# ANALITICAL SOLUTIONS OF AN OSCILLATING ELASTIC CABLE LOADED WITH A CONCENTRATED MASS

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*Summary* We present a new analytical study of the oscillations of an elastic cable, loaded with a point mass in the center of the span. The solutions of the equations of motion represent a generalization of the linear theory of Irvine [**?**], [**?**] for the empty elastic cable, i.e. we find the transcendental equations to determine the eigenfrequencies for symmetric/antisymmetric modes of oscillations, as well as the eigenfunctions to determine the displacement of the cable. To test our analytical results we used a simple model of discrete mass points in the gravitational potential, connected by massless elastic rods.

## INTRODUCTION

We generalize the theory of Irvine which treats oscillations of empty suspended elastic cables with small midspan catenary sag. Here we consider an elastic cable loaded with a point mass in the center of the span and investigate the in–plane oscillations of this system. Problems of this kind arise in many practical situations, for example when one studies the motion and in particular the vehicle–oscillations of ropeways. For the authors the content of this paper represents a tentative of an analytic approach to the problem of oscillations of cables in presence of a concentrated mass fixed in a given point of the cable using the theory and the approximations of Irvine.

## ELEMENTS OF THE THEORY

We introduce a material point of mass $m$, fixed in the center of the cable. The coordinates of the suspension points are $(x = 0, y = 0)$, $(x = l, y = 0)$, and the point mass, at the equilibrium, has coordinates $(x = l/2, y = D$, see Fig. **??**). Denoting (improperly) with $K$ the horizontal component of the cable tension, constant along the whole cable, the static configuration of the cable is described by the formula

$$y_0(x) = \frac{\rho g(l + \frac{m}{\rho})}{2K}\left[x - \frac{x^2}{(l + \frac{m}{\rho})}\right], \quad x \in [0, l/2] \tag{1}$$

(symmetric for $x \in (\frac{l}{2}, l]$), in which the relation between $K$ and the sag $D$ is given by

$$D = y_0\left(\frac{l}{2}\right) = \frac{\rho g l^2}{8K}\left(1 + \frac{2m}{\rho l}\right) . \tag{2}$$

Concerning dynamics, let's denote with $x = x_0(s)$, $y = y_0(s)$ and $T(s)$ the parametric representation of the cable and the tension in the equilibrium configuration, and with $u = u(t, s)$, $v = v(t, s)$ and $T(s) + \tau(t, s)$ the horizontal and vertical displacements of the points of the cable and the tension at the point of curvilinear parameter $s$ at time $t$. The functions $u = u(t, s)$, $v = v(t, s)$ and $\tau(t, s)$ satisfy the linearized equations of motion:

$$\frac{\partial}{\partial s}\left(T(s)\frac{\partial u(t,s)}{\partial s} + \frac{dx_0(s)}{ds}\tau(t,s)\right) = \rho\frac{\partial^2 u(t,s)}{\partial t^2} \quad s \neq s^* , \tag{3}$$

$$\frac{\partial}{\partial s}\left(T(s)\frac{\partial v(t,s)}{\partial s} + \frac{dy_0(s)}{ds}\tau(t,s)\right) = \rho\frac{\partial^2 v(t,s)}{\partial t^2} \quad s \neq s^* , \tag{4}$$

$$\left[\frac{dx_0(s)}{ds}\tau(t,s) + T(s)\frac{\partial u(t,s)}{\partial s}\right]_{|(t,s^*)} = m\frac{\partial^2 u(t,s^*)}{\partial t^2} , \tag{5}$$

$$\left[\frac{dy_0(s)}{ds}\tau(t,s) + T(s)\frac{\partial v(t,s)}{\partial s}\right]_{|(t,s^*)} = m\frac{\partial^2 v(t,s^*)}{\partial t^2} , \tag{6}$$

$[F(t, s)]_{|(t,s^*)} := F(t,(s^*)^+) - F(t,(s^*)^-)$ denoting the jump of the function $F(t, s)$ at $s^*$, namely at the point in which the concentrated mass $m$ is present.

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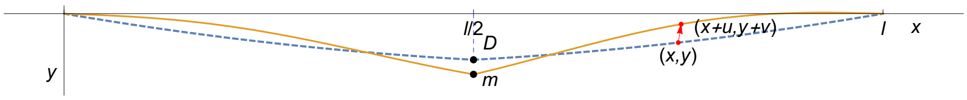


Figure 1: Sketch of the system under consideration. Dashed line: equilibrium configuration $y_0(x)$. Solid line: cable out of equilibrium, where a point undergoes the displacement $(x, y) \to (x + u(t, x), y + v(t, x))$.

## RESULTS - SYMMMETRIC AND ANTISYMMETRIC MODES OF OISCILLATIONS

Adopting the definition of Irvine a mode of oscillation is called *symmetric (resp. antisymmetric)* if the vertical displacement $v(t, x)$ is symmetric (resp. antisymmetric) with respect to $x = \frac{l}{2}$. We have no space to present here in details the analytic approach and we give the results of a comparison of the analytic approach with a numerical example, using the following parameters: linear density $\rho = 1$ kg/m, span $l = 100$ m, $A_c E_c = 10^5$ N. The static midspan sag for the empty cable is $d =$ 5m and the horizontal component of the tension is $K = 2450$ N. We compare our analytical results with the ones obtained by evaluation of a model, consisting of a set of discrete mass points in a gravitational potential (each with a mass $m_p = 2.396$ kg) connected by massless rods with appropriate elastic constant and rest length. We consider the case of an empty cable, and with a concentrated load $m = 6m_p = 14.37$ kg. The resulting values are collected in the tables that appear below, together with the numerical values for the different loading cases and modes (symmetric (s) and antisymmetric (a)) of oscillation. Unfortunately, we have no space to exhibit the corresponding eigenfunctions.

## A NUMERICAL EXAMPLE

The presented tables show a comparison of both approaches for the empty and the loaded cable respectively.

| $n$ | 1 (s) | 2 (a) | 3 (s) | 4 (a) | 5 (s) | 6 (a) |
|---|---|---|---|---|---|---|
| $\omega_n$ analytical | 1.92 | 3.11 | 4.68 | 6.22 | 7.78 | 9.33 |
| $\omega_n$ numerical | 1.92 | 3.06 | 4.63 | 6.14 | 7.68 | 9.18 |

Table 1: First 6 eigenfrequencies (Hz) for the empty cable ($m = m_p$). The analytical results agree with the theory of Irvine. The letters (a), (s) denote antisymmetric/symmetric modes respectively.

| $n$ | 1 (s) | 2 (a) | 3 (s) | 4 (a) | 5 (s) | 6 (a) |
|---|---|---|---|---|---|---|
| $\omega_n$ analytical | 1.81 | 3.19 | 4.34 | 6.40 | 7.28 | 9.60 |
| $\omega_n$ numerical | 1.81 | 3.14 | 4.31 | 6.34 | 7.21 | 8.97 |

Table 2: First 6 eigenfrequencies (Hz) for the loaded cable with $m = 6m_p$.

## References

[1] Irvine H. M., and Caughey T. K., The linear theory of free vibrations of a suspended cable, *Proc. R. Soc. London A* **341**: 299-315, 1974.
[2] Irvine H. M., Studies in the statics and dynamics of simple cable systems DYNL-108, California Institute of Technology, 1974.