A study on the anonymity of pairwise comparisons in group decision making [1]

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Abstract

Pairwise comparisons between criteria/alternatives are a well-established methodology for group decision making. Both in the literature and in real-world applications, it is common practice to average the opinions of various experts expressed as pairwise comparisons to find a compromise solution. This paper dwells on the inverse problem: given a decision maker who knows his own preferences and the aggregate ones, we introduce and study some optimization problems to help him infer, in the form of intervals, the preferences of the other decision makers. Since the possibility of inferring the other participants preferences violates the requirement of anonymity, this paper also reports the results of some numerical simulation examining the relation between the number of participants in the decision process and their anonymity. In addition to numerical simulations with randomly generated data, we will also experiment with a dataset of preferences collected in a real-world survey.

Keywords: Pairwise comparisons, group decision making, aggregation, multi-criteria decision making
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1. Introduction

Very often, real-world decision processes are nontrivial and involve a variety of criteria and alternatives. In these situations, decision makers and analysts can model and solve the problem by means of Multi-Criteria Decision Making (MCDM) methods such as Multi-Attribute Value Theory (MAVT) (Keeney and Raiffa, 1976) and the Analytic Hierarchy Process (AHP) (Saaty, 1977). Interestingly, one *trait d'union* among many methods is the pivotal role played by pairwise comparisons between criteria/alternatives. In fact, making comparisons over pairs of criteria allows to decompose the problem into smaller subproblems and thus makes the entire decision process cognitively simpler and

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more transparent. More precisely, the technique of pairwise comparisons has been widely used in decision support systems (DSSs) to express the intensities of preferences of decision makers on a set of criteria/alternatives. When expressed in their multiplicative form, pairwise comparisons can also be interpreted as rates of substitution between criteria (Choo et al., 1999).

To consider experts' opinions on a subject matter it is recommended that individual decision making be extended to *group decision making* (Kilgour and Eden, 2010). In this latter case, stakeholders, committees or groups of experts are involved in the decision making process and a plurality of opinions in the form of pairwise comparisons should be accounted for. Among others, Dyer and Forman (1992) pointed out that "the fundamental model of DSS—the lonely decision maker striding down the hall at high noon to make a decision—is true only in rare cases".

At this point, it is a common approach to average the pairwise comparisons provided by different decision makers to find a compromise solution. Such a solution can be used to make the final decision, but also as a starting point for future rounds of negotiations, as is for instance advocated by the Delphi method (Dalkey and Helmer, 1963). Either way, it is possible to envision that individual decision makers, knowing their preferences and the aggregated ones, be strategically interested in inferring (if possible) the preferences of the other participants to the group decision process and use this additional knowledge to their own advantage. Was this possible, then the widely accepted anonymity requirement would be, at least to some extent, violated.

The scope of this paper is to present and study some optimization problems whose optimal values yield upper and lower bounds for the judgments which can have been expressed by the decision makers. Since these optimization problems show the possibility of learning other decision makers' opinions, in its second part the paper presents the results of experiments—both numerical and empirical—showing to what extent a greater number of decision makers can "anonymize" the judgments of the participants in the group decision process.

The paper is organized as follows. Section 2 introduces the necessary formalism regarding preferences expresses as pairwise comparisons and their aggregation. Section 3 illustrates some optimization problems whose solutions determine upper and lower bounds for the opinions which could have been expressed by an arbitrary decision maker participating in the group decision. Section 4 formalizes some properties of the solutions to the above mentioned optimization problems. Section 5 argues in favor of the anonymity requirement and reports the results of some numerical and empirical tests. Section 6 changes slightly the setting of the problem and analyzes the case when the group weight vector is known, instead of the group pairwise comparison matrix. Section 7 shows how the results of this paper can be extended to other types of representations of valued preferences. Finally, the last section draws some conclusions and discusses possible ramifications.

2. Pairwise comparisons and group decision making

We consider a set of pairwise comparisons between the relative weights of the elements of a non-empty finite set $X = \{x_1, \ldots, x_n\}$. Elements $x_1, \ldots, x_n$ can be, for example, the criteria used to judge alternatives in MCDM problems. In this setting, the value of a given pairwise comparison $a_{ij} > 0$ is the subjective estimation of the ratio between the weights of x_i and x_j. This interpretation is in agreement with both MAVT with additive value function and the AHP. A pairwise comparison matrix is a mathematical structure used to collect and analyze the set of pairwise comparisons a_{ij} $\forall i, j$. More formally, a *pairwise comparison matrix* (PCM) is a positive square matrix $\mathbf{A} = (a_{ij})_{n \times n}$ such that $a_{ii} = 1 \, \forall i$ and $a_{ij} = 1/a_{ji} \, \forall i, j$. Hence,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{pmatrix}.$$

We need further formalism to represent the pairwise comparisons of a set of decision makers. To this purpose, given a finite non-empty set of m decision makers $\{d_1, \ldots, d_m\}$, we will use the notation

$$\mathbf{A}_k = \begin{pmatrix} a_{11k} & a_{12k} & \ldots & a_{1nk} \\ a_{21k} & a_{22k} & \ldots & a_{2nk} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1k} & a_{n2k} & \ldots & a_{nnk} \end{pmatrix} = \begin{pmatrix} 1 & a_{12k} & \cdots & a_{1nk} \\ \frac{1}{a_{12k}} & 1 & \cdots & a_{2nk} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1nk}} & \frac{1}{a_{2nk}} & \cdots & 1 \end{pmatrix}.$$

to denote the pairwise comparison matrix containing the judgments expressed by the kth decision maker, i.e. d_k. It has been shown (Aczél and Alsina, 1986; Aczél and Saaty, 1983) that the weighted geometric mean is the most suitable function to aggregate individual pairwise comparisons into single representative values. Namely, if we consider the individual pairwise comparison matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ provided by $d_1, \ldots, d_m$, then the matrix containing their aggregate judgments is $\mathbf{A}^\star = (a^\star_{ij})_{n \times n}$, where

$$a^\star_{ij} = \prod_{k=1}^{m} a_{ijk}^{\lambda_k} \tag{1}$$

with $\lambda_k > 0 \, \forall k$ and $\lambda_1 + \cdots + \lambda_m = 1$. It is common to interpret the weights $\lambda_1, \ldots, \lambda_m$ as the degrees of importance of the decision makers. The particular case $\lambda_1 = \cdots = \lambda_m = 1/m$ indicates that all the decision makers have the same importance. In this case, the previous formula can be simplified into

$$a^\star_{ij} = \left(\prod_{k=1}^{m} a_{ijk} \right)^{\frac{1}{m}}.$$

The use of the aggregation of individual judgments (AIJ) described so far has been especially advocated whenever the decision makers, albeit expressing subjective and therefore possibly discordant opinions, form a homogeneous group and share a common purpose (Forman and Peniwati, 1998; Ossadnik et al., 2016). Among other applications, the AIJ technique has been successfully employed to aggregate responses to questionnaires about technology adoption (Nikou and Mezei, 2013), stakeholders' preferences related to regional forest planning (Ananda and Herath, 2008), and experts' opinions on the best emergency treatment in the case of chemical pollution accident (Shi et al., 2014).

At this point, bearing in mind that anonymity of decision makers' judgments has often been considered a *desiderata* in social choice theory and at times even be proposed as an axiom (May, 1952), we can formulate an initial research question.

RQ: Can a decision maker d_k, with the knowledge of his own comparisons $\mathbf{A}_k$ and the aggregate $\mathbf{A}^\star$, infer the judgments given by another decision maker d_h?

A positive answer to the research question would pose serious limits to the attainability of the requirements of anonymity in group decision making when individual preferences are aggregated.

3. Optimization problems

In the rest of the paper, without loss of generality, we will assume that the decision maker d_1 wants to know what judgments were given by the decision maker d_m. Decision maker d_1 could try to deduce upper and lower bounds for the possible values of the entries of $\mathbf{A}_m$. Furthermore, as commonly done in many practical applications, we will also assume that the pairwise comparisons take values from the bounded positive interval $[l, u]$. From these premises, it follows that a lower bound of a_{ijm} can be found by solving the optimization problem

$$\begin{aligned} a^-_{ijm} = \underset{a_{ij2},\ldots,a_{ijm}}{\text{minimize}} \quad & a_{ijm} \\ \text{subject to} \quad & l \leq a_{ijk} \leq u, \quad k = 2, \ldots, m \\ & a_{ij1}^{\lambda_1} a_{ij2}^{\lambda_2} \cdots a_{ijm}^{\lambda_m} = a^\star_{ij} \,. \end{aligned} \tag{2}$$

The first set of constraints specifies that the values of the variables a_{ijk}'s must lie in the interval $[l, u]$. The last constraint stipulates the coherence between the individual judgments and their aggregate according to the geometric mean method (1). Indeed, a_{ij1}, $a^\star_{ij}$, $a^\star_{ij}$ and the λ_j's are parameters of the problem, because they are both known to d_1. Following the same line of reasoning we can find the upper bound of a_{ijm}. That is,

$$\begin{aligned} a^+_{ijm} = \underset{a_{ij2},\dots,a_{ijm}}{\text{maximize}} \quad & a_{ijm} \\ \text{subject to} \quad & l \le a_{ijk} \le u, \quad k = 2,\dots,m \\ & a^{\lambda_1}_{ij1} a^{\lambda_2}_{ij2} \cdots a^{\lambda_m}_{ijm} = a^\star_{ij} \,. \end{aligned} \tag{3}$$

In the present form, the optimization problems (2) and (3) are nonconvex due to the nonlinearity of the equality constraint[2]. Nevertheless, they can be linearized. First, if we consider (2), we need to apply the transformation $a_{ijk} \mapsto e^{b_{ijk}} \quad \forall i, j$ and thus change variables,

$$\begin{aligned} b^-_{ijm} = \underset{b_{ij2},\dots,b_{ijm}}{\text{minimize}} \quad & e^{b_{ijm}} \\ \text{subject to} \quad & l \le e^{b_{ijk}} \le u, \quad k = 2,\dots,m \\ & e^{\lambda_1 b_{ij1} + \cdots + \lambda_m b_{ijm}} = a^\star_{ij} \,. \end{aligned}$$

Next, by applying the logarithmic function to the different parts of this last optimization problem we recover

$$\begin{aligned} b^-_{ijm} = \underset{b_{ij2},\dots,b_{ijm}}{\text{minimize}} \quad & b_{ijm} \\ \text{subject to} \quad & \ln l \le b_{ijk} \le \ln u, \quad k = 2,\dots,m \\ & \lambda_1 b_{ij1} + \cdots + \lambda_m b_{ijm} = \ln a^\star_{ij} \,. \end{aligned} \tag{4}$$

This last optimization model is a linear optimization problem and therefore it can be solved straightforwardly by linear programming. Then, it is sufficient to solve it and recover the optimal value of (2) by applying the inverse transformation $a^-_{ijk} = e^{b^-_{ijk}}$. Similarly, we can obtain the optimal value of (3) by solving

$$\begin{aligned} b^+_{ijm} = \underset{b_{ij2},\dots,b_{ijm}}{\text{maximize}} \quad & b_{ijm} \\ \text{subject to} \quad & \ln l \le b_{ijk} \le \ln u, \quad k = 2,\dots,m \\ & \lambda_1 b_{ij1} + \cdots + \lambda_m b_{ijm} = \ln a^\star_{ij} \,. \end{aligned} \tag{5}$$

and applying the inverse transformation $a^+_{ijk} = e^{b^+_{ijk}}$.

The so obtained information can then be collected in a matrix whose entries are intervals according to the next definition.

Definition 1 (Induced matrix)**.** *We say that* $\bar{\mathbf{A}}_m = (\bar{a}_{ijm})_{n\times n}$ *is* induced *(by* $\mathbf{A}_1$ *and* $\mathbf{A}^\star$*) if* $\bar{a}_{iim} = [1, 1]\ \forall i$ *and* $\bar{a}_{ijm} = [a^-_{ijm}, a^+_{ijm}]\ \forall i \neq j$*.*

The induced matrix $\bar{\mathbf{A}}_m$ has a clear interpretation: when d_1 knows $\mathbf{A}_1$ and $\mathbf{A}^\star$, the entries of $\bar{\mathbf{A}}_m$ are the ranges containing the values of the pairwise

[2]More formally, optimization problems (2) and (3) are instances of signomial programming which, in turn, is a generalization of geometric programming (Boyd et al., 2007).

comparisons which could have been expressed by d_m. Moreover, if we know that a *discrete* scale $S \subset [l, u]$ was used, then we can further restrict the intervals and obtain a matrix $\tilde{\mathbf{A}}_m = [\tilde{a}^-_{ijm}, \tilde{a}^+_{ijm}]_{n\times n}$ where

$$\tilde{a}^-_{ijm} = \min\{x \in S | x \geq a^-_{ijm}\} \tag{6}$$

$$\tilde{a}^+_{ijm} = \max\{x \in S | x \leq a^+_{ijm}\} \tag{7}$$

3.1. Illustrative example

Let us consider the four PCMs already used by **?**.

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 4 & 6 & 7 \\ 1/4 & 1 & 3 & 4 \\ 1/6 & 1/3 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 5 & 7 & 9 \\ 1/5 & 1 & 4 & 6 \\ 1/7 & 1/4 & 1 & 2 \\ 1/9 & 1/6 & 1/2 & 1 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} 1 & 3 & 5 & 8 \\ 1/3 & 1 & 4 & 5 \\ 1/5 & 1/4 & 1 & 2 \\ 1/8 & 1/5 & 1/2 & 1 \end{pmatrix} \qquad \mathbf{A}_4 = \begin{pmatrix} 1 & 4 & 5 & 6 \\ 1/4 & 1 & 3 & 3 \\ 1/5 & 1/3 & 1 & 2 \\ 1/6 & 1/3 & 1/2 & 1 \end{pmatrix}$$

? assumed that the entries are expressed on the scale $[1/9, 9]$, that is $l = 1/9$ and $u = 9$, and that the decision makers are equally important, e.g. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$, so that their aggregation through (1) produces the following pairwise comparison matrix

$$\mathbf{A}^\star \approx \begin{pmatrix} 1 & 3.93598 & 5.69243 & 7.41559 \\ 0.254066 & 1 & 3.4641 & 4.35588 \\ 0.175672 & 0.288675 & 1 & 2 \\ 0.134851 & 0.229575 & 0.5 & 1 \end{pmatrix}.$$

Next, by solving the optimization problems (2) and (3) $\forall i \neq j$ and collecting the results we obtain the following induced matrix (Def. 1).

$$\bar{\mathbf{A}}_4 = ([a^-_{ij4}, a^+_{ij4}])_{4\times 4} \approx \begin{pmatrix} [1,1] & [0.7407, 9] & [2.1605, 9] & [5.333, 9] \\ [1/9, 1.35] & [1,1] & [0.5926, 9] & [10/9, 9] \\ [1/9, 0.4629] & [1/9, 1.6875] & [1,1] & [1/9, 9] \\ [1/9, 0.1875] & [1/9, 0.9] & [1/9, 9] & [1,1] \end{pmatrix} \tag{8}$$

It can be remarked that $\bar{\mathbf{A}}_4$ contains d_1's interval estimations of the preferences given by d_4 under the circumstance that d_1 knows only his own preferences $\mathbf{A}_1$ and the group ones $\mathbf{A}^\star$. To give a concrete numerical instance, the value

$a_{124}^{-} \approx 0.7407$ is the optimal solution of the optimization problem

$$
\begin{aligned}
a_{124}^{-} = \underset{a_{122},a_{123},a_{124}}{\text{minimize}} \quad & a_{124} \\
\text{subject to} \quad & 1/9 \leq a_{122} \leq 9, \\
& 1/9 \leq a_{123} \leq 9, \\
& 1/9 \leq a_{124} \leq 9, \\
& (4a_{122}a_{123}a_{124})^{\frac{1}{4}} = 3.93598\,.
\end{aligned}
$$

Additionally, if we consider the discrete scale $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\}$, we can use formulas (6) and (7) to further restrict the intervals and recover

$$
\tilde{\mathbf{A}}_4 = \begin{pmatrix} [1,1] & [1,9] & [3,9] & [6,9] \\ [1/9,1] & [1,1] & [1,9] & [2,9] \\ [1/9,1/3] & [1/9,1] & [1,1] & [1/9,9] \\ [1/9,1/6] & [1/9,1/2] & [1/9,9] & [1,1] \end{pmatrix}
$$

from which, d_1 infers that $a_{144} \in \{6, 7, 8, 9\}$. Namely, d_1, knows that d_4 must have judged the weight of x_1 between 6 and 9 times greater than the one of x_4.

4. Some theoretical results

The structure of the induced matrix (8) suggests a connection with interval pairwise comparison matrices, which have been widely studied in the literature (Li et al., 2016) and used to solve real-world decision problems.

Definition 2 (Interval pairwise comparison matrix (Salo and Hämäläinen, 1995))**.** *An interval pairwise comparison matrix is a square matrix* $\bar{\mathbf{A}} = ([a_{ij}^{-}, a_{ij}^{+}])_{n\times n}$ *with entries being positive intervals and* $a_{ji}^{-} = 1/a_{ij}^{+}$ *for all* i, j.

Proposition 1. *If* $l = 1/u$, *then the induced matrix* $\bar{\mathbf{A}}_m = ([a_{ijm}^{-}, a_{ijm}^{+}])_{n\times n}$ *is an interval pairwise comparison matrix.*

Proof. We know that the entries of $\bar{\mathbf{A}}$ are positive intervals, so it remains to prove that $a_{ijm}^{-} = 1/a_{jim}^{+}$ holds for all i, j. This is true, by definition, for $i = j$. For the case $i \neq j$, we can consider

$$
\begin{aligned}
b_{jim}^{+} = \underset{b_{ji2},\ldots,b_{jim}}{\text{maximize}} \quad & b_{jim} \\
\text{subject to} \quad & \ln l \leq b_{jik} \leq \ln u, \quad k = 2, \ldots, m \\
& \lambda_1 b_{ji1} + \cdots + \lambda_m b_{jim} = \ln a_{ji}^{\star}\,.
\end{aligned} \tag{9}
$$

and show that its optimum is the same of (4) but with opposite sign. This would correspond, after proper transformation, to $a_{ijm}^{-} = 1/a_{jim}^{+}$. We can consider (5) and apply the minus sign to both sides of the equality constraint and by reckoning that, since $l = 1/u$, we have $\ln l = -\ln u$, in the inequality constraint we can replace b_{ijm} with $-b_{ijm}$. In addition, we know that reciprocity of the individual pairwise comparisons implies $a^\star_{ji} = 1/a^\star_{ij}$. Finally, since we know that $\max f = -\min(-f)$, we can reach an equivalent formulation of the previous optimization problem

$$\begin{aligned} b^+_{jim} = -\underset{b_{ji2},\dots,b_{jim}}{\text{minimize}} \quad & -b_{jim} \\ \text{subject to} \quad & \ln l \le -b_{jik} \le \ln u, \quad k = 2,\dots,m \\ & -\lambda_1 b_{ji1} - \dots - \lambda_m b_{jim} = \ln a^\star_{ij} \,. \end{aligned}$$

Now it is sufficient to use reciprocity and operate the substitution $-b_{jim} \mapsto b_{ijm}$ to obtain $b^+_{jim} = -b^-_{ijm}$ and thus $a^+_{jim} = 1/a^-_{ijm}$. □

Remark 1. *If $l = 1/u$ and the discrete scale is symmetric in a multiplicative sense, i.e. $S = \{1/s, 1/(s-1), \dots, 1, \dots, (s-1), s\}$, then also the matrix $\tilde{\mathbf{A}}_m$ obtained by applying* (6) *and* (7) *to the induced matrix $\bar{\mathbf{A}}_m$ is an interval PCM.*

Since the entries on the diagonal of $\bar{\mathbf{A}}_m$ are, by definition, $[1, 1]$, Proposition 1 and Remark 1 imply that, under extremely weak assumptions, it is sufficient to solve $n(n-1)$ linear programs to determine the induced matrix $\bar{\mathbf{A}}_m$ and its restriction $\tilde{\mathbf{A}}_m$ in the case of a discrete scale. Furthermore, we can determine some conditions under which the matrix $\bar{\mathbf{A}}_m = ([a^-_{ijm}, a^+_{ijm}])_{n\times n}$ collapses into a real valued PCM, thus leaving no uncertainty on the preferences of the other decision makers. This happens when there are only two decision makers ($m = 2$) or when the preferences of the decision makers are polarized and equal for every pair of alternatives (i, j).

Proposition 2. *If $m = 2$ or $a^\star_{ij} \in \{l, u\}$, then $a^-_{ijm} = a^+_{ijm}$ and therefore $\bar{a}_{ijm} = [a^-_{ijm}, a^+_{ijm}]$ collapses into a singleton.*

Proof. If $m = 2$ it is possible to determine the exact value of a_{ijm}. Hence, $a^-_{ijm} = a^+_{ijm}$. For the second condition, we consider an arbitrary pair (i, j), if $a^\star_{ij} = u$ then because $a_{ijk} \le u$ for all k and the idempotency of the geometric mean, we know that $a_{ij1} = \dots = a_{ijm} = u$. Hence, $a^-_{ijm} = a^+_{ijm} = u$. A similar line of reasoning holds if $a^\star_{ij} = l$. □

It is possible to formalize further limits to the possibility of inferring other decision makers judgments.

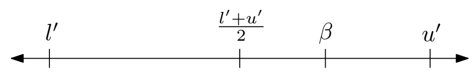
Proposition 3. *For $m \ge 3$, if $\lambda_2 + \dots + \lambda_{m-1} \ge \lambda_m$, then at least one between $a^-_{ijk} = l$ and $a^+_{ijk} = u$ holds.*

Proof. First of all, we can split the general case into two subcases: $a^\star_{ij} \in \{l, u\}$ and $a^\star_{ij} \notin \{l, u\}$. If $a^\star_{ij} \in \{l, u\}$ then either $a^-_{ijm} = a^+_{ijm} = l$ or $a^-_{ijm} = a^+_{ijm} = u$. Conversely, if $= a^\star_{ij} \notin \{l, u\}$, we can consider $a^{\lambda_1}_{ij1} \cdots a^{\lambda_m}_{ijm} = a^\star_{ij}$. With $a^\star_{ij}/a^{\lambda_1}_{ij1} = \alpha$ we can write $a^{\lambda_2}_{ij2} \cdots a^{\lambda_m}_{ijm} = \alpha$. For simplicity, we can use the logarithmic function to linearize it and so, by setting $b_{ijk} = \ln a_{ijk}$, $\beta = \ln \alpha$,

$l' = \ln l$ and $u' = \ln u$, it boils down to analyze the solutions of

$$\begin{cases} \lambda_2 b_{ij2} + \cdots + \lambda_m b_{ijm} = \beta \\ l' \leq b_{ijk} \leq u' \quad \forall k = 2, \ldots, m \end{cases}$$

Now we shall consider two cases:
$\beta \geq (u'+l')/2$): In this case, since $\lambda_2 + \cdots + \lambda_{m-1} \geq \lambda_m$, when $b_{ij2}, \ldots, b_{i\,j\,m-1}$ tend to l', b_{ijm} grows and reaches u' before $b_{ij2}, \ldots, b_{i\,j\,m-1}$ can attain the value l'. Thus, $b^+_{ijm} = u'$ and $a^+_{ijm} = u$. The initial condition $\beta \geq (u'+l')/2$ can be sketched



$\beta \leq (u'+l')/2$): In this case, since $\lambda_2 + \cdots + \lambda_{m-1} \geq \lambda_m$, when $b_{ij2}, \ldots, b_{i\,j\,m-1}$ tend to u', b_{ijm} decreases and reaches l' before $b_{ij2}, \ldots, b_{i\,j\,m-1}$ can attain the value u'. Thus, $b^-_{ijm} = l'$ and $a^-_{ijm} = l$. □

For $m \geq 3$, $\lambda_1 = \cdots = \lambda_m = 1/m$ implies $\lambda_1 + \cdots + \lambda_{m-1} \geq \lambda_m$. Consequently, the case of equally important decision makers is a special case of the previous proposition.

5. Experimental results

Very often, group DSS call for rounds of negotiations and confrontations, where intermediate results are shown to the decision makers. Let us for instance consider the Delphi method, often used together with questionnaires in the form of pairwise comparisons (Khorramshahgol and Moustakis, 1988; Vidal et al., 2011). In its traditional form, the Delphi method is structured as a series of panel discussions led by a discussion leader and based on anonymous questionnaires filled by the experts/panelists. In each round of discussion the panelists discuss—the discussion is guided by showing them the aggregate of their opinions—and in the next round they can revise their judgments. The process iterates until a sufficient level of consensus, or a maximum number of iterations, has been reached. Showing intermediate results is not an exclusive characteristic of Delphi and it is shared by many other methods; Islei and Lockett (1991) argued that the practice of group decision making has shown that a group DSS should allow for frequent feedback and consequent preference development.

In this context anonymity plays a central role. According to Rauch (1979), anonymity serves (i) to avoid the influence of personal reputation on the acceptance of ones opinions, (ii) to make it possible to take extreme views and to protect the panelists and (iii) to help attain objectivity and emotive neutrality. Similarly, Dalkey et al. (1969) considered anonymity "a way of reducing the effect of dominant individuals", and in MCDM methods, anonymity allows participants to be more frank with their opinions (Bose et al., 1997). Given its importance, and the results obtained in the previous section, we may be concerned that, in cases with a small number of decision makers, the requirement of anonymity be compromised. This, in turn, could jeopardize the fairness of the decision process.

5.1. Numerical results

To investigate the extent of the violation of anonymity, this subsection presents the results of a *numerical* study. We randomly generated 1000 sets of m PCMs for an increasing number of decision makers $m \in \{3, \ldots, 10\}$ and entries from the scale $\{1/9, \ldots, 9\}$. Fixing an m, for each one of the 1000 sets of PCMs we used the optimization problems (2)–(3) to find the induced pairwise comparison matrix $\bar{\mathbf{A}}_m$. At this point, we measured how indeterminate each one of the so generated 1000 induced PCMs was. A measure of indeterminacy is a quantification of the amplitude of the interval-valued entries of positive interval-valued matrices and therefore can be used as a proxy measure of the degree of anonymity of preferences. To this aim, and to recover an index of global indeterminacy, we used the formula

$$I(\bar{\mathbf{A}}) = \prod_{i<j} \left(\frac{a_{ij}^{+}}{a_{ij}^{-}} \right)^{\frac{2}{n(n-1)}}$$

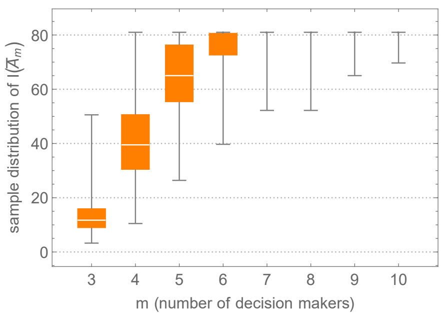
which was already used by Li et al. (2016) to measure the indeterminacy level of interval PCMs. Indeed, the greater the indeterminacy, the more anonymous the judgments are. Let us note that, if we consider a discrete scale $\{1/s, \ldots, 1, \ldots s\}$, we have $I(\bar{\mathbf{A}}) \in [1, s^2]$. In particular $I(\bar{\mathbf{A}}) = 1$ indicates that all the intervals in the matrix collapsed into real values, whereas $I(\bar{\mathbf{A}}) = s^2$ indicates that all the non-diagonal entries are as wide as possible, i.e. $[1/s, s]$. For the very well-known case of the $\{1/9, \ldots, 9\}$ scale, $I(\bar{\mathbf{A}}) \in [1, 81]$.

Figure 1 reports box plots synthesizing the distributions of indeterminacy levels. To study the effects of using a different scale the same figure also reports the results of the experiments when entries are sampled from scales $\{1/s, \ldots, s\}$ with $s = 3, 5, 7$.

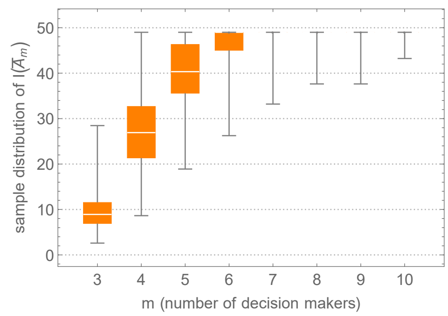
It appears that, although there is always the possibility of disclosing opinions, already with more than seven decision makers the probability and the extent of this event is drastically reduced. By comparing Figures 1a–1d, we note that, using reduced scales has only a slight effect on the degree of anonymity of individual pairwise comparisons.

5.2. Empirical results

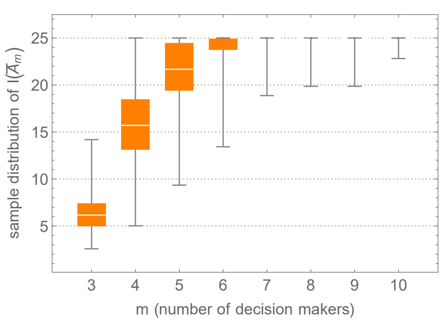
It is reasonable to question the results obtained from randomly generated PCMs, since it often happens that, in group decision making, preferences are far from being random and could, instead, contain patterns and regularities. For example, randomly generated matrices do not account for the case of two criteria such that one dominates the other so much that the large majority of the decision makers prefer the former over the latter. This section presents an



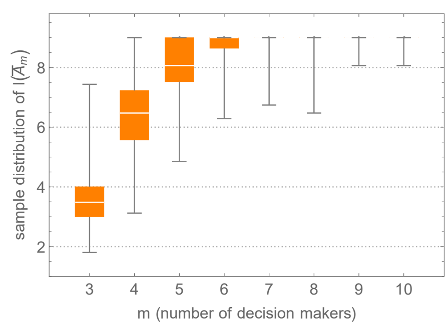
(a) Indeterminacy analysis for random PCMs with scale $\{1/9, \ldots, 9\}$.



(b) Indeterminacy analysis for random PCMs with scale $\{1/7, \ldots, 7\}$.

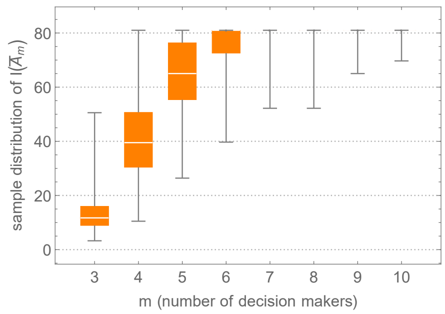


(c) Indeterminacy analysis for random PCMs with scale $\{1/5, \ldots, 5\}$.

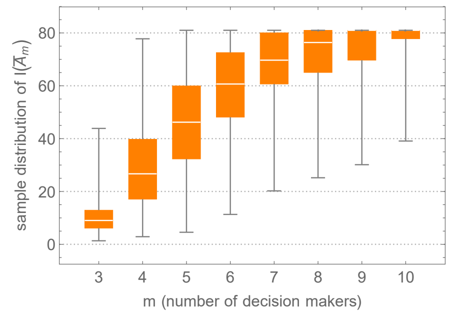


(d) Indeterminacy analysis for random PCMs with scale $\{1/3, \ldots, 3\}$.

Figure 1: Indeterminacy analysis of induced matrices from randomly generated PCMs on different scales.



(a) Indeterminacy analysis for *random* PCMs with scale $\{1/9, \ldots, 1, \ldots, 9\}$.



(b) Indeterminacy analysis for *empirical* PCMs with scale $\{1/9, \ldots, 1, \ldots, 9\}$.

Figure 2: Indeterminacy analysis of induced matrices from different types of judgments.

empirical analysis of some real-world PCMs. Particularly, it refers to the PCMs collected by Nikou and Mezei (2013) to study the factors determining the adoption of mobile services. In their survey, Nikou and Mezei collected 66 PCMs comparing five criteria (Communication, Entertainment, Information, Web 2.0, Transaction) on a scale $\{1/9, \ldots, 9\}$. To study the effect of an increasing number of decision makers on real-world preferences we sampled 1000 subsets of cardinality m from the set of 66 PCMs collected by Nikou and Mezei, and iterate the process for $m = 3, \ldots, 10$. Thereafter, the analysis proceeds similarly to the one presented in the previous section.

Figure 2 compares the results obtained from empirical data with randomly generated ones making it visible that, at least when compared to these empirical data, randomly generated matrices tend to underestimate the extent of the violation of the anonymity principle.

Since Figures 1 and 2 only offered snapshots, more details are reported in Table 1. In particular, Table 1 reports the frequencies with which the induced matrix was fully undetermined, thus granting full anonymity to the decision maker. In light of these results it seems that, using reduced scales slightly favors anonymity. However, this phenomenon does not seem significant. More important is the difference between the last two rows, which signals that, most likely, the numerical results on random matrices are underestimating the loss of anonymity in real-world contexts.

6. Induced matrix from the group weight vector

The last stage of decision making with pairwise comparisons if very often the elicitation of a weight vector. In the case of group decision making, this adds one more stage to the entire process: the elicitation of the weight vector $\mathbf{w}^\star = (w_1^\star, \ldots, w_n^\star)$ from $\mathbf{A}^\star$. A common procedure is to use the geometric mean

Data	Scale	Number of decision makers (m)							
		3	4	5	6	7	8	9	10
random	3	0	.021	.293	.669	**.904**	**.962**	**.991**	**.998**
random	5	0	.006	.202	.634	.859	**.961**	**.988**	**.998**
random	7	0	.007	.182	.579	.835	**.956**	**.990**	**.994**
random	9	0	.006	.173	.556	.850	**.943**	**.985**	**.996**
empirical	9	0	0	.024	.108	.241	.401	.514	.673

Table 1: Percentage of fully undetermined induced matrices. Values greater than 0.9 are highlighted in boldface.

method, so that

$$w_i^\star = \left(\prod_{j=1}^{n} a_{ij}^\star \right)^{\frac{1}{n}} \qquad i = 1, \ldots, n\,. \tag{10}$$

and the entire aggregation be summarized as follows

$$\mathbf{A}_1, \ldots, \mathbf{A}_m \xrightarrow{(1)} \mathbf{A}^\star \xrightarrow{(10)} \mathbf{w}^\star \tag{11}$$

It is possible to envision that, to facilitate the discussion, during the decision making activity individual decision makers could be informed about $\mathbf{w}^\star$ instead of $\mathbf{A}^\star$. Since weight vector $\mathbf{w}^\star$ is a synthesis of the preferences contained in $\mathbf{A}^\star$, it is reasonable to suppose that the knowledge of $\mathbf{w}^\star$ yields less information on the opinions of the different decision makers than the knowledge of $\mathbf{A}^\star$. To investigate this possibility it is necessary to devise optimization problems which can estimate upper bounds for the pairwise comparisons expressed by d_m as functions of the entries of $\mathbf{A}_1$ and the components of $\mathbf{w}^\star$.

$$\begin{aligned} a_{ijm}^- = \text{minimize} \quad & a_{ijm} \\ \text{subject to} \quad & l \le a_{ijk} \le u, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\ & a_{ijk} = 1/a_{jik} \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\ & \prod_{j=1}^{n} \left(\prod_{k=1}^{m} a_{ijk}^{\lambda_k} \right)^{\frac{1}{n}} = w_i^\star \quad i = 1,\ldots,n \end{aligned} \tag{12}$$

$$\begin{aligned} a_{ijm}^+ = \text{maximize} \quad & a_{ijm} \\ \text{subject to} \quad & l \le a_{ijk} \le u, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\ & a_{ijk} = 1/a_{jik} \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\ & \prod_{j=1}^{n} \left(\prod_{k=1}^{m} a_{ijk}^{\lambda_k} \right)^{\frac{1}{n}} = w_i^\star \quad i = 1,\ldots,n \end{aligned} \tag{13}$$

In each of the previous two optimization problems, the first two sets of constraints ensure that pairwise comparisons expressed by $d_2, \ldots, d_m$ be reciprocal and within the interval $[l, u]$. The last set of constraints stipulates the coherence of the judgments with the group weight vector. Using the logarithmic transformation, they can be reformulated as linear programs and easily solved,

$$
\begin{aligned}
b^-_{ijm} = \text{minimize} \quad & b_{ijm} \\
\text{subject to} \quad & \ln l \le b_{ijk} \le \ln u, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\
& b_{ijk} = -b_{jik}, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\
& \frac{1}{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\lambda_k b_{ijk}\right) = \ln w^\star_i \quad i = 1,\ldots,n.
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
b^+_{ijm} = \text{maximize} \quad & b_{ijm} \\
\text{subject to} \quad & \ln l \le b_{ijk} \le \ln u, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\
& b_{ijk} = -b_{jik}, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\
& \frac{1}{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\lambda_k b_{ijk}\right) = \ln w^\star_i \quad i = 1,\ldots,n.
\end{aligned} \tag{15}
$$

We can formulate a proposition similar to Proposition 1.

Proposition 4. *If $l = 1/u$, then the induced matrix $\bar{\mathbf{A}}_m = ([a^-_{ijm}, a^+_{ijm}])_{n\times n}$ (from $\mathbf{A}_1$ and $\mathbf{w}^\star$) by solving the optimization problems* (12) *and* (13) *is an interval pairwise comparison matrix.*

Proof. Similarly to the proof of 1 we intend to show that $b^+_{jik} = -b^-_{ijk}$. Hence we start considering

$$
\begin{aligned}
b^+_{jim} = \text{maximize} \quad & b_{jim} \\
\text{subject to} \quad & \ln l \le b_{jik} \le \ln u, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\
& b_{jik} = -b_{ijk}, \quad i,j = 1,\ldots,n,\ k = 2,\ldots,m \\
& \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{m}\lambda_k b_{jik}\right) = \ln w^\star_j \quad j = 1,\ldots,n.
\end{aligned}
$$

Bearing in mind that $\max f = -\min(-f)$ and $\ln l = -\ln u$ we can modify the objective function and the first set of constraint. Furthermore, also the other sets of constraints can be modified by inverting the indices. It results in

$$
\begin{aligned}
b^+_{jim} = -\text{minimize} \quad & -b_{jim} \\
\text{subject to} \quad & \ln l \leq b_{ijk} \leq \ln u, \quad i,j = 1,\dots,n,\ k = 2,\dots,m \\
& b_{ijk} = -b_{jik}, \quad i,j = 1,\dots,n,\ k = 2,\dots,m \\
& \frac{1}{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\lambda_k b_{ijk}\right) = \ln w^\star_i \quad i = 1,\dots,n.
\end{aligned}
$$

Now by substituting $-b_{jim} \mapsto b_{ijm}$ in the objective function we obtain the optimization problem (14), leading to the conclusion that $a^+_{jik} = 1/a^-_{ijk}$ $\square$

Using these linear optimization problems we can run simulations, similar to the previous ones, to estimate the degree of indeterminacy of the matrix $\bar{\mathbf{A}}_m$ induced by $\mathbf{A}_m$ and $\mathbf{w}^\star$ instead of from $\mathbf{A}_m$ and $\mathbf{A}^\star$.

For sake of brevity results of these simulations are not reported here, since they simply indicate that, except for the special case of three decision makers and three alternatives ($n, m = 3$), the induced matrix is almost always fully undetermined and thus the anonymity of the decision maker is preserved.

7. Other representations of preferences

Pairwise comparison matrices are not the only framework for expressing real-valued preference. It is safe to say that, the other foremost representation of preferences is represented by *reciprocal preference relations*. Unlike a PCM, a reciprocal preference relation $\mathbf{R} = (r_{ij})_{n\times n}$ allows the decision maker to express his preferences in the unit interval $]0,1[$ (sometimes $[0,1]$), with indifference represented by the value 0.5 and reciprocity $r_{ij} + r_{ji} = 0.5$. In the literature, reciprocal preference relations have often been called fuzzy preference relations (Herrera-Viedma et al., 2004; Kacprzyk, 1986; Tanino, 1984).

Reciprocal preference relations are related to pairwise comparison matrices by means of the isomorphism $r_{ij} = g(a_{ij}) = \frac{a_{ij}}{1+a_{ij}}$ and its inverse $a_{ij} = g^{-1}(r_{ij}) = \frac{r_{ij}}{r_{ij}}$ (Xu and Da, 2003). In addition, when the structure of a PCM is mapped into a reciprocal relation by means of the isomorphism g, we obtain that, in $]0,1[$, the so called three-Π-uninorm (Grabisch et al., 2009),

$$
r^\star_{ij} = \frac{\left(\prod_{k=1}^{m} r_{ijk}\right)^{\frac{1}{m}}}{\left(\prod_{k=1}^{m} r_{ijk}\right)^{\frac{1}{m}} + \left(\prod_{k=1}^{m}(1-r_{ijk})\right)^{\frac{1}{m}}}
$$

is the equivalent of the geometric mean.

Clearly, any equality constraint in the form of this equation makes an optimization problem nonconvex. However, it is possible to use the isomorphism

g^{-1} to map a reciprocal preference relation into a pairwise comparison matrix and then, similarly to the optimization problems (2) and (3), linearize the optimization problems to find upper and lower bounds also for the case of reciprocal preference relations too. Therefore, although the previous sections referred to pairwise comparison matrices, the results should not be interpreted in a restrictive sense as they can be straightforwardly generalized to other preference structures, as long as there exists an isomorphism between them. One further example is the additive representation of preferences which was indirectly obtained by linearizing, by means of the logarithmic function, the optimization problems. In fact an additive preference relation is a preference relation where entries are expressed as real numbers and reciprocity is expressed as $b_{ij}+b_{ji}=0$.

8. Discussion and conclusions

According to recent literature, there is still a lack of studies on the optimal number of judges in group decision making (Saaty and Özdemir, 2014). Although the answer to this question is certainly context-dependent, this study hopes to have shed some light on this issue by discussing and raising awareness on the possibility of indirectly violating the requirement of anonymity. Specifically, it occurs that, although pairwise comparisons are given in anonymous form, a decision maker can partially discover the preferences of another one.

To this scope we devised some optimization problems which can restrict the range of possible values given by decision makers and analyzed how, by increasing the number of decision makers one can also increase their degree of anonymity. Although each case is different and a final word cannot be said, empirical experiments with real-world pairwise comparison matrices revealed that already with more than eight decision makers the degree of anonymity of the decision makers' preferences is quite high. Hence, from the perspective of anonymity, the more decision makers the better. However, we cannot claim this in general. In fact, in some contexts, it was hypothesized that a too large number of decision makers does not provide sufficient individual incentives to accurately ponder the decision problem (Mukhopadhaya, 2003). Future research could use additional empirical data to verify the robustness of the results obtained in this paper.

Another issue emerging from this analysis is the discrepancy, presented in Figure 2, between the results obtained with random matrices and those obtained with empirical matrices. Such discrepancy shows that not all the facets of pairwise comparison matrices can be fruitfully analyzed by means of randomly generated matrices. This suggests increasing the use of empirical preferences in the study of preference relations, as done by Bozóki et al. (2013).

Finally, we can sketch a possible connection between group decision making with pairwise comparisons and game theory, whose full development would go beyond the scope of this paper, but might inspire extensions of this research. The basic idea is to assume that a decision maker attaches a value to both (i) influencing the final outcome of the decision process and (ii) maintaining his judgments anonymous. In other words, the utility function of a decision maker may depend on both factors. If we exclude the case with two decision makers ($m = 2$), and we focus on larger groups, we can see that the degree to which a decision maker loses anonymity depends on his pairwise comparisons as well as to the pairwise comparisons of the other decision makers. Hence, in some situations, he might not be willing to express his real preferences in order to retain some of his anonymity. This, in turn, depends on his expectations on the other players stated preferences, and therefore it appears to be a game theoretic problem.

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