and  $\beta \in \text{Aut}(G)$ . Applying this to (6), we obtain that for  $h, k \in G$

$$(9) \quad \gamma(kh) = \gamma(k^{\gamma(h)^{-1}})\gamma(h) = \gamma(k)^{\gamma(h)^{-1}}\gamma(h) = \gamma(h)\gamma(k),$$

that is,  $\gamma : G \rightarrow \text{Aut}(G)$  is an antihomomorphism.

Now note that (5) follows from (8) and (9), as

$$\gamma(g^{\gamma(h)}h) = \gamma(h)\gamma(g)^{\gamma(h)} = \gamma(g)\gamma(h).$$

We have obtained

**Theorem 5.2.** *Let  $G$  be a finite group. The following data are equivalent.*

- (1) *A regular subgroup  $N \trianglelefteq \text{Hol}(G)$ , that is, an element of  $\mathcal{J}(G)$ .*
- (2) *A map  $\gamma : G \rightarrow \text{Aut}(G)$  such that for  $g, h \in G$  and  $\beta \in \text{Aut}(G)$*

$$(10) \quad \begin{cases} \gamma(gh) = \gamma(h)\gamma(g) \\ \gamma(g^\beta) = \gamma(g)^\beta. \end{cases}$$

Moreover, under these assumptions

- (a) *the assignment*

$$g \circ h = g^{\gamma(h)}h.$$

*for  $g, h \in G$ , defines a group structure  $(G, \circ)$  with the same unity as that of  $G$ .*

- (b) *There is an isomorphism  $\nu : (G, \circ) \rightarrow N$ .*
- (c) *For  $g, h \in G$ , one has*

$$g^{\nu(h)} = g \circ h.$$

- (d) *Every automorphism of  $G$  is also an automorphism of  $(G, \circ)$ .*

**Remark 5.3.** Note that under the hypotheses of Theorem 4.1,  $G$  becomes a skew right brace (for which see [13]) under the operations  $\cdot$  and  $\circ$ , that is,  $G$  is a group with respect to both operations, which are connected by

$$(gh) \circ k = (g \circ k)k^{-1}(h \circ k), \quad \text{for } g, h, k \in G.$$

The braces which correspond to the normal regular subgroups of  $\text{Hol}(G)$  satisfy the additional condition of Theorem 5.2(d).

In the following, when dealing with  $N \in \mathcal{J}(G)$ , we will be using the notation of Theorem 5.2 without further mention.

*Proof.* The last statement follows from

$$(g \circ h)^\beta = (g^{\gamma(h)}h)^\beta = (g^\beta)^{\gamma(h)^\beta} h^\beta = (g^\beta)^{\gamma(h^\beta)} h^\beta = g^\beta \circ h^\beta,$$

for  $g, h \in G$  and  $\beta \in \text{Aut}(G)$ .  $\square$

Let us exemplify the above for the case of the left regular representation. Consider the morphism

$$\begin{aligned} \iota : G &\rightarrow \text{Aut}(G) \\ y &\mapsto (x \mapsto y^{-1}xy), \end{aligned}$$

that is,  $\iota(y) \in \text{Inn}(G)$  is conjugacy by  $y$ . If  $N = \lambda(G)$ , then we have for  $y \in G$

$$\lambda(y) = \iota(y^{-1})\rho(y),$$

as for  $z, y \in G$  we have

$$z^{\iota(y^{-1})\rho(y)} = yzy^{-1}y = yz = z^{\lambda(y)}.$$

Therefore  $\gamma(y) = \iota(y^{-1})$ , and

$$x \circ y = x^{\iota(y^{-1})}y = yx,$$

that is,  $(G, \circ)$  is the opposite group of  $G$ .

Also, in [7] S. Carnahan and L. Childs prove that if  $G$  is a non-abelian finite simple group, then  $\mathcal{H}(G) = \{\rho(G), \lambda(G)\}$ . In our context, this can be proved as follows. If  $G$  is a non-abelian finite simple group, and  $N \in \mathcal{H}(G)$ , then the normal subgroup  $\ker(\gamma)$  of  $G$  can only be either  $G$  or  $\{1\}$ . In the first case we have  $x \circ y = x^{\gamma(y)}y = xy$  for  $x, y \in G$ , so that

$$x^{\nu(y)} = x \circ y = xy = x^{\rho(y)},$$

and  $N = \rho(G)$ . In the second case,  $\gamma$  is injective. Since we have

$$(11) \quad \gamma(x \circ y) = \gamma(x)\gamma(y) = \gamma(yx),$$

we obtain  $x \circ y = yx$ , so  $N = \lambda(G)$  as we have just seen.

## 6. COMMUTATORS

In this section we assume we are in the situation of Theorem 5.2.

Let  $\beta \in \text{Aut}(G)$ ,  $g \in G$ , and consider the commutator  $[\beta, g^{-1}] = g^\beta g^{-1}$  taken in  $\text{Aut}(G)G$ . Using (10), we get

$$(12) \quad \gamma([\beta, g^{-1}]) = \gamma(g^\beta g^{-1}) = \gamma(g)^{-1} \gamma(g)^\beta = [\gamma(g), \beta].$$

In the particular case when  $\beta = \iota(h)$ , for some  $h \in G$ , we obtain

$$\gamma([h, g^{-1}]) = \gamma([\iota(h), g^{-1}]) = [\gamma(g), \iota(h)] = \iota([\gamma(g), h]),$$

that is

$$(13) \quad \gamma([h, g^{-1}]) = \iota([\gamma(g), h]).$$

From this identity we obtain

$$\begin{aligned} \iota([\gamma(g), h]) &= \gamma([h, g^{-1}]) = \gamma([g^{-1}, h]^{-1}) = \\ &= \iota([\gamma(h^{-1}), g^{-1}]^{-1}) = \iota([g^{-1}, \gamma(h^{-1})]), \end{aligned}$$

that is,

$$(14) \quad [\gamma(g), h] \equiv [g^{-1}, \gamma(h^{-1})] \pmod{Z(G)}$$

for all  $g, h \in G$ .

In the rest of the paper we will deal with the case of finite perfect groups, that is, those finite groups  $G$  such that  $G' = G$ . In this case, according to (13), we have  $\gamma(G) \leq \text{Inn}(G)$ .

## 7. PERFECT GROUPS

Let  $G$  be a non-trivial, finite, perfect group. We will determine  $\mathcal{J}(G)$ , and then discuss its relationship to  $\mathcal{H}(G)$ .

Recall that an automorphism  $\beta$  of a group  $G$  is said to be *central* if  $[x, \beta] = x^{-1}x^\beta \in Z(G)$  for all  $x \in G$ . In other words, an automorphism of  $G$  is central if it induces the identity on  $G/Z(G)$ .

We record for later usage a couple of elementary, well-known facts.

**Lemma 7.1.** *Let  $G$  be a finite perfect group.*

- (1)  $Z_2(G) = Z(G)$ .
- (2) A central automorphism of  $G$  is trivial.

*Proof.* The first part is Grün's Lemma [12].

For the second part, if  $\beta$  is a central automorphism of  $G$ , then

$$x \mapsto [x, \beta]$$

is a homomorphism from  $G$  to  $Z(G)$ . Since  $G = G'$ , this homomorphism maps  $G$  onto the identity.  $\square$

We now show that an element  $N \in \mathcal{J}(G)$  yields a direct product decomposition of  $\text{Inn}(G)$ .

**Proposition 7.2.** *Let  $G$  be a finite, perfect group, and  $N \in \mathcal{J}(G)$ .*

- (1)  $Z(G) \leq \ker(\gamma)$ .  
(2)  $\text{Inn}(G) = \gamma(G) \times \iota(\ker(\gamma))$ .

Later we will lift the direct product decomposition (2) of  $\text{Inn}(G)$  to a central product decomposition of  $G$  (Theorem 7.5(1)).

*Proof.* For the first part, let  $g \in Z(G)$ . (14) yields  $[\gamma(g), h] \in Z(G)$  for all  $h \in G$ , that is,  $\gamma(g)$  is a central automorphism of  $G$ . By Lemma 7.1(2),  $\gamma(g) = 1$ .

For the second part, we first show that  $\gamma(G)$  and  $\iota(\ker(\gamma))$  commute elementwise. Let  $g \in G$  and  $k \in \ker(\gamma)$ . The results of Section 6 yield

$$[\gamma(g), \iota(k)] = \iota([\gamma(g), k]) = \iota([g^{-1}, \gamma(k^{-1})]) = 1.$$

We now show that  $\gamma(G) \cap \iota(\ker(\gamma)) = 1$ . Write an element of the perfect group  $G$  as

$$x = \prod_{i=1}^n [h_i, g_i^{-1}],$$

for suitable  $g_i, h_i \in G$ .

Using the first identity of (10) we get first

$$(15) \quad \gamma(x) = \prod_{i=n}^1 \gamma([h_i, g_i^{-1}]) = \prod_{i=n}^1 [\gamma(g_i), \gamma(h_i^{-1})]$$

(note that the order of the product has been inverted by the application of  $\gamma$ ).

Using (13) and the first identity of (10) we also get

$$\gamma(x) = \prod_{i=n}^1 \gamma([h_i, g_i^{-1}]) = \prod_{i=n}^1 \iota([\gamma(g_i), h_i]) = \iota\left(\prod_{i=n}^1 [\gamma(g_i), h_i]\right).$$

Now if  $\gamma(x) \in \gamma(G) \cap \iota(\ker(\gamma))$ , part (1) yields

$$\prod_{i=n}^1 [\gamma(g_i), h_i] \in \ker(\gamma).$$

We thus have, using (10) and (12)

$$1 = \gamma\left(\prod_{i=n}^1 [\gamma(g_i), h_i]\right) = \prod_{i=1}^n \gamma([\gamma(g_i), h_i]) = \prod_{i=1}^n [\gamma(h_i^{-1}), \gamma(g_i)] = \gamma(x)^{-1},$$

according to (15). Therefore  $\gamma(x) = 1$ , as claimed.

Finally we have, keeping in mind part (1),

$$|\gamma(G) \times \iota(\ker(\gamma))| = |\gamma(G)| \cdot \frac{|\ker(\gamma)|}{|Z(G)|} = |G/Z(G)| = |\text{Inn}(G)|,$$

so that  $\gamma(G) \times \iota(\ker(\gamma)) = \text{Inn}(G)$ .  $\square$

Regarding  $\text{Inn}(G)$  and  $G$  as groups with operator group  $\text{Aut}(G)$ , we note that the second equation of (10) implies that both  $\gamma(G)$  and  $\iota(\ker(\gamma))$  are  $\text{Aut}(G)$ -invariant, and so are  $H = \iota^{-1}(\gamma(G))$  and  $\ker(\gamma)$ . (Clearly the latter statement is the same as saying that  $H$  and  $\ker(\gamma)$

are characteristic subgroups of  $G$ , but we prefer to use the same terminology of groups with  $\text{Aut}(G)$  as a group of operators for both  $G$  and  $\text{Inn}(G)$ .)

We have  $G = H \ker(\gamma)$ . We claim that  $[H, \ker(\gamma)] = 1$ , that is,  $G$  is the central product of  $H$  and  $\ker(\gamma)$ , amalgamating  $Z(G)$ . We will need the following simple Lemma, which is hinted at by Joshua A. Grochow and Youming Qiao in [11, Remark 7.6].

**Lemma 7.3.** *Let  $G$  be a group, and  $H, K \leq G$  such that*

$$G/Z(G) = HZ(G)/Z(G) \times KZ(G)/Z(G).$$

*Suppose  $KZ(G)/Z(G)$  is perfect.*

*Then*

- (1)  $K'$  is perfect, and
- (2)  $[H, K] = 1$ .

*Proof.* Since  $KZ(G)/Z(G)$  is perfect, we have  $KZ(G) = K'Z(G)$ , so that  $K' = K''$ , and  $K'$  is perfect. As  $[H, K'] = [H, K'Z(G)] = [H, KZ(G)] = [H, K] \leq Z(G)$ ,  $H$  induces by conjugation central automorphisms on  $K'$ , so that by Lemma 7.1(2)  $[H, K] = [H, K'] = 1$ .  $\square$

In our situation, take  $K = \ker(\gamma)$ . We have that  $KZ(G)/Z(G) \cong \iota(K)$  is perfect, as a direct factor of the perfect group  $G/Z(G)$ . Then Lemma 7.3(2) implies that  $G$  is the central product of  $H$  and  $\ker(\gamma)$ , amalgamating  $Z(G)$ .

We claim

**Lemma 7.4.**  $\gamma(y) = \iota(y^{-1})$  for  $y \in H$ .

*Proof.* Let  $y \in H$ . We claim that  $\gamma(y)\iota(y)$  is a central automorphism of the perfect group  $G$ , so that by 7.1(2)  $\gamma(y) = \iota(y^{-1})$ .

If  $x \in K = \ker(\gamma)$ , we have from (14)

$$[\gamma(y), x] \equiv [y^{-1}, \gamma(x^{-1})] \equiv 1 \pmod{Z(G)},$$

so that  $\gamma(y)$  induces an automorphism of the characteristic subgroup  $K$  which is the identity modulo  $Z(G)$ , and so does  $\gamma(y)\iota(y)$ , as  $[H, K] = 1$ .

Let now  $x \in H$ . Consider first the special case when  $G = H \times K$ . Then  $\gamma$  is injective on  $H$ , so that (11) implies  $x \circ y = yx$ , and thus

$$x^{-1}x^{\gamma(y)\iota(y)} = x^{-1}y^{-1}(x \circ y)y^{-1}y = x^{-1}y^{-1}(yx)y^{-1}y = 1,$$

that is,  $\gamma(y)\iota(y)$  is the identity on  $H$ .

In the general case, (11) implies that  $x \circ y \equiv yx \pmod{K}$ , so that as above

$$x^{-1}x^{\gamma(y)\iota(y)} \equiv x^{-1}y^{-1}(x \circ y)y^{-1}y \equiv 1 \pmod{K},$$

that is,  $x^{-1}x^{\gamma(y)\iota(y)} \in K$ . Clearly  $x^{-1}x^{\gamma(y)\iota(y)} \in H$ , as  $H$  is characteristic in  $G$ . Therefore  $x^{-1}x^{\gamma(y)\iota(y)} \in H \cap K \leq Z(G)$ , so that  $\gamma(y)\iota(y)$  induces an automorphism of  $H$  which is the identity modulo  $Z(G)$ .

It follows that  $\gamma(y)\iota(y)$  is a central automorphism of  $G = HK$ , as claimed.  $\square$

For  $y \in H$  and  $x \in \ker(\gamma)$  we have

$$x \circ y = x^{\gamma(y)}y = x^{\iota(y^{-1})}y = yxy^{-1}y = yx = y^{\gamma(x)}x = y \circ x.$$

Also, if  $x, y \in H$  we have  $x \circ y = x^{\gamma(y)}y = x^{y^{-1}}y = yx$ .

In the following we will be writing the elements of  $G$  as pairs in  $H \times \ker(\gamma)$ , understanding that a pair represents an equivalence class with respect to the central product equivalence relation which identifies  $(xz, y)$  with  $(x, zy)$ , for  $z \in Z(G)$ .

We have obtained

**Theorem 7.5.** *Let  $G$  be a finite perfect group.*

- (1) *If  $N \in \mathcal{J}(G)$ , then  $G$  is a central product of its subgroups  $H = \iota^{-1}(\gamma(G))'$  and  $K = \ker(\gamma)$ . Both  $H$  and  $K$  are  $\text{Aut}(G)$ -subgroups of  $G$ .*
- (2) *For  $x \in H$  we have  $\gamma(x) = \iota(x^{-1})$ .*
- (3)  *$(G, \circ)$  is also a central product of the same subgroups. If we represent the elements of  $G$  as (equivalence classes of) pairs in  $H \times K$ , then*

$$(16) \quad (x, y) \circ (a, b) = (ax, yb).$$

- (4) *For  $(a, b) \in G$ , the action of  $\nu(a, b)$  on  $(x, y) \in G$  is given by*

$$(x, y)^{\nu(a, b)} = (x, y) \circ (a, b) = (ax, yb),$$

*that is,  $N$  induces the right regular representation on  $K$ , and the left regular representation on  $H$ .*

We note the following analogue of Proposition 2.4(1) and (4).

**Proposition 7.6.** *Let  $G$  be a finite perfect group, and let  $G = HK$  be a central decomposition, with  $\text{Aut}(G)$ -invariant subgroups  $H, K$ . Consider the following two elements of  $\mathcal{J}(G)$ .*

- (1)  $N_1$ , for which  $\ker(\gamma_1) = KZ(G)$  and  $H = \iota^{-1}(\gamma_1(G))'$ , with  $\gamma_1(x) = \iota(x^{-1})$  for  $x \in H$ , and associated group operation  $(x, y) \circ_1 (a, b) = (ax, yb)$ .
- (2)  $N_2$ , for which  $\ker(\gamma_2) = HZ(G)$  and  $K = \iota^{-1}(\gamma_2(G))'$ , with  $\gamma_2(x) = \iota(x^{-1})$  for  $x \in K$ , and associated group operation  $(x, y) \circ_2 (a, b) = (xa, by)$ .

*Then*

- (1)  $N_1^{\text{inv}} = N_2$ .
- (2)  $\text{inv} : (N_1, \circ_1) \rightarrow (N_2, \circ_2)$  is an isomorphism.
- (3)  $C_{S(G)}(N_1) = N_2$  and  $C_{S(G)}(N_2) = N_1$ .

*Proof.* The proof is straightforward. If  $N_i = \{\nu_i(a, b) : (a, b) \in G\}$  as in Section 4, we have

$$\begin{aligned} (x, y)^{\text{inv} \nu_1(a, b) \text{inv}} &= (x^{-1}, y^{-1})^{\nu_1(a, b) \text{inv}} = \\ &= (ax^{-1}, y^{-1}b)^{\text{inv}} = (xa^{-1}, b^{-1}y) = (x, y)^{\nu_2((a, b)^{\text{inv}})}, \end{aligned}$$

and then, as in Lemma 4.2,  $\text{inv} : (N_1, \circ_1) \rightarrow (N_2, \circ_2)$  is an isomorphism.  $\square$

We now give a description of all possible central product decompositions of the perfect group  $G$  as in Theorem 7.5.

We deal first with the particular case when  $Z(G) = 1$ , where we are able to show that  $\mathcal{J}(G) = \mathcal{I}(G) = \mathcal{H}(G)$  and determine this set, and the group  $T(G) = \text{NHol}(G)/\text{Hol}(G)$ . When  $Z(G)$  is allowed to be non-trivial, we are able to determine  $\mathcal{J}(G)$ . However, examples show that in this case  $\mathcal{H}(G)$ ,  $\mathcal{I}(G)$  and  $\mathcal{J}(G)$  can be distinct, and we are unable at the moment to describe  $T(G)$ .

**7.1. The centerless case.** Suppose  $Z(G) = 1$ , so that  $\iota : G \rightarrow \text{Inn}(G)$  is an isomorphism of  $\text{Aut}(G)$ -groups.

Consider a Krull-Remak-Schmidt decomposition

$$G = A_1 \times A_2 \times \cdots \times A_n$$

of  $G$  as an  $\text{Aut}(G)$ -group. Since  $Z(G) = 1$ , this is unique [18, 3.3.8, p. 83]. Therefore the only way to decompose  $G$  as the ordered direct product of two characteristic subgroups  $H, K$  is by grouping together the  $A_i$ , so that there are  $2^n$  ways of doing this. If  $G = H \times K$  is one of these ordered decompositions, define an antihomomorphism  $\gamma : G \rightarrow \text{Aut}(G)$  by  $\gamma(k) = 1$  for  $k \in K$ , and  $\gamma(h) = \iota(h^{-1})$ , for  $h \in H$ . Then  $\gamma$  satisfies also the second identity of (10), and we have obtained an element  $N \in \mathcal{J}(G)$  as in Theorem (7.5)(3). The involution  $\vartheta \in \text{NHol}(G)$  given by  $(h, k)^\vartheta = (h^{-1}, k)$ , for  $h \in H$  and  $k \in K$  is an isomorphism  $G \rightarrow (G, \circ)$ . We have obtained

**Theorem 7.7.** *Let  $G$  be a finite perfect group with  $Z(G) = 1$ .*

- (1) *If  $N \in \mathcal{J}(G)$ , that is,  $N \trianglelefteq \text{Hol}(G)$  is regular, then  $N \in \mathcal{H}(G)$ , that is,  $N \cong G$ .*
- (2) *If  $n$  is the length of a Krull-Remak-Schmidt decomposition of  $G$  as an  $\text{Aut}(G)$ -group, then  $\mathcal{H}(G)$  has  $2^n$  elements.*
- (3)  *$T(G)$  is an elementary abelian group of order  $2^n$ .*

**7.2. Non-trivial center.** We now consider the situation when  $Z(G)$  is (allowed to be) non-trivial.

We describe the elements  $N$  of  $\mathcal{J}(G)$ , in analogy with the centerless case.

As in the centerless case, we may consider the Krull-Remak-Schmidt decomposition

$$(17) \quad \text{Inn}(G) = A_1 \times A_2 \times \cdots \times A_n$$

of  $\text{Inn}(G)$  as an  $\text{Aut}(G)$ -group. This corresponds uniquely to the central product decomposition of  $G$

$$(18) \quad G = B_1 B_2 \cdots B_n,$$

where  $B_i = \iota^{-1}(A_i)'$  are *perfect*  $\text{Aut}(G)$ -subgroups, which are centrally indecomposable as  $\text{Aut}(G)$ -subgroups. Therefore the central product decomposition (18) is also unique. (Recall also that the Krull-Remak-Schmidt of  $G$  in terms of indecomposable  $\text{Aut}(G)$ -subgroups is unique, because of [18, 3.3.8, p. 83] and Lemma 7.1.(2).)

As in the centerless case, we obtain that every decomposition  $G = H \ker(\gamma)$  as in Theorem 7.5 can be obtained by grouping together the  $B_i$  in two subgroups  $H$  and  $K$ , and then defining an antihomomorphism  $\gamma : G \rightarrow \text{Aut}(G)$  by  $\gamma(k) = 1$  for  $k \in KZ(G)$ , and  $\gamma(h) = \iota(h^{-1})$ , for  $h \in H$ , and then  $\circ$  as in (16). As in the centerless case, this yields an element  $N \in \mathcal{J}(G)$ . Moreover  $(G, \circ)$  is still a central product of  $H$  and  $KZ(G)$ , with  $\circ$  as in Theorem 7.5(3).

We have obtained the following weaker analogue of Theorem 7.7.

**Theorem 7.8.** *Let  $G$  be a finite perfect group.*

*If  $n$  is the length of a Krull-Remak-Schmidt decomposition of  $\text{Inn}(G)$  as an  $\text{Aut}(G)$ -group, then  $\mathcal{J}(G)$  has  $2^n$  elements, that is, there are  $2^n$  regular subgroups  $N \trianglelefteq \text{Hol}(G)$ .*

Write  $Y = H \cap KZ(G)$ . As the subgroups  $H$  and  $KZ(G)$  are characteristic in  $G$ , we obtain that the elements of  $\text{Aut}(G)$  can be described via the set of pairs

$$\{(\sigma, \tau) : \sigma \in \text{Aut}(H), \tau \in \text{Aut}(KZ(G)), \sigma|_Y = \tau|_Y\}.$$

Theorem 5.2(d) states that  $\text{Aut}(G) \leq \text{Aut}(G, \circ)$ . However, the latter group might well be bigger than the former. This is shown by the following

**Proposition 7.9.** *There exist perfect and centrally indecomposable groups  $Q_1, Q_2, Q_3$ , and a central product  $G = Q_1 Q_2 Q_3$ , such that*

- (1) *each  $Q_i$  is characteristic in  $G$ ,*
- (2) *in the group  $(G, \circ)$  obtained by replacing  $Q_1$  with its opposite, the subgroups  $Q_1$  and  $Q_2$  are exchanged by an automorphism of  $(G, \circ)$ , and thus are not characteristic in  $(G, \circ)$ .*

Clearly the automorphism of the second condition lies in  $\text{Aut}(G, \circ) \setminus \text{Aut}(G)$ . This example shows that  $\mathcal{I}(G)$  may well be a proper subset of  $\mathcal{J}(G)$ .

Clearly  $N \in \mathcal{I}(G)$  if and only if  $\text{Aut}(G) = \text{Aut}(G, \circ)$ . However, in this general situation  $\mathcal{H}(G)$  might be a proper subset of  $\mathcal{I}(G)$ . This is shown by the following

**Proposition 7.10.** *There exist perfect and centrally indecomposable groups  $Q_1, Q_2$ , and a central product  $G = Q_1 Q_2$  such that*

- (1)  $Q_1, Q_2$  are characteristic in  $G$ ;
- (2) the group  $(G, \circ)$  obtained from  $G$  by replacing  $Q_1$  with its opposite  $G$  is not isomorphic to  $G$ .

Moreover,  $Q_1$  and  $Q_2$  are still characteristic in  $(G, \circ)$ .

Thus if  $N$  is the regular subgroup corresponding to  $(G, \circ)$  of this Proposition, we have  $N \in \mathcal{I}(G) \setminus \mathcal{H}(G)$ .

So on the one hand not all central product decompositions lead to regular subgroups  $N$  which are isomorphic to  $G$ . And even when  $N$  is isomorphic to  $G$ , it is not clear whether  $\rho(G)$  and  $N$  are conjugate under an involution in  $T(G)$ , and therefore it is not clear to us at the moment whether  $T(G)$  is elementary abelian in this general case.

To construct the groups of Proposition 7.9 and 7.10, we rely on the following family of examples, based on a construction that we have learned from Derek Holt.

**Proposition 7.11.** *There exists a family of groups  $L_p$ , for  $p \equiv 1 \pmod{3}$  a prime, with the following properties:*

- (1) the groups  $L_p/Z(L_p)$  are pairwise non-isomorphic,
- (2) the groups  $L_p$  are perfect, and centrally indecomposable,
- (3)  $Z(L_p)$  is of order 3, and
- (4)  $\text{Aut}(L_p)$  acts trivially on  $Z(L_p)$ .

This is proved in Section 8.

*Proof of Proposition 7.9.* Let  $Q_1, Q_2$  be two isomorphic copies of one of the groups of Proposition 7.11, and  $Q_3$  another group as in Proposition 7.11, not isomorphic to  $Q_1, Q_2$ .

Fix an isomorphism  $\zeta : Q_1 \rightarrow Q_2$ . If  $Z(Q_1) = \langle a_1 \rangle$ , let  $a_2 = a_1^\zeta$ . Let  $Z(Q_3) = \langle b \rangle$ .

Consider the central product  $G = Q_1 Q_2 Q_3$ , amalgamating  $a_2 = a_1^{-1} = b$ .

Consider the quotient  $G/Z(G)$ . Since the groups of Proposition 7.11 are centrally indecomposable, and have pairwise non-isomorphic central quotients, Krull-Remak-Schmidt implies that  $Q_3/Z(G)$  is characteristic in  $G/Z(G)$ , and so is  $Q_1 Q_2/Z(G)$ . Therefore  $Q_3$  and  $Q_1 Q_2$  are characteristic in  $G$ . Moreover, if there is an automorphism  $\alpha$  of  $G$  that does not map  $Q_1$  to itself, then applying to  $Q_1 Q_2/Z(G)$  either Krull-Remak-Schmidt, or the results of [2, Theorem 3.1], and using either the fact that  $Q_1/Z(Q_1)$  and  $Q_2/Z(Q_2)$  are perfect, or that they are centerless, we obtain that  $\alpha$  exchanges  $Q_1$  and  $Q_2$ .

Therefore  $\alpha|_{Q_1} \zeta^{-1}$  is an automorphism of  $Q_1$ , and thus maps  $a_1$  to  $a_1$ . Therefore  $\alpha$  maps  $a_1$  to  $a_1^\zeta = a_2 = a_1^{-1}$ . But  $\alpha|_{Q_3}$  is an automorphism of  $Q_3$ , and thus fixes  $b = a_1$ , a contradiction.

Consider now the group  $(G, \circ)$  obtained by replacing  $Q_1$  with its opposite. Now the map which is the identity on  $Q_3$ ,  $\zeta$  inv on  $Q_1$  and

$\zeta^{-1} \text{inv}$  on  $Q_2$  induces an automorphism of  $(G, \circ)$  which exchanges  $Q_1$  and  $Q_2$ . In fact we have for  $x, y \in Q_1$

$$(x \circ y)^{\zeta \text{inv}} = (yx)^{\zeta \text{inv}} = (y^\zeta x^\zeta)^{\text{inv}} = x^{\zeta \text{inv}} y^{\zeta \text{inv}},$$

and for  $x, y \in Q_2$

$$\begin{aligned} (x \circ y)^{\zeta^{-1} \text{inv}} &= (xy)^{\zeta^{-1} \text{inv}} = (x^{\zeta^{-1}} y^{\zeta^{-1}})^{\text{inv}} = \\ &= y^{\zeta^{-1} \text{inv}} x^{\zeta^{-1} \text{inv}} = x^{\zeta^{-1} \text{inv}} \circ y^{\zeta^{-1} \text{inv}}. \end{aligned}$$

Moreover  $a_1^{\zeta \text{inv}} = a_2^{\text{inv}} = a_2^{-1} = a_1$ , and  $a_2^{\zeta^{-1} \text{inv}} = a_1^{\text{inv}} = a_1^{-1} = a_2$ , which is compatible with the identity on  $Q_3$ .  $\square$

*Proof of Proposition 7.10.* Let  $Q_1, Q_2$  to be two non-isomorphic groups as in Proposition 7.11, and let  $G = Q_1 Q_2$ , amalgamating the centers. Consider the group  $(G, \circ)$  obtained by replacing  $Q_1$  with its opposite. If the map  $\vartheta : G \rightarrow G$  is an isomorphism of  $G$  onto  $(G, \circ)$ , by the arguments of the previous proofs it has to map each  $Q_i$  to itself. Then  $\vartheta$  induces an anti-automorphism on  $Q_1$ , thus inverting  $Z(G)$ , and an automorphism on  $Q_2$ , thus fixing  $Z(G)$  elementwise, a contradiction.  $\square$

## 8. PROOF OF PROPOSITION 7.11

Consider the groups  $T_p = \text{PSL}(3, p)$ , where  $p \equiv 1 \pmod{3}$  is a prime, and let  $\mathbb{F}_p$  be the field with  $p$  elements.

It is well known ([20, Theorem 3.2]) that the outer automorphism group of  $T_p$  is isomorphic to  $S_3$ , where an automorphism  $\Delta$  of order 3 is diagonal, obtained via conjugation with a suitable  $\delta \in \text{PGL}(3, p)$ , and one of the involutions is the transpose inverse automorphism  $\top$ .

Moreover, the Schur multiplier  $M(T_p)$  of  $T_p$  has order 3 [20, 3.3.6], it is inverted by  $\top$ , and clearly centralized by  $\Delta$ .

Let  $P$  be the natural  $\mathbb{F}_p$ -permutation module of  $T_p$  in its action on the points of the projective plane.  $P$  is the direct sum of a copy of the trivial module, and of a module  $N$ . The structure of  $N$  is investigated in [21], [1]. In particular, it is shown in [1] that  $N$  has a unique composition series  $\{0\} \subset N_1 \subset N$ , such that  $N_1$  and  $N_2 = N/N_1$  are dual to each other, exchanged by  $\top$ , but not isomorphic to each other. Note that  $\Delta$  still acts on  $P$ , as it comes from conjugation with an element  $\delta \in \text{PGL}(3, p)$ , and thus also acts on  $N_1$  and  $N_2$ .

Consider the natural semidirect product  $S_p$  of  $N_1$  by  $T_p$ . It has been shown by K.I. Tahara [19] that the Schur multiplier of  $T_p$  is a direct summand of the Schur multiplier of  $S_p$ , that is,  $M(S_p) = M(T_p) \oplus K$  for some  $K$ . We may thus consider the central extension  $L_p$  of  $M(T_p)$  by  $S_p$ , which is the quotient of the covering group of  $S_p$  by  $K$ . Thus  $Z(L_p) = M(T_p)$  (Derek Holt has shown to us calculations for small primes, based on the description of [19], which appear to indicate that actually  $K = \{0\}$  here.)

An automorphism  $\sigma$  of  $L_p$  induces automorphisms  $\alpha$  of  $N_1$  and  $\beta$  of  $T_p$ . Abusing notation slightly, we have, for  $n \in N_1$  and  $h \in T_p$ ,

$$(n^h)^\alpha = (n^h)^\sigma = (n^\sigma)^{h^\sigma} = (n^\alpha)^{h^\beta}.$$

If  $\beta$  is an involution in  $\text{Out}(T_p)$ , say  $\beta = \Delta^{-1}\top\Delta$ , then we have

$$h^\beta = \delta^{-2}h^\top\delta^2,$$

so that

$$(n^h)^{\alpha\Delta^{-2}} = (n^{\alpha\Delta^{-2}})^{h^\top},$$

that is,  $\alpha\Delta^{-2}$  is an isomorphism of  $N_1$  with its dual, a contradiction. Then  $\text{Aut}(L_p)$  induces on  $T_p$  only inner automorphisms and at most outer automorphisms of order 3, all of which centralize  $M(T_p) = Z(L_p)$ .

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