HETEROGENEOUS ELASTIC PLATES WITH IN-PLANE MODULATION OF THE TARGET CURVATURE AND APPLICATIONS TO THIN GEL SHEETS

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Abstract. We rigorously derive a Kirchhoff plate theory, *via* Γ-convergence, from a three-dimensional model that describes the finite elasticity of an elastically heterogeneous, thin sheet. The heterogeneity in the elastic properties of the material results in a spontaneous strain that depends on both the thickness and the plane variables x'. At the same time, the spontaneous strain is h-close to the identity, where h is the small parameter quantifying the thickness. The 2D Kirchhoff limiting model is constrained to the set of isometric immersions of the mid-plane of the plate into $\mathbb{R}^3$, with a corresponding energy that penalizes deviations of the curvature tensor associated with a deformation from a x'-dependent target curvature tensor. A discussion on the 2D minimizers is provided in the case where the target curvature tensor is piecewise constant. Finally, we apply the derived plate theory to the modeling of swelling-induced shape changes in heterogeneous thin gel sheets.

Contents

1. Introduction

Plants [6, 15] and other natural systems [7, 8] are able to perform complex shape changes that produce curved configurations, often starting from flat initial states. These shape changes usually involve thin structures, such as membranes, plates or shells, and exploit some internal activation or the responsiveness of the material to non-mechanical external triggers, such as changes in humidity. By mimicking natural behaviors and architectures, synthetic, polymer-based thin sheets have been

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fabricated that can spontaneously deform in response to non-mechanical stimuli. In particular, in these systems curvature arises from heterogeneous in-plane [16, 25, 26, 37, 45] or through-the-thickness strains [1, 2, 38, 39, 40, 44], which are induced by heterogeneous material properties, including variable anisotropy. Thus, to study and control the emerging shapes, the derivation of plate theories for materials with heterogeneous response to external stimuli has become as a topic of interest in both the mathematical and the physical literature [5, 9, 10, 33, 34, 43].

In this framework, we wish to contribute by drawing attention to some new plate theories and corresponding mathematical problems in dimension reduction inspired by shape morphing applications involving polymer gels. Specifically, in these applications one wants to program the material properties of a thin gel sheet $\Omega_h = \omega \times (-h/2, h/2)$, where $\omega \subseteq \mathbb{R}^2$ is the mid-plane and $0 < h \ll 1$ is the thickness, in order to endow it with a controlled curvature that is realizable, upon swelling, at the minimum energy cost. Practically, curvature of the sheet can be obtained by imprinting a heterogeneous density N of polymer chains, which corresponds to a heterogeneous shear modulus of the polymer network. For concreteness, we consider the case where N is a (small) perturbation of order h of the average value $\overline{N}$, that is,

$$N = \overline{N}^h(z) = \overline{N} + hg\left(z', \frac{z_3}{h}\right), \qquad z = (z', z_3) \in \Omega_h, \tag{1.1}$$

for some bounded function $g : \omega \times (-1/2, 1/2) \to \mathbb{R}$. Referring to the classical Flory-Rehner model [17] for isotropic polymer gels, we obtain as a consequence of the above assumption on N that the free energy density $\overline{W}^h$ associated with the system is minimized at

$$\big(\alpha + hb(z', z_3/h)\big)\mathrm{SO}(3), \qquad b = \Theta g, \tag{1.2}$$

where the (dimensionless) constants α and Θ are functions of the material parameters appearing in the expression of $\overline{W}^h$, including $\overline{N}$. We refer the reader to (4.4) for the explicit expression of $\overline{W}^h$ and to the whole Section 4 for more details on this 3D model. We recall that α corresponds to the free-swelling stretch of a homogeneous gel (i.e. $g = 0$ in (1.1)) with respect to its dry state, and is hence greater than one.

Intuitively, the connection between the density of polymer chains in (1.1) and the energy minimizers in (1.2) offers a way to program minimum-energy strain fields that, as we will see in the following, induce a *target* curvature for the system. This mechanism of generation of curvature through heterogeneous elastic properties is poorly explored in the mathematical literature of active or "pre-strained" materials, and thus constitutes a novel ingredient of our theory. Tipically, these materials are modeled by 3D energy densities of the form (see, for example, [42])

$$\overline{W}^h(z, F) = W\big(FV^h(z', z_3/h)\big), \tag{1.3}$$

for a certain (frame indifferent) homogeneous energy density W minimized at SO(3), where the "pre-stretch" V^h is generally a smooth, invertible tensor field that represents (the inverse of) an active stretch, growth, plasticity or other inelastic phenomena. In these models, V^h plays the role of a parameter that is externally controlled, without any dependence on the elastic properties of the system (or on other parameters of the energy). Instead, in models based on the Flory-Rehner energy and on the relations (1.1)-(1.2), there is an intimate connection between material parameters and minimum energy deformations.

Another interesting feature of the Flory-Rehner energy and, correspondingly, of the family of energy densities we consider, is that they are *not* representable in the pre-stretch form (1.3), with $V^h = \big(1 + hb/\alpha\big)^{-1}\mathbb{I}_3$ and with energy density W minimized at $\alpha\mathrm{SO}(3)$ (see Remark 4.1). This feature depends on the different structure of such an energy with respect to the models based on the representation (1.3), which originates from physical considerations. More precisely, in the latter models $\overline{W}^h$ has the physical meaning of purely elastic energy, while in models for polymer gels $\overline{W}^h$ is the sum of two energy contributions (elastic and mixing energies) that concurrently define the energy minimum, but none of them is separately minimized at $\mathrm{SO}(3)(V^h)^{-1}$. However, the corresponding rescaled densities converge uniformly to some homogeneous density W.

Motivated by the above observations and discarding for the moment the scalar parameter α, which can be accounted for by a simple change of variable, we thus consider the slightly more general setting of a material characterized by a *spontaneous stretch distribution* $\overline{U}^h$ of the form

$$\overline{U}^h(z) \,=\, \mathbb{I}_3 + h\, B\left(z', \frac{z_3}{h}\right), \tag{1.4}$$

where $B : \omega \times (-1/2, 1/2) \to \mathrm{Sym}(3)$ is a given (bounded) strain distribution.

The term "spontaneous" for the distribution $\overline{U}^h$ refers to the tendency of the system to deform, at each point z, according to a deformation whose gradient coincides with $\overline{U}^h(z)$, in order to attain the energy minimum pointwise. However, generally there is no (orientation-preserving) deformation defined globally in Ω_h, whose gradient coincides with $\overline{U}^h$ in the whole of Ω_h. Equivalently, in the words of Mechanics, $\overline{U}^h$ is not kinematically compatible, or, in the words of Riemannian geometry, the Riemann curvature tensor associated with $\overline{U}^h$ does not vanish identically throughout Ω_h.

It is now appropriate to notice that the 3D setting just described can be seen as a generalization of the setting considered in [42] (see also [41]), where the pre-stretch is of the same form as in (1.4), except that the z'-dependence is not considered. At the same time, the relevant case where the pre-stretch in (1.3) is only z'-dependent has been addressed in [31] and [10] and has given rise to the fortunate route of the mathematical treatment of the "non-Euclidean plate theories" (see also [28]), introduced from a physical and mechanical view point by the pioneering work of Sharon and coauthors in [18] and [26]. All in all, our theory stands between those of [42], on one hand, and of [31] and [10], on the other hand, and, to the best of our knowledge, represents the first attempt to considering Kirchhoff plate theories originated by 3D energies characterized by pre-stretches or spontaneous stretches which are heterogeneous in plane as well as along the thickness. Pre-stretches of the form (1.4) have been very recently treated in [13] and [27] to derive corresponding rod models with misfit. Moreover, similar prestretches have been considered in [30] to obtain 2D models in the case of scaling orders higher than the Kirchhoff one.

The central result of this paper is the derivation of a Kirchhoff plate theory from the 3D model outlined above. With abuse of notation, we again denote by $\overline{W}^h$ the energy density associated with this system, which is minimized, for every $z \in \Omega_h$, at $\mathrm{SO}(3)\overline{U}^h(z)$. Hence, the total free energy associated with a deformation $v : \Omega_h \to \mathbb{R}^3$ is

$$\overline{\mathcal{E}}^h(v) \,=\, \int_{\Omega_h} \overline{W}^h\big(z, \nabla v(z)\big)\,\mathrm{d}z.$$

Then, the same arguments as in [42] (which are in turn a slight variant of those employed in the seminal work [20]) can be used to find the corresponding limiting Kirchhoff plate model, under the assumption that

$$\operatorname{curl}\big(\operatorname{curl}\check{D}\big) \;=\; 0, \qquad \text{with} \qquad \check{D}(z') := \int_{-1/2}^{1/2} \check{B}(z',t)\,\mathrm{d}t \qquad \text{for a.e. } z' \in \omega, \tag{1.5}$$

where $\check{B} : \omega \to \mathrm{Sym}(2)$ is obtained from the spontaneous strain distribution B appearing in (1.4) by omitting the third row and the third column. Condition (1.5) deserves some comments. It guarantees that $\check{D}$ is a symmetrized gradient, and in turn allows for the construction of a standard ansatz for the recovery sequence. When instead condition (1.5) is violated, usual arguments such as local modifications or perturbation arguments seem insufficient to prove the same Γ-limit. In fact, we believe that the general Γ-limit has to include a nonlocal term, which can be interpreted as a "first order stretching term". To conclude the comments on condition (1.5), let us add a trivial but important observation: the difficulties one encounter in removing the compatibility assumption on the matrix field $\check{D}$ do not originate from the dependence of the spontaneous strain on the thickness variable, since they persist even in the case where such a dependence is absent.

The Kirchhoff model resulting from the dimension reduction is governed by the energy functional

$$\mathcal{E}^0(y) \;=\; \frac{1}{24}\int_\omega Q_2\left(\mathrm{A}_y(z') - \overline{A}(z')\right)\,\mathrm{d}z' + \text{ad.t.}, \tag{1.6}$$

on each $\mathrm{W}^{2,2}$-isometry y, where ad.t. stays for "additional terms" not depending on y.

In the above expression, the quadratic form Q_2 is defined via a standard relaxation of the second differential of the limiting density W at $\mathbb{I}_3$ (see formulas (2.6) and (2.7)), the symbol A_y stands for the pull-back of the second fundamental form associated with $y(\omega)$ (see (2.21)), and the *target curvature tensor* $\overline{A}$ is defined as

$$\overline{A}(z') \;=\; 12\int_{-1/2}^{1/2} t\check{B}(z',t)\,\mathrm{d}t, \qquad \text{for a.e. } z' \in \omega.$$

It is readily seen that in the case where the prestretch depends only on the thickness variable, a constant target cuvature $\overline{A}$ is produced. For polymer gels, this expression makes the anticipated connection between heterogeneous density N of polymer chains (encoded by $\check{B}$) and curvature more evident, even if not fully explicit. Further, under some approximations or using numerical methods, such a relation can be made explicit and thus can be actually employed in the design of shape morphing gel plates. In general, our derivation, which relies on an accurate description of the 3D swelling energy, offers an advantage over theories based on purely elastic energies with "pre-stretch", where such a connection must be plugged in artificially.

It is worth mentioning that beam theories derived from 2D energies of the form (1.6), in the limit as $\varepsilon \to 0$ when $\omega = (-\ell/2, \ell/2) \times (\varepsilon/2 \times \varepsilon/2)$, can be found in [3] for the case $\overline{A}$ constant and in [19] in the case $\overline{A} = \overline{A}(x_1)$. To use a common terminology, these 1D theories may describe *narrow ribbons* of soft active materials.

To give some insight on the minimizers of the derived 2D model (1.6), we focus on the case where the spontaneous strain B is an odd function of the thickness variable (which trivially fulfills condition (1.5)), being at the same time a piecewise constant function of the planar variable. This case leads in turn to a piecewise constant target curvature tensor. In Section 3, we recall that in the

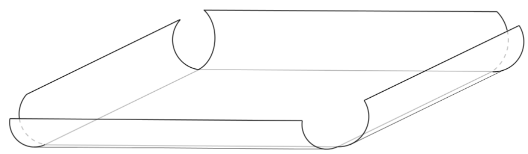

FIGURE 1. An example of a 2D minimum energy configuration.

case where $\overline{A}$ is constant, then a minimizer of the 2D energy $\mathcal{E}^0$ actually minimizes the integrand function pointwise, and the corresponding deformed configuration is a piece of cylindrical surface (see Lemma 3.3 and the discussion preceding it). In the case of a piecewise constant $\overline{A}$, some conditions (specified in Theorem 3.9) under which cylindrical surfaces can be patched together resulting into an isometry must be fulfilled for the pointwise minimizer to exist. When these conditions hold, an example of minimum energy configuration, a patchwork of cylindrical surfaces, is sketched in Figure 1.

The paper is structured as follows: we deal with the theoretical results concerning dimension reduction in Sections 2 and 3, and then we apply them to the case of thin gel sheets, in Section 4. In the final section, we draw some conclusions.

We end this section by introducing some general notation which will be used throughout the paper.

1.1. **Notation.** For fixed $n \in \mathbb{N}$ we will denote by

- $\mathbb{R}^{n\times n}$ the vector space of real $n \times n$ matrices and by $\mathbb{I}_n \in \mathbb{R}^{n\times n}$ the identity matrix,
- $\mathrm{Sym}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal = M\}$ the vector space of symmetric matrices, where by $M^\intercal \in \mathbb{R}^{n\times n}$ we denote the transpose of the matrix $M \in \mathbb{R}^{n\times n}$,
- $\mathrm{Skew}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal = -M\}$ the set of skew-symmetric matrices,
- $\mathrm{SO}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal M = \mathbb{I}_n, \det(M) = 1\}$ the set of all rotations of $\mathbb{R}^n$,
- $\mathrm{Orth}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal M = \mathbb{I}_n\}$ the set of all orthogonal transformations of $\mathbb{R}^n$,
- $\mathrm{Trs}(n) := \{T_v := \cdot + v : v \in \mathbb{R}^n\}$ the set of all translations in $\mathbb{R}^n$. Sometimes, to distinguish between translations in $\mathbb{R}^2$ and $\mathbb{R}^3$, we will denote by τ_v the elements of $\mathrm{Trs}(2)$,
- $M_{\mathrm{sym}} := \frac{M+M^\intercal}{2}$ the symmetric part of the matrix $M \in \mathbb{R}^{n\times n}$,
- $\mathrm{tr}\, M$ the trace of the matrix M and $\mathrm{tr}^2 M := (\mathrm{tr} M)^2$,
- $|M| := \sqrt{\sum_{i,j=1}^n |m_{ij}|^2} = \sqrt{\mathrm{tr}(M^\intercal M)}$, Frobenius norm of a matrix $M = [m_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$,
- $\mathcal{L}^n$ the n-dimensional Lebesgue measure,
- $\mathcal{H}^n$ the n-dimensional Hausdorff measure.

Furthermore, we give the following definitions:

- $\check{F} \in \mathbb{R}^{2\times 2}$ is the 2×2 submatrix of $F \in \mathbb{R}^{3\times 3}$ obtained by omitting the last row and the last column of F,
- given $G \in \mathbb{R}^{2\times 2}$, the matrix $\hat{G} \in \mathbb{R}^{3\times 3}$ associated to G is defined as

$$\hat{G} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{G}} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

We denote by $\{\mathsf{e}_1, \mathsf{e}_2\}$ the standard basis of $\mathbb{R}^2$ and by $\{\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3\}$ the standard basis of $\mathbb{R}^3$. An open connected subset of $\mathbb{R}^2$ will be called *domain*. Sometimes, for the sake of brevity, an open subset of $\mathbb{R}^2$ with Lischitz boundary will be called a *Lipschitz subset* of $\mathbb{R}^2$. The closure of a set $S \subseteq \mathbb{R}^2$ is denoted by $\overline{S}$ or by $\mathrm{cl}(S)$.

2. Three-dimensional model and derivation of the corresponding Kirchhoff plate model

Throughout the paper $\omega \subseteq \mathbb{R}^2$ will be a simply-connected, bounded domain with Lipschitz boundary satisfying the following condition:

$$\begin{aligned}&\text{there exists a closed subset } \Sigma \subset \partial\omega \text{ with } \mathcal{H}^1(\Sigma) = 0 \text{ such that}\\&\text{the outer unit normal exists and is continuous on } \partial\omega \setminus \Sigma.\end{aligned} \tag{2.1}$$

The requirement that ω is a simply-connected domain has to do with the "compatibility" condition of Theorem 2.8 below, which is imposed on the tensor-valued map $D_{\min}$ defined by (2.2) and (2.9). The condition (2.1) is a standard requirement on the domain in order to have some density results for the space of $\mathrm{W}^{2,2}$-isometric immersions of ω into $\mathbb{R}^3$ (see, e.g., [23] and [24]).

We are interested in a thin sheet $\Omega_h := \omega \times (-h/2, h/2)$, with $0 < h \ll 1$, of a material characterized by a *spontaneous stretch* given at each point of Ω_h in the form $\overline{U}^h(z) = \mathbb{I}_3 + hB\big(z', \frac{z_3}{h}\big)$, for a suitable *spontaneous strain* $B \in \mathrm{L}^\infty(\Omega, \mathrm{Sym}(3))$. The stretch $\overline{U}^h$ being *spontaneous* for the material is modeled by introducing a energy density whose minimum state is precisely $\overline{U}^h(z)$ at each point z, modulo superposed rigid body rotations. We denote by U^h the spontaneous stretch given in terms of the rescaled variable $x \in \Omega := \Omega_1$. Namely, $U^h(x) = \overline{U}^h(x', hx_3)$ so that $U^h = \mathbb{I}_3 + hB$.

More in general, we consider a family $\mathcal{B} = \{B^h\}_{h\geq 0}$ of spontaneous strains such that

$$B^h \to B^0 =: B \qquad \text{in } \ \mathrm{L}^\infty(\Omega, \mathrm{Sym}(3)), \ \text{as } h \to 0, \tag{2.2}$$

the corresponding family $\{U^h\}_{h\geq 0}$ of spontaneous stretches defined as

$$U^h(x) := \mathbb{I}_3 + hB^h(x) \qquad \text{for a.e. } x \in \Omega \text{ and for every } h \geq 0, \tag{2.3}$$

and the associated family $\{W^h\}_{h>0}$ of (rescaled) energy density functions $W^h : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$, which we suppose to be Borel functions satisfying the following properties:

(i) for a.e. $x \in \Omega$, the map $W^h(x, \cdot)$ is frame indifferent, i.e.

$$W^h(x, F) = W^h(x, RF) \qquad \text{for every } F \in \mathbb{R}^{3\times 3} \text{ and every } R \in \mathrm{SO}(3);$$

(ii) for a.e. $x \in \Omega$, $W^h(x, \cdot)$ is minimized precisely at $\mathrm{SO}(3)U^h(x)$;

(iii) there exists an open neighbourhood $\mathcal{U}$ of $\mathrm{SO}(3)$ and $W \in C^2(\overline{\mathcal{U}})$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} \big\|W^h(x, \cdot) - W\big\|_{C^2(\overline{\mathcal{U}})} \to 0, \qquad \text{as } h \to 0; \tag{2.4}$$

(iv) there exists a constant $C > 0$, independent of h, such that for a.e. $x \in \Omega$ it holds that

$$W^h(x, F) \geq C \mathrm{dist}^2\big(F, \mathrm{SO}(3)U^h(x)\big), \qquad \text{for every } F \in \mathbb{R}^{3\times 3}. \tag{2.5}$$

The most interesting scenarios occur when the *Cauchy-Green* distribution C^h associated with the spontaneous stretch distribution U^h – namely, $C^h(x) := (U^h(x))^2$ – is not kinematically compatible, i.e. there is no orientation-preserving deformation $v^h : \Omega \to \mathbb{R}^3$ such that $(\nabla v^h)^\intercal \nabla v^h = C^h$ in Ω. We also recall that, since $C^h(x)$ is a positive definite symmetric matrix, the distribution C^h can be interpreted as a metric on Ω and that, in this framework, the kinematic compatibility of C^h is equivalent to the condition that the Riemann curvature tensor associated with C^h vanishes identically in Ω (see [11] and [31]).

Definition 2.1 (Admissible family of free-energy densities)**.** *Given* $\mathcal{B} = \{B^h\}_{h\geq 0}$ *satisfying* (2.2) *and the associated family* $\{U^h\}_{h\geq 0}$ *defined in* (2.3)*, we call* $\mathcal{B}$-admissible *a family* $\{W^h\}_{h>0}$ *of Borel functions from* $\Omega \times \mathbb{R}^{3\times 3}$ *to* $[0, +\infty]$ *fulfilling* (i)-(iv)*.*

Given a $\mathcal{B}$-admissible family $\{W^h\}_{h>0}$ of free-energy densities, with associated limiting density function W, using a standard notation we define the following quadratic form:

$$Q_3(F) := D^2W(\mathbb{I}_3)[F, F], \qquad \text{for every } F \in \mathbb{R}^{3\times 3}. \tag{2.6}$$

Moreover, for every $G \in \mathbb{R}^{2\times 2}$, we set

$$Q_2(G) := \min_{d\in\mathbb{R}^3} Q_3\big(\hat{G} + d \otimes \mathsf{f}_3\big), \tag{2.7}$$

referring to Subsection 1.1 for the notation $\hat{G}$. Observe that the limiting density W inherits properties (i), (ii) and (iv) from convergence (2.4). From this fact one can deduce that Q_2 is indeed a quadratic form and that Q_k, for $k = 2, 3$, has the following properties:

- Q_k is positive semi-definite on $\mathbb{R}^{k\times k}$ and positive definite when restricted to $\mathrm{Sym}(k)$,
- $Q_k(F) = 0$ for every $F \in \mathrm{Skew}(k)$,
- Q_k is strictly convex on $\mathrm{Sym}(k)$.

The proof of some of the listed properties can be found for instance in [10, 20]. We also refer to [14, Proposition 11.9] for a useful characterization of quadratic forms.

Our limiting 2D model will be related to the 2D density function $\overline{Q}_2 : \omega \times \mathbb{R}^{2\times 2} \to [0, +\infty)$ defined as

$$\overline{Q}_2(x', G) := \min_{D\in\mathbb{R}^{2\times 2}} \int_{-1/2}^{1/2} Q_2\big(D + tG - \check{B}(x', t)\big)\,\mathrm{d}t,$$

for a.e. $x' \in \omega$ and every $G \in \mathbb{R}^{2\times 2}$, where $\check{B}$ is related to the 3D model through (2.2), using the notation introduced in Subsection 1.1. Since Q_2 does not depend on the skew-symmetric part of its argument,

we can think of $\overline{Q}_2$ to be defined only on $\omega \times \mathrm{Sym}(2)$ as

$$\overline{Q}_2(x', G) = \min_{D\in\mathrm{Sym}(2)} \int_{-1/2}^{1/2} Q_2\big(D + tG - \check{B}(x', t)\big)\,\mathrm{d}t. \tag{2.8}$$

This minimum problem can be solved explicitly, as stated by the following lemma.

Lemma 2.2. *For a.e.* $x' \in \omega$ *and every* $G \in \mathrm{Sym}(2)$*, the minimizer in* (2.8) *is unique and coincides with*

$$D_{\min}(x') := \int_{-1/2}^{1/2} \check{B}(x', t)dt. \tag{2.9}$$

In other words, we have that

$$\overline{Q}_2(x', G) = \int_{-1/2}^{1/2} Q_2\left(\int_{-1/2}^{1/2} \check{B}(x', s)\,\mathrm{d}s + tG - \check{B}(x', t)\right)\mathrm{d}t \tag{2.10}$$

for a.e. $x' \in \omega$ *and every* $G \in \mathrm{Sym}(2)$*.*

Proof. By using the bilinear form associated with Q_2 it is easy to see that for a.e. $x' \in \omega$ and every $G \in \mathrm{Sym}(2)$ it holds

$$\begin{aligned}&\min_{D\in\mathrm{Sym}(2)}\int_{-1/2}^{1/2} Q_2(D+tG-\check{B}(x',t))\,\mathrm{d}t\\ &=\int_{-1/2}^{1/2} Q_2\big(tG-\check{B}(x',t)\big)\,\mathrm{d}t - Q_2\big(D_{\min}(x')\big) + \min_{D\in\mathrm{Sym}(2)} Q_2\big(D-D_{\min}(x')\big).\end{aligned}$$

From this equality, the thesis trivially follows. □

Note that $D_{\min}$, which is in principle dependent on G from its definition, turns out to be independent of G in the end. This is not the case when, e.g., the limiting density function W depends explicitly on x_3, not just through its spontaneous stretch, see [42]. Note also from hypothesis (2.2) that $D_{\min} \in \mathrm{L}^\infty(\omega, \mathrm{Sym}(2))$. Finally, observe for future reference that from (2.10) one can rewrite $\overline{Q}_2$ in the more explicit form

$$\begin{aligned}\overline{Q}_2(x',G) \;=\; &\frac{1}{12}Q_2\left(G-12\int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t\right)+\int_{-1/2}^{1/2} Q_2\big(\check{B}(x',t)\big)\,\mathrm{d}t\\ &- Q_2\left(\int_{-1/2}^{1/2}\check{B}(x',t)\,\mathrm{d}t\right) - 12Q_2\left(\int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t\right),\end{aligned} \tag{2.11}$$

for a.e. $x' \in \omega$ and every $G \in \mathrm{Sym}(2)$.

Before passing to the rigorous derivation of the 2D model, we provide a technical lemma consisting in two estimates for the family $\{W^h\}$ of energy densities and for its uniform limit W defined in a neighbourhood $\mathcal{U}$ of $\mathrm{SO}(3)$. They are elementary consequences of properties (ii) and (iii) of Definition 2.1 (hence we omit their proof). These estimates will be used in the proof of the Γ-$\liminf$ and the Γ-$\limsup$.

Lemma 2.3. *Let $\bar{r} > 0$ be such that $B_{2\bar{r}}(\mathbb{I}_3)$ is contained in $\mathcal{U}$. Then for every $\varepsilon > 0$ there exists $h_\varepsilon > 0$ and $C > 0$ such that for a.e. $x \in \Omega$, every $F \in B_{\bar{r}}(0)$ and every $h \in (0, h_\varepsilon]$ it holds that*

$$\left|W^h\Big(x, U^h(x)+F\Big) - W\Big(\mathbb{I}_3+F\Big)\right| \le \varepsilon|F|^2, \tag{2.12}$$

$$\left|W^h\Big(x, U^h(x)+F\Big)\right| \le C|F|^2. \tag{2.13}$$

We also introduce two auxiliary functions that will be used in the proof of Γ-convergence result. Letting $\bar{r} > 0$ be as in Lemma 2.3 above, we define

$$\rho^0(F) := W(\mathbb{I}_3+F) - \frac{1}{2}D^2W(\mathbb{I}_3)[F]^2 \quad\text{ and }\quad \rho(s) := \sup_{|F|\le s}|\rho^0(F)| \tag{2.14}$$

for every $F \in B_{\bar{r}}(0)$ and every $s > 0$.

As a direct consequence of the regularity of W, we have that

$$\rho(s)/s^2 \to 0 \quad \text{as } s \to 0. \tag{2.15}$$

In order to state and prove the following compactness and Γ-convergence results we will use the standard notation

$$\nabla' y := \Big(\partial_1 y \Big| \partial_2 y\Big) \qquad \text{and} \qquad \nabla_h y := \Big(\nabla' y \Big| \frac{1}{h}\partial_3 y\Big).$$

Moreover, given a $\mathcal{B}$-admissible family $\{W^h\}_{h>0}$ of energy densities in the sense of Definition 2.1, for every $h>0$ we define the rescaled *free energy functional* $\mathcal{E}^h : \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3) \to [0,+\infty]$ as

$$\mathcal{E}^h(y) := \int_\Omega W^h\big(x, \nabla_h y(x)\big)\,\mathrm{d}x, \qquad \text{for every } y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3). \tag{2.16}$$

Theorem 2.4 (Compactness). *Let $\{y^h\}_{h>0} \subseteq \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ be a sequence which satisfies*

$$\limsup_{h\to 0} \frac{1}{h^2}\mathcal{E}^h(y^h) < +\infty. \tag{2.17}$$

Then $\{\nabla_h y^h\}_{h>0}$ is precompact in $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3})$, that is: there exists a (not relabeled) subsequence such that $\nabla_h y^h \to (\nabla' y|\nu)$ in $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3})$, where $\nu(x) := \partial_1 y(x) \wedge \partial_2 y(x)$. Moreover, the limit $(\nabla' y|\nu)$ has the following properties:

(i) $(\nabla' y|\nu)(x) \in \mathrm{SO}(3)$ *for a.e.* $x \in \Omega$,
(ii) $(\nabla' y|\nu) \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^{3\times 3})$ *and*
(iii) $(\nabla' y|\nu)$ *is independent of* x_3.

In other words, the limiting deformation y belongs to the class $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ defined as in (2.20).

To prove this compactness result, we can use the same argument as in the proof of the corresponding result in [20] where the spontaneous stretch is $\mathbb{I}_3$ in place of our $U^h = \mathbb{I}_3 + hB$. Note that the same argument holds in the case of spontaneous stretch of the form $\mathbb{I}_3 + h^\alpha B$ with $\alpha \geq 1$.

Proof of Theorem 2.4. We will show that the sequence $\{\nabla_h y^h\}_{h>0} \subseteq \mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3})$ satisfies

$$\limsup_{h\to 0} \frac{1}{h^2}\int_\Omega \operatorname{dist}^2\Big(\nabla_h y^h(x), \mathrm{SO}(3)\Big)\,\mathrm{d}x < +\infty. \tag{2.18}$$

The thesis then directly follows by applying Theorem 4.1 from [20]. Fix $h>0$ and $F \in \mathbb{R}^{3\times 3}$. For a.e. $x \in \Omega$ there exists $R_{h,F}(x) \in \mathrm{SO}(3)$ such that

$$\operatorname{dist}\Big(F, \mathrm{SO}(3)\big(\mathbb{I}_3 + hB^h(x)\big)\Big) = \big|F - R_{h,F}(x)(\mathbb{I}_3 + hB^h(x))\big|.$$

We have the following estimate:

$$\begin{aligned} \operatorname{dist}^2(F,\mathrm{SO}(3)) &\leq \big|F - R_{h,F}(x)\big|^2 \leq 2\Big|F - R_{h,F}(x)\big(\mathbb{I}_3 + hB^h(x)\big)\Big|^2 + 2\Big|hR_{h,F}(x)B^h(x)\Big|^2 \\ &\overset{(2.5)}{\leq} \frac{2}{C}W^h(x,F) + 6h^2\big|B^h(x)\big|^2 \end{aligned} \tag{2.19}$$

for a.e. $x \in \Omega$. By (2.17) and (2.19) we have that (2.18) holds true. □

Given a bounded Lipschitz domain $S \subset \mathbb{R}^2$, the class of the isometries of S into $\mathbb{R}^3$ is denoted by

$$\mathrm{W}^{2,2}_{\mathrm{iso}}(S,\mathbb{R}^3) = \Big\{y \in \mathrm{W}^{2,2}(S,\mathbb{R}^3) : |\partial_1 y| = |\partial_2 y| = 1,\ \partial_1 y \cdot \partial_2 y = 0\Big\}. \tag{2.20}$$

For the sake of brevity, we equivalently use the symbol $\mathrm{W}^{2,2}_{\mathrm{iso}}(S)$. We recall that for a given $y \in \mathrm{W}^{2,2}\big(\omega,\mathbb{R}^3\big)$ the pull-back of the second fundamental form of $y(\omega)$ at the point $y(x')$ is given by

$$\mathrm{A}_y(x') := (\nabla' y(x'))^{\mathsf{T}}\nabla'\nu(x'), \qquad \text{where } \nu(x') := \partial_1 y(x') \wedge \partial_2 y(x') \quad \text{for a.e. } x' \in \omega. \tag{2.21}$$

As we are going to see, the 2D limiting model will depend on deformations $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega,\mathbb{R}^3)$ through A_y.

More precisely, the limiting model
will be described by the energy functional $\mathcal{E}^0 : \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3) \to [0,+\infty]$ defined as

$$\mathcal{E}^0(y) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\omega \overline{Q}_2(x',\mathrm{A}_y(x'))\,\mathrm{d}x', & \text{for } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.22}$$

where $\overline{Q}_2$ is defined through (2.2) and (2.10).

We also recall that smooth functions are dense in the class of $\mathrm{W}^{2,2}$-isometric immersions, as stated in the following theorem proved in [23].

Theorem 2.5. *Assume that $S \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain which satisfies (2.1). Then $\mathrm{W}^{2,2}_{\mathrm{iso}}(S) \cap C^\infty(\overline{S},\mathbb{R}^3)$ is $\mathrm{W}^{2,2}$-strongly dense in $\mathrm{W}^{2,2}_{\mathrm{iso}}(S)$.*

This density result will be used for the construction of the recovery sequence in the proof of the Γ-$\limsup$ convergence result below.

Theorem 2.6 (Γ-limit)**.** *The following convergence results hold true:*

(i) Γ-$\liminf$: *for every sequence $\{y^h\}_{h>0}$ and every y such that $y^h \rightharpoonup y$ weakly in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$, it holds*

$$\mathcal{E}^0(y) \le \liminf_{h\to 0} \frac{1}{h^2}\mathcal{E}^h(y^h),$$

(ii) Γ-$\limsup$: *under the hypothesis*

$$\operatorname{curl}\big(\operatorname{curl} D_{\min}\big) = 0 \quad in \quad \mathrm{W}^{-2,2}\big(\omega,\mathrm{Sym}(2)\big), \tag{2.23}$$

with $D_{\min}$ defined by (2.2) and (2.9), we have that for every $y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ there exists a sequence $\{y^h\}_{h>0}$ such that $y^h \to y$ in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$, fulfilling

$$\mathcal{E}^0(y) = \lim_{h\to 0} \frac{1}{h^2}\mathcal{E}^h(y^h).$$

The convergence results of the previous theorem amount to saying that the sequence of energy functionals $\frac{1}{h^2}\mathcal{E}^h$ Γ-converge to $\mathcal{E}^0$, as $h \to 0$, in the strong and weak topology of $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$. The operator curl inside the parenthesis in condition (2.23) acts on a 2×2 matrix by taking the curl of each row, giving as a result a two-dimensional vector. We postpone the proof of the theorem after the following example.

Example 2.7. Note that when $D_{\min}$ is constant, condition (2.23) is trivially satisfied. In particular, recalling definition (2.9), condition (2.23) is trivially satisfied whenever the map $x \mapsto \check{B}$ is constant in x'. At the same time, the same condition is satisfied with $D_{\min} \equiv 0$ by every map $x \mapsto \check{B}(x)$ which is nothing but odd in x_3. We also note that it is possible to realize $D_{\min} \equiv C \neq 0$ through a map

$x \mapsto \check{B}(x)$ which is not constant in x'. To construct such an example, one can fix $B_m \in \mathrm{Sym}(2) \setminus \{0\}$ and define

$$\check{B}(x) := \sum_{i=1}^{N} \check{B}_i(x_3)\chi_{\omega_i}(x'), \qquad \text{for a.e. } x \in \Omega,$$

where $\{\omega_i\}_{i=1}^N$ is a partition of ω and $\{\check{B}_i\}_{i=1}^N \subseteq \mathrm{L}^\infty\big((-1/2,1/2),\mathbb{R}^{3\times 3}\big)$ is a family of functions satisfying

$$\int_{-1/2}^{1/2} \check{B}_i(x_3)\,\mathrm{d}x_3 = B_m, \qquad \text{for every } i = 1,\dots,N,$$

and such that $\check{B}_i(x_3) \neq \check{B}_j(x_3)$ for every $i \neq j$ and every x_3. This gives rise to $\check{B}$ which is piecewise constant in x' (but not constant in the same variable), and in turn to

$$D_{\min}(x') \;=\; \sum_{i=1}^{N} \chi_{\omega_i}(x') \int_{-1/2}^{1/2} \check{B}_i(t)\,\mathrm{d}t \;=\; B_m.$$

Note also that the above defined map $\check{B}$ can give rise to a non-constant tensor valued map $x' \mapsto \int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t$, which is interpreted in Section 3 (in each point x') as the target curvature tensor which appears in the 2D limiting model. Indeed, in the case of $N=2$, by choosing $\check{B}_1(x_3) := (x_3+1)\mathbb{I}_2$ and $\check{B}_2(x_3) := (x_3^3+1)\mathbb{I}_2$ for all $x_3 \in (-1/2,1/2)$, we obtain a simple example of $\check{B}$ for which $D_{\min}$ is constant, while the tensor-valued map $x' \mapsto \int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t$ is piecewise constant.

$\triangle$

The proof of the Γ-$\liminf$ is a straightforward adaptation to the case of a family of energy densities $\{W^h\}$ with wells $\mathrm{SO}(3)\big(\mathbb{I}_3 + hB^h\big)$, of the corresponding result in [20] pertaining the case of a homogeneous W (minimized at $\mathrm{SO}(3)$). For the construction of the recovery sequence in the proof of the Γ-$\limsup$ one has instead to add an additional term with respect to the classical construction (see the third summand on the right-hand side of (2.26)). Such additional term gives rise, in the limit as $h \to 0$, to a symmetrized gradient (see formula (2.29)), in a position where the map $D_{\min}$ should appear in order to match the Γ-limit (cfr. (2.8) and (2.22)). For this purpose, condition (2.23) guarantees that the map $D_{\min}$ is a symmetrized gradient, thanks to Theorem 2.8. Throughout the following proof $\overline{C}$ is a generic positive constant, varying form line to line and independent of all other quantities.

Proof of Theorem 2.6. (i) Γ-$\liminf$: Let $y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ and $\{y^h\}$ be such that $y^h \rightharpoonup y$ weakly in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$. Assume that $\liminf_{h\to 0} \mathcal{E}^h\big(y^h\big)/h^2 < +\infty$, otherwise the proof is trivial. Then, as shown in [20] and up to a (not relabeled) subsequence, there exists a family of piecewise constant maps $R^h : Q_h \to \mathrm{SO}(3)$ such that

$$\int_{Q_h\times(-1/2,1/2)} \big|\nabla_h y^h(x) - R^h(x')\big|^2\,\mathrm{d}x \le \overline{C}h^2, \tag{2.24}$$

and $R^h \to (\nabla' y|\nu)$ in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ as $h \to 0$, where $Q_h := \bigcup_{Q_{a,3h}\subseteq\omega} Q_{a,h}$ and $Q_{a,h} := a + (-h/2,h/2)^2$ for every $h > 0$ and $a \in h\mathbb{Z}^2$. Moreover, the sequence $G^h : \Omega \to \mathbb{R}^{3\times 3}$ defined by

$$G^h(x',x_3) := \begin{cases} \dfrac{R^h(x')^\intercal \nabla_h y^h(x',x_3) - \mathbb{I}_3}{h} & \text{for } x \in Q_h \times (-1/2,1/2), \\ 0 & \text{elswhere in } \Omega, \end{cases} \tag{2.25}$$

converges weakly in $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$, as $h\to0$, to some $G\in\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$ such that

$$\check{G}(x)=\check{G}(x',0)+x_3\mathrm{A}_y(x'),\qquad \text{for a.e. } x\in\Omega.$$

Letting χ_h be the characteristic function of the set $Q_h\cap\big\{|G^h(x)|\le 1/\sqrt{h}\big\}$ we also have that $\chi_h G^h\rightharpoonup G$ in $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$ as $h\to0$. Now, by denoting $A^h:=G^h-B^h$ and by using also the convergence in (2.2) we have

$$A^h\rightharpoonup G-B \text{ in } \mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})\quad\text{and}\quad \|hA^h\|_{\mathrm{L}^\infty\left(Q_h\cap\{|G^h(x)|\le1/\sqrt{h}\}\right)}\to0.$$

By using frame indifference of W^h, (2.15) and the estimate (2.12) from Lemma 2.3, we have that for a fixed $\varepsilon>0$, there exists $\bar h>0$ such that the following estimates hold for every $h\in(0,\bar h)$:

$$\begin{aligned}
\frac{1}{h^2}\int_\Omega W^h(x,\nabla_h y^h(x))\,\mathrm{d}x\ge&\frac{1}{h^2}\int_\Omega \chi_h W^h(x,R^h(x')^\top\nabla_h y^h(x))\,\mathrm{d}x\\
=&\frac{1}{h^2}\int_\Omega\chi_h W^h\Big(x,\big(\mathbb{I}_3+hB^h(x)\big)+hA^h(x)\Big)\,\mathrm{d}x\\
\ge&\frac{1}{h^2}\int_\Omega\chi_h\frac12 D^2W(\mathbb{I}_3)\big[hA^h(x)\big]^2-\chi_h\varepsilon|hA^h(x)|^2+\chi_h\rho^0\big(hA^h(x)\big)\,\mathrm{d}x\\
\ge&\int_\Omega\chi_h\frac12 Q_3\big(A^h(x)\big)-\chi_h\varepsilon|A^h(x)|^2-\chi_h\frac{\rho\big(|hA^h(x)|\big)}{|hA^h(x)|^2}|A^h(x)|^2\,\mathrm{d}x,
\end{aligned}$$

where ρ^0 and ρ are defined in (2.14) . Since Q_3 is lower semicontinuous in the weak topology of $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$ and since (2.15) holds , passing to lim inf as $h\to0$ in the above inequality we obtain

$$\liminf_{h\to0}\frac{1}{h^2}\int_\Omega W^h(x,\nabla_h y^h(x))\,\mathrm{d}x\ge\int_\Omega\frac12 Q_3\big(G(x)-B(x)\big)\,\mathrm{d}x-\overline{C}\varepsilon,$$

where $\overline{C}>0$ is such that $||A^h||_{\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})}\le\overline{C}$. Finally, by letting $\varepsilon\to0$ and by using the fact that $Q_3(F)\ge Q_2(\check F)$ for every $F\in\mathbb{R}^{3\times3}$ we get that

$$\begin{aligned}
\liminf_{h\to0}\frac{1}{h^2}\int_\Omega W^h(x,\nabla_h y^h(x))\,\mathrm{d}x\ge&\frac12\int_\Omega Q_2\big(\check G(x',0)+x_3\mathrm{A}_y(x')-\check B(x',x_3)\big)\,\mathrm{d}x\\
\ge&\frac12\int_\omega\overline{Q}_2\big(x',\mathrm{A}_y(x')\big)\,\mathrm{d}x',
\end{aligned}$$

which proves Γ- lim inf inequality.

(ii) Γ- lim sup: Let us prove Γ- lim sup inequality for a given $y\in\mathrm{W}^{2,2}_{\mathrm{iso},0}(\omega):=\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)\cap C^\infty(\overline{\omega},\mathbb{R}^3)$. Once we have proved it, Γ-lim sup inequality will follow for any $y\in\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ by the density result of Theorem 2.5 and the continuity of the limiting functional $\mathcal{E}^0$ with respect to $\mathrm{W}^{2,2}$ convergence. Suppose that $\mathcal{E}^0(y)<+\infty$ (otherwise the proof is trivial). Let $d\in C^\infty_c(\Omega,\mathbb{R}^3)$ and define $D:\Omega\to\mathbb{R}^3$ by

$$D(x',x_3):=\int_0^{x_3}d(x',t)\,\mathrm{d}t,\qquad\text{for every } (x',x_3)\in\omega\times(-1/2,1/2)=\Omega.$$

Let $\tilde g:=(\tilde g_1,\tilde g_2)\in C^\infty_c(\mathbb{R}^2,\mathbb{R}^2)$. We consider the family of functions y^h of the form

$$y^h(x):=y(x')+h\big[x_3\nu(x')+\nabla'y(x')\tilde g(x')\big]+h^2D(x',x_3),\tag{2.26}$$

for every $x\in\Omega$ and every $h>0$, whose (h-rescaled) gradient $\nabla_h y^h$ reads as

$$\nabla_h y^h(x)=(\nabla'y(x')\big|\nu(x'))+h\big(\nabla'\big[x_3\nu(x')+\nabla'y(x')\tilde g(x')\big]\big|d(x)\big)+h^2\big(\nabla'D(x)\big|0\big),$$

for every $x \in \Omega$ and every $h > 0$, One can easily verify that, in particular, $\{y^h\}_{h>0} \subseteq \mathrm{W}^{2,\infty}(\Omega, \mathbb{R}^3)$ and that it converges in $\mathrm{W}^{1,2}$ to y, as $h \to 0$. Denote by $R(x') := \big(\nabla' y(x') \big| \nu(x')\big)$ for every $x' \in \omega$. Set

$$C^h(x) := R^\intercal(x')\Big(\big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\big|d(x)\big) + h\big(\nabla' D(x)\big|0\big)\Big) - B^h(x), \qquad \text{for a.e. } x \in \Omega,$$

and note that C^h converges in $\mathrm{L}^\infty(\Omega, \mathbb{R}^{3\times 3})$ to the function

$$\Omega \ni x \mapsto R^\intercal(x')\big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\big|d(x)\big) - B(x) \in \mathbb{R}^{3\times 3}.$$

With this notation, we have that $R^\intercal(x')\nabla_h y^h(x) = U^h(x) + hC^h(x)$ for a.e. $x \in \Omega$ with U^h given by (2.3). By the frame indifference of $W^h(x, \cdot)$, boundedness of C^h and B^h in L^∞-norm and the estimates (2.12) and (2.13) from Lemma 2.3, there exists $\overline{C}, \bar{h} > 0$ such that

$$\frac{1}{h^2} W^h\Big(x, \nabla_h y^h(x)\Big) = \frac{1}{h^2} W^h\Big(x, R^\intercal(x')\nabla_h y^h(x)\Big) = \frac{1}{h^2} W^h\Big(x, U^h(x) + hC^h(x)\Big) \le \overline{C},$$

for a.e. $x \in \Omega$ and every $0 < h \le \overline{h}$. Moreover,

$$\frac{1}{h^2} W^h\Big(x, \nabla_h y^h(x)\Big) \to \frac{1}{2} Q_3\Big(R^\intercal(x')\big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\big|d(x)\big) - B(x)\Big)$$

pointwise almost everywhere in Ω, as $h \to 0$. Then, by applying dominated convergence theorem we get that

$$\frac{1}{h^2}\int_\Omega W^h\Big(x, \nabla_h y^h(x)\Big)\,\mathrm{d}x \to \frac{1}{2}\int_\Omega Q_3\Big(R^\intercal(x')\big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\big|d(x)\big) - B(x)\Big)\,\mathrm{d}x$$

as $h \to 0$. To proceed, for a.e. $x \in \Omega$ we denote by $\overline{F}(x)$ the 2×2 part of the argument of Q_3 in the above integral. Now let $\ell : \mathrm{Sym}(2) \to \mathbb{R}^3$ be the map that associates to every $F \in \mathrm{Sym}(2)$ the unique element of $\operatorname{argmin}_{c\in\mathbb{R}^3} Q_3\big(\hat{F} + (c \otimes \mathsf{f}_3)_{\mathrm{sym}}\big)$. By writing down the first order necessary condition for the minimum problem defining $\ell(F)$, one can easily deduce that the map ℓ is linear. Define $\overline{d} : \Omega \to \mathbb{R}^3$ as

$$\overline{d}(x) := R(x')\left(\ell\big(\overline{F}_{\mathrm{sym}}(x)\big) + (2B_{13}\ \ 2B_{23}\ \ B_{33})^\intercal(x) - \begin{pmatrix} \big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\big)^\intercal \nu(x') \\ 0 \end{pmatrix}\right),$$

for a.e. $x \in \Omega$. Since $\overline{F}_{\mathrm{sym}} \in \mathrm{L}^\infty(\Omega, \mathrm{Sym}(2))$, $B \in \mathrm{L}^\infty(\Omega, \mathrm{Sym}(3))$ and y and $\tilde{g}$ are smooth vector fields, it follows that $\bar{d}$ belongs to $\mathrm{L}^2(\Omega, \mathbb{R}^3)$. By choosing d to be equal to $\bar{d}$, one can readily check that

$$\Big(R^\intercal(x')\big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\big|\bar{d}(x)\big) - B(x)\Big)_{\mathrm{sym}} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\overline{F}_{\mathrm{sym}}(x)$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array}\right) + \Big(\ell\big(\overline{F}_{\mathrm{sym}}(x)\big) \otimes \mathsf{f}_3\Big)_{\mathrm{sym}}. \tag{2.27}$$

Observe further that

$$\begin{aligned} (\nabla' y)\tilde{g} &= \tilde{g}_1\partial_1 y + \tilde{g}_2\partial_2 y, \\ \nabla'\big((\nabla' y)\tilde{g}\big) &= \big(\tilde{g}_1\partial_1\partial_1 y + \tilde{g}_2\partial_1\partial_2 y\big|\, \tilde{g}_1\partial_2\partial_1 y + \tilde{g}_2\partial_2\partial_2 y\big) + \nabla' y \nabla' \tilde{g}. \end{aligned}$$

Since $y \in \mathrm{W}^{2,2}_{\mathrm{iso},0}(\omega)$, it holds that $\partial_i y \cdot \partial_j y = \delta_{ij}$ and $\partial_i\partial_j y \cdot \partial_k y = 0$ for every $i, j, k = 1, 2$. In turn, we have that

$$(\nabla' y)^\intercal\, \nabla'\big((\nabla' y)\tilde{g}\big) = \nabla' \tilde{g}.$$

Now, by direct computation we obtain

$$\overline{F}_{\mathrm{sym}}(x) = x_3 \mathrm{A}_y(x') + \nabla'_{\mathrm{sym}} \tilde{g}(x') - \check{B}(x), \qquad \text{for a.e. } x \in \Omega. \tag{2.28}$$

Finally, by definition of Q_2, (2.27) and (2.28) it holds that

$$\begin{aligned}&\int_\Omega Q_3\Big(R^\intercal(x')\big(\nabla'\big[x_3\nu(x') + \nabla' y(x')\tilde{g}(x')\big]\,\big|\,\overline{d}(x)\big) - B(x)\Big)\,\mathrm{d}x\\ &= \int_\Omega Q_2\big(x_3\mathrm{A}_y(x') + \nabla'_{\mathrm{sym}}\tilde{g}(x') - \check{B}(x)\big)\,\mathrm{d}x.\end{aligned}$$

Therefore, the density of $C_c^\infty(\Omega,\mathbb{R}^3)$ in $\mathrm{L}^2(\Omega,\mathbb{R}^3)$ and a diagonal argument give us that

$$\limsup_{h\to 0}\frac{1}{h^2}\int_\Omega W^h(x,\nabla_h y^h(x))\,\mathrm{d}x = \int_\omega \frac{1}{2}\int_{-1/2}^{1/2} Q_2\big(x_3\mathrm{A}_y(x') + \nabla'_{\mathrm{sym}}\tilde{g}(x') - \check{B}(x',x_3)\big)\,\mathrm{d}x_3\,\mathrm{d}x'. \tag{2.29}$$

The compatibility assumption (2.23) on $D_{\min}$ and Theorem 2.8 guarantee the existence of the map $w \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^2)$ such that $D_{\min}(x') = \nabla_{\mathrm{sym}} w(x')$ for a.e. $x' \in \omega$. Thus, by using the density of $C_c^\infty(\mathbb{R}^2,\mathbb{R}^2)$ (when restricted to ω) in $\mathrm{W}^{1,2}(\omega,\mathbb{R}^2)$ and a diagonal argument one more time, we prove Γ-lim sup inequality for a given $y \in \mathrm{W}^{2,2}_{\mathrm{iso},0}(\omega)$. □

The following result is used in the proof of the Γ-lim sup. It can be found in [12, Theorem 3.2] and has to do with the so-called Saint-Venant compatibility condition in L^2 .

Theorem 2.8. *Let $S \subseteq \mathbb{R}^2$ be a simply-connected bounded domain with Lipschitz boundary and let $A \in \mathrm{L}^2(S,\mathrm{Sym}(2))$. Then*

$$\mathrm{curl}\big(\mathrm{curl}\,A\big) = 0 \ \text{ in } \ \mathrm{W}^{-2,2}(S,\mathrm{Sym}(2)) \quad \Longleftrightarrow \quad A = \nabla_{\mathrm{sym}} w \ \text{ for some } \ w \in \mathrm{W}^{1,2}(S,\mathbb{R}^2). \tag{2.30}$$

Moreover w is unique up to rigid displacements.

Remark 2.9. By standard arguments of Γ-convergence it can be shown that the above analysis holds also in the case when the appropriate body forces are present. More precisely, the above results can be applied to the sequence of functionals $\{\mathcal{F}^h\}_{h>0}$ defined by

$$\mathcal{F}^h(y) = \mathcal{E}^h(y) - \int_\Omega f^h(x)\cdot y(x)\,\mathrm{d}x, \qquad \text{for every } y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3),$$

where $\{f^h\}_{h\geq 0} \subseteq \mathrm{L}^2(\Omega,\mathbb{R}^3)$ is the family of body forces such that

$$\frac{f^h}{h^2} \rightharpoonup f^0 \quad \text{weakly in } \mathrm{L}^2(\Omega,\mathbb{R}^3) \quad \text{and} \quad \int_\Omega f^h(x)\,\mathrm{d}x = 0 \ \text{ for every } h \geq 0.$$

The sequence $\{\mathcal{F}^h\}$ $\Gamma-$converges, as $h \to 0$, to

$$\mathcal{F}^0(y) := \begin{cases} \mathcal{E}^0(y) - \displaystyle\int_\omega f(x')\cdot y(x')\,\mathrm{d}x', & \text{for } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega),\\ +\infty, & \text{otherwise.}\end{cases}$$

where $f(x') := \int_{-1/2}^{1/2} f^0(x',t)\,\mathrm{d}t$ for a.e. $x' \in \omega$. ■

3. 2D ENERGY MINIMIZERS

3.1. x'-dependent target curvature tensor $\overline{A}$ and pointwise minimizers. In this section, we discuss the minimizers of the derived $2D$ model in some special cases. Recall that the $2D$ limiting energy functional $\mathcal{E}^0$ is given by

$$\mathcal{E}^0(y) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\omega \overline{Q}_2\big(x', \mathrm{A}_y(x')\big)\,\mathrm{d}x', & \text{for } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ is the set of $\mathrm{W}^{2,2}$-isometric immersions of ω into $\mathbb{R}^3$, defined by (2.20). From formula (2.11), we have that

$$\mathcal{E}^0(y) \;=\; \frac{1}{24}\int_\omega Q_2\bigg(\mathrm{A}_y(x') - 12\int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t\bigg)\,\mathrm{d}x' + \text{ad.t.} \tag{3.1}$$

for every $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, where ad.t. stays for "additional terms" (not depending on y) and is given by

$$\text{ad.t.} := \frac{1}{2}\int_\omega \int_{-1/2}^{1/2} Q_2\big(\check{B}(x',t)\big)\,\mathrm{d}t - Q_2\left(\int_{-1/2}^{1/2}\check{B}(x',t)\,\mathrm{d}t\right) - 12 Q_2\left(\int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t\right)\mathrm{d}x'. \tag{3.2}$$

Recall that A_y is the pull-back of the second fundamental form associated with $y(\omega)$ (see (2.21)), hence it gives information on the curvature *realized* by the deformation y. On the other hand, when reading the expression for $\mathcal{E}^0$, it is natural to define the *target* curvature tensor

$$\overline{A}(x') := 12\int_{-1/2}^{1/2} t\check{B}(x',t)\,\mathrm{d}t, \qquad \text{for a.e. } x' \in \omega, \tag{3.3}$$

which encodes the spontaneous curvature of the system. While, for a.e. x', the tensor $\overline{A}(x')$ (which depends on $\check{B}$ and in turn on the family of spontaneous strains $\{B^h\}$, see formula (2.2)) is a given $2{\times}2$ symmetric matrix with possibly nonzero determinat, it is a well known result of differential geometry that every smooth $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ satisfies $\det \mathrm{A}_y = 0$ in ω. From [36, Lemma 2.5], one can deduce that the same property holds for any arbitrary $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, a.e. in ω. Our aim is to determine explicitly some classes of minimizers. More precisely, introducing the notation

$$\mathcal{F} \;:=\; \big\{F \in \mathrm{Sym}(2) : \det F = 0\big\}, \tag{3.4}$$

and having in mind the inequality

$$\min_{\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)} \mathcal{E}^0 \;\geq\; \frac{1}{24}\int_\omega \min_{F\in\mathcal{F}} Q_2\big(F - \overline{A}(x')\big)\,\mathrm{d}x' + \text{ad.t.},$$

we will focus our attention on *pointwise minimizers* of $\mathcal{E}^0$. Namely, on those $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ such that

$$\mathcal{E}^0(y) \;=\; \frac{1}{24}\int_\omega \min_{F\in\mathcal{F}} Q_2\big(F - \overline{A}(x')\big)\,\mathrm{d}x' + \text{ad.t.} \;=\; \min_{\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)} \mathcal{E}^0. \tag{3.5}$$

To go on, let us consider the set

$$\mathcal{N}(x') := \operatorname*{argmin}_{F\in\mathcal{F}} Q_2\big(F - \overline{A}(x')\big), \tag{3.6}$$

for a.e. $x' \in \omega$. Note that $\mathcal{N}(x') \neq \emptyset$ for a.e. $x' \in \omega$, because Q_2 is a positive definite quadratic form (when restricted to $\mathrm{Sym}(2)$) and $\mathcal{F}$ is a closed subset of $\mathrm{Sym}(2)$. To accomplish our program, we would like to have some explicit representation of the elements of $\mathcal{N}(x')$, for a.e. $x' \in \omega$, also in view of the application which motivates our analysis (see Section 4.). Therefore, we restrict our attention to case of W *isotropic*, i.e. such that

$$W(RFP) = W(F), \qquad \text{for every } F \in \mathbb{R}^{3\times 3} \text{ and every } R, P \in \mathrm{SO}(3).$$

This implies the existence of constants $\lambda \in \mathbb{R}$ and $\mu > 0$, called Lamé moduli, such that

$$Q_3(F) \,:=\, D^2W(\mathbb{I}_3)[F,F] \,=\, 2\mu|F_{\mathrm{sym}}|^2 + \lambda\,\mathrm{tr}^2 F,$$

for every $F \in \mathbb{R}^{3\times 3}$ (see [22]). In turn, from this expression one can easily show that

$$Q_2(F) \,:=\, \min_{d\in\mathbb{R}^3} Q_3\big(\hat{F} + d\otimes \mathsf{f}_3\big) \,=\, 2\mu\left(|F_{\mathrm{sym}}|^2 + \beta\,\mathrm{tr}^2 F\right), \qquad \text{for every } F \in \mathbb{R}^{2\times 2}, \tag{3.7}$$

where β has the expression

$$\beta \,=\, \frac{\lambda}{2\mu+\lambda}. \tag{3.8}$$

Since Q_3 is positive definite by its very definition, then we have that $\mu > 0$ and $2\mu + 3\lambda > 0$. In turn, it holds that $\beta > -1/2$ and hence that Q_2 is positive definite. This fact guarantees in particular that the quantities appearing in the statement of Lemma 3.1 below are well defined.

Note that in the case when $\overline{A}$ is constant in ω, pointwise minimizers of $\mathcal{E}^0$ always exist. More precisely, as noticed in [41] and [42] (see Lemma 3.3 below), any minimizer y of $\mathcal{E}^0$ with $\overline{A}$ constant is characterized by the property $\mathrm{A}_y(x') \equiv \text{const.} \in \mathcal{N}$ for a.e. $x' \in \omega$, where

$$\mathcal{N} \,:=\, \operatorname*{argmin}_{F\in\mathcal{F}} Q_2\big(F - \overline{A}\big).$$

Clearly, in the case of nonconstant $\overline{A}$, this is not always true. Now, while the analysis of the *minimizers* of $\mathcal{E}^0$, with an arbitrary nonconstant $\overline{A}$, is behind the scope of the present paper, it is natural in our context to try to understand under which conditions the existence of *pointwise minimizers* of $\mathcal{E}^0$ is guaranteed. In Subsection 3.2 we answer this question in the case when $\overline{A}$ is piecewise constant. To do this, we need a structure result for the set $\mathcal{N}$ in the case of constant $\overline{A}$. This is the content of the following lemma.

Lemma 3.1. *Let a and b be two real numbers and let β be given by (3.8). The following implications hold:*

(*i*) *If* $\overline{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ *then* $\mathcal{N} = \left\{ \rho^{\mathsf{T}} \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix} \rho \,:\, \rho \in \mathrm{SO}(2) \right\}$ *with* $\mathfrak{r} = a\frac{1+2\beta}{1+\beta}$.

(*ii*) *If* $\overline{A} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ *then* $\mathcal{N} = \left\{ \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -\mathfrak{r} \end{pmatrix} \right\}$ *with* $\mathfrak{r} = \frac{a}{1+\beta}$.

(*iii*) *If* $\overline{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $|a| > |b|$ *then* $\mathcal{N} = \left\{ \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix} \right\}$ *with* $\mathfrak{r} = a + \frac{b\beta}{1+\beta}$.

(*iv*) *If* $\overline{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $|b| > |a|$ *then* $\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{r} \end{pmatrix} \right\}$ *with* $\mathfrak{r} = b + \frac{a\beta}{1+\beta}$.

Before giving the proof of the above statement, let us make a couple of comments. First, note that the lemma, though restricted to the case of $\overline{A}$ diagonal, covers all the interesting cases, from the simple observation that, with abuse of notation, $\mathcal{N}_{\overline{A}} = \bar{\rho}\mathcal{N}_{\overline{D}}\,\bar{\rho}^{\mathsf{T}}$, where $\bar{\rho} \in \mathrm{Orth}(2)$ is such that $\bar{\rho}^{\mathsf{T}}\,\overline{A}\,\bar{\rho}$ coincides with the diagonal matrix $\overline{D}$. Moreover, interpreting the elements of $\mathcal{N}$ as second fundamental forms of *cylinders* (see the discussion below), the parameter $\mathfrak{r}$, when different from zero, corresponds to the nonzero principal curvature. In this case, observe also that, with abuse of notation, the set $\mathcal{N}_{(ii)}$ is never a subset of $\mathcal{N}_{(i)}$ and that, as for the (two) elements of $\mathcal{N}_{(ii)}$, the elements of $\mathcal{N}_{(i)}$ are pairwise linearly independent. This can be easily read off from the simple fact that

$$\mathcal{N}_{(i)} \;=\; \mathfrak{r}\left\{n \otimes n \,:\, n \in \mathbb{R}^2 \text{ with } |n| = 1\right\}.$$

Finally, the set of the directions corresponding to $\pm\mathfrak{r}$ in the cases (i), (ii), (iii), and (iv) is given by $\{\rho\,\mathsf{e}_1 : \rho \in \mathrm{SO}(2)\}, \{\mathsf{e}_1, \mathsf{e}_2\}, \{\mathsf{e}_1\}, \{\mathsf{e}_2\}$, respectively. This fact can be interpreted saying that, in order to reduce the energy, while in case (i) rolling up along all the possible directions is equally favorable, in the remaining cases the system rolls up along the direction corresponding to the greater (in modulus) eigenvalue of the target curvature tensor $\overline{A}$.

Proof of Lemma 3.1. Let $a, b \in \mathbb{R}$ and let $\overline{A} = \mathrm{diag}(a, b)$. By representing any $F \in \mathrm{Sym}(2)$ by $\left(\begin{smallmatrix} \xi & \zeta \\ \zeta & \upsilon \end{smallmatrix}\right)$, $\zeta, \xi, \upsilon \in \mathbb{R}$ and recalling that Q_2 is of the form (3.7), the minimization problem to be solved is:

$$\min_{F \in \mathcal{F}} \left\{\left|F - \overline{A}\right|^2 + \beta\, \mathrm{tr}^2\!\left(F - \overline{A}\right)\right\} = \min_{\substack{(\xi,\upsilon)\in\mathbb{R}^2,\, \zeta\in\mathbb{R} \\ \xi\upsilon = \zeta^2}} \left\{\left|\begin{pmatrix} \xi - a & \zeta \\ \zeta & \upsilon - b \end{pmatrix}\right|^2 + \beta\, \mathrm{tr}^2 \begin{pmatrix} \xi - a & \zeta \\ \zeta & \upsilon - b \end{pmatrix}\right\}$$

Denote $P := \left\{(\xi, \upsilon) \in \mathbb{R}^2 \,\middle|\, \xi\upsilon \geq 0\right\}$ and define for every $(\xi, \upsilon) \in P$ the function

$$f(\xi, \upsilon) := (1 + \beta)(\xi + \upsilon)^2 - 2\big(a(1 + \beta) + b\beta\big)\xi - 2\big(b(1 + \beta) + a\beta\big)\upsilon + a^2 + b^2 + \beta(a + b)^2,$$

so that the minimization problem becomes $\min_{(\xi,\upsilon)\in P} f(\xi, \upsilon)$. In the case when $a \neq b$, f attains its minimum on $\partial P = \left\{(\xi, \upsilon) \in \mathbb{R}^2 \,|\, \xi\upsilon = 0\right\}$. With this said, (ii), (iii) and (iv) easily follow by straightforward computations. To prove (i), we first note that the set of stationary points of f in $\mathrm{int}(P)$ is given by

$$\left\{\left(\eta^{\pm}_{\zeta}, \eta^{\mp}_{\zeta}\right) \in \mathbb{R}^2 \,:\, \zeta \in \left[-\frac{|\mathfrak{r}|}{2}, \frac{|\mathfrak{r}|}{2}\right] \setminus \{0\}\right\},$$

where

$$\mathfrak{r} = \frac{a(1 + 2\beta)}{(1 + \beta)} \quad \text{and} \quad \eta^{\pm}_{\zeta} := \frac{\mathfrak{r}}{2} \pm \frac{\sqrt{\mathfrak{r}^2 - 4\zeta^2}}{2}, \qquad \text{for every } \zeta \in \left[-\frac{|\mathfrak{r}|}{2}, \frac{|\mathfrak{r}|}{2}\right] \setminus \{0\}.$$

Moreover, the value of f at these stationary points coincides with the value of f at the boundary of P. In turn,

$$\mathcal{N} \;=\; \left\{\begin{pmatrix} \eta^{\pm}_{\zeta} & \zeta \\ \zeta & \eta^{\mp}_{\zeta} \end{pmatrix} : |\zeta| \leq \frac{|\mathfrak{r}|}{2}\right\} = \left\{\rho^{\mathsf{T}} \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix} \rho \,:\, \rho \in \mathrm{SO}(2)\right\},$$

concluding the result of point (i), and thus proving the lemma.

□

To conclude the section, we give some definitions which will be useful later on. They regard the sub-class of $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ consisting of *cylinders*. Given $r \in (0, +\infty]$, we define the map $C_r : \mathbb{R}^2 \to \mathbb{R}^3$ as

$$C_r(x') \;:=\; \begin{cases} \Big(r\big(\cos(x_1/r) - 1\big), r\sin(x_1/r), x_2\Big)^{\mathsf{T}}, & r \in (0, +\infty), \\ \big(0, x_1, x_2\big)^{\mathsf{T}}, & r = +\infty, \end{cases}$$

for every $x' = (x_1, x_2) \in \mathbb{R}^2$. Then we define the family of maps

$$\mathrm{Cyl} \;:=\; \left\{T_v \circ R \circ C_r \circ \rho : \mathbb{R}^2 \to \mathbb{R}^3 \,\middle|\, r \in (0, +\infty],\ T_v \in \mathrm{Trs}(3),\ R \in \mathrm{SO}(3) \text{ and } \rho \in \mathrm{Orth}(2)\right\} \tag{3.9}$$

and we call its elements *cylinders*. Note that the above defined family of cylinders includes also *planes* - the elements of Cyl with $r = +\infty$.

Remark 3.2. Observe that any cylinder $y = T_v \circ R \circ C_r \circ \rho$ maps lines parallel to $\rho^\mathsf{T}\mathsf{e}_2$ to the lines of zero curvature - rulings. More in general, direct computations give

$$\nabla y(x') \;=\; R\,\nabla C_r\big(\rho(x')\big)\,\rho = R \begin{pmatrix} -\sin\left(\frac{x'\cdot\rho^\mathsf{T}\mathsf{e}_1}{r}\right) & 0 \\ \cos\left(\frac{x'\cdot\rho^\mathsf{T}\mathsf{e}_1}{r}\right) & 0 \\ 0 & 1 \end{pmatrix} \rho, \qquad \text{for all } x' \in \mathbb{R}^2, \tag{3.10}$$

so that

$$\nabla y(\lambda\rho^\mathsf{T}\mathsf{e}_2) = R \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \rho, \qquad \text{for every } \lambda \in \mathbb{R}.$$

■

By direct computations one can see that a map $y = T \circ R \circ C_r \circ \rho \in \mathrm{Cyl}$ is an isometry whose second fundamental form is given by

$$\mathrm{A}_y(x') \;=\; (\det\rho)\,\rho^\mathsf{T} \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & 0 \end{pmatrix} \rho, \qquad \text{for every } x' \in \mathbb{R}^2. \tag{3.11}$$

Now, let us go back to the set $\mathcal{F}$ defined in (3.4). From (3.11) and from the simple observation that $\mathcal{F}$ can be equivalently represented as

$$\mathcal{F} \;=\; \mathbb{R}\left\{n \otimes n : \, n \in \mathbb{R}^2 \text{ with } |n| = 1\right\} \;=\; \mathbb{R}\left\{\rho^\mathsf{T} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho \,:\, \rho \in \mathrm{SO}(2)\right\},$$

one can prove that the set $\mathcal{F}$ coincides with the set of (constant) second fundamental forms of cylinders. This fact can in turn be used to show, in the case where the target curvature tensor $\overline{A}$ is constant, that

$$y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega) \text{ is a minimizer of } \mathcal{E}^0 \text{ if and only if } y \text{ is a pointwise minimizer.} \tag{3.12}$$

This is the first step of the proof of Lemma 3.3 below. The second part of the proof consists then in showing that

$$\mathrm{A}_y(x') \in \mathcal{N} \quad \text{for a.e. } x' \in \omega \quad \Longrightarrow \quad \mathrm{A}_y \equiv \text{const.}. \tag{3.13}$$

This property is at the core of our investigations in the following subsection and can be proved using some fine properties of isometric immersions

([23], [24] and [36]). The proof of the following lemma can be found in [42, Proposition 4.2].

Lemma 3.3. *Let $\overline{A}$ be constant (cfr. (3.1)–(3.3)) and let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ be a minimizer of $\mathcal{E}^0$. Then $y = v|_\omega$ for some $v \in \mathrm{Cyl}$. In particular, y has constant second fundamental form.*

3.2. **The case of piecewise constant $\overline{A}$.** In this subsection, we consider the case where the target curvature is a piecewise constant tensor valued map $x' \mapsto \overline{A}(x')$. More precisely, given $n \in \mathbb{N}$, $n \geq 2$, we say that the map $\overline{A} \in \mathrm{L}^\infty\big(\omega, \mathbb{R}^{2\times 2}\big)$ is *piecewise constant* if it is of the form

$$\overline{A} \;=\; \sum_{k=1}^{n} \overline{A}_k\,\chi_{\omega_k} \quad \text{a.e. in } \omega, \qquad \text{with } \overline{A}_k = \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix}, \quad a_k, b_k \in \mathbb{R}, \tag{3.14}$$

where $\{\omega_k\}_{k=1}^n$ is a partition of ω made of Lipschitz subdomains ω_k according to Definition 3.4 . Clearly, it is convenient distinguishing between two different neighboring subdomain only when the corresponding spontaneous curvature are different from each other. Namely, we suppose that $\overline{A}_k \neq \overline{A}_j$ for every $k \neq j$ such that $\partial\omega_j \cap \partial\omega_k \neq \emptyset$. With such target curvature, our 2D energy functional takes the form

$$\mathcal{E}^0(y) = \frac{1}{24}\sum_{k=1}^n \int_{\omega_k} Q_2\big(\mathrm{A}_y(x') - \overline{A}_k\big)\,\mathrm{d}x' + \text{ad.t.}, \qquad \text{for every } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega).$$

We want to determine the conditions the map $x' \mapsto \overline{A}(x')$ has to satisfy in order to guarantee the existence of pointwise minimizers of $\mathcal{E}^0$, i.e. to guarantee that there exists $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ such that $\mathrm{A}_y(x') \in \mathcal{N}(x')$ for a.e. $x' \in \omega$, where $\mathcal{N}(x')$ is defined by (3.6). In view of (3.14), we equivalently look for the necessary and sufficient conditions such that

$$\text{there exists } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega) \text{ such that } \mathrm{A}_y(x') \in \mathcal{N}_k \text{ for a.e. } x' \in \omega_k, \text{ for all } k = 1,\dots,n, \tag{3.15}$$

where

$$\mathcal{N}_k := \operatorname{argmin}_{F\in\mathcal{F}} Q_2\big(F - \overline{A}_k\big), \qquad \text{for every } k = 1,\dots,n. \tag{3.16}$$

Note from (3.13) that a deformation satisfying (3.15) is, roughly speaking, a "patchwork" of cylinders. Therefore, conditions on $\overline{A}$ guaranteeing (3.15) translates into conditions under which cylinders can be patched together resulting into an isometry. This is the content of the main result of the present section, namely of Theorem 3.9 below. In order to state and prove it, we need a definition and a preliminary lemma.

Definition 3.4 (Lipschitz n-subdivision)**.** *Fix $n \in \mathbb{N}$, $n \geq 2$. A family $\{\omega_k\}_{k=1}^n$ of open, bounded and connected subsets of $\mathbb{R}^2$ is said to be a* Lipschitz n-subdivision *of ω provided it can be obtained* via *the following procedure:*

- *Call $\omega'_1 := \omega$.*
- *Suppose that for every $k = 1,\dots,n-1$ there exists a continuous injective curve $\gamma_k : [0,1] \to \mathrm{cl}(\omega'_k)$ such that $\partial\omega'_k \cap [\gamma_k] = \big\{\gamma_k(0),\gamma_k(1)\big\}$ (note that $\gamma_k(0) \neq \gamma_k(1)$) and the two connected components of $\omega'_k \setminus [\gamma_k]$ are Lipschitz. Then call ω'_{k+1} one of such connected components.*
- *Once the domains $\omega'_1,\dots,\omega'_n$ are defined, let $\omega_k := \omega'_k \setminus \mathrm{cl}(\omega'_{k+1})$ for every $k = 1,\dots,n-1$ and let $\omega_n := \omega'_n$.*

In particular, the subdomains $\omega_1,\dots,\omega_n$ of ω are Lipschitz domains such that

$$\omega = \bigcup_{k=1}^n \omega_k \cup \bigcup_{k=1}^{n-1} \gamma_k\big((0,1)\big).$$

Remark 3.5. Since each ω_k is a Lipschitz domain, one has that its boundary $\partial\omega_k$ has null $\mathcal{L}^2$-measure. In particular, we deduce that $\mathcal{L}^2\big(\omega \setminus \bigcup_{k=1}^n \omega_k\big) = 0$. ■

The following Lemma 3.6 will be the main ingredient for the proof of Theorem 3.9. It gives a "recipe" on how two cylinders can be patched together. We refer to Remark 3.8 below for the notation and the properties of roto-translations used in this section. We point out that the proof of Theorem 3.9 will be achieved via an induction argument, which relies upon Lemma 3.6 and the definition of Lipschitz subdivision of ω.

We remark that, more in general, the very fact that y is a $\mathrm{W}^{2,2}$-isometry is sufficient to deduce that ω consists, up to a null set, of finitely many subdomains (touching each other on a finite union of line segments) on which y is either a plane, or a cylinder, or a cone or "tangent developable". This description can be obtained as a consequence of some fine properties of the class $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ - see [36] for ω convex and [23], [24] for a more general ω.

Lemma 3.6. *Let $\gamma : [0,1] \to \mathrm{cl}(\omega)$ be a continuous injective curve such that $[\gamma] \cap \partial\omega = \{\gamma(0), \gamma(1)\}$ and such that two connected components ω_1 and ω_2 of $\omega \setminus [\gamma]$ are Lipschitz. Let $y_1, y_2 \in \mathrm{Cyl}$, say $y_1 = T_{v_1} \circ R_1 \circ C_{r_1} \circ \rho_1$ and $y_2 = T_{v_2} \circ R_2 \circ C_{r_2} \circ \rho_2$, with $r_1, r_2 \in (0, +\infty)$ such that $\det\rho_1 = -\det\rho_2$ whenever $r_1 = r_2$. The map defined as*

$$y := y_1\chi_{\omega_1} + y_2\chi_{\omega_2}, \qquad a.e. \text{ in } \omega,$$

belongs to $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ if and only if the following conditions hold:

(i) $[\gamma]$ is a line segment spanned by some $\mathsf{e} \in \mathbb{R}^2 \setminus \{0\}$;

(ii) $\rho_1^\top \mathsf{e}_2$ and $\rho_2^\top \mathsf{e}_2$ are parallel to e. This in particular implies that $\rho_1\rho_2^\top = \mathrm{diag}(\sigma_1, \sigma_2)$, for some $\sigma_1, \sigma_2 \in \{\pm 1\}$;

(iii) Setting $w_k := \rho_k\big(\gamma(0) - (0,0)\big)$ and $\theta_k := (w_k \cdot \mathsf{e}_1)/r_k$, for $k = 1, 2$, we have

$$\big(R_1\hat{R}_{\theta_1}\big)^\top \big(R_2\hat{R}_{\theta_2}\big) = \mathrm{diag}\big(\sigma_1\,\sigma_2, \sigma_1, \sigma_2\big) \quad \text{and} \quad v_1 + R_1C_{r_1}(w_1) = v_2 + R_2C_{r_2}(w_2). \tag{3.17}$$

Proof. **(Necessity)** Here, we show that if the deformation $y := y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$ is in $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, then it complies with conditions (i), (ii) and (iii). First of all, we recall from [35, Proposition 5] that the very condition $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ implies $y \in C^1(\omega, \mathbb{R}^3)$. At the same time, from the specific expression of y we have that $\nabla y = \nabla y_k$ in ω_k for $k = 1, 2$, where

$$\nabla y_k = R_k \begin{pmatrix} -\sin\left(\frac{x'\cdot\rho_k^\top \mathsf{e}_1}{r_k}\right) & 0 \\ \cos\left(\frac{x'\cdot\rho_k^\top \mathsf{e}_1}{r_k}\right) & 0 \\ 0 & 1 \end{pmatrix} \rho_k. \tag{3.18}$$

This expression says in particular that ∇y is bounded and in turn that $y \in C^1(\overline{\omega}, \mathbb{R}^3)$. Let us first prove the necessity of the conditions (i), (ii) and (iii) in the case when $\gamma(0) = (0,0)$. The continuity of y and ∇y at the point $(0,0)$ gives, respectively, that $v_1 = v_2$ (obtained by imposing $y_1(0,0) = y_2(0,0)$), and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \rho_1\rho_2^\top = R_1^\top R_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow R_1^\top R_2 = \left(\begin{array}{c|c} \det(\rho_1\rho_2^\top) & 0 \quad 0 \\ \hline 0 & \\ 0 & \rho_1\rho_2^\top \end{array}\right) \tag{3.19}$$

(obtained from $\nabla y_1(0,0) = \nabla y_2(0,0)$ and from expression (3.18)), which proves (iii) . The continuity of ∇y gives also that $\nabla y_1(\gamma(t)) = \nabla y_2(\gamma(t))$ for each $t \in [0,1]$, that is

$$\begin{pmatrix} -\sin\left(\frac{\gamma(t)\cdot\rho_1^\top \mathsf{e}_1}{r_1}\right) & 0 \\ \cos\left(\frac{\gamma(t)\cdot\rho_1^\top \mathsf{e}_1}{r_1}\right) & 0 \\ 0 & 1 \end{pmatrix} \rho_1\rho_2^\top = R_1^\top R_2 \begin{pmatrix} -\sin\left(\frac{\gamma(t)\cdot\rho_2^\top \mathsf{e}_1}{r_2}\right) & 0 \\ \cos\left(\frac{\gamma(t)\cdot\rho_2^\top \mathsf{e}_1}{r_2}\right) & 0 \\ 0 & 1 \end{pmatrix}.$$

In turn, using the second condition in (3.19) and the notation $\rho_1\rho_2^\mathsf{T} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, we have

$$\begin{pmatrix} -m_1 \sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & -m_2 \sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) \\ m_1 \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & m_2 \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} -\det(\rho_1\rho_2^\mathsf{T}) \sin\left(\frac{\gamma(t)\cdot\rho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ m_1 \cos\left(\frac{\gamma(t)\cdot\rho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & m_2 \\ m_3 \cos\left(\frac{\gamma(t)\cdot\rho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & m_4 \end{pmatrix}. \tag{3.20}$$

By the equality between the elements of the first row in the above expression one deduces that $\rho_1^\mathsf{T}\mathsf{e}_2$ and $\rho_2^\mathsf{T}\mathsf{e}_2$ must be parallel. This proves one part of the statement in (ii) and implies, in particular, that $\rho_1\rho_2^\mathsf{T} = \mathrm{diag}(m_1, m_4)$ with $m_1, m_4 \in \{\pm 1\}$. In order to conclude the proof of (ii) and in the same time prove (i) , we need to show that $[\gamma]$ is a line segment parallel to $\rho_1^\mathsf{T}\mathsf{e}_2$ (and to $\rho_2^\mathsf{T}\mathsf{e}_2$). Observe that $\rho_1\rho_2^\mathsf{T} = \mathrm{diag}(m_1, m_4)$ implies $\rho_2^\mathsf{T}\mathsf{e}_1 = m_1\rho_1^\mathsf{T}\mathsf{e}_1$, so that the equation (3.20) simplifies to

$$\begin{pmatrix} -m_1 \sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & 0 \\ m_1 \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & 0 \\ 0 & m_4 \end{pmatrix} = \begin{pmatrix} -m_4 \sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ m_1 \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ 0 & m_4 \end{pmatrix}, \tag{3.21}$$

for every $t \in [0,1]$. By differentiating the above equality restricted to the first elements of the first and second rows one gets

$$\begin{aligned} m_1 \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right)\frac{\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1} &= m_4 \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right)\frac{\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_2} \\ \sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right)\frac{\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1} &= \sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right)\frac{\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_2} \end{aligned} \tag{3.22}$$

It turns out that (3.21) and (3.22) can be satisfied only if

$$\dot\gamma(t)\cdot\left(\rho_1^\mathsf{T}\mathsf{e}_1\right) = 0, \qquad \text{for every } t \in [0,1], \tag{3.23}$$

which implies that $[\gamma]$ is a line segment parallel to $\rho_1^\mathsf{T}\mathsf{e}_2$ (thus accordingly also to $\rho_2^\mathsf{T}\mathsf{e}_2$). To prove previous assertion, we distinguish two cases:

- if $r_1 \neq r_2$, call $s := m_1/m_4$ and fix $t \in [0,1]$. Condition (3.21) grants that $\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1/r_1$ and $s\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1/r_2$ have the same sine and cosine. Then, since sine and cosine cannot simultaneously vanish, (3.22) yields $\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1/r_1 = s\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1/r_2$, whence necessarily $\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1 = 0$.
- if $r_1 = r_2$, by hypotheses we have that $\det\rho_1 = -\det\rho_2$. Since $\rho_1\rho_2^\mathsf{T} = \mathrm{diag}(m_1, m_4)$, we conclude that $m_1 m_4 = -1$, or equivalently $m_1 = -m_4$. Now the first condition in (3.22) gives
$$\frac{d}{dt}\sin\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) = \cos\left(\frac{\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right)\frac{\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1}{r_1} = 0,$$
so that the map $t \mapsto \gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1/r_1$ is constant and accordingly that $\dot\gamma(t)\cdot\rho_1^\mathsf{T}\mathsf{e}_1 = 0$ for every $t \in [0,1]$.

This concludes the proof of the necessary condition of the lemma in the case where $\gamma(0) = (0,0)$.

Considering now the case $v := \gamma(0) - (0,0) \neq 0$, define $\hat{\omega} := \omega - v$ and $\hat{y}_k := y_k \circ \tau_v$, $k = 1,2$ (recall form Section 1.1 that $\tau_v := \cdot + v \in \mathrm{Trs}(2)$). By Remark 3.8, one can easily verify that

$$\hat{y}_k = T_{u_k} \circ R_k \circ \hat{R}_{\theta_k} \circ C_{r_k} \circ \rho_k, \tag{3.24}$$

where $\theta_k := (w_k \cdot \mathsf{e}_1)/r_k$ and $u_k := v_k + R_k \circ C_{r_k}(w_k)$, with $w_k := \rho_k\big(\gamma(0) - (0,0)\big)$, for $k = 1,2$. Observe that the domain $\hat{\omega}$ is partitioned into $\hat{\omega}_1$ and $\hat{\omega}_2$ by the subdivision curve $[\gamma] - v$ which satisfies the condition $\gamma(0) - v = (0,0)$. It is now clear that $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ implies $\hat{y} := y \circ \tau_v = \hat{y}_1\chi_{\hat{\omega}_1} + \hat{y}_2\chi_{\hat{\omega}_2} \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\hat{\omega})$, which further implies that $[\gamma] - v$ (and hence $[\gamma]$) is a line segment parallel to $\rho_1^\intercal \mathsf{e}_2$ and $\rho_2^\intercal \mathsf{e}_2$, implying $\rho_1\rho_2^\intercal = \mathrm{diag}(\sigma_1, \sigma_2)$, for some $\sigma_1, \sigma_2 \in \{\pm 1\}$, and that

$$v_1 + R_1 \circ C_{r_1}(w_1) = v_2 + R_2 \circ C_{r_2}(w_2) \quad \text{and} \quad \big(R_1\hat{R}_{\theta_1}\big)^\intercal\big(R_2\hat{R}_{\theta_2}\big) = \mathrm{diag}\big(\sigma_1\,\sigma_2, \sigma_1, \sigma_2\big),$$

which are precisely conditions (i), (ii) and (iii).

(Sufficiency) Let $y_1, y_2 \in \mathrm{Cyl}$ satisfy conditions (ii) and (iii). Let $v := \gamma(0) - (0,0)$ and let $\rho \in \mathrm{SO}(2)$ be a rotation which brings the line segment $[\gamma] - v$ to the vertical position. Let $\underline{y}_k := y_k \circ \tau_v \circ \rho^\intercal$. By denoting $u := v_1 + R_1 \circ C_{r_1}(w_1)$ and $R := R_1 \circ \hat{R}_{\theta_1}$ we have by (iii) that $\underline{y} := \underline{y}_1\chi_{\omega_1} + \underline{y}_2\chi_{\omega_2}$ is of the form

$$\underline{y}(x_1, x_2) = \begin{cases} T_u R\Big(r_1\big(\cos(x_1/r_1) - 1\big), \sigma_1^1 r_1 \sin(x_1/r_1), \sigma_2^1 x_2\Big)^\intercal, & x_1 \leq 0, \\ T_u R\Big(\sigma_1\sigma_2\, r_2\big(\cos(x_1/r_2) - 1\big), \sigma_1^1 r_2 \sin(x_1/r_2), \sigma_2^1 x_2\Big)^\intercal, & x_1 > 0, \end{cases} \tag{3.25}$$

where $\sigma_k^1 \in \{\pm 1\}$ are such that $\rho_1\rho^\intercal = \mathrm{diag}(\sigma_1^1, \sigma_2^1)$ (which follows form the fact that $\rho_1^\intercal \mathsf{e}_2 \parallel [\gamma]$). By construction, $\bar{y} \in C^1(\underline{\omega}, \mathbb{R}^3)$ with $\underline{\omega} = \rho(\omega - v)$. Simple computations give $\partial_1\underline{y}, \partial_2\underline{y} \in \mathrm{W}^{1,2}(\underline{\omega}, \mathbb{R}^3)$, which implies that $\underline{y} \in \mathrm{W}^{2,2}(\underline{\omega}, \mathbb{R}^3)$. Note also that $\nabla\underline{y}(x')^\intercal\,\nabla\underline{y}(x') = \mathbb{I}_3$ for a.e. $x' \in \underline{\omega}$. Therefore $\underline{y} \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\underline{\omega})$, thus accordingly $y := \underline{y} \circ \rho \circ \tau_{-v} \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$. □

Remark 3.7. Observe that the condition "$\det\rho_1 = -\det\rho_2$ whenever $r_1 = r_2$" permits to exclude the trivial case where we patch together pieces of cylinders y_1 and y_2 having the same curvatures (i.e. $\det\rho_1/r_1 = \det\rho_2/r_2$, according to formula (3.11)). Clearly, this case does not force any condition on $[\gamma]$.

Moreover, an argument similar to that in the proof of Lemma 3.6 allows to prove necessary and sufficient conditions for having $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ of the form $y = y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$ with, say, y_2 affine (using our terminology, a cylinder with $r_2 = +\infty$). In this case, condition (i) remains the same and condition (ii) reduces to $\rho_1^\intercal \mathsf{e}_2 \parallel [\gamma]$ (while $\rho_2 \in \mathrm{Orth}(2)$ can be arbitrarily chosen). Moreover, for a chosen $\rho_2 \in \mathrm{Orth}(2)$, condition (iii) becomes

$$(R_1\hat{R}_{\theta_1})^\intercal R_2\hat{R}_{\theta_2} = \left(\begin{array}{c|cc} \det(\rho_1\rho_2^\intercal) & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\rho_1\rho_2^\intercal$}} \\ 0 & & \end{array}\right) \quad \text{and} \quad v_1 + R_1C_{r_1}(w_1) = v_2 + R_2C_{r_2}(w_2)$$

with $w_k := \rho_k\big(\gamma(0) - (0,0)\big)$ and $\theta_k := w_k \cdot \mathsf{e}_1/r_k$. ■

Remark 3.8 (Properties of "roto-translations"). The following two properties, regarding the composition of cylinders, translations and rotations, can be easily proved.

(i) Fix $R \in \mathrm{SO}(3)$ and $T_w \in \mathrm{Trs}(3)$. Then $R \circ T_w = T_{Rw} \circ R$.

(ii) Let $\tau_v \in \mathrm{Trs}(2)$ and $\hat{R}_\theta \in \mathrm{SO}(3)$ be defined by

$$\hat{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $C_r \circ \tau_v = T_{C_r(v)} \circ \hat{R}_{(v\cdot \mathsf{e}_1)/r} \circ C_r$, for every positive real number r.

In particular, property (ii) justifies the choice of the representation used for the elements in Cyl and it is useful for the proof of Lemma 3.6. ■

Given a piecewise constant $\overline{A}$ and referring to Lemma 3.1 (see also the discussion after its statement), we set

$$\mathfrak{r}_k := \begin{cases} \dfrac{a_k(1+2\beta)}{1+\beta}, & \text{if} \quad b_k = a_k, \\ \dfrac{a_k}{1+\beta}, & \text{if} \quad b_k = -a_k, \\ a_k + \dfrac{b_k\,\beta}{1+\beta}, & \text{if} \quad |a_k| > |b_k|, \\ b_k + \dfrac{a_k\beta}{1+\beta}, & \text{if} \quad |b_k| > |a_k|, \end{cases} \qquad \text{for every } k = 1,\dots,n. \tag{3.26}$$

Recall that $\{0, \pm\mathfrak{r}_k\}$ are the eigenvalues (principal curvatures) of the (constant) curvature tensors ranging in $\mathcal{N}_k$.

Theorem 3.9. *Let $\overline{A}$ be of the form (3.14). Assume that $\mathfrak{r}_k \neq \mathfrak{r}_j$ for all $1 \le k < j \le n$ such that $\mathcal{H}^1\big(\partial\omega_k \cap \partial\omega_j\big) > 0$. Then there exists a pointwise minimizer $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ of $\mathcal{E}^0$ if and only if the following conditions are satisfied:*

- *(a) $[\gamma_k]$ is a line segment with $\gamma_k(0), \gamma_k(1) \in \partial\omega$, for every $k = 1,\dots,n-1$;*
- *(b) $\gamma_k\big((0,1)\big) \cap \gamma_j\big((0,1)\big) = \emptyset$ for all $k \neq j = 1,\dots,n-1$;*
- *(c) every* non flat *region ω_k, i.e. ω_k with corresponding $\mathfrak{r}_k \neq 0$, satisfies: $\partial\omega_k \cap \omega$ consists of connected components which are orthogonal to some eigenvector (principal curvature direction) of the matrices of $\mathcal{N}_k$ corresponding to $\mathfrak{r}_k$.*

Proof. The sufficiency part of the statement follows by straightforward computations, as in the proof of Lemma 3.6. In order to prove necessity, we focus on the case $n = 2$, when ω is subdivided into two Lipschitz subdomains ω_1 and ω_2 by a curve $\gamma := \gamma_1$ as in Definition 3.4, since the general case can be achieved by an induction argument as a consequence of our definition of Lipschitz subdivision of the domain ω.

Let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ be a pointwise minimizer of $\mathcal{E}^0$. Note that on both subdomains ω_1 and ω_2 the target curvature tensor $\overline{A}$ is constant. Then by the definition of pointwise minimizers, by Lemma 3.3 and Lemma 3.1 we deduce that $y = y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$, with $y_k = T_{v_k} \circ R_k \circ C_{1/|\mathfrak{r}_k|} \circ \rho_k \in \mathrm{Cyl}$, $k = 1,2$, where $\mathfrak{r}_k$ is given by (3.26) and ρ_k is such that $\mathrm{A}_{y_k} \equiv (\det\rho_k)\,\rho_k^{\mathsf{T}}\,\mathrm{diag}\big(|\mathfrak{r}_k|, 0\big)\rho_k \in \mathcal{N}_k$. Since $\mathfrak{r}_1 \neq \mathfrak{r}_2$, by Lemma 3.6 and Remark 3.7 we obtain that $[\gamma]$ must be a line segment and that $\rho_k^{\mathsf{T}}\mathsf{e}_2$ must be parallel to $[\gamma]$ (or equivalently that the eigenvector $\rho_k^{\mathsf{T}}\mathsf{e}_1$ of A_{y_k} is orthogonal to $[\gamma]$) whenever $\mathfrak{r}_k \neq 0$, $k = 1,2$, which is precisely the statement of (a) and (c) in the case in which $n = 2$.

□

Remark 3.10. Let k and j be such that $\mathcal{H}(\partial\omega_k \cap \partial\omega_j) > 0$. Observe that when $\mathfrak{r}_k = \mathfrak{r}_j$ (this may happen, though $\overline{A}_k \neq \overline{A}_j$), this condition does not impose that $\partial\omega_k \cap \partial\omega_j$ is a line segment. Indeed, when $\mathfrak{r}_k = \mathfrak{r}_j$, a pointwise minimizer y, when restricted to ω_k and ω_j, will be given by some cylinders y_k and y_j with $r_k = 1/|\mathfrak{r}_k|$ and $r_j = 1/|\mathfrak{r}_j|$, respectively, which have the same curvatures $\det \rho_k |\mathfrak{r}_k| = \mathfrak{r}_k = \mathfrak{r}_j = \det \rho_j |\mathfrak{r}_j|$. This fact, as observed in Remark 3.7, does not impose any further conditions on $\partial\omega_k \cap \partial\omega_j$. ■

Note that, if the target curvature does not induce any flat region, the presence of a pointwise minimizer forces the subdivision lines $[\gamma_k]$ to be all parallel (see Figure 2, (A) and (B)). When instead a flat region is present in the subdivision, this can give rise to a pointwise minimizer, even if the $[\gamma_k]$ are not mutually parallel (see Figure 2, (C) and (D)). Finally, observe that in this case a subdomain of type (iii) and (iv) can coexist (tough they cannot be neighbors).

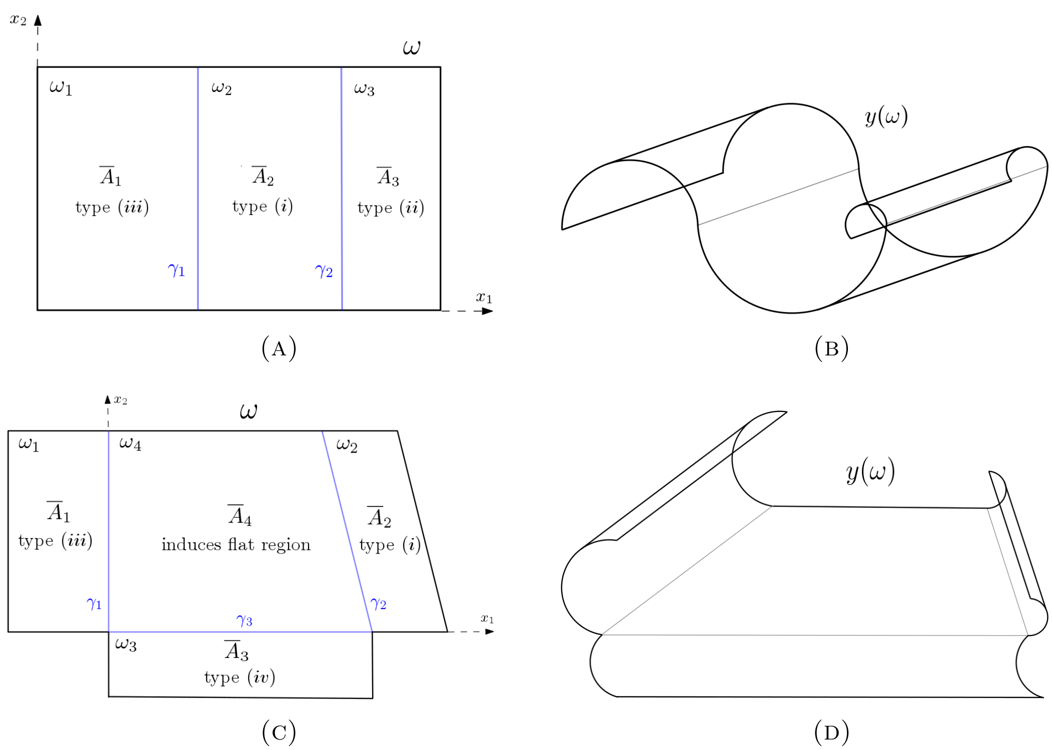

FIGURE 2. Examples of reference domains with given target curvature $\overline{A} = \sum_{k=1}^{3} \overline{A}_k \chi_{\omega_k}$, which guarantees the existence of a pointwise minimizer y in the case when there are no flat regions induced (figure (A)) and in the case when a flat regions are present (figure (C)). Corresponding examples of $y(\omega)$ are illustrated in pictures (B) and (D), respectively.

Point (c) above implies that for every k and j such that ω_k and ω_j are neighbor (i.e. share a piece of boundary, in symbols $\mathcal{H}^1(\partial\omega_k \cap \partial\omega_j) > 0$) it cannot be that A_k is of type (iii) (see Lemma 3.1) and A_j is of type (iv) at the same time. This is because, if not so, from point (c) above it would follow that the line segment $[\gamma] = \partial\omega_k \cap \partial\omega_j$ is simultaneously parallel to $\mathtt{e}_2$ and to $\mathtt{e}_1$, which is absurd. Hence, a reference domain endowed with target curvature as in Figure 3 does not admit a pointwise minimizer.

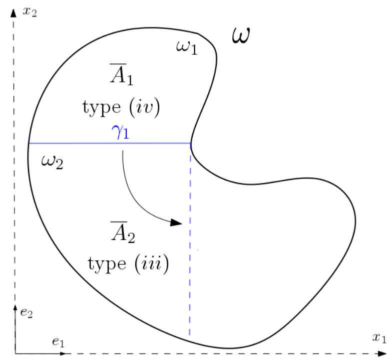

FIGURE 3. An example of reference domain with given target curvature $\overline{A} = \overline{A}_1\chi_{\omega_1} + \overline{A}_2\chi_{\omega_2}$ which does not allow for a pointwise minimizer y. This is because $\overline{A}_1$ of type (iv) forces $[\gamma_1]$ to be parallel to e_1, while $\overline{A}_2$ of type (iii) forces $[\gamma_1]$ to be parallel to e_2.

4. Application to thin gel sheets

In this section, we apply the reduced model derived in Section 2 to the study of thin sheets of polymer gel. In the present context, a polymer gel is a network of cross-linked polymer chains swollen with a liquid solvent. Denote by v the volume per solvent molecule, by $\overline{N}$ the density of polymer chains in the reference volume and define $\mathbb{R}^{3\times3}_1 := \{F \in \mathbb{R}^{3\times3} : \det F \geq 1\}$. The dimensionless free-energy density for isotropic

and homogeneous

polymer gels is of Flory-Rehner type (see [17]) and is given by the function $W : \mathbb{R}^{3\times3}_1 \to \mathbb{R}$ defined as

$$W(F) := \frac{v\overline{N}}{2}\big(|F|^2 - 3\big) + W^{\chi}_{vol}(\det F) + \delta(\det F - 1), \qquad \text{for every } F \in \mathbb{R}^{3\times3}_1. \tag{4.1}$$

Here $\chi \in (0, 1/2]$ and $\delta \geq 0$ are fixed dimensionless constants depending on the physical and chemical properties of the material and on environmental conditions, respectively. The function $W^{\chi}_{vol} : [1, +\infty) \to (-\infty, 0]$ is of class C^∞ on $(1, +\infty)$ and (right-) continuous at 1, with

$$W^{\chi}_{vol}(1) = 0, \quad \frac{d}{dt}W^{\chi}_{vol}(t) < 0 \text{ for every } t \in (1, +\infty) \quad \text{and} \quad \inf_{t\in[1,+\infty)} W^{\chi}_{vol}(t) = \chi - 1.$$

Our attention is in particular focussed on a *heterogeneous* thin gel sheet occupying the reference configuration Ω_h. More precisely, we suppose that the sheet is characterized by a z-dependent cross-linking density, which in turn determines a z-dependent density $\overline{N}^h$ of polymer chains. At the same time, we suppose that $\overline{N}^h$ is a perturbation of a constant value $\overline{N}$, namely

$$\overline{N}^h(z) := \overline{N} + hg\left(z', \frac{z_3}{h}\right), \qquad \text{for a.e. } z \in \Omega_h \text{ and every } 0 < h \ll 1, \tag{4.2}$$

where $g \in \mathrm{L}^\infty(\Omega)$ and

$$\fint_{-h/2}^{h/2} \overline{N}^h(z', z_3)\,\mathrm{d}z_3 = \overline{N}, \qquad \text{for a.e. } z' \in \omega. \tag{4.3}$$

Observe that the condition (4.3) is equivalent to $\int_{-1/2}^{1/2} g(x',t)\,\mathrm{d}t = 0$ for a.e. $x' \in \omega$. Using the model energy density (4.1), we can describe this heterogeneous system via the family of densities

$$\overline{W}^h(z,F) := \frac{v}{2}\big(\overline{N} + hg(z', z_3/h)\big)\big(|F|^2 - 3\big) + W^\chi_{vol}(\mathrm{det}F) + \delta(\mathrm{det}F - 1) \tag{4.4}$$

for a.e. $z \in \Omega_h$, every $F \in \mathbb{R}^{3\times 3}_1$ and every $h > 0$. Letting $\{W^h\}$ be the associated family of rescaled densities $W^h : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$W^h(x,F) := \overline{W}^h\big((x', hx_3), F\big), \qquad \text{for a.e. } x \in \Omega \text{ and every } F \in \mathbb{R}^{3\times 3}_1 \tag{4.5}$$

and declaring it to be equal $+\infty$ on $\mathbb{R}^{3\times 3} \setminus \mathbb{R}^{3\times 3}_1$, one can show (see the details in [4]) that there exist constants $\alpha > 1$ and $\Theta \in \mathbb{R} \setminus \{0\}$, depending on v, $\overline{N}$, χ and δ, such that for a.e. $x \in \Omega$ it holds

$$W^h(x,F) \;=\; \min_{\mathbb{R}^{3\times 3}} W^h(x,\cdot) \quad \text{ if and only if } \quad F \in \big(\alpha + hb(x)\big)\mathrm{SO}(3) \;\text{ with }\; b := \Theta g. \tag{4.6}$$

Moreover, one can show that $\{W^h\}$ is a family of frame indifferent functions that uniformly converges (in the sense of (iii) in Definition 2.1) to W and have quadratic growth. The hypothesis (4.3) ensures that the spontaneous strain $B = b\mathbb{I}_3$ in this case satisfies the assumption (2.23).

By a suitable change of variable in order to switch from the energy wells that are h-close to $\alpha\mathbb{I}_3$ to those that are h-close to $\mathbb{I}_3$ and than using the theory developed in Section 2, we obtain the corresponding 2D Kirchhoff model, which is in this case given by the energy functional

$$\mathcal{E}^0(y) := \frac{1}{24}\int_\omega Q_2\big(\mathrm{A}_y(x') - \overline{A}(x')\big)\,\mathrm{d}x' + \frac{1}{2}\int_\Omega Q_2\big(b(x)\mathbb{I}_2\big)\,\mathrm{d}x - \frac{1}{24}\int_\omega Q_2\big(\overline{A}(x')\big)\,\mathrm{d}x', \tag{4.7}$$

for every $y \in \mathrm{W}^{2,2}(\omega,\mathbb{R}^3)$ satisfying $(\nabla y)^\mathsf{T}\nabla y = \alpha^2\mathbb{I}_2$ a.e. in ω (that we will briefly call an α-isometry), and $+\infty$ otherwise in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$. The relation with the initial 3D model can be seen trough the target curvature tensor $\overline{A}$, given by

$$\overline{A} = 12\int_{-1/2}^{1/2} x_3\, b(\cdot,x_3)\,\mathrm{d}x_3\,\mathbb{I}_2, \qquad \text{a.e. in } \omega, \qquad b = \Theta g,$$

and through the quadratic form Q_2 defined by

$$Q_2(F) := \min_{d\in\mathbb{R}^3} D^2W(\alpha\mathbb{I}_3)[\hat{F} + d\otimes \mathsf{f}_3]^2, \qquad \text{for every } F \in \mathbb{R}^{2\times 2},$$

and explicitly reads as

$$Q_2(F) = 2G|F_{\mathrm{sym}}|^2 + \Lambda(\alpha)\,\mathrm{tr}^2 F, \qquad \text{for every } F \in \mathbb{R}^{2\times 2}, \tag{4.8}$$

where G and $\Lambda(\alpha)$ are positive constants depending only on the (fixed) physical properties of the material.

Remark 4.1. We remark that the (rescaled) energy densities $W^h(x,\cdot)$ defined by (4.4) are minimized on

$$\big(\alpha + hb(x)\big)\mathrm{SO}(3), \qquad \text{for every } h > 0 \text{ and a.e. } x \in \Omega,$$

for some $\alpha > 1$. Moreover, they uniformly converge to W given by (4.1), which is minimized at $\alpha\,\mathrm{SO}(3)$. However, by directly confronting formulas (4.1) and (4.4), one can check that the densities W^h cannot be rewritten in the "prestretch" form

$$W^h(x,F) = W\left(\left(1 + h\frac{b(x)}{\alpha}\right)^{-1} F\right).$$

■

Example 4.2. Consider a thin film made of polymeric gel occupying the domain Ω_h where $\omega = (-d,d)\times(0,\ell)$ and with associated family of energy densities $\{\overline{W}^h\}$ given by (4.4). Suppose that the variation of the number of polymeric chains $\overline{N}^h$ given by (4.2) is such that the associated function g is of the form

$$g(x',x_3) := \begin{cases} g_1(x_3), & \text{if } x' \in (-d,0]\times(0,\ell) \\ g_2(x_3), & \text{if } x' \in (0,d)\times(0,\ell), \end{cases}$$

with $g_1, g_2 \in \mathrm{L}^\infty(-1/2,1/2)$, satisfying $\int_{-1/2}^{1/2} g_1(t)\,\mathrm{d}t = \int_{-1/2}^{1/2} g_2(t)\,\mathrm{d}t = 0$, and

$$a_1 := 12\int_{-1/2}^{1/2} x_3\,\Theta g_1(x_3)\,\mathrm{d}x_3 \neq 12\int_{-1/2}^{1/2} x_3\,\Theta g_2(x_3)\,\mathrm{d}x_3 =: a_2$$

with a_1, a_2 non zero .

In turn, the limiting 2D model is characterized by the target curvature tensor $\overline{A}$ that equals $a(x')\mathbb{I}_2$ at each $x' \in \omega$, where $a(x') = a_1$ if $x' \in (-d,0]\times(0,\ell)$ and $a(x') = a_2$ if $x' \in (0,d)\times(0,\ell)$.

Finally, using the results of Lemma 3.1 and Theorem 3.9 , we can determine the minimizers of the limiting energy $\mathcal{E}^0$. Note that we are in the case of Lipschitz 2-subdivision of ω into subdomains $\omega_1 := (-d,0)\times(0,\ell)$ and $\omega_2 := (0,d)\times(0,\ell)$. Given that the subdivision curve is $[\gamma_1] = \partial\omega_1 \cap \partial\omega_2 = [0,\ell]$ (a line segment parallel to e_2) and $\overline{A}$ is of type (i) (see Lemma 3.1), a pointwise minimizer of $\mathcal{E}^0$ exists and is any (up to rotations and translations in $\mathbb{R}^3$) α-isometry $y := y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$, with y_1, y_2 given for every $(x_1,x_2) \in \omega$ by

$$\begin{aligned} y_1(x_1,x_2) &:= \alpha\left(\alpha r_1\Big(\cos\big(x_1/(\alpha r_1)\big) - 1\Big), \sigma_1\alpha r_1 \sin\big(x_1/(\alpha r_1)\big), \sigma_2 x_2\right)^\mathsf{T}, \\ y_2(x_1,x_2) &:= \alpha\left(\sigma_0\alpha r_2\Big(\cos\big(x_1/(\alpha r_2)\big) - 1\Big), \sigma_1\alpha r_2 \sin\big(x_1/(\alpha r_2)\big), \sigma_2 x_2\right)^\mathsf{T}, \end{aligned} \tag{4.9}$$

with

$$r_k := \frac{1}{|\mathfrak{r}_k|} \quad \text{and} \quad \mathfrak{r}_k = a_k\frac{2G + 2\Lambda(\alpha)}{2G + \Lambda(\alpha)}, \quad k = 1,2,$$

(according to Lemma 3.1 and (4.8)) and appropriate choice (depending on the sign of $\mathfrak{r}_k$, $k = 1,2$) of $\sigma_i \in \{-1,1\}, i = 0,1,2$.

Since the pull-back of the second fundamental form associated with $y_1(\omega_1)$ and $y_2(\omega_2)$ respectively, is given by

$$\mathrm{A}_{y_1} = \sigma_1\sigma_2\begin{pmatrix} |\mathfrak{r}_1| & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{A}_{y_2} = \sigma_0\sigma_1\sigma_2\begin{pmatrix} |\mathfrak{r}_2| & 0 \\ 0 & 0 \end{pmatrix},$$

it is clear that there exists two different (up to rotations and translation in $\mathbb{R}^3$) minimizing surfaces $y(\omega)$. The choice of $\sigma_1 \in \{-1,1\}$ determines one of the two possible options for y_1, represented by a dashed or a full black line in Figure 4 below. For any chosen value of σ_1, the values of σ_2 and σ_0 are immediately determined by the sign of $\mathfrak{r}_1$ and $\mathfrak{r}_2$, respectively.

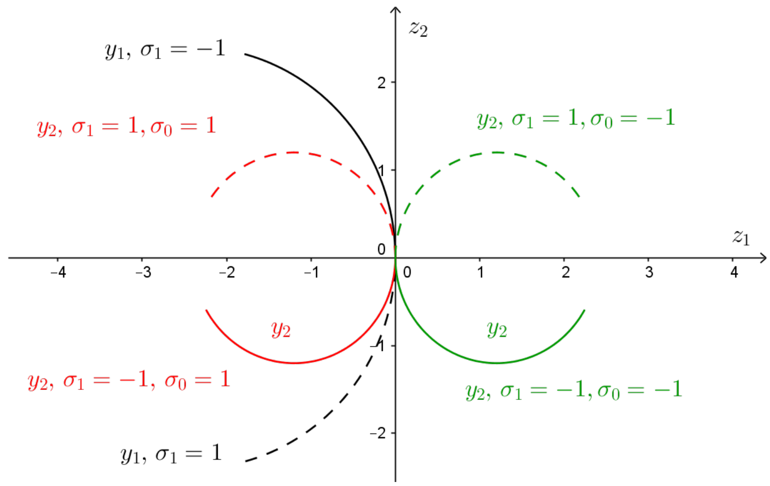

FIGURE 4. Intersection with (z_1, z_2)-plane in $\mathbb{R}^3$ of two possible (up to roto-translations) minimizing surfaces $y(\omega)$. One corresponds to a full line (by choosing $\sigma_1 = -1$) and the other one to a dashed line (by choosing $\sigma_1 = 1$). For both choices of σ_1, the value of the target curvature $\mathfrak{r}_2$ uniquely determines the value of σ_0 and thus "decides" whether (both) intersections are black-red (if $\sigma_0 = 1$) or black-green (if $\sigma_0 = -1$) lines.

△

5. Conclusions

In this paper, we have considered a family of 3D energy functionals that is relevant from the viewpoint of applications to shape morphing materials, especially in the context of swelling gels. As remarked in the Introduction, the starting 3D model (2.2)–(2.5) (which reduces to (4.4)–(4.6) in the specific gel case), which may be employed to accurately describe the swelling of polymer gels, is characterized by spontaneous stretches but not in general representable in the "pre-stretch form". Another peculiarity of such a family of 3D energies is that the spontaneous stretches are naturally related to the elastic parameters of the material. Hence, heterogeneities in the stiffness can be exploited to program the target shape of the system.

Having in mind applications to free-swelling, thin gel sheets with heterogeneous stiffness, we have derived by dimension reduction from the aforementioned 3D model a Kirchhoff plate theory (Sections 2–3). This plate model, whose governing equations are (3.1)–(3.2), is then specialized to thin gel sheets in Section 4. A central result of the theory is the expression of the spontaneous curvature as a function of parameters that can be traced back to the three-dimensional stiffness field. The derivation of the limiting model is restricted to the case where the compatibility condition (1.5) is fulfilled by the spontaneous strain. As explained in the introduction, this fact allows us to perform a rigorous dimension reduction with standard arguments. Even though it is possible to realize experimentally simple systems that fulfill such condition, this paper raises and leaves open a mathematically relevant problem, that is, finding the general limiting Kirchhoff model without the restriction (1.5), whose complete solution would potentially give new insight into the dimension reduction from 3D elasticity to plate theory.

We have then investigated the pointwise minimizers of the 2D model, restricting the attention to the case where the target curvature is piecewise constant (see Figure 1, Figure 2 (B) and (D), for some sketches of the configurations which occur in this case). The interest in this special class of minimizers is twofold. On the one hand, such a class corresponds to some of the simplest structures that can be realized experimentally, which yet can find interesting engineering applications (i.e. foldable structures, see the forthcoming [4]). On the other hand, it opens the way to the study of a huge class of open minimum problems (that is the minimization of the functional (3.1)–(3.3) in $W^{2,2}_{\mathrm{iso}}(\omega)$, given an arbitrary bounded $\overline{A} : \omega \to \mathrm{Sym}(2)$), for which ready-made analytical tools are not yet available.

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