Boundary properties of the inconsistency of pairwise comparisons in group decisions

Matteo Brunelli

Department of Mathematics and Systems Analysis

Aalto University, P.O. box 11100, 00076 Aalto, Finland

e-mail: `matteo.brunelli@aalto.fi`

Michele Fedrizzi

Department of Industrial Engineering

University of Trento, Via Sommarive 9, I-38123 Trento, Italy

e-mail: `michele.fedrizzi@unitn.it`

June 9, 2018

Abstract

This paper proposes an analysis of the effects of consensus and preference aggregation on the consistency of pairwise comparisons. We define some boundary properties for the inconsistency of group preferences and investigate their relation with different inconsistency indices. Some results are presented on more general dependencies between properties of inconsistency indices and the satisfaction of boundary properties. In the end, given three boundary properties and nine indices among the most relevant ones, we will be able to present a complete analysis of what indices satisfy what properties and offer a reflection on the interpretation of the inconsistency of group preferences.

Keywords: Pairwise comparison matrix, inconsistency indices, boundary properties, group decision making, analytic hierarchy process.

1 Introduction

In a wide range of decision making problems, it occurs that an expert, or a group of experts, is asked to rate some alternatives. Selecting the best alternative is trivial when

the number of considered alternatives is very small, but complexity arises as the number of alternatives, and criteria with respect to which alternatives are judged, grows. Techniques based on pairwise comparisons allow the expert to discriminate between two alternatives at a time, thus decomposing the problem into more simple and easily tractable sub-problems.

The Analytic Hierarchy Process (AHP) by Saaty (1977, 1980) is probably the best known among all the methods using pairwise comparisons. In a recent survey by Ishizaka and Labib (2011) on the latest developments of the AHP, consistency of preferences and group decisions have been considered hot topics, and the possibility of estimating inconsistency been regarded as a valuable asset for techniques adopting pairwise comparison matrices.

Consistency has been widely regarded as a desirable, yet hardly ever achievable, property of preferences in decision making problems. Following the thesis of Irwin (1958), which links the concepts of preference and discrimination, being consistent in expressing preferences means being rational in discriminating between alternatives. Although one might argue that consistency does not necessarily imply expertise of the decision maker (consistent preferences could possibly be obtained randomly), it is undebatable that a good expert should always be able to state his preferences in a non-contradictory way. Hence, although consistency alone does not guarantee the expertise of a decision maker, the existence of inconsistencies should be symptomatic of the decision maker's scarce preparation or lack of knowledge of the problem. Going back in time, even in the fundamental contribution by Savage (1972) consistency of preferences was regarded as a *desideratum.* More recently, Gass (2005) recalled that also Luce and Raiffa (1957) and Fishburn (1991) regarded transitivity of preferences, and consequently their consistency, as an auspicable, but not necessary, condition for the preferences of a decision maker.

Note that the use of the concept of consistency has not been limited to the quantification of inconsistency. For example, just to cite the most recent results, it has been used to improve consistency of preferences in the framework of a model based on Hadamard product between matrices (Kou et al., 2014), to detect the most inconsistent comparisons (Ergu et al., 2011), and to derive the priority vector by means of a geometric similarity measure (Kou and Lin, 2014). On the other hand, also the concept of group decisions with pairwise comparisons has been studied; for example Altuzarra et al. (2010) proposed a statistical model for consensus in the AHP including assumptions on group consistency, and Bernasconi et al. (2014) studied, from the empirical point of view, different aggregation methods for preferences.

Consistency has been a widely studied research topic in decision sciences, but, in spite of the growing effort in studying group decisions, most of the research on consistency of preference relations has focused on the reliable assessment of the degree of inconsistency of single pairwise comparison matrices. Only few, and more recent, studies (Escobar et al., 2004; Grošelj and Zadnik Stirn, 2012; Lin et al., 2008; Liu et al., 2012) have tried to extend the issue of inconsistency quantification to the case with multiple decision makers by examining single measures of inconsistency but never proposing a more general reasoning on the matter. Furthermore, very few studies have in-

vestigated the connection between consensus and consistency. In a qualitative study, Weiss and Shanteau (2004) highlighted how consensus alone does not necessarily lead to better decisions and, instead, emphasized the fundamental role of the expertise of decision makers, which they called consistency.

In this paper we will provide further results on the connection between group decisions and inconsistency, in particular on how the former affects the latter. More specifically we shall define some general boundary properties for the inconsistency of a group of decision makers and see whether different inconsistency indices satisfy them or not. In doing so, we shall be able to derive and use some more general results starting from some axiomatic properties of inconsistency indices (Brunelli and Fedrizzi, 2013).

The paper is outlined as follows. In Section 2 we recall the definitions of pairwise comparison matrices and inconsistency, and we summarize the axioms which were proposed to characterize inconsistency indices. In Section 3 we define the boundary properties and, within the same section, in Subsections 3.1 and 3.2 we study the satisfaction of lower and upper boundary properties. In Section 4 we discuss the implications of the results, and in Section 5 we draw the conclusions.

2 Pairwise comparison matrices and inconsistency indices

The intensity of pairwise preferences of a decision maker can be represented on bipolar scales. The approach proposed by Fishburn (1991) based on skew symmetric additive preferences considers the opinions of a decision maker to be expressed on the real line with the value 0 representing indifference between two alternatives. Conversely, Luce and Suppes (1965) and part of the fuzzy sets community (De Baets et al., 2006) studied judgments expressed on the scale $]0,1[$ with the indifference represented by the value 0.5. Hereafter, due to its popularity, we shall instead consider the approach offered by Saaty (1977, 1980), where pairwise judgments are expressed as entries of positive reciprocal matrices, often called pairwise comparison matrices. Even so, our conclusions should not lose in generality as it was proven that all the above mentioned approaches are group isomorphic to each other (Cavallo and D'Apuzzo, 2009). Given a set of n alternatives, a *pairwise comparison matrix* is a positive square matrix $\mathbf{A} = (a_{ij})_{n \times n}$ such that $a_{ij}a_{ji} = 1 \, \forall i,j$, where a_{ij} is the subjective assessment of the relative importance of the i-th alternative with respect to the j-th. For instance, $a_{ij} = 2$ means that, for the decision maker, the i-th alternative is two times better than the j-th. A pairwise comparison matrix is *consistent* if and only if

$$a_{ik} = a_{ij}a_{jk} \quad \forall i,j,k. \tag{1}$$

Furthermore, if and only if a pairwise comparison matrix $\mathbf{A}$ is consistent, there exists a vector $\mathbf{w} = (w_1, \ldots, w_n)$ such that

$$a_{ij} = \frac{w_i}{w_j} \quad \forall i,j.$$

3

A possible way of finding vector $\mathbf{w}$ from a consistent pairwise comparison matrix is the geometric mean method

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}}.$$

The same method is also commonly used to estimate reliable vectors from inconsistent pairwise comparison matrices.

For notational convenience, we define the set of all pairwise comparison matrices as

$$\mathcal{A} = \left\{ \mathbf{A} = (a_{ij})_{n \times n} | a_{ij} > 0, a_{ij}a_{ji} = 1 \; \forall i, j, \; n > 2 \right\}.$$

Similarly, the set of all *consistent* pairwise comparison matrices $\mathcal{A}^* \subset \mathcal{A}$ is defined as

$$\mathcal{A}^* = \left\{ \mathbf{A} = (a_{ij})_{n \times n} | \mathbf{A} \in \mathcal{A}, a_{ik} = a_{ij}a_{jk} \; \forall i, j, k \right\}$$

A very precise definition of consistency was given in (1), but in the literature there is not a meeting of minds on how inconsistency should be quantified. In fact, inconsistency is generally regarded as a lack of consistency, i.e. a deviation from (1), but there is not a unique formula to quantify it. To overcome this problem, inconsistency indices have been introduced. Inconsistency indices are functions

$$I : \mathcal{A} \to \mathbb{R}, \tag{2}$$

where the value $I(\mathbf{A})$ is an estimation of the degree of inconsistency of the pairwise comparison matrix $\mathbf{A}$. It is important to note that each inconsistency index is in fact a different *definition* of inconsistency measuring. Establishing if a function returns a reasonable estimation of inconsistency—i.e. if a function is a good definition of inconsistency—is a crucial point indeed. In fact, there exist infinitely many functions (2) whose behavior is obviously meaningless when it comes to estimate the degree of inconsistency of a pairwise comparison matrix. This is the reason which motivated the introduction of some minimal reasonable properties that any inconsistency index should satisfy (Brunelli and Fedrizzi, 2013). The five axiomatic properties are summarized and justified in the following and will later be used to derive some results. We refer to the original paper (Brunelli and Fedrizzi, 2013) for more detailed descriptions and comments.

A1: There exists a unique $\nu \in \mathbb{R}$ representing the situation of full consistency, i.e.

$$\exists! \nu \in \mathbb{R} \text{ such that } I(\mathbf{A}) = \nu \Leftrightarrow \mathbf{A} \in \mathcal{A}^*.$$

Hence, every inconsistency index should at least be able to distinguish between fully consistent and inconsistent matrices.

A2: Changing the order of the alternatives does not affect the inconsistency of preferences. That is,

$$I(\mathbf{P}\mathbf{A}\mathbf{P}^T) = I(\mathbf{A})$$

for any permutation matrix $\mathbf{P}$. Thus, inconsistency remains unchanged when the names of alternatives are exchanged.

4

A3: If preferences of a matrix $\mathbf{A}$ are properly intensified obtaining a new matrix which we denote by $\mathbf{A}(b)$, then the inconsistency of $\mathbf{A}(b)$ cannot be smaller than the inconsistency of $\mathbf{A}$. In fact, if all the expressed preferences indicate indifference between alternatives, it is $a_{ij} = 1 \forall i, j$, and $\mathbf{A}$ is consistent. Going farther from this uniformity means having sharper and stronger judgments, and this should not make their possible inconsistency less evident. In other words, intensifying the preferences (pushing them away from indifference) should not de-emphasize the characteristics of these preferences and their possible contradictions. More formally, it was proved by Saaty (1977) that the exponential is the only non-trivial function preserving reciprocity and consistency (when $\mathbf{A}$ is consistent). Thus, the intensification of preferences is obtained defining $\mathbf{A}(b) = \left(a_{ij}^b \right)_{n \times n}$ with $b > 1$. Then, the property is as follows

$$I(\mathbf{A}(b)) \geq I(\mathbf{A}) \quad \forall \mathbf{A} \in \mathcal{A}, \ \ b > 1.$$

A4: Given a *consistent* pairwise comparison matrix and considering a single arbitrary comparison between two alternatives, then as we push its value far from its original one, we clearly increase the distance from consistency. Axiom 4 requires that the inconsistency of the matrix should not decrease. More formally, given a consistent matrix $\mathbf{A} \in \mathcal{A}^*$, and considering any arbitrary non-diagonal element a_{pq} (and its reciprocal a_{qp}) such that $a_{pq} \neq 1$, let $\mathbf{A}(\delta)$ be the inconsistent matrix obtained from $\mathbf{A}$ by replacing the entry a_{pq} with a_{pq}^δ, where $\delta \neq 1$. Necessarily, a_{qp} must be replaced by a_{qp}^δ in order to preserve reciprocity. Let $\mathbf{A}(\delta')$ be the inconsistent matrix obtained from $\mathbf{A}$ by replacing entries a_{pq} and a_{qp} with $a_{pq}^{\delta'}$ and $a_{qp}^{\delta'}$ respectively. The property can then be formulated for all $\mathbf{A} \in \mathcal{A}^*$ as

$$\begin{aligned} \delta' > \delta > 1 &\Rightarrow I(\mathbf{A}(\delta')) \geq I(\mathbf{A}(\delta)) \\ \delta' < \delta < 1 &\Rightarrow I(\mathbf{A}(\delta')) \geq I(\mathbf{A}(\delta)). \end{aligned} \tag{3}$$

A5: Function I is continuous with respect to the entries of $\mathbf{A}$. This is required, as infinitesimally small variations of the preferences should cause infinitesimally small changes of the value of the inconsistency.

We remark that all the indices described in this paper measure the *inconsistency* of the preferences. Nevertheless, in order to avoid multiple labels, we maintain the original names for the most popular indices, e.g. 'Consistency Index' and 'Geometric Consistency Index'.

3 The effects of consensus on the consistency of preferences

In the typical group decision framework, there are m $(m \geq 2)$ decision makers, each associated with a set of preferences. Hence, there exists m pairwise comparison matrices in the form $\mathbf{A}_1 = (a_{ij}^{(1)})_{n \times n}, \ldots, \mathbf{A}_m = (a_{ij}^{(m)})_{n \times n}$.

In our opinion, it would be interesting to study if, and how, the inconsistency of the pairwise comparisons changes under some circumstances. When studying the connection between consensus and inconsistency, some natural questions could be the following:

- How does the inconsistency of the preferences of the single decision makers react when they negotiate and their preferences converge to a consensual solution?
- Is the group inconsistency of the aggregated preferences a weighted mean of the inconsistencies of the original preference relations or *systematically* higher/lower?

In some recent papers, the problem of computing an upper bound for group inconsistency has been addressed only for a couple of inconsistency indices. In particular, Xu (2000) allegedly proved that Saaty's Consistency Index CI of a combination of pairwise comparison matrices cannot be greater that the maximum CI of the single pairwise comparison matrices. However, Lin et al. (2008) showed that the proof was not satisfactory and that Xu's result was a conjecture. Finally, Liu et al. (2012) provided a proof showing that Xu's conjecture was, in effect, true. Grošelj and Zadnik Stirn (2012) noted that the whole controversy was based on the unawareness that a more general problem had already been solved before by Elsner et al. (1988). Some studies have been proposed for another inconsistency index; Escobar et al. (2004) investigated the upper boundary of the Geometric Consistency Index, GCI.

The question of how the pairwise comparison matrices of different decision makers should be aggregated was answered when Aczel and Saaty (1983) proved that the weighted geometric mean is the only reasonable function to do so. Hence, when it comes to synthesize m pairwise comparison matrices into a single one $\mathbf{A}^\star = (a_{ij}^\star)_{n \times n}$, then its entries should be obtained as the weighted geometric means of the corresponding entries of the decision makers' pairwise comparison matrices, e.g.

$$a_{ij}^\star = \prod_{h=1}^{m} a_{ij}^{(h)\lambda_h} \tag{4}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_h \geq 0 \ \forall h$ and $\sum_{h=1}^{m} \lambda_h = 1$ is the weight vector of relative importances of the decision makers. We use $\mathcal{L}_m$ to denote the set of all the weight vectors with m components. That is,

$$\mathcal{L}_m = \left\{ (\lambda_1, \ldots, \lambda_m) \middle| \lambda_h \geq 0 \ \forall h, \ \sum_{h=1}^{m} \lambda_h = 1 \right\}. \tag{5}$$

Let us then define some properties which will play a pivotal role in the rest of the paper.

Definition 1 (Boundary properties). *Let $\mathbf{A}^\star$ be the aggregated pairwise comparison matrix as in (4). A function $I : \mathcal{A} \to \mathbb{R}$ is lower bounded (w.r.t the geometric mean) if:*

$$I(\mathbf{A}^\star) \geq \min \{I(\mathbf{A}_1), \ldots, I(\mathbf{A}_m)\} \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \ \boldsymbol{\lambda} \in \mathcal{L}_m \tag{6}$$

and upper bounded *(w.r.t. the geometric mean) if*

$$I(\mathbf{A}^\star) \leq \max \{I(\mathbf{A}_1), \ldots, I(\mathbf{A}_m)\} \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \ \boldsymbol{\lambda} \in \mathcal{L}_m \tag{7}$$

A function $I : \mathcal{A} \to \mathbb{R}$ is strongly upper bounded (w.r.t the geometric mean) if and only if

$$I(\mathbf{A}^\star) \leq \sum_{h=1}^{m} \lambda_h I(\mathbf{A}_h) \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \quad \boldsymbol{\lambda} \in \mathcal{L}_m \tag{8}$$

Note that, even if not specified, the previous definition is for all $n, m \geq 2$. Furthermore, Figure 1 provides a graphical interpretation of these properties in the case of two pairwise comparison matrices.
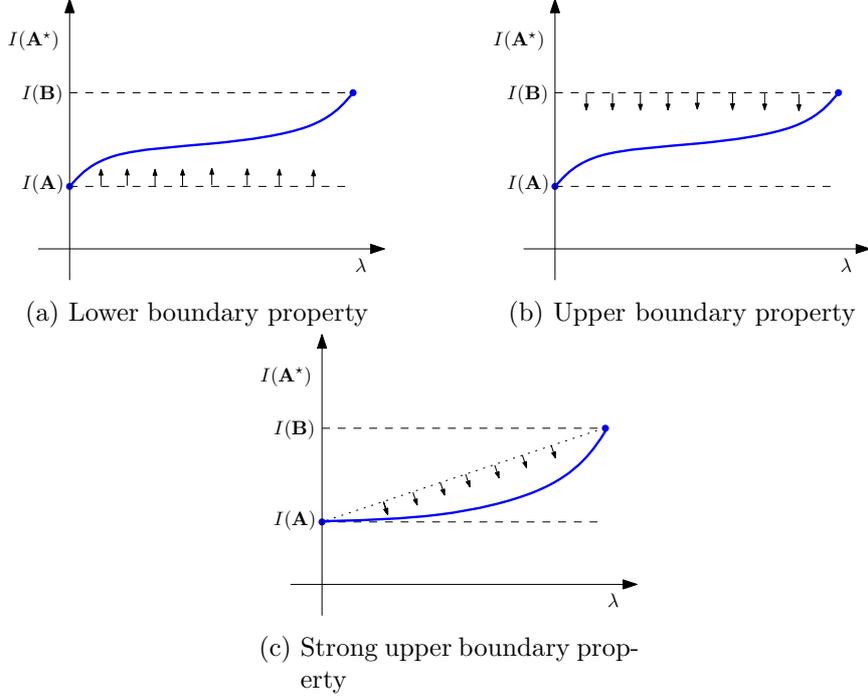

(a) Lower boundary property

(b) Upper boundary property

(c) Strong upper boundary property

Figure 1: Interpretations of boundary properties in the case of two decision makers with pairwise comparison matrices $\mathbf{A}$ and $\mathbf{B}$. We use the convention $\lambda_1 = \lambda$ and $\lambda_2 = 1 - \lambda$.

Remark 1. *Given a set of m matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$, if a function I is strongly upper bounded, then it is also convex w.r.t. the variables $(\lambda_1, \ldots, \lambda_m) \in \mathcal{L}_m$. In fact, if a function I is not convex, then, there exists a subset $\{\mathbf{A}_1, \ldots, \mathbf{A}_s\} \subset \mathcal{A}$ and a vector $(\lambda_1', \ldots, \lambda_s') \in \mathcal{L}_s$ such that $I(\mathbf{A}^\star) > \sum_{i=1}^{s} \lambda_i I(\mathbf{A}_i)$, which contradicts the definition of strongly upper bounded inconsistency index.*

We believe that it is important to study the phenomenon of inconsistency in the wider context of group decisions and raise the level of the discussion from single indices to their properties A1–A5. In fact, at this point some questions may arise concerning the connection between the axiomatic properties A1–A5, here recalled in Section 2, and the boundary properties of Definition 1. Namely, before trying to prove whether or

not each single inconsistency index satisfies the boundary properties, we shall use the axiomatic framework A1–A5 to derive more general results. For instance, one type of more general result would be that, if some axiomatic properties hold for one index I, then it is guaranteed that also some of the boundary conditions in Definition 1 must hold/not hold for I.

3.1 Lower boundary of inconsistency indices

If an inconsistency index is lower bounded, then it is impossible to achieve a group inconsistency lower than the lowest inconsistency of all the decision makers, by aggregating their preferences. With the following proposition, we shall show that an inconsistency index satisfying the simple axiom A1 cannot be lower bounded. Perhaps due to the previous lack of an axiomatic framework which allowed the derivation of general results, to our best knowledge, this result has never been explicitly spelled out.

Proposition 1. *If an inconsistency index satisfies A1, then it cannot be lower bounded.*

Proof. It is sufficient to consider a matrix $\mathbf{A} \notin \mathcal{A}^*$ and its transpose $\mathbf{A}^T$. Since they are both inconsistent, A1 implies $I(\mathbf{A}) > 0$ and $I(\mathbf{A}^T) > 0$, but the new matrix obtained as their element-wise geometric mean $\left(\sqrt{a_{ij}a_{ji}}\right)_{n \times n}$ is always consistent. As an inconsistency index respects A1, then the result that $I\left(\left(\sqrt{a_{ij}a_{ji}}\right)_{n \times n}\right) = 0 < \min\{I(\mathbf{A}), I(\mathbf{A}^T)\} \ \forall \mathbf{A} \notin \mathcal{A}^*$ concludes the proof. $\qquad \square$

Let us remark the importance of this result: for any inconsistency index satisfying the extremely weak axiom A1, it is possible that, given some pairwise comparison matrices, the inconsistency index calculated on their geometric mean-based combination be smaller than the minimum of their inconsistencies. This suggests that negotiation and convergence towards consensus might have a good effect on consistency of preferences. Furthermore, it can happen that two decision makers, initially completely inconsistent could, by negotiating, converge towards a consensual and fully consistent solution. Consider, for instance, the case

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1/9 & 9 \\ 9 & 1 & 1/9 \\ 1/9 & 9 & 1 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 9 & 1/9 \\ 1/9 & 1 & 9 \\ 9 & 1/9 & 1 \end{pmatrix}, \tag{9}$$

whose combination with weights $\lambda_1 = \lambda_2 = 0.5$ is the matrix

$$\mathbf{A}^\star = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathcal{A}^*. \tag{10}$$

This should make clear that, if the reliability of a decision should depend on the expertise of the decision makers, the evaluation of inconsistency of a consensual matrix should *not* be considered as a substitute for the evaluation of single pairwise comparison matrices.

Therefore, if what we want to asses by means of inconsistency tests is the ability of a decision maker, then the inconsistency of their initial preferences is more appropriate than the inconsistency of their modified/aggregated preferences. Clearly, moving towards consensus, or to an ever greater extent having his preferences aggregated with someone else's, does not make the decision maker a better and more capable expert.

To further dwell on this topic, we shall now focus on upper boundaries of inconsistency indices.

3.2 Upper boundaries of inconsistency indices

We start examining the properties of upper boundary and strong upper boundary by considering their connections with the five axioms A1–A5.

Proposition 2. *If an index satisfies A1 but not A3 or A4, then the index is not upper bounded.*

Proof. Let us assume that index I satisfies A1 but not A3. Then there exists a matrix $\mathbf{A}$ such that $I(\mathbf{A}(b')) < I(\mathbf{A}(b)) > I(\mathbf{A}(b''))$ for some $b' < b < b''$. On the other side, $\mathbf{A}(b)$ can be written as the weighted geometric mean of $\mathbf{A}(b')$ and $\mathbf{A}(b'')$, i.e.

$$\mathbf{A}(b) = \left(a_{ij}^b\right) = \left(a_{ij}^{(\lambda b' + (1-\lambda)b'')}\right) = \left(a_{ij}^{b'\,\lambda} a_{ij}^{b''\,1-\lambda}\right),$$

for a suitable $\lambda \in]0,1[$ as in (4). Then it is $I(\mathbf{A}(b)) > \max\{I(\mathbf{A}(b')), I(\mathbf{A}(b''))\}$ and (7) is violated. Therefore, I is not upper bounded.
The proof for the case of A1 holding and A4 not holding is similar, and thus omitted. $\qquad\square$

In other words, Proposition 2 states that axioms A3 and A4 are *necessary* conditions for an index to be upper bounded. Moreover, considering that the strong upper boundary property is tighter than the upper boundary property, we can formulate the following corollary.

Corollary 1. *If an index satisfies A1 but not A3 or A4, then the index is not strongly upper bounded.*

Building on Proposition 2, and drawing on previous results showing that a number of indices do not satisfy A3 or A4 (see Brunelli and Fedrizzi, 2013), we can formalize the fact that these same indices are *not* upper bounded. For brevity, their definitions are omitted and the reader can refer to the original works as well as to a survey paper by Brunelli et al. (2013a).

Corollary 2. *Indices GW (Golden and Wang, 1989), HCI (Stein and Mizzi, 2007), NI_n^σ (Ramík and Korviny, 2010) and RE (Barzilai, 1998) are not upper bounded.*

Example 1. Consider two decision makers and the corresponding two pairwise comparison matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1 \\ 1/4 & 1 & 1 \\ 1/1 & 1 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1/2 & 1 & 7 \\ 1 & 1/7 & 1 \end{pmatrix}.$$

Their preferences are aggregated according to (4) obtaining the aggregated pairwise comparison matrix $\mathbf{A}^\star(\lambda) = \left(a_{ij}^\lambda b_{ij}^{1-\lambda} \right)$. Then, the index RE proposed by Barzilai (1998) is computed for $\lambda \in [0,1]$. In Figure 2 is reported the plot of this inconsistency index, i.e. $RE\left(\mathbf{A}^\star(\lambda)\right)$, as a function of λ. It can be noted, for example, that the inconsistency of the aggregated preferences $RE\left(\mathbf{A}^\star(0.5)\right)$ is greater than the inconsistency of $\mathbf{A}$ and greater than the inconsistency of $\mathbf{B}$, i.e. $RE\left(\mathbf{A}^\star(0.5)\right) > \max\{RE(\mathbf{A}), RE(\mathbf{B})\}$. Thus, in this example, group preferences can be more inconsistent than the preferences of all decision makers, as index RE is not upper bounded.
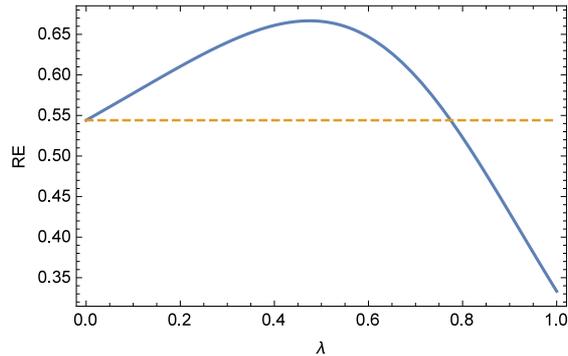

Figure 2: Example of a index which is not upper bounded: RE.

Hereafter, until the end of the section, we shall analyze the upper boundaries of some other inconsistency indices. Saaty's CI remains the most popular inconsistency index in the literature and thus any analysis which does not take it into account, would be incomplete.

Definition 2 (Consistency Index CI). *Given a pairwise comparison matrix $\mathbf{A}$ of order n, its Consistency Index (CI) is defined as:*

$$CI(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A}) - n}{n - 1}. \tag{11}$$

where $\lambda_{\max}(\mathbf{A})$ is the maximum eigenvalue of $\mathbf{A}$.

To avoid possible confusion, we remark that the symbol $\lambda_{\max}$ has been used here, as usual, to denote the largest eigenvalue and should not be confused with the coefficients λ_h.

Liu et al. (2012) already proved that CI is upper bounded. Here we extend their findings and show that it is strongly upper bounded. Let us first recall, in the following Lemma, a result by Elsner et al. (1988).

Lemma 1 ((Elsner et al., 1988)). *Given m matrices $\mathbf{M}_1, \ldots, \mathbf{M}_m \in \mathbb{R}_+^{n \times n}$ and a vector $\boldsymbol{\lambda} \in \mathcal{L}_m$, then*

$$\rho(\mathbf{M}^\star) \leq \prod_{h=1}^m \rho(\mathbf{M}_h)^{\lambda_h} \tag{12}$$

where ρ denotes the spectral radius, and $\mathbf{M}^\star$ is the weighted geometric mean of $\mathbf{M}_1, \ldots, \mathbf{M}_m$ as in (4).

Taking into account Lemma 1, and the inequality between arithmetic and geometric means, one derives that

$$\rho(\mathbf{A}^\star) \leq \sum_{h=1}^{m} \lambda_h \rho(\mathbf{A}_h). \tag{13}$$

Since the Perron-Frobenius theorem guarantees that the spectral radius of a positive matrix equals its maximum eigenvalue, $\rho(\mathbf{A}) = \lambda_{\max}(\mathbf{A})$, by noting that $CI(\mathbf{A})$ is a positive affine transformation of $\lambda_{\max}(\mathbf{A})$, the following corollary, stating the strong upper boundedness of CI, can be derived.

Corollary 3. *Index CI is strongly upper bounded, i.e. the pairwise comparison matrix $\mathbf{A}^\star$ as in (4) satisfies*

$$CI(\mathbf{A}^\star) \leq \sum_{h=1}^{m} \lambda_h CI(\mathbf{A}_h) \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \quad \boldsymbol{\lambda} \in \mathcal{L}_m. \tag{14}$$

Among other indices, the Geometric Consistency Index (GCI) has probably been the most widely studied.

Definition 3 (Geometric Consistency Index GCI (Crawford and Williams, 1985))**.** *Given a pairwise comparison matrix $\mathbf{A}$ of order n, its Geometric Consistency Index (GCI) is defined as:*

$$GCI(\mathbf{A}) = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \ln^2 \left(a_{ij} \frac{(\prod_{k=1}^{n} a_{jk})^{\frac{1}{n}}}{(\prod_{k=1}^{n} a_{ik})^{\frac{1}{n}}} \right). \tag{15}$$

Escobar et al. (2004) proved that index GCI is upper bounded. Nevertheless, here we shall prove that GCI is *strongly* upper bounded.

Proposition 3. *Index GCI is strongly upper bounded, i.e. the pairwise comparison matrix $\mathbf{A}^\star$ as in (4) satisfies*

$$GCI(\mathbf{A}^\star) \leq \sum_{h=1}^{m} \lambda_h GCI(\mathbf{A}_h) \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \quad \boldsymbol{\lambda} \in \mathcal{L}_m. \tag{16}$$

Proof. As observed by Brunelli et al. (2013b), $GCI(\mathbf{A})$ is proportional to the following quantity

$$\sum_{i<j<k}^{n} \left(\ln a_{ij} a_{jk} a_{ki} \right)^2. \tag{17}$$

11

Therefore, by expanding and rearranging (16) in the form (17) one obtains

$$\sum_{i<j<k}^{n} \left(\ln \left(\prod_{h=1}^{m} a_{ij}^{(h)\lambda_h} \prod_{h=1}^{m} a_{jk}^{(h)\lambda_h} \prod_{h=1}^{m} a_{ki}^{(h)\lambda_h} \right) \right)^2 \leq \sum_{i<j<k}^{n} \sum_{h=1}^{m} \lambda_h \left(\ln \left(a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)} \right) \right)^2 .$$

(18)

If we analyze a single transitivity (i, j, k), then we shall drop $\sum_{i<j<k}^{n}$ and, for notational convenience, use $x_h := a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}$. Then, we can rewrite it as

$$\left(\ln \prod_{h=1}^{m} x_h^{\lambda_h} \right)^2 \leq \sum_{h=1}^{m} \lambda_h \left(\ln x_h \right)^2$$

(19)

Considering that $\ln a + \ln b = \ln ab$ and $\ln a^b = b \ln a$, it becomes

$$\left(\sum_{h=1}^{m} \lambda_h \ln x_h \right)^2 \leq \sum_{h=1}^{m} \lambda_h \left(\ln x_h \right)^2 .$$

Now by putting $y_h := \ln x_h$, we can rewrite it as

$$\left(\sum_{h=1}^{m} \lambda_h y_h \right)^2 \leq \sum_{h=1}^{m} \lambda_h y_h^2.$$

which holds thanks to the convexity of the quadratic function. Thus, (19) is satisfied. Extending it to the sum for all the transitivity is straightforward as we know that, as it holds for all transitivities, then it must hold for their sum too. $\square$

Another index, CI^*, was proposed by Peláez and Lamata (2003) and used to improve the consistency of judgments.

Definition 4 (CI^* (Peláez and Lamata, 2003))**.** *Given a pairwise comparison matrix* **A** *of order n, CI^* is defined as:*

$$CI^*(\mathbf{A}) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} - 2 \right) \bigg/ \binom{n}{3}.$$

(20)

The validity of such an index has been corroborated by the fact that Shiraishi et al. (1998) independently suggested another inconsistency index (c_3) which was in effect proved to be proportional to CI^* by Brunelli et al. (2013b). Notably, both indices CI^* and c_3 were also used as an objective function in order to estimate missing elements in incomplete pairwise comparison matrices (Shiraishi et al., 1999). With the following proposition we prove that CI^*, and consequently also c_3, are strongly upper bounded.

Proposition 4. *Index CI^* is strongly upper bounded, i.e. the pairwise comparison matrix* $\mathbf{A}^\star$ *as in (4) satisfies*

$$CI^*(\mathbf{A}^\star) \leq \sum_{h=1}^{m} \lambda_h CI^*(\mathbf{A}_h) \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \quad \boldsymbol{\lambda} \in \mathcal{L}_m.$$

(21)

Proof. By expanding and rearranging (21) one obtains

$$\sum_{i<j<k}^{n} \left(\prod_{h=1}^{m} a_{ij}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{jk}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{ki}^{(h)^{\lambda_h}} + \frac{1}{\prod_{h=1}^{m} a_{ij}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{jk}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{ki}^{(h)^{\lambda_h}}} - 2 \right) \le$$
$$\sum_{h=1}^{m} \sum_{i<j<k}^{n} \lambda_h \left(a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)} + \frac{1}{a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}} - 2 \right). \tag{22}$$

Now, similarly to what was done in the previous proof, we consider the single transitivity (i, j, k) and rename the comparisons as follows: $x_h := a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}$. Thus, we obtain

$$\left(\prod_{h=1}^{m} x_h^{\lambda_h} + \frac{1}{\prod_{h=1}^{m} x_h^{\lambda_h}} - 2 \right) \le \sum_{h=1}^{m} \lambda_h \left(x_h + \frac{1}{x_h} - 2 \right) \tag{23}$$

which can be rewritten as

$$\underbrace{\prod_{h=1}^{m} x_h^{\lambda_h}}_{\text{I}} + \underbrace{\prod_{h=1}^{m} \left(\frac{1}{x_h} \right)^{\lambda_h}}_{\text{II}} \le \underbrace{\sum_{h=1}^{m} \lambda_h x_h}_{\text{III}} + \underbrace{\sum_{h=1}^{m} \lambda_h \frac{1}{x_h}}_{\text{IV}}. \tag{24}$$

Then, it is sufficient to prove that I $\le$ III and II $\le$ IV. Namely, the following two inequalities must hold simultaneously.

$$\prod_{h=1}^{m} x_h^{\lambda_h} \le \sum_{h=1}^{m} \lambda_h x_h \quad \text{and} \quad \prod_{h=1}^{m} \left(\frac{1}{x_h} \right)^{\lambda_h} \le \sum_{h=1}^{m} \lambda_h \frac{1}{x_h}.$$

The fact that they hold derives from the inequality between the geometric and the arithmetic mean. At this stage it is proved that inequality (23) is true for single transitivities, but its extension for the sum $\sum_{i<j<k}^{n}$ is straightforward. □

More recently, another index, here denoted by I_{CD}, was proposed and grounded in the theory of Abelian linearly ordered groups (Cavallo and D'Apuzzo, 2009, 2010). This index can be equivalently formulated for different types of preference relations where judgments are expressed on different scales.

Definition 5 (I_{CD} (Cavallo and D'Apuzzo, 2009)). *Given a pairwise comparison matrix* **A** *of order n,* I_{CD} *is defined as:*

$$I_{CD}(\mathbf{A}) = \prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^{n} \left(\max \left\{ \frac{a_{ij} a_{jk}}{a_{ik}}, \frac{a_{ik}}{a_{ij} a_{jk}} \right\} \right)^{\frac{1}{\binom{n}{3}}}. \tag{25}$$

With the following proposition, we state that also index I_{CD} is strongly upper bounded.

13

Proposition 5. *Index I_{CD} is strongly upper bounded, i.e. the pairwise comparison matrix $\mathbf{A}^{\star}$ as in (4) satisfies*

$$I_{CD}(\mathbf{A}^{\star}) \leq \sum_{h=1}^{m} \lambda_h I_{CD}(\mathbf{A}_h) \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \;\; \boldsymbol{\lambda} \in \mathcal{L}_m. \tag{26}$$

Proof. If we neglect the exponent, then we can expand and rearrange (26) as follows:

$$\prod_{i<j<k}^{n} \max\left\{ \prod_{h=1}^{m} a_{ij}^{(h)\lambda_h} \prod_{h=1}^{m} a_{jk}^{(h)\lambda_h} \prod_{h=1}^{m} a_{ki}^{(h)\lambda_h}, \frac{1}{\prod_{h=1}^{m} a_{ij}^{(h)\lambda_h} \prod_{h=1}^{m} a_{jk}^{(h)\lambda_h} \prod_{h=1}^{m} a_{ki}^{(h)\lambda_h}} \right\} \leq$$
$$\sum_{h=1}^{m} \lambda_h \prod_{i<j<k}^{n} \max\left\{ a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}, \frac{1}{a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}} \right\}. \tag{27}$$

Now, by dropping $\prod_{i<j<k}^{n}$ we consider the single transitivity (i,j,k). Moreover, by means of $x_h := a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}$ we briefly obtain

$$\max\left\{ \prod_{h=1}^{m} x_h^{\lambda_h}, \prod_{h=1}^{m} \left(\frac{1}{x_h}\right)^{\lambda_h} \right\} \leq \sum_{h=1}^{m} \lambda_h \max\left\{ x_h, \frac{1}{x_h} \right\}. \tag{28}$$

Now, in order to prove the previous inequality, we conjecture that there exists a quantity as follows

$$\max\left\{ \sum_{h=1}^{m} \lambda_h x_h, \sum_{h=1}^{m} \lambda_h \frac{1}{x_h} \right\} \tag{29}$$

whose value is always bounded by the left and right hand sides of (28), i.e.

$$\max\left\{ \prod_{h=1}^{m} x_h^{\lambda_h}, \prod_{h=1}^{m} \left(\frac{1}{x_h}\right)^{\lambda_h} \right\} \leq (29) \leq \sum_{h=1}^{m} \lambda_h \max\left\{ x_h, \frac{1}{x_h} \right\}.$$

Hence, it is sufficient to prove the following two inequalities,

$$\max\left\{ \prod_{h=1}^{m} x_h^{\lambda_h}, \prod_{h=1}^{m} \left(\frac{1}{x_h}\right)^{\lambda_h} \right\} \leq \max\left\{ \sum_{h=1}^{m} \lambda_h x_h, \sum_{h=1}^{m} \lambda_h \frac{1}{x_h} \right\}, \tag{30}$$

$$\max\left\{ \sum_{h=1}^{m} \lambda_h x_h, \sum_{h=1}^{m} \lambda_h \frac{1}{x_h} \right\} \leq \sum_{h=1}^{m} \lambda_h \max\left\{ x_h, \frac{1}{x_h} \right\}. \tag{31}$$

Inequality (30) holds thanks to the inequality between arithmetic and geometric mean and arguments similar to those used in the proof of (24). Conversely, (31) holds since it can be verified that both the arguments of the max operator in the left hand side of (31) cannot be greater than the right hand side of the same inequality. $\quad\square$

Another index was introduced by Koczkodaj (1993) and later studied (Duszak and Koczkodaj, 1994) and compared with other indices, for example with CI (Bozóki and Rapcsák, 2008).

Definition 6 (Index K (Duszak and Koczkodaj, 1994))**.** *Given a pairwise comparison matrix* $\mathbf{A}$ *or order* n, *the inconsistency index* K *is defined as:*

$$K(\mathbf{A}) = \max_{i<j<k} \left\{ \min \left\{ \left|1 - a_{ij}a_{jk}a_{ki}\right|, \left|1 - \frac{1}{a_{ij}a_{jk}a_{ki}}\right| \right\} \right\}. \tag{32}$$

Proposition 6. *The inconsistency index* K *is upper bounded, i.e. the pairwise comparison matrix* $\mathbf{A}^\star$ *as in (4) satisfies*

$$K(\mathbf{A}^\star) \leq \max\{K(\mathbf{A}_1), \ldots, K(\mathbf{A}_m)\} \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \;\; \boldsymbol{\lambda} \in \mathcal{L}_m.$$

Proof. Here again we consider the case with only one transitivity (i, j, k). Then we can start by writing

$$\min \left\{ \left|1 - \prod_{h=1}^{m} a_{ij}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{jk}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{ki}^{(h)^{\lambda_h}}\right|, \left|1 - \frac{1}{\prod_{h=1}^{m} a_{ij}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{jk}^{(h)^{\lambda_h}} \prod_{h=1}^{m} a_{ki}^{(h)^{\lambda_h}}}\right| \right\} \leq$$
$$\max_{h} \min \left\{ \left|1 - a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}\right|, \left|1 - \frac{1}{a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}}\right| \right\}, \tag{33}$$

now with $x_h := a_{ij}^{(h)} a_{jk}^{(h)} a_{ki}^{(h)}$, we have

$$\min \left\{ \left|1 - \prod_{h=1}^{m} x_h^{\lambda_h}\right|, \left|1 - \frac{1}{\prod_{h=1}^{m} x_h^{\lambda_h}}\right| \right\} \leq \max_{h} \min \left\{ \left|1 - x_h\right|, \left|1 - \frac{1}{x_h}\right| \right\}. \tag{34}$$

By using the following notational change, $f(x) := \min \left\{ \left|1 - x\right|, \left|1 - \frac{1}{x}\right| \right\}$, it is possible to rewrite the previous inequality as

$$f\left(\prod_{h=1}^{m} x_h^{\lambda_h}\right) \leq \max \left\{f(x_1), \ldots, f(x_m)\right\}. \tag{35}$$

Now we shall prove it by contradiction. Let us suppose that the proposition is wrong and that there exist $x_1, \ldots, x_m$ and $\lambda_1, \ldots, \lambda_m$ such that

$$f\left(\prod_{h=1}^{m} x_h^{\lambda_h}\right) > \max \left\{f(x_1), \ldots, f(x_m)\right\}.$$

Then, by examining the function $f(x)$ one can see that it is decreasing for $x \in]0, 1[$ and increasing for $x \in]1, \infty[$ with a minimum in 1 and then deduce that either $\prod_{h=1}^{m} x_h^{\lambda_h} >$

$\max\{x_1, \ldots, x_m\}$ or $\prod_{h=1}^{m} x_h{}^{\lambda_h} < \min\{x_1, \ldots, x_m\}$. However, this cannot be possible, as we know that the weighted geometric mean is an averaging aggregation function, i.e. $\min\{x_1, \ldots, x_m\} \leq \prod_{h=1}^{m} x_h^{\lambda_h} \leq \max\{x_1, \ldots, x_m\}$. Having proved (33) for each triplet (i, j, k), the original inequality in the proposition holds in particular for the maximum, since the maximum of quasiconvex functions is quasiconvex. $\qquad\square$

Remark 2. *Although index K is upper bounded, it is not strongly upper bounded. Namely, given a pairwise comparison matrix $\mathbf{A}^\star$ as in (4) the following property does not hold*

$$K(\mathbf{A}^\star) \leq \sum_{h=1}^{m} \lambda_h K(\mathbf{A}_h) \quad \forall \mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathcal{A}, \quad \boldsymbol{\lambda} \in \mathcal{L}_m. \tag{36}$$

In order to prove that (36) does not hold in general, let us present the following counterexample. Consider two decision makers and the corresponding two pairwise comparison matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1/9 & 9 \\ 9 & 1 & 1/9 \\ 1/9 & 9 & 1 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 3 & 9 \\ 1/3 & 1 & 3 \\ 1/9 & 1/3 & 1 \end{pmatrix}.$$

As in Example 1, the preferences are aggregated according to (4) obtaining matrix $\mathbf{A}^\star(\lambda)$. Then, the index $K(\mathbf{A}^\star(\lambda))$ is computed for $\lambda \in [0, 1]$. As illustrated in Figure 3, K is not strongly upper bounded. For instance, taking $\lambda = 0.5$,

$$K(\mathbf{A}^\star(0.5)) \approx 0.96 > 0.5K(\mathbf{A}_1) + 0.5K(\mathbf{A}_2) \approx 0.5.$$

Thus, in this example, the inconsistency of the aggregated preferences is greater than the average inconsistency of the decision makers, as index K is not strongly upper bounded.
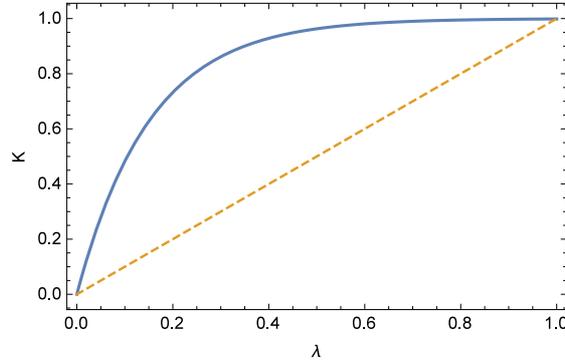

Figure 3: Index K is upper bounded but not strongly upper bounded.

4 Discussion

In the previous section we analyzed some boundary properties and whether they are respected or not by some well-known inconsistency indices. To do so, we also related

these new properties with the axioms A1–A5 introduced by Brunelli and Fedrizzi (2013) and discussed some implications. One result is that the inconsistency of a combination of pairwise comparison matrices cannot be lower bounded by the value of inconsistency of the least inconsistent matrix. We have in fact shown that an inconsistency index satisfying A1 is never lower bounded with respect to the aggregation of the preferences of different decision makers. Moreover, results show that, more often than not, inconsistency indices are, instead, upper bounded. Table 1 summarizes the results obtained in the previous section.

Index / Property	LB	UB	S-UB
CI	✗	<u>✓</u>	✓
RE	✗	✗	✗
CI^*, c_3	✗	✓	✓
GCI	✗	<u>✓</u>	✓
HCI	✗	✗	✗
GW	✗	✗	✗
I_{CD}	✗	✓	✓
K	✗	✓	✗
NI_n^σ	✗	✗	✗

Table 1: Summary table. LB = lower bounded, UB = upper bounded, S-UB = strongly upper bounded, ✓= the property is satisfied, ✗= the property is not satisfied. The underlined results were already known in the literature.

Comparing the results summarized in Table 1 with the satisfaction of the axioms A1–A5, it appears that all the indices which satisfy the axioms A1–A5 are also upper bounded. From this, it is only natural to hypothesize that the axioms A1–A5 mathematically imply the satisfaction of the upper boundary condition. However, this is not true and can be formalized in the following proposition.

Proposition 7. *If an inconsistency index satisfies the axiomatic system A1–A5, then it is not necessarily upper bounded.*

To prove the proposition, it is sufficient to show a counterexample; that is, an index satisfying A1–A5 which is not upper bounded. The following index is an example:

$$I_M(\mathbf{A}) = \min_{i<j<k} \left| \ln \frac{a_{ij}a_{jk}}{a_{ik}} \right| + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left| \ln \frac{a_{ij}a_{jk}}{a_{ik}} \right|. \tag{37}$$

It is easy to check that I_M satisfies the five axioms but, when applied to the following two matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1 & 1 \\ 1/4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 1/4 & 1 \end{pmatrix},$$

17

we obtain $I_M(\mathbf{A}) = I_M(\mathbf{B}) \approx 2.77$, but if we consider that $I_M(a_{ij}^{0.5} b_{ij}^{0.5}) \approx 3.46574$, it appears that the proposition is correct. A graphical representation of this case is presented in Figure 4.
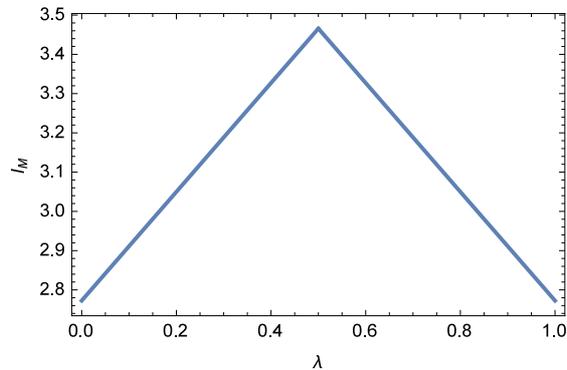

Figure 4: Index I_M satisfies A1–A5 but is not upper bounded.

Another matter, this time of a more qualitative debate, relates with the meaning attached to the consistency of group preferences. If inconsistency indices are used to estimate the rationality of decision makers and detect those which are too irrational, then the meaning of the inconsistency of the aggregated preferences vanishes. This is due to the fact that the link between rationality of a decision maker and consistency of preferences, which is generally assumed in the practice of the AHP, holds only for the initial preferences of a decision maker and breaks down as the preferences are aggregated, as already discussed in Subsection 3.1. We believe that we can learn a lot from the boundary properties: for instance we know, a priori, that when evaluated by an upper bounded inconsistency index, if the pairwise preferences of the most inconsistent decision maker move towards a consensual solution, at least at the beginning, they become more consistent. Nevertheless, we refrain—and we advise other researchers to do the same—from considering the inconsistency of the group preferences as a global measure of the inconsistency of various decision makers.

5 Conclusions

From our investigation on the connection between consensus and inconsistency, we can summarize some relevant findings. First, the effect of the preference aggregation among different decision makers on the inconsistency of the group preferences depends crucially on the inconsistency index that is used. More precisely, we proved that a certain number of already known indices satisfy the upper boundary property and/or the strong upper boundary property. As a consequence, for some indices the inconsistency of the aggregated preferences is always lower than a weighted mean of the inconsistencies of the original preference relations. This effect can be synthesized by saying that preference aggregation is consistency improving. On the other hand, if the inconsistency is

evaluated by means of other indices, then the opposite result can be obtained. Therefore, as pointed out in Brunelli and Fedrizzi (2013), a suitable choice of an inconsistency index is a crucial phase in decision-making processes, since the use of different methods for measuring consistency can lead to different conclusions and can affect the decision outcome in practical applications. Interestingly, some more general results have been derived from the axiomatic system proposed by Brunelli and Fedrizzi (2013). If it is true that one of the merits of an axiomatic system is its fertility, i.e. the capacity to produce propositions, then Propositions 1 and 2 and Corollaries 1 and 2 are positive signs in this direction and seem to indicate that the axiomatic system can be used to derive interesting results. Hence, we believe that future research on inconsistency of pairwise comparisons (i) can build upon the axiomatic system A1–A5 and (ii) further investigate the relation between consensus and consistency, perhaps clarifying the open issues exposed in the previous section.

Acknowledgements

References

Aczel, J., & Saaty, T. L. (1983). Procedures for synthesizing ratio judgements. *Journal of Mathematical Psychology*, 27, 93–102.

Altuzarra, A., Moreno-Jiménez, J. M., & Salvador, M. (2010). Consensus building in AHP-group decision making: A Bayesian approach. *Operations Research*, 58, 1755–1773.

Barzilai. J. (1998). Consistency measures for pairwise comparison matrices. *Journal of Multi-Criteria Decision Analysis*, 7, 123–132.

Bernasconi, M., Choirat, C., & Seri R. (2014) Empirical properties of group preference aggregation methods in AHP: Theory and evidence. *European Journal of Operational Research*, 232, 584–592.

Bozóki, S. & Rapcsák, T. (2008). On Saaty's and Koczkodaj's inconsistencies of pairwise comparison matrices. *Journal of Global Optimization*, 42, 157–175.

Brunelli, M., Canal, L., & Fedrizzi, M. (2013a). Inconsistency indices for pairwise comparison matrices: a numerical study. *Annals of Operations Research*, 211, 493–509.

Brunelli, M., Critch, A. & Fedrizzi, M., (2013b). A note on the proportionality between some consistency indices in the AHP. *Applied Mathematics and Computation*, 219(14), 7901–7906.

Brunelli, M., & Fedrizzi, M. (2013). Axiomatic properties of inconsistency indices for pairwise comparisons, *Journal of the Operational Research Society*. Advance online publication. doi:10.1057/jors.2013.135

Cavallo, B., & D'Apuzzo, L. (2009). A General unified framework for pairwise comparison matrices in multicriterial methods. *International Journal of Intelligent Systems*, 24, 377–398.

Cavallo, B., & D'Apuzzo, L. (2010). Characterizations of consistent pairwise comparison matrices over Abelian linearly ordered groups. *International Journal of Intelligent Systems*, 25, 1035–1059.

Crawford, G., & Williams, C. (1985). A note on the analysis of subjective judgement matrices. *Journal of Mathematical Psychology*, 29, 387–405.

De Baets, B., De Meyer, H., De Schuymer, B., & Jenei, S., Cyclic evaluation of transitivity of reciprocal relations. *Social Choice and Welfare*, 26, 217–238.

Duszak, Z. & Koczkodaj, W. W. (1994). Generalization of a new definition of consistency for pairwise comparisons. *Information Processing Letters*, 52, 273–276.

Elsner, L., Johnson, C. R., & Dias da Silva, J. A. (1988). The Perron root of a weighted geometric mean of nonnegative matrices. *Linear and Multilinear Algebra*, 24, 1–13.

Ergu, D., Kou, G., Peng, Y. & Shi, Y. (2011). A simple method to improve the consistency ratio of the pair-wise comparison matrix in ANP. *European Journal of Operational Research*, 213, 246–259.

Escobar, M. T., Aguarón, J., & Moreno-Jiménez, J. M. (2004). A note on AHP group consistency for the row geometric mean prioritization procedure. *European Journal of Operational Research*, 153, 318–322.

Fishburn, P. C. (1991). Nontransitive preferences in decision theory. *Journal of Risk and Uncertainty*, 4, 113–134.

Gass, S. I. (2005). Model world: The great debate — MAUT versus AHP. *Interfaces*, 35, 308–312.

Golden, B. L., & Wang, Q. (1989). An alternate measure of consistency, In Golden, B. L., Wasil, E. A., & Harker P. T. (Eds.), *The Analytic Hierarchy Process, Applications and Studies* (pp. 68–81). Berlin-Heidelberg: Springer-Verlag.

Grošelj, P., & Zadnik Stirn, L. (2012). Acceptable consistency of aggregated comparison matrices in analytic hierarchy process. *European Journal of Operational Research*, 223, 417–420.

Irwin, F. W. (1958). An analysis of the concepts of discrimination and preference. *The American Journal of Psychology*, 71, 152–163.

Ishizaka, A., & Labib, A. (2011). Review of the main developments in the analytic hierarchy process, *Expert Systems and Applications* 38(11) 14336–14345 (2011)

Koczkodaj, W. W. (1993). A new definition of consistency of pairwise comparisons. *Mathematical and Computer Modelling*, 18, 79–84.

Kou, G., & Lin, C. (2014). A cosine maximization method for the priority vector derivation in AHP. *European Journal of Operational Research*, 235, 225–232.

Kou, G., Ergu, D., & Shang, J. (2014). Enhancing data consistency in decision matrix: Adapting Hadamard model to mitigate judgment contradiction. *European Journal of Operational Research*, 236, 261–271.

Lamata, M. T., & Peláez, J. I. (2002). A method for improving the consistency of judgements. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10, 677–686.

Lin, R., Lin, J. S.-J., Chang, J., Tang, D., Chao, H., & Julian, P. C. (2008). Note on group consistency in analytic hierarchy process. *European Journal of Operational Research*, 190, 672–678.

Liu, F., Zhang, W.-G., & Wang, Z.-X. (2012). A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making. *European Journal of Operational Research*, 218, 747–754.

Luce, R. D., & Raiffa, H. (1957). *Games and Decisions.* John Wiley and Sons: New York.

Luce, R. D., & Suppes, P. (1965). Preference, Utility and Subjective Probability. In Luce R. D., Bush R. R., & Galanter E. (Eds.), *Handbook of Mathematical Psychology* (pp. 249–410) (vol. 3). John Wiley and Sons: New York.

Peláez, J. I., & Lamata, M. T. (2003). A new measure of consistency for positive reciprocal matrices, *Computer and Mathematics with Applications*, 46, 1839–1845.

Ramík, J., & Korviny, P. (2010). Inconsistency of pair-wise comparison matrix with fuzzy elements based on geometric mean. *Fuzzy Sets and Systems*, 161, 1604–1613.

Saaty, T. L. (1977). A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, 15, 234–281.

Saaty, T. L. (1980). *The Analytic Hierarchy Process.* McGraw Hill: New York.

Savage L. J. (1972). *The Foundations of Statistics* (2nd ed.) Dover Publications: New York.

Shiraishi, S., Obata, T., & Daigo, M. (1998). Properties of a positive reciprocal matrix and their application to AHP. *Journal of the Operations Research Society of Japan*, 41, 404–414.

Shiraishi, S., Obata, T., Daigo, M., & Nakajima, N. (1999). Assessment for an incomplete matrix and improvement of the inconsistent comparison: computational experiments. *Proceedings of ISAHP 1999* (pp. 200–205). Kobe, Japan.

Stein, W. E., & Mizzi, P. J. (2007). The harmonic consistency index for the analytic hierarchy process. *European Journal of Operational Research*, 177, 488–497.

Weiss, D. J., & Shanteau, J. (2004). The vice of consensus and the virtue of consistency. In Shanteau J., Johnson, P., Smith C. (Eds.), *Psychological Investigations of Competent Decision Making* (pp. 200–205). Cambridge University Press: New York.

Xu, Z. (2000). On consistency of the weighted geometric mean complex judgement matrix in AHP. *European Journal of Operational Research*, 126, 683–687.