A general formulation for some inconsistency indices of pairwise comparisons *

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Abstract

In this paper, we propose a unifying approach to the problem of measuring the inconsistency of judgments. More precisely, we define a general framework that allow several well-known inconsistency indices to be expressed as special cases of our formulation. Our proposal defines an inconsistency index as an aggregation of 'local', i.e. triple-based, inconsistencies. We prove that some simple and reasonable assumptions on the local inconsistency evaluation and on the aggregation rules guarantee a set of good properties for the obtained general inconsistency index. We show that OWA functions and t-conorms are suitable examples of aggregation functions for the local inconsistencies. We argue that the flexibility of our proposal is a relevant property for inconsistency evaluation that allows a suitable tuning of some important characteristics of the index. For example, by using different types of OWA functions, it is possible to obtain the desired balance between an averaging behavior and a 'largest inconsistency focused' behavior. We show that our proposal can be formulated in an equivalent way for the additive representation of local inconsistencies too. Under this representation, we prove a property of Pareto-efficiency for our general, aggregation-based inconsistency index.

Keywords: Pairwise comparisons; multiplicative preference relations; consistency; inconsistency indices; analytic hierarchy process; aggregation functions; OWA operators.

1 Introduction

In various decision making problems, a fundamental step relates with the determination of the weights of a finite set of alternatives/criteria. A widely accepted methodology to analyze alternatives and criteria and eventually derive a weight vector is to pairwise compare them. Namely, by pairwise comparing alternatives one can decompose an otherwise cognitively too large problem into more tractable sub-problems and tackle these latter ones.

Although a carefully selected set of $(n-1)$ pairwise comparisons is sufficient to elicitate a set of weights (Bozóki et al, 2010), to enhance the robustness of the set of weights, the expert is

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often asked more questions in the form of pairwise comparisons. Different methodologies seem to agree on this specific point. For example, in the Analytic Hierarchy Process (AHP), $\frac{n(n-1)}{2}$ pairwise comparisons are normally required, and a similar principle has also been advocated in Multi-Attribute Value Theory (MAVT) where, referring to the case with only $(n-1)$ comparisons, Keeney and Raiffa (1976) claimed that "it may be desirable to ask additional questions thereby getting an over-determined system of equations, fully expecting that the set of responses would be inconsistent in practice". This last statement captures a fundamental problem in the theory of pairwise comparisons: more comparisons help improve the robustness of the results, but an expert can hardly ever be completely rational in expressing them. The same idea was also supported by (Belton and Stewart, 2002, Sec. 5.4.4). Furthermore, the capacity of preferences expressed as ratio statements to represent ratios between criteria weights in MCDM problems where the value function is additive has recently been validated by Pajala et al (2017).

Consistent preferences do not automatically imply the rationality and qualification of the decision maker (Temesi, 2011). On the other hand, it is reasonable to assume that highly inconsistent preferences might be a symptom of the decision maker's incapacity in discriminating between alternatives (Irwin, 1958) or of a lack of attention and concentration in the elicitation phase, possibly due to information overload (Carmone Jr et al, 1997). Again, different theories in decision analysis seem to agree on the importance of allowing for inconsistencies but, at the same time, limiting their extent. In the AHP, Saaty (1977) claimed that preferences should be 'close' to consistency, and in MAVT Keeney and Raiffa (1976) recommended that inconsistencies be reduced to a 'nominal level'.

Given the fact that judgments which are not too inconsistent can be accepted whereas too inconsistent ones should be revised, the estimation of inconsistency concretely affects the decision process and consequently it becomes crucial to rely on fair estimations of inconsistency. To this end, a wide range of inconsistency indices have been proposed in the literature to quantify the deviation of the preferences of a decision maker from a fully consistent form. Namely, such indices act as indicators of the inconsistency level of pairwise comparisons. Some numerical comparative studies analyzed the most relevant indices (Brunelli et al, 2013a; Bozóki and Rapcsák, 2008; Grzybowski, 2016; Kazibudzki, 2016) while, recently, formal studies on inconsistency indices have been proposed in the literature through axiomatic approaches (Brunelli, 2017; Brunelli and Fedrizzi, 2015; Csató, 2017; Koczkodaj and Szwarc, 2014).

Having established these facts, the scope of this paper is that of constructing a general framework for inconsistency indices and then study under what conditions the indices generated within it satisfy some basic reasonable properties. The definition of such a general framework will (i) push forward the formal analysis of inconsistency indices, (ii) clarify the essence of inconsistency estimation, and (iii) be used to devise new and customized indices.

This paper is organized as follows. Section 2 introduces the necessary formalism on pairwise comparisons and inconsistency indices, also recalling the recent literature on the subject matter and the definitions of some indices used in the rest of the analysis. Section 3 shows that some well-known inconsistency indices share a common algebraic structure and proves under what conditions a general framework generates reasonable inconsistency indices. Continuing in the same direction, Section 4 proposes the use of aggregation functions to model the general formulation of indices. Finally, Section 5 contains a discussion and the conclusions.

2 Pairwise comparisons and inconsistency indices

In the following, we shall consider a finite non-empty set of entities, hereafter also called 'alternatives', $X = \{x_1, \ldots, x_n\}$ and a set of pairwise comparisons between them, i.e. a valued preference relation on the set X. According to the multiplicative scheme, the degree of preference of x_i to x_j is represented by a positive real value $a_{ij} > 0$ which is the numerical estimation of the ratio ω_i/ω_j where ω_i and ω_j are the weights of x_i and x_j, respectively. It is common to assume reciprocity, i.e. $a_{ij} = 1/a_{ji}$ $\forall i, j$. In this setting, a *pairwise comparison matrix* $\mathbf{A} = (a_{ij})_{n \times n}$ is defined as a positive and reciprocal matrix, i.e. a convenient mathematical structure where the comparisons are collected. A desirable property of pairwise comparison matrices is called *consistency* and resembles an extension of the concept of transitivity to the case of valued preferences. Namely, a pairwise comparison matrix is *consistent* (Saaty, 1977) if

$$a_{ik} = a_{ij} a_{jk} \quad \forall i, j, k. \tag{1}$$

This means that each direct comparison a_{ik} is supported by all indirect comparisons $a_{ij}a_{jk}$ via x_j $\forall j$ (Saaty, 1977). For later convenience, the set of all pairwise comparison matrices is defined as

$$\mathcal{A} = \{\mathbf{A} = (a_{ij})_{n\times n} | a_{ij} > 0,\ a_{ij}a_{ji} = 1\ \forall i, j, \quad n > 2\}.$$

Similarly, the set of all *consistent* pairwise comparison matrices $\mathcal{A}^* \subset \mathcal{A}$ is the following,

$$\mathcal{A}^* = \{\mathbf{A} = (a_{ij})_{n\times n} | \mathbf{A} \in \mathcal{A},\ a_{ik} = a_{ij}a_{jk}\ \forall i, j, k\}.$$

Inconsistency indices have been introduced to estimate the extent of the violation of consistency—i.e. the deviation from (1)—in pairwise comparison matrices. Formally, an inconsistency index is a function I mapping pairwise comparison matrices into real numbers, such that the value $I(\mathbf{A}) \in \mathbb{R}$ represents the degree of inconsistency of $\mathbf{A}$ (Brunelli and Fedrizzi, 2015). Originally, Saaty (1977) introduced an inconsistency index based on the Perron-Frobenius eigenvalue of $\mathbf{A}$. However, in recent years new indices have proliferated in the literature. In this paper, we consider only some of them, and the reader can refer to a survey (Brunelli et al, 2013a) for a broader overview. Besides Saaty's index, the most widely known and studied index is, probably, the Geometric Consistency Index (Crawford and Williams, 1985),

$$GCI(\mathbf{A}) = \frac{2}{(n-1)(n-2)} \sum_{1 \leq i < j \leq n} \ln^2 \left(a_{ij} \frac{\left(\prod_{k=1}^{n} a_{jk}\right)^{\frac{1}{n}}}{\left(\prod_{k=1}^{n} a_{ik}\right)^{\frac{1}{n}}} \right).$$

For later convenience, we remark that it was shown (Brunelli et al, 2013b) that GCI can be rewritten as follows,

$$GCI(\mathbf{A}) = \chi_n \sum_{1 \leq i < j < k \leq n} \left(\ln \frac{a_{ij}a_{jk}}{a_{ik}} \right)^2, \tag{2}$$

where χ_n is a normalization factor which depends on the order n of the matrix $\mathbf{A}$. In the following, we shall adopt formulation (2). Index GCI has been studied (Aguarón and Moreno-Jiménez, 2003; Aguarón et al, 2016) and extended to other representations of preferences as, for instance, interval-valued pairwise comparisons (Meng et al, 2015; Wang, 2015).

Peláez and Lamata (2003) proposed another index in the following form,

$$CI^*(\mathbf{A}) = \frac{1}{\binom{n}{3}} \sum_{1 \le i < j < k \le n} \left(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} - 2 \right). \tag{3}$$

Such index was used by the same authors (Lamata and Peláez, 2002) as the objective function to be minimized in the estimation of missing comparisons. Remarkably, another index was proposed to estimate inconsistency (Shiraishi et al, 1998) and to estimate missing preferences (Obata et al, 1999) and later proven to be proportional, and thus equivalent, to CI^* (Brunelli et al, 2013b). This double interpretation of (3) seems to corroborate its soundness.

Cavallo and D'Apuzzo (2009, 2010) studied pairwise comparison matrices using group theory and derived the following index

$$I_{CD}(\mathbf{A}) = \prod_{1 \le i < j < k \le n} \left(\max \left\{ \frac{a_{ij}a_{jk}}{a_{ik}}, \frac{a_{ik}}{a_{ij}a_{jk}} \right\} \right)^{\frac{1}{\binom{n}{3}}}, \tag{4}$$

whose range is the interval $[1, +\infty[$. It is also interesting to note that I_{CD} has been proven functionally related to another important index used in the framework of reciprocal preference relations (Brunelli, 2016). This fact certainly increases the relevance of index I_{CD}.

Inspired by an original proposal by Koczkodaj (1993), Duszak and Koczkodaj (1994) introduced a max-min inconsistency index,

$$K(\mathbf{A}) = \max_{1 \le i < j < k \le n} \left\{ \min \left\{ \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right|, \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right| \right\} \right\}, \tag{5}$$

which was recently characterized by Csató (2017). Additionally, index K was used to estimate missing comparisons (Koczkodaj et al, 1999) and employed in real-world decision analysis (Koczkodaj et al, 2014). More recently, Grzybowski (2016) proposed and justified a modification of index K, where the maximum is replaced by the arithmetic mean,

$$ATI(\mathbf{A}) = \frac{1}{\binom{n}{3}} \sum_{1 \le i < j < k \le n} \min \left\{ \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right|, \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right| \right\}. \tag{6}$$

Let us note that, unlike the Consistency Index by Saaty, which requires the estimation of the spectral radius of the pairwise comparison matrix $\mathbf{A}$, the indices GCI, CI^*, I_{CD}, K and ATI are closed-form expressions of the entries of $\mathbf{A}$, and therefore especially transparent and appealing.

Additionally, recently some properties have been introduced by Brunelli and Fedrizzi (2015), Brunelli (2017), and Koczkodaj and Szwarc (2014) to help define the concept of inconsistency index. In view of the fact that the properties proposed by Brunelli and Fedrizzi are seemingly more restrictive than those by Koczkodaj and Szwarc, we shall focus on these former to derive stronger results. Since they are going to be used later, it is here the case to shortly recall the above mentioned axiomatic properties.

Property 1 (P1)**.** *Index I attains its minimum value $\nu \in \mathbb{R}$ if and only if $\mathbf{A}$ is consistent, i.e. $I(\mathbf{A}) = \nu \Leftrightarrow \mathbf{A} \in \mathcal{A}^* \; \forall \mathbf{A} \in \mathcal{A}$.*

Property 2 (P2)**.** *Index I is invariant under permutation of alternatives, i.e. $I(\mathbf{A}) = I(\mathbf{PAP}^T) \; \forall \mathbf{A} \in \mathcal{A}$ and for all permutation matrices $\mathbf{P}$.*

Property 3 (P3)**.** *As the preferences are intensified, the inconsistency cannot decrease. Define $\mathbf{A}(b) = (a_{ij}^b)_{n \times n}$. Formally, $I(\mathbf{A}(b)) \ge I(\mathbf{A}) \; \forall \mathbf{A} \in \mathcal{A}$ and $\forall b > 1$.*

Property 4 (P4)**.** *Consider a matrix* $\mathbf{A} \in \mathcal{A}^*$ *and the matrix* $\mathbf{A}_{pq}(\delta)$ *which is the same as* $\mathbf{A}$ *except for entries* a_{pq} *and* a_{qp} *which are replaced by* a_{pq}^{δ} *and* a_{qp}^{δ}*, respectively. Then,* $I(\mathbf{A}_{pq}(\delta))$ *is a quasi-convex function of* $\delta \in]0, \infty[$ *with minimum in* $\delta = 1$ *for all* $p, q = 1, \ldots, n$*.*

Property 5 (P5)**.** *Index* I *is a continuous function of the entries of* $\mathbf{A}$ *for all* $\mathbf{A} \in \mathcal{A}$*.*

Property 6 (P6)**.** *Index* I *is invariant under inversion of preferences, i.e.* $I(\mathbf{A}) = I(\mathbf{A}^T)\ \forall \mathbf{A} \in \mathcal{A}$*.*

Note that, albeit lengthy in its formulation, P4 concerns changing a single comparison and its reciprocal in a consistent matrix and simply states that the larger the change, the larger the inconsistency.

3 Decomposition of some transitivity-based indices

From a first inspection, it appears that indices GCI, CI^*, I_{CD}, K and ATI share a common structure. Firstly, in all of them there are terms $\frac{a_{ij}a_{jk}}{a_{ik}}$ or, equivalently, thanks to reciprocity, $a_{ij}a_{jk}a_{ki}$. Hereafter, we shorten them with $\alpha_{ijk} := a_{ij}a_{jk}a_{ki}$. Secondly, all these indices are based on an evaluation of the local inconsistency of triple (i, j, k) for $1 \leq i < j < k \leq n$. This evaluation is a real valued function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ mapping positive values α_{ijk}'s into real values $f(\alpha_{ijk})$ acting as estimations of local inconsistency. Thirdly, another function, $\bigoplus : \mathbb{R}^{\binom{n}{3}} \rightarrow \mathbb{R}$, is used to aggregate the contributions of local inconsistencies, $f(\alpha_{ijk})$, to the global inconsistency. Thus, indices GCI, CI^*, I_{CD}, K and ATI can be rewritten in the more general form

$$I(\mathbf{A}) = \bigoplus_{1 \leq i < j < k \leq n} f(\alpha_{ijk}). \tag{7}$$

The decompositions of the above mentioned indices into the more general form (7) is summarized in Table 1.

Index	$\bigoplus : \mathbb{R}^{\binom{n}{3}} \longrightarrow \mathbb{R}$	$f : \mathbb{R}_+ \longrightarrow \mathbb{R}$
GCI	$\chi_n \sum_{1 \leq i < j < k \leq n}$	$(\ln \alpha_{ijk})^2$
CI^*	$\frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n}$	$\alpha_{ijk} + \frac{1}{\alpha_{ijk}} - 2$
I_{CD}	$\prod_{1 \leq i < j < k \leq n} (\cdot)^{1/\binom{n}{3}}$	$\max\left\{\alpha_{ijk}, \frac{1}{\alpha_{ijk}}\right\}$
K	$\max_{1 \leq i < j < k \leq n}$	$\min\left\{\left\|1 - \alpha_{ijk}\right\|, \left\|1 - \frac{1}{\alpha_{ijk}}\right\|\right\}$
ATI	$\frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n}$	$\min\left\{\left\|1 - \alpha_{ijk}\right\|, \left\|1 - \frac{1}{\alpha_{ijk}}\right\|\right\}$

Table 1: Decomposition of inconsistency indices GCI, CI^*, I_{CD}, K and ATI.

Figure 1 offers a snapshot of the rationale behind the decomposition of the indices. We shall note that an heuristic attempt to see indices as an aggregation of local inconsistencies was proposed by Siraj et al (2015).

Inconsistency indices based on values $\alpha_{ijk} = a_{ij}a_{jk}a_{ki}$ have some interesting properties which will later on become useful. Let us now investigate some properties of the general quantity α_{ijk}.

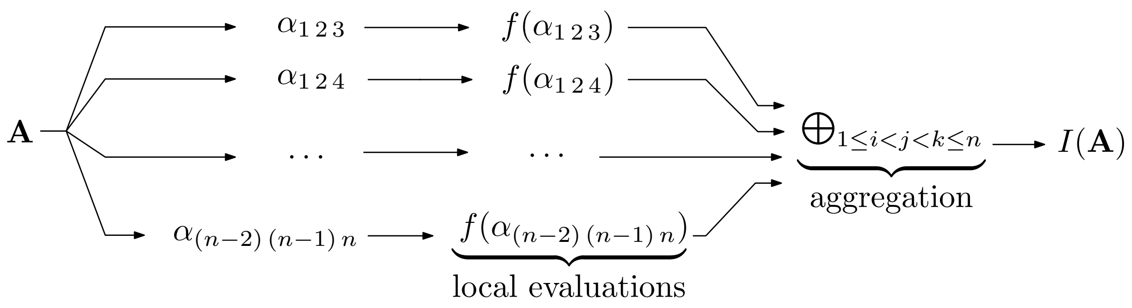


Figure 1: Inconsistency indices (2)–(6) evaluate and then aggregate local contributions into a value of global inconsistency.

The first property is that, given a pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n\times n}$ and an arbitrary α_{ijk},

$$\alpha_{\sigma(i)\sigma(j)\sigma(k)} \in \left\{\alpha_{ijk}, \frac{1}{\alpha_{ijk}}\right\}, \tag{8}$$

for all permutations $\sigma : \{i,j,k\} \to \{i,j,k\}$. In fact, by inspection, we have

$$\alpha_{ijk} = \alpha_{jki} = \alpha_{kij} = \frac{1}{\alpha_{jik}} = \frac{1}{\alpha_{ikj}} = \frac{1}{\alpha_{kji}}.$$

The second property is that,

$$a_{ij}^b a_{jk}^b a_{ki}^b = (a_{ij}a_{jk}a_{ki})^b = \alpha_{ijk}^b \quad \forall i,j,k, \tag{9}$$

and for all $b \in \mathbb{R}$. Let us also note that $\alpha_{ijk} = 1$ means that the alternatives x_i, x_j and x_k are compared in a perfectly consistent way. Given the existence of properties defining the concept of inconsistency index, one natural research direction would be that of finding some requirements which, when imposed to f and $\bigoplus$, ensure that the induced inconsistency index I is an inconsistency index satisfying properties P1–P6. The next theorem provides results in this sense.

Theorem 1. *Consider an inconsistency index I in the form (7). Then, if*

- *Function f is quasi-convex in $]0,\infty[$ with strict minimum in $\alpha_{ijk} = 1$, and such that*

$$f(x) = f(1/x) \quad \text{for all } x > 0. \tag{10}$$

- *Function $\bigoplus$ is symmetric, monotone increasing in all arguments, and strictly monotone increasing w.r.t. the greatest argument.*
- *Both f and $\bigoplus$ are continuous,*

then I satisfies the properties P1–P6.

Proof. First, let us observe that from (10) and (8) it follows

$$f(\alpha_{\sigma(i)\sigma(j)\sigma(k)}) = f(\alpha_{ijk}). \tag{11}$$

Property (11) states that $f(\alpha_{ijk})$ is invariant under indices permutation. Now, we shall analyze how the single properties P1–P6 are implied by the restrictions imposed to f and $\bigoplus$.

P1 We shall prove the implication in both directions.

$\Rightarrow$) By contrapositive, this can be equivalently written $\mathbf{A} \notin \mathcal{A}^* \Rightarrow I(\mathbf{A}) \neq \nu$. If $\mathbf{A} \notin \mathcal{A}^*$, then there exists a triple (i',j',k') for which $\alpha_{i'j'k'} \neq 1$. Hence, there exists a $f(\alpha_{i'j'k'})$

with value strictly greater than the minimum attainable by f. Since we required that $\bigoplus$ be strictly monotone in its greatest argument, it follows that $\mathbf{A} \notin \mathcal{A}^* \Rightarrow I(\mathbf{A}) > \nu$.
$\Leftarrow$) Functions f have strict global minimum for $\alpha_{ijk} = 1$. Hence, if $\mathbf{A} \in \mathcal{A}^*$, then all the $f(\alpha_{ijk})$ are minimized. Since they are all minimized and $\bigoplus$ is a monotone increasing function, it also reaches its minimum ν.

P2 We shall consider that the operation $\mathbf{PAP}^T$ maps the original argument α_{ijk} into $\alpha_{\sigma(i)\sigma(j)\sigma(k)}$ where σ is a permutation of $\{1, \ldots, n\}$. Hence, thanks to (11), the operation $\mathbf{PAP}^T$ induces a reordering of the arguments of $\bigoplus$, but this is irrelevant since we required $\bigoplus$ to be symmetric.

P3 Thanks to (9) and the quasi-convexity of f we know that $f(\alpha_{ijk}^b)$ is monotone increasing w.r.t $b \geq 1$. Given that $\bigoplus$ is also monotone increasing, we deduce that P3 is satisfied.

P4 Let us consider $I(\mathbf{A}_{pq}(\delta))$. Each term $f(\alpha_{ijk})$ in (7) not containing a_{pq} or a_{qp} remains unchanged as in $I(\mathbf{A})$, i.e. $f(\alpha_{ijk}) = f(1)$. The terms α_{ijk} containing both indices p and q can be written in the form $\alpha_{pqk} = a_{pq}^{\delta} a_{qk} a_{kp}$. Since $\mathbf{A} \in \mathcal{A}^*$, it is $\alpha_{pqk} = a_{pq} a_{qk} a_{kp} = 1$. Then, $\alpha_{pqk} = a_{pq}^{\delta} a_{qk} a_{kp}$ is an increasing function of δ if $a_{pq} > 1$ and a decreasing function of δ if $0 < a_{pq} < 1$. Then, satisfaction of P4 follows from the assumptions on functions f and $\bigoplus$.

P5 Functions f and $\bigoplus$ are continuous and therefore their composition, I, is also continuous.

P6 With the operation of transposition, for all $i < j < k$ each α_{ijk} is transformed into $\frac{1}{\alpha_{ijk}} = \alpha_{kji}$. Given (10), we know that $I(\mathbf{A}) = I(\mathbf{A}^T)$.

□

The next theorem follows easily from Theorem 1.

Theorem 2. *Indices GCI, CI^*, I_{CD}, K and ATI satisfy properties P1–P6.*

Proof. According to Table 1, the indices mentioned above are particular cases of (7). Then, it is sufficient to prove that the corresponding functions f and $\bigoplus$ satisfy the assumptions of Theorem 1. By inspection of Table 1, it is easy to verify that all functions f corresponding to the considered indices are continuous quasi-convex functions in $]0, \infty[$, such that $f(x) = f(1/x)$, with strict minimum in $\alpha_{ijk} = 1$, see Figure 3. Functions $\bigoplus$ from Table 1 are based on sums, products of positive numbers, or max functions. We conclude that GCI, CI^*, I_{CD}, K and ATI satisfy the assumptions of Theorem 1 and consequently they satisfy the properties P1–P6. □

Figure 2 explains the role played by Theorems 1 and 2. In addition, we note that the satisfaction of properties P1–P6 by indices GCI, CI^*, I_{CD} and K was already proven (Brunelli, 2017; Brunelli and Fedrizzi, 2015) separately for each index. Clearly, individual proofs are much more onerous than the one of Theorem 2 and therefore we believe that general formulations, such as the one presented in Theorem 1, could help simplify the analysis of inconsistency indices.

3.1 Pareto efficiency

Let us introduce the following notation

$$\bar{\alpha}_{ijk} := \max \left\{ \alpha_{ijk} \, , \, \frac{1}{\alpha_{ijk}} \right\}. \tag{12}$$

Definition 1 (Pareto efficiency)**.** *Given $\mathbf{A} = (a_{ij})_{n \times n} \in \mathcal{A}$ and $\mathbf{A}' = (a'_{ij})_{n \times n} \in \mathcal{A}$, let $\alpha_{ijk} := a_{ij} a_{jk} a_{ki}$, $\alpha'_{ijk} := a'_{ij} a'_{jk} a'_{ki}$. Moreover, let $\bar{\alpha}_{ijk}$ and $\bar{\alpha}'_{ijk}$ be defined according to (12), as well as*

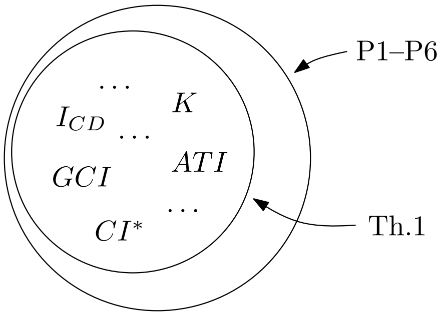


Figure 2: Theorem 1 implicitly defines a family of inconsistency indices satisfying P1–P6. Theorem 2 proves that indices GCI, CI^*, I_{CD} and K can be generated by the general form (7) and satisfy properties P1–P6.

$I(\mathbf{A})$ defined by (7). If

$$\bar{\alpha}'_{ijk} \geq \bar{\alpha}_{ijk} \quad \forall\, i < j < k \implies I(\mathbf{A}') \geq I(\mathbf{A})\,, \tag{13}$$

then index I is called Pareto efficient.

Note that Definition 1 emphasizes the role of the basic local inconsistency terms $\bar{\alpha}_{ijk}$ from the point of view of Pareto efficiency of inconsistency indices (7). The intuitive interpretation is the following: if all the local violations of the inconsistency of a matrix $\mathbf{A}'$ are greater or equal than those in $\mathbf{A}$, then also the global inconsistency of $\mathbf{A}'$ is greater or equal than the global inconsistency of $\mathbf{A}$[1]. The following theorem proves that Pareto efficiency is guaranteed under assumptions even weaker than those of Theorem 1.

Theorem 3. *Consider an inconsistency index I in the form (7). Then, if*

- *Function f is quasi-convex in $]0,\infty[$ with strict minimum in $\alpha_{ijk} = 1$, and such that*

$$f\,(x) = f\,(1/x) \quad \text{for all } x > 0 \tag{14}$$

- *Function $\bigoplus$ is monotone increasing in all arguments*

then, I is a Pareto–efficient index.

Proof. From the assumptions on f and definition (12), it follows that

$$f(\alpha_{ijk}) = f(\bar{\alpha}_{ijk}) \quad \forall\, i < j < k.$$

Since $\bar{\alpha}_{ijk} \geq 1$, then

$$\bar{\alpha}'_{ijk} \geq \bar{\alpha}_{ijk} \quad \implies \quad f(\bar{\alpha}'_{ijk}) \geq f(\bar{\alpha}_{ijk}).$$

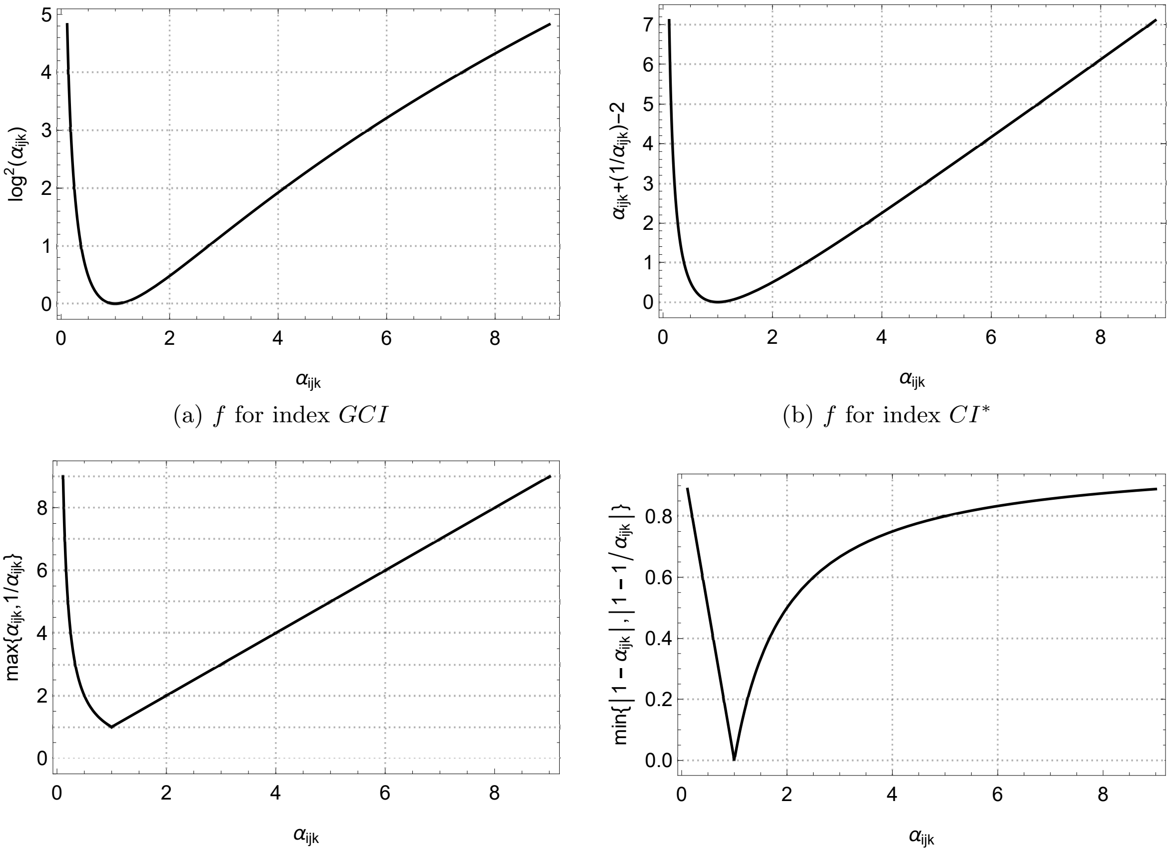
As $\bigoplus$ is monotone increasing in all arguments, then

$$\bar{\alpha}'_{ijk} \geq \bar{\alpha}_{ijk} \quad \forall\, i < j < k \implies I(\mathbf{A}') \geq I(\mathbf{A}),$$

and I is Pareto efficient. □

Corollary 1. *Indices GCI, CI^*, I_{CD}, K and ATI are Pareto efficient.*

[1] A similar property, called *locality*, was formulated in "W.W. Koczkodaj and J. Szybowski, Axiomatization of inconsistency indicators for pairwise comparisons matrices revisited, arXiv:1509.03781v1".



(a) f for index GCI

(b) f for index CI^*

(c) f for index I_{CD}

(d) f for indices K and ATI

Figure 3: Plots of functions f for indices GCI, CI^*, I_{CD}, K and ATI. They are quasi-convex functions in $]0, \infty[$ such that $f(\alpha) = f(1/\alpha)$ with strict minimum in $\alpha = 1$.

Proof. The proof immediately follows from Theorem 2. □

It is also possible to relate the lack of Pareto efficiency with the non-satisfaction of some properties. In fact, it can be proven that, if an index I does not satisfy P3 or P4, then it cannot be Pareto efficient. Since a significant number of inconsistency indices do not satisfy P3 and P4 (Brunelli, 2017), Pareto efficiency is not a trivial property of inconsistency indices.

4 Some general forms for inconsistency indices

As pointed out in the previous section, different functions f are used by the inconsistency indices studied in this paper and they all fit the requirements of Theorem 1. A graphical illustration of such functions lends itself well to the study of their behavior. For this scope, Figure 3 illustrates the plots of the functions f for the indices GCI, CI^*, I_{CD}, K and ATI. Although very similar, with the exception of K, different indices treat deviations from consistency in different ways. For example, when $\alpha > 1$, the increase in the inconsistency of a triple is linear for I_{CD}, whereas K penalizes deviations close to the consistent situation more than those happening when the triple is already quite inconsistent.

4.1 OWA functions

To bridge different views and create a common framework, in this section we propose some aggregation functions which can be used to aggregate the local contributions to the global inconsistency. To achieve this goal, some general families of averaging operators are recalled and later on used.

Definition 2 (Quasi-Ordered Weighted Averaging Function (Grabisch et al, 2009))**.** *A quasi-ordered weighted averaging ($QOWA_{\psi,\mathbf{w}}$) function of dimension m is a mapping that has an associated weighting vector $\mathbf{w} = (w_1, \ldots, w_m)$ with $w_1 + \cdots + w_m = 1$, $0 \leq w_i \leq 1$, $i = 1, \ldots, m$ such that*

$$QOWA_{\psi,\mathbf{w}}(a_1, \ldots, a_m) = \psi^{-1}\left(\sum_{i=1}^{m} w_i \psi(b_i)\right),$$

where b_j is the jth largest element of the multiset $\{a_1, \ldots, a_m\}$, and where $\psi : \mathbb{I} \to \mathbb{R}$ is a continuous and strictly monotone increasing function on an interval $\mathbb{I} \subseteq \mathbb{R}$.

QOWA functions resemble quasi-arithmetic means, but with the difference that the weights are not associated to the position of the argument but to its value compared with the others'. Quasi-arithmetic means were used, among others, by Aczél and Saaty (1983) to aggregate pairwise comparisons in group decision making. In the following definitions, two special cases of $QOWA_{\psi,\mathbf{w}}$ are recalled and will be used to rewrite inconsistency indices in equivalent forms.

Definition 3 (Ordered weighted averaging (OWA) functions (Yager, 1988))**.** *When ψ is the identity function, the QOWA function collapses in the following,*

$$OWA_{\mathbf{w}}(a_1, \ldots, a_m) = \sum_{i=1}^{m} w_i b_i,$$

and it is simply called ordered weighted averaging *(OWA) function.*

The arithmetic mean and the maximum are well-known examples of OWA functions. Namely,

$$OWA_{(1/m,\ldots,1/m)}(a_1, \ldots, a_m) = \frac{1}{m}\sum_{i=1}^{m} a_i$$
$$OWA_{(1,0,\ldots,0)}(a_1, \ldots, a_m) = \max\{a_1, \ldots, a_m\}.$$

Definition 4 (Ordered weighted geometric averaging (OWGA) functions (Chiclana et al, 2000))**.** *When $\psi = \log$, the QOWA function collapses in the following,*

$$OWGA_{\mathbf{w}}(a_1, \ldots, a_m) = \prod_{i=1}^{m} b_i^{w_i},$$

and is simply called ordered weighted geometric averaging *(OWGA) function.*

The geometric mean and the maximum are examples of OWGA functions. Namely,

$$OWGA_{(1/m,\ldots,1/m)}(a_1, \ldots, a_m) = \left(\prod_{i=1}^{m} a_i\right)^{\frac{1}{m}}$$
$$OWGA_{(1,0,\ldots,0)}(a_1, \ldots, a_m) = \max\{a_1, \ldots, a_m\}.$$

By using the definitions of OWA and OWGA functions we can rewrite the five inconsistency indices GCI, CI^*, I_{CD}, K and ATI in the following equivalent forms.

$$GCI(\mathbf{A}) = \chi_n OWA_{(1/\binom{n}{3},\dots,1/\binom{n}{3})}\left(\ln \frac{a_{ij}a_{jk}}{a_{ik}}\right)^2 \tag{15}$$

$$CI^*(\mathbf{A}) = OWA_{(1/\binom{n}{3},\dots,1/\binom{n}{3})}\left(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} - 2\right) \tag{16}$$

$$I_{CD}(\mathbf{A}) = OWGA_{(1/\binom{n}{3},\dots,1/\binom{n}{3})} \max\left\{\frac{a_{ik}}{a_{ij}a_{jk}}, \frac{a_{ij}a_{jk}}{a_{ik}}\right\} \tag{17}$$

$$K(\mathbf{A}) = OWA_{(1,0,\dots,0)} \min\left\{\left|1 - \frac{a_{ik}}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}}\right|\right\} \tag{18}$$

$$ATI(\mathbf{A}) = OWA_{(1/\binom{n}{3},\dots,1/\binom{n}{3})} \min\left\{\left|1 - \frac{a_{ik}}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}}\right|\right\}. \tag{19}$$

Note that, by writing the five indices in the forms (15)–(19), we used a simplified notation that needs a clarification. We wrote a single argument, but the number of arguments of the above OWA functions is in fact $m = \binom{n}{3}$. More precisely, each one of the arguments corresponds to a combination of three different indices (i, j, k) among n.

For sake of simplicity, in the rest of the discussion we shall consider OWA functions, which are more popular in the literature and in applications. Nevertheless, extensions to the OWGA are straightforward. The attitudinal character of an OWA can be represented by some indices, the most important of which are orness and entropy (H),

$$orness(\mathbf{w}) = \sum_{i=1}^{m} \frac{m-i}{m-1} w_i, \tag{20}$$

$$H(\mathbf{w}) = -\sum_{i=1}^{m} w_i \ln w_i \tag{21}$$

Specifically, *orness* is the degree to which the OWA function resembles the maximum operator. When orness is equal to 1, the OWA is the maximum. Conversely, when orness is equal to 0, then the OWA is the minimum. Values between 0 and 1 represent trade-offs between minimum and maximum. *Entropy* measures the dispersion of weights and it reaches its maximum when all the weights are equal to $1/m$. One family of methods for finding the most appropriate weight vector considers the weight vector as the optimal solution of a given optimization problem. The most popular optimization method was proposed by O'Hagan (1988) to find the vector $\mathbf{w}$ with *a priori* specified $orness(\mathbf{w}) = \gamma \in [0, 1]$ and maximum entropy. That is, the following optimization problem has to be solved.

$$\begin{aligned} \underset{(w_1,\dots,w_m)}{\text{maximize}} \quad & H(\mathbf{w}) \\ \text{subject to} \quad & orness(\mathbf{w}) = \gamma \\ & w_1 + \cdots + w_m = 1 \\ & w_1, \dots, w_m \geq 0\,. \end{aligned} \tag{22}$$

The weight vectors $(1/m, \dots, 1/m)$, representing the average, and $(1, 0, \dots, 0)$, representing the maximum, can be obtained by solving the optimization problem (22) with values $\gamma = 0.5$ and $\gamma = 1$, respectively. Fullér and Majlender (2001) showed that when (22) is solved with $\gamma \in [0.5, 1]$ the weights of the optimal solution respect the relation $w_1 \geq \cdots \geq w_n$, which is the same condition discussed by Can (2014) under the name of "monotonically decreasing vector". We observe that the condition $w_1 \geq ... \geq w_n$ corresponds to giving more importance to larger local inconsistencies.

It is worth noting that there is not a consensus in the literature on whether local inconsistencies should be averaged, as done by indices GCI, CI^*, I_{CD} and ATI, or instead, only the largest one be considered, as done by index K. This latter view has been pushed forward by Koczkodaj and Szwarc (2014) in contrast to approaches based on the averaging approach (Cavallo and D'Apuzzo, 2009; Peláez and Lamata, 2003). As illustrated in Figure 4, by using OWA functions we offer a flexible and unifying framework which provides a continuous trade-off between these two different approaches. We hope, in this way, to build a bridge between different points of view.

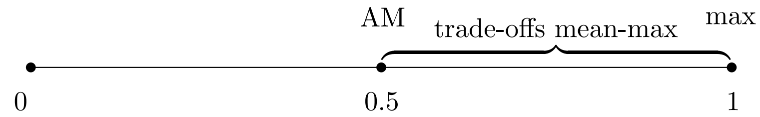


Figure 4: AM = arithmetic mean. Using levels of orness between 1/2 and 1 allows to trade-off between averaging local inconsistencies and considering only the largest.

An interesting trade-off between the average and the maximum could be, for instance, the OWA function with the weight vector solving (22) with orness equal to 0.75. The weights obtained with this value of orness would represent a halfway compromise between the average and the maximum. More generally, a suitable value of γ can be chosen depending on the faced problem. The obtained inconsistency index will accordingly take care of both the mean inconsistency among the $\binom{n}{3}$ triples (i, j, k) and the requirement of avoiding large local inconsistencies. The following example shows that differently balancing the relevance of the two objectives may lead to different consistency rankings.

Example 1. *Consider the two pairwise comparison matrices*

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 4 & 9/2 & 9 \\ 1/4 & 1 & 3 & 6 \\ 2/9 & 1/3 & 1 & 2 \\ 1/9 & 1/6 & 1/2 & 1 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 3 & 9/2 & 9 \\ 1/3 & 1 & 3 & 6 \\ 2/9 & 1/3 & 1 & 5 \\ 1/9 & 1/6 & 1/5 & 1 \end{pmatrix}$$

and let us evaluate their inconsistency by means of an index (7) where $\bigoplus$ is an OWA operator with associated weighting vector $\mathbf{w} = (w_1, \ldots, w_{\binom{4}{3}})$ obtained by solving the maximum entropy optimization problem (22) for different levels of orness. We further assume that the local inconsistencies are modeled by means of the following function,

$$f(\alpha_{ijk}) = \alpha_{ijk} + \frac{1}{\alpha_{ijk}} - 2\,, \tag{23}$$

which coincides with the the function f used by index CI^*. In Figure 5, we consider a so constructed inconsistency index and we plot its value for $\mathbf{A}_1$, $\mathbf{A}_2$ and different levels of orness, γ. The plot shows that the two matrices are differently ranked according to their inconsistency depending on the orness of the OWA function. More precisely, if we focus on the presence of a single high local inconsistency, thus using a high value of orness, $\mathbf{A}_1$ is classified as more inconsistent than $\mathbf{A}_2$. Conversely, if we mainly focus on average of local inconsistencies, thus choosing γ close to 0.5, $\mathbf{A}_2$ is considered more inconsistent than $\mathbf{A}_1$.

Needless to say, not all the OWA functions can be used to generate new inconsistency indices satisfying P1–P6. For example, following the requirements of Theorem 1, and bearing in mind that the OWA functions replace $\bigoplus$, one deduces that the OWA function should have $w_1 > 0$ so that it is strictly monotone increasing in its maximum argument. We are then ready to formalize our findings in the following corollary.

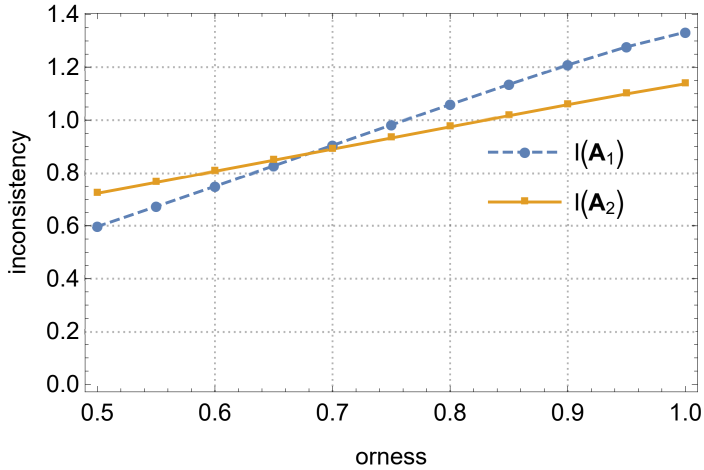


Figure 5: Inconsistency of matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ as a function of the orness of the maximum entropy OWA when f is defined as in (23).

Corollary 2. *If $\bigoplus$ is an OWA function with $w_1 > 0$ and f respects the properties listed in Theorem 1, then a function in the form $\bigoplus_{1 \leq i \leq j \leq k \leq n} f(\alpha_{ijk})$ is an inconsistency index satisfying the properties P1–P6.*

4.2 Triangular conorms

Triangular-norms (t-norms) and triangular conorms (t-conorms) were proposed as metrics in probabilistic metric spaces. Nonetheless, the use of t-norm and t-conorms has gone beyond its original purpose and nowadays they are widely employed in other fields, most notably in fuzzy sets theory (Klir and Yuan, 1995) and as aggregation functions (Grabisch et al, 2009).

Definition 5 (t-conorm)**.** *A t-conorm is a function $\perp : [0,1]^2 \rightarrow [0,1]$ with the following properties:*

- $\perp(a, 0) = 0$ *(neutral element 1)*
- $\perp(a, b) = \perp(b, a)$ *(commutativity)*
- $\perp(a, \perp(b, c)) = \perp(\perp(a, b), c)$ *(associativity)*
- $\perp(a, b) \geq \perp(c, d)$ *if $a \geq c$ and $b \geq d$ (monotonicity)*

Definition 6 (Strictly monotone t-conorm)**.** *A t-conorm is* strictly monotone *if $a \geq c$ and $b \geq d$ and at least one of these two inequalities is strict, implies that $\perp(a, b) > \perp(c, d)$.*

Triangular conorms are, in their basic form, functions $\perp : [0,1]^2 \rightarrow [0,1]$, but thanks to associativity they can be extended to map points in the m-ary unit cube into the interval $[0,1]$. It can be shown that, with some caution, some t-conorms can be used to model inconsistency indices.

Corollary 3. *If $\bigoplus$ is a continuous and strictly monotone t-conorm, f respects the properties listed in Theorem 1 and the image of f is in $[0,1]$, then a function in the form $\bigoplus_{1 \leq i \leq j \leq k \leq n} f(\alpha_{ijk})$ is an inconsistency index satisfying the properties P1–P6.*

Example 2. *We have previously seen that the index K can be rewritten by means of OWA functions. Index K is a special case of*

$$K(\mathbf{A}) = \perp_{i,j,k} \min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right|, \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right| \right\}$$

where $\perp = \max$. Different results, for example differentiability, can be obtained by replacing max *with another t-conorm.*

5 Discussion and conclusions

In the current scientific debate, conflicting points of view have been expressed on the concept of inconsistency index. Many authors (Cavallo and D'Apuzzo, 2009; Lamata and Peláez, 2002; Siraj et al, 2015) proposed approaches to average local inconsistencies whereas some others claimed that only the inconsistency of the most inconsistent triple should matter (Duszak and Koczkodaj, 1994). We believe that both approaches can coexist and be applied to capture different facets of the concept of inconsistency. The results presented in this paper will hopefully serve as a *trait d'union* between the two aforementioned viewpoints. Furthermore, besides unifying different approaches, this research could also constitute a bridge between the study of inconsistency of preferences and the theory of aggregation functions (Beliakov et al, 2007; Grabisch et al, 2009).

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