Global Observability Analysis of a Nonholonomic Robot using Range Sensors

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***Abstract*— The global observability property of nonholonomic mobile robots equipped with a ranging systems is considered in this paper in relation to the localisation problem. The discrete time nature of the problem, induced by the fixed sampling frequency of the sensor, turns the problem into a set of algebraic conditions, whose solution naturally maps onto the control space of the robot. To the best of the Authors' knowledge, this is the first solution providing global observability results. The solution is reported for two different mobile robots, a unicycle-like and an unmanned fixed-wing robot.**

I. INTRODUCTION

Modern applications of autonomous systems require the ability to interface with the available sensing systems and to deal with their limitations and constraints [1]. One of the most relevant problems is to retrieve the actual state space configuration in order to apply properly designed feedback control laws. Hence, any (linear or nonlinear) observer or filter conceived to this purpose can actually be implemented if the system is observable or at least detectable. Since many actually available sensors generate a measurement value that is non-linearly related to the state (e.g. a distance, an angle, etc.), nonlinear observability tools should be adopted. This is even more relevant if the system dynamic is also nonlinear, which is often the case when dealing with robots.

Related Work. It is unsurprising that observability analysis is usually considered for state-feedback robot control in many research papers dealing with nonlinear system models, such as the popular unicycle-like vehicles. Among the others, an observability analysis has been proposed in [2], where collaborative localisation with generic nonlinear sensors is considered to analyse the best linearisation point to apply an Extended Kalman Filter (EKF). In [3] the observability analysis for leader-follower formations is carried out considering the availability of visual data. Of particular interest is the coupling of nonlinear observability with motion control algorithms, as in [4], where ranging data are considered as exteroceptive sensors for the design of a nonlinear observer. Reference [5] presents the sensibility analysis of the localisation problem when multiple landmarks are adopted and when the relative angle measure is considered. The work in [6], similarly to [7], proposes a planar bearing-only observability analysis for unicycles using a nonlinear continuous-time derivation, while [8] deals with a similar problem in the presence of unknown but constant disturbance in the vehicle dynamics. Another observability analysis is proposed in [9] for a ranging sensor with unknown but constant offset associated with RFID tags.

All the previously discussed literature results, while witnessing the relevance and the interest that this topic still has, establish a clear link between the system observability requirement and the localisation of a robot with respect to a predefined fixed reference frame. For localisation purposes, the state space corresponds to the set of generalised coordinates needed to uniquely identify the robot position in the environment, which is exactly the topic treated in this paper for robots with nonholonomic dynamics. The localisation problem as a specific instantiation of the observability condition has been also used in [10] for a unicycle-like vehicle with only one landmark and using nonlinear tools for observability analysis in a continuous time domain [11]. A continuous time analysis in the same spirit is also proposed in [12] for Autonomous Underwater Vehicles (UAVs) and ranging measurements.

Contributions of the paper. In this paper, we consider the problem of vehicle observability using ranging measurements, which means that the output function is the distance of the robot with respect to a number of ranging sensors (landmarks). We consider a scenario in which the robot can measure its own velocities, in essence a self localisation problem. Contrary to most of the literature, we adopt a discrete–time formulation for the dynamics of the robot. As well as being the most natural setting for a realistic localisation system (induced by the presence of a digital platform), the discrete–time formulation lends itself to an easy and natural analysis of the observability problem, putting us in condition to discover some insightful and noteworthy facts. Specifically, we show that: i) If we use only two landmarks the system's state is globally unobservable as long as the robot moves along straight lines; on the other hand, it is sufficient that the robot turns for two sampling periods in a row in order for its state to be observable; ii)If we use at least three landmarks, then the state of the system can be reconstructed in one step. Therefore its state is globally observable whatever the trajectory it follows.

We would like to stress that the simplicity of the discrete time formulation allows us to obtain global observability, i.e. valid in the entire state space of the robot. This result is often very difficult to achieve with standard tools [11] in continuous time, given the nonlinearity of both the system and of the output function. Importantly, it is possible to apply the same algebraic machinery to a different but related problem: creating a map of the position of the landmarks

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with respect to the initial position of the robot. In particular, we show that it is possible to construct a map if each landmark is seen at least twice in two different steps along the robot motion. Finally, even if most of our results have been found for a robot moving in a 2D environment, we show how it is possible to generalise them also to the case of vehicles moving in a 3D environment (e.g., a fixed wing drone).

The paper is organised as follows. In Section II, we set up the problem introducing the model and the notation adopted. In Section III, we present our observability study for two landmarks and more, while, in Section IV, we show how our framework is applicable to the inverse problem: generating a map of the landmarks from the range measurements and from the motion information of the robot. The proposed analysis is then extended in Section V to vehicles moving in a 3D space. Finally, in Section VI, we present our conclusions and announce future work directions.

II. Problem

Consider a unicycle vehicle with dynamics.

$$\dot{x} = v\cos\theta,\ \dot{y} = v\sin\theta,\ \dot{\theta} = \omega, \tag{1}$$

where $s(t) = (x,\, y,\, \theta)$ is the state of the system (θ is the orientation with respect to the X_w axis of the reference frame $\langle W\rangle$), while v (the forward velocity) and ω (the angular velocity) are the input variables. Assuming that the system is sampled with period T_s and that the command variable v and ω are held constant throughout the sampling period (a customary assumption for digital control), it is possible to find the following discrete time ZoH equivalent dynamics:

$$\begin{aligned} x((k+1)T_s) &= x(kT_s) + \int_{kT_s}^{(k+1)T_s} v\cos(\theta(\tau))d\tau,\\ y((k+1)T_s) &= y(kT_s) + \int_{kT_s}^{(k+1)T_s} v\sin(\theta(\tau))d\tau,\\ \theta((k+1)T_s) &= \theta(kT_s) + \omega T_s. \end{aligned} \tag{2}$$

Introducing for a generic function $f(\cdot)$ the notation $f_k = f(kT_s)$, using the simple variable transformation $\tau' = \tau - kT_s$, we obtain:

$$\begin{aligned} &\int_0^{T_s} v\cos(\theta(\tau' + kT_s))d\tau' = \\ &= \begin{cases} v_k T_s \cos\theta_k & \text{if } \omega = 0,\\ 2\frac{v_k}{\omega_k}\sin\left(\frac{\omega_k}{2}T_s\right)\cos\left(\theta_k + \frac{\omega_k}{2}T_s\right) & \text{otherwise.}\end{cases} \end{aligned}$$

Using the simplified notation $x_{k+1} = x((k+1)T_s)$ (i.e., $s_{k+1} = [x_{k+1}, y_{k+1}, \theta_{k+1}]^T$) and applying the same argument to the second integral of (2), we finally have

$$\begin{aligned} x_{k+1} &= \begin{cases} x_k + v_k T_s\cos\theta_k & \text{if } \omega_k = 0,\\ x_k + 2\frac{v_k}{\omega_k}\sin\left(\frac{\omega_k}{2}T_s\right)\cos\left(\theta_k + \frac{\omega_k}{2}T_s\right) & \text{otherwise,}\end{cases}\\ y_{k+1} &= \begin{cases} y_k + v_k T_s\sin\theta_k & \text{if } \omega_k = 0,\\ y_k + 2\frac{v_k}{\omega_k}\sin\left(\frac{\omega_k}{2}T_s\right)\sin\left(\theta_k + \frac{\omega_k}{2}T_s\right) & \text{otherwise,}\end{cases}\\ \theta_{k+1} &= \theta_k + \omega_k T_s. \end{aligned} \tag{3}$$

The vehicle moves across a space instrumented with a number of electromagnetic landmarks (e.g., UWB nodes). Each landmark is deployed at coordinate (X_i, Y_i), which are known to the vehicle. At each time step kT_s, the vehicle receives a signal from each landmark from which it is possible to reconstruct the distance between the vehicle and the landmark. The readings are collected in the vector $\mathcal{Y}_k$:

$$\mathcal{Y}_k = \begin{bmatrix} \mathcal{Y}_{1,\,k}\\ \dots\\ \mathcal{Y}_{M,\,k}\end{bmatrix} = \begin{bmatrix} \sqrt{(x_k - X_1)^2 + (y_k - Y_1)^2}\\ \dots\\ \sqrt{(x_k - X_M)^2 + (y_k - Y_M)^2}\end{bmatrix}. \tag{4}$$

Through the direct application of the definition, it is possible to derive the relation between two consecutive observations, i.e.

$$\mathcal{Y}_{i,k+1}^2 = \mathcal{Y}_{i,k}^2 + A_k^2 + 2A_kC_k\,(x_k - X_i) + 2A_kS_k\,(y_k - Y_i) \tag{5}$$

where

$$\begin{aligned} A_k &= \begin{cases} v_kT_s & \text{if } \omega_k = 0,\\ 2\frac{v_k}{\omega_k}\sin\left(\frac{\omega_k}{2}T_s\right) & \text{if } \omega_k \neq 0,\end{cases}\\ C_k &= \begin{cases} \cos\theta_k & \text{if } \omega_k = 0,\\ \cos\left(\theta_k + \frac{\omega_k}{2}T_s\right) & \text{if } \omega_k \neq 0,\end{cases}\\ S_k &= \begin{cases} \sin\theta_k & \text{if } \omega_k = 0,\\ \sin\left(\theta_k + \frac{\omega_k}{2}T_s\right) & \text{if } \omega_k \neq 0.\end{cases} \end{aligned} \tag{6}$$

Our objective is to study under which conditions it is possible for the robot to reconstruct its state s_k, starting from any initial state s_0, assuming the knowledge of the sequence of the distance vector $\mathcal{Y}_k$ and of the velocities v_k and ω_k, which is an *observability problem*. It is worth noting that, for non-linear systems, observability is not a structural property of the system as a whole but of its trajectories. Therefore, our results will make a distinction between rectilinear trajectories (for which ω_k is zero) and bended trajectories (for which ω_k is allowed to be non-zero for some values of k).

The inverse problem is the *mapping problem* and can be described in the following terms. Suppose that the robot does not know the coordinate (X_i, Y_i) of the landmarks, but it can measure the distance from each landmark. As customary for mapping problems, we assume that the robot starts from the origin of the reference frame $\langle W\rangle$ (or that the reference frame origin is set on the robot initial position), with its orientation aligned to the X_w-axis, i.e. $s_0 = [0, 0, 0]^T$. The mapping problem amounts to finding the position of the landmarks from the sequence of measurements.

Notation and useful relations. In the paper we will use the symbol $\mathbb{S}_k$ if the robot at step k moves along a rectilinear path ($\omega_k = 0$) and $\mathbb{T}_k$ if it bends at step k ($\omega_k \neq 0$). DWe also denote $C_w = \cos\omega T_s$ and $S_w = \sin\omega T_s$. By using simple trigonometric formulae, we get from (6)

$$\begin{aligned} C_{k+1} &= \begin{cases} C_k & \mathbb{S}_k,\\ C_kC_w - S_kS_w & \mathbb{T}_k,\end{cases}\\ S_{k+1} &= \begin{cases} S_k & \mathbb{S}_k,\\ S_kC_w + S_wC_k & \mathbb{T}_k,\end{cases} \end{aligned} \tag{7}$$

which will prove useful below.

III. Observability

Our objective in this section is to show under which conditions it is possible to reconstruct the state of the robot minimising the number of landmarks. We will consider two types of trajectories: 1. rectilinear trajectories ($\mathbb{S}_k, \forall k$), 2. bended trajectories ($\mathbb{T}_k$ for some k). Clearly, if the forward velocity of the robot is $v_k = 0$ (i.e., the robot is still or rotates around its axis), its configuration will be unobservable regardless of the number of landmarks. Therefore we will henceforth *assume* that $v_k \neq 0, \forall k$.

A first simple result concerns the use of a single landmark. In this case, any trajectory is unobservable because of the radial symmetry of the distance, as formally shown in the following:

Theorem 1: Consider a robot with kinematics (3), with output function (4) and $M = 1$ (one landmark) and moving with non-null forward velocity $v_k \neq 0$. The system state is unobservable for any trajectory.

Proof: Suppose, for the sake of simplicity and without loss of generality that the landmark is placed in the origin: $X_1 = 0$, $Y_1 = 0$ and that the initial state $s_0 = [x_0, y_0, \theta_0]$. Consider as a different initial state $s_0' = [x_0', y_0', \theta_0']$ such that **a)** $x_0^2 + y_0^2 = x_o'^2 + y_0'^2$, **b)** $x_0 C_0 + y_0 S_0 = x_0' C_0' + y_0' S_0'$ and **c)** $C_0 Y_0 - S_0 X_0 = C_0' Y_0' - S_0' X_0'$, where C_k and S_k are defined as in (6). It is easy to see that the initial distance from the landmark is the same for both point $\mathcal{Y}_{1,0} = \mathcal{Y}_{1,0}'$. In view of (5), the relations **a)** and **b)** imply that the distances will remain equal after one step: $\mathcal{Y}_{1,1} = \mathcal{Y}_{1,1}'$. Furthermore the use of the dynamic model (3) and of (7) shows, after some manipulation, that if conditions **b)** and **c)** hold true at step 0, they will also hold true at step 1: $x_1 C_1 + y_1 S_1 = x_1' C_1' + y_1' S_1'$ and $C_1 Y_1 - S_1 X_1 = C_1' Y_1' - S_1' X_1'$. A simple inductive argument leads us to the conclusion that the two trajectories starting from s_0 and s_0' will produce the same output at every step, making the state unobservable from the evolution of the measurements. ■

A. The case of two landmarks

Our intuition suggests that the simultaneous measurements of two distances could be more promising than the measurement of a single distance. Unfortunately, this is not true for rectilinear trajectories, as stated in the following.

Theorem 2: Consider a robot with kinematics (3), with output function (4), $M = 2$ (two landmarks) and moving with non-null forward velocity $v_k \neq 0$. If the system follows rectilinear trajectories (i.e., $\mathbb{S}_k, \forall k$), then its state is not globally observable.

Proof: Consider the case in which two tags are deployed parallel to the Y_w axis. This is not a loss of generality because any system can be reduced to this condition through a simple transformation of coordinates. With reference to Fig. 1, suppose the robot lies in point U_1 at step $k-1$ and moves to point U_2 at step k (green line). The output functions $\mathcal{Y}_{i,k}$, for $i = 1, 2$ in (4), will be given by the length of the

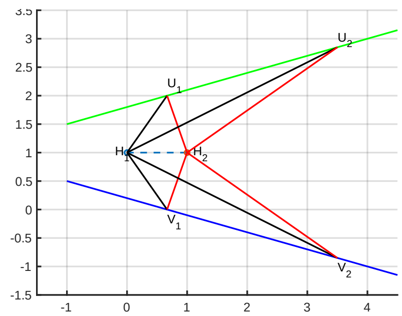

Fig. 1. Evolution of the system along a linear trajectory (green line) and its alias trajectory (blue line) producing the same measurement values for $\mathcal{Y}_k$, and $\mathcal{Y}_{k+1}$.

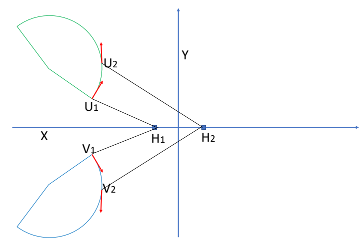

Fig. 2. Evolution of the system along a circular trajectory (green line) and its alias trajectory (blue line) producing the same measurement values

two segments $\overline{H_1 U_2}$ and $\overline{H_2 U_2}$. As we can see, the motion along an alias trajectory (blue line) from point V_1 to point V_2 produces the same readings, being the length of the two segments $\overline{H_1 V_2}$ and $\overline{H_2 V_2}$ respectively equal to $\overline{H_1 U_2}$ and $\overline{H_2 U_2}$. The same situation will be repeated at the next steps from points U_2 and V_2 onward. In summary, if the robot moves along a line, a specular motion along the symmetric to the line joining the two tags produces the same readings and is therefore indistinguishable. ■

Despite this negative result, the situation for two landmarks is not nearly as bad as it is for one. Indeed, as discussed in the proof, the possible ambiguity is reduced to a pair of symmetric lines, and by using standard results of nonlinear system analysis [11], it is possible to see that the system is locally observable [7]. For curvilinear motion, it is even possible to recover global observability. Indeed, the argument leading to the non observability of linear trajectories hinges on the point that linear trajectories that move symmetrically to the line joining the two tags generate the same outputs and are, therefore, globally unobservable. Following the same line of reasoning, we could conjecture that the same problem occurs for circular trajectories, i.e. $\omega \neq 0$ in (3). In fact, it is easy to see that the only way to obtain the same measurements is a circular trajectory that is symmetric to the actual trajectory with respect to the line joining the two landmarks. This is shown in Figure 2. If the system moves from U_1 to U_2 the distance from the two landmarks (i.e., the length of the segments $\overline{H_1 U_1}$ and $\overline{H_2 U_2}$) will be the same for the system moving on the "alias" arc of circle from the symmetric point V_1 and V_2. Luckily this is not an observability loss since the direction of motion has to be opposite (e.g., clockwise on the actual trajectory and counter-clockwise on the alias trajectory or viceversa), and the direction of the circular motion can be inferred from the sign of ω, which is assumed an available input in any observability analysis. This is formalised in the following.

Theorem 3: Consider a robot with kinematic (3), with output function (4), $M = 2$ (two landmarks) and moving with non-null forward velocity $v_k \neq 0$. Suppose that for two consecutive steps, the robot executes turns, e.g., it executes the sequence $\mathbb{T}_0\mathbb{T}_1$. Then the state of the system is globally observable if $\omega_0 T_s \neq h\pi$, and $\omega_1 T_s \neq h\pi$, for $h \in \mathbb{N}$.

Proof: Consider (5) with $i = 1, 2$ and $k = 0, 1$. Subtracting $\mathcal{Y}_{1,k}^2$ from $\mathcal{Y}_{1,k+1}^2$ (two consecutive readings of the $i = 1$ landmark) and $\mathcal{Y}_{2,k}^2$ from $\mathcal{Y}_{2,k+1}^2$ (two consecutive readings of the $i = 2$ landmark), we get:

$$\begin{aligned}
&\frac{1}{2}\left(\mathcal{Y}_{1,1}^2 - \mathcal{Y}_{2,1}^2 - \mathcal{Y}_{1,0}^2 + \mathcal{Y}_{2,0}^2\right) = \\
&\quad = 2A_0C_0(X_2 - X_1) + 2A_0S_0(Y_2 - Y_1) \\
&\frac{1}{2}\left(\mathcal{Y}_{1,2}^2 - \mathcal{Y}_{2,2}^2 - \mathcal{Y}_{1,1}^2 + \mathcal{Y}_{2,1}^2\right) = \\
&\quad = 2A_1C_1(X_2 - X_1) + 2A_1S_1(Y_2 - Y_1).
\end{aligned}$$

If the velocities remain constant, from (6) we have $A_0 = A_1$. Exploiting Equality (7), we can write

$$\begin{aligned}
&\begin{bmatrix} A_0(X_2 - X_1) & A_0(Y_2 - Y_1) \\ A_1\Sigma_1 & A_1\Sigma_2 \end{bmatrix} \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \\
&= \frac{1}{2}\begin{bmatrix} \left(\mathcal{Y}_{1,1}^2 - \mathcal{Y}_{2,1}^2 - \mathcal{Y}_{1,0}^2 + \mathcal{Y}_{2,0}^2\right) \\ \left(\mathcal{Y}_{1,2}^2 - \mathcal{Y}_{2,2}^2 - \mathcal{Y}_{1,1}^2 + \mathcal{Y}_{2,1}^2\right) \end{bmatrix},
\end{aligned}$$

where $\Sigma_1 = C_w(X_2 - X_1) + S_w(Y_2 - Y_1)$ and $\Sigma_2 = C_w(Y_2 - Y_1) - S_w(Y_2 - Y_1)$. Observe that the second row of the matrix is given by the first one rotated by ωT_s and scaled. Under the assumption that $\omega T_s \neq h\pi$, we have $A_0 \neq 0$, $A_1 \neq 0$ and that the second row of the matrix is linearly independent from the first. Therefore the matrix is invertible and the pair (C_0, S_0) is uniquely defined, which leads us to the angle $\theta_0 = \text{atan2}\,(S_0, C_0)$. Moreover, observe that:

$$\begin{aligned}
\mathcal{Y}_{1,0}^2 &= (x_0 - X_1)^2 + (y_0 - Y_1)^2 = \\
&= X_1^2 + x_0^2 + y_0^2 + Y_1^2 - 2X_1x_0 - 2Y_1y_0, \\
\mathcal{Y}_{2,0}^2 &= (x_0 - X_2)^2 + (y_0 - Y_2)^2 = \\
&= X_2^2 + x_0^2 + y_0^2 + Y_2^2 - 2X_2x_0 - 2Y_2y_0.
\end{aligned}$$

Subtracting the second equation from the first, we get:

$$\mathcal{Y}_{1,0}^2 - \mathcal{Y}_{2,0} - X_1^2 + x_2^2 - Y_1^2 + Y_2^2 = 2(X_2 - X_1)x_0 + 2(Y_2 - Y_1)y_0,$$

while another equation comes from (5):

$$\mathcal{Y}_{1,1}^2 - \mathcal{Y}_{1,0} - A_0^2 = 2A_0C_0\left(x_0 - X_1\right) + 2A_0S_0\left(y_0 - Y_1\right).$$

Therefore x_0 and y_0 can be found as the solution of a linear system whose matrix is

$$\begin{bmatrix} (X_2 - X_1) & (Y_2 - Y_1) \\ A_0C_0 & A_0S_0 \end{bmatrix},$$

which is invertible if the initial orientation is not aligned along the direction joining the two landmarks. Should this unfortunate situation happen, another step is needed. ■

The bottom line of this discussion is that if we use two landmarks, the vehicle can disambiguate its position as far as it turns for at least two time steps.

B. The case of three landmarks

The use of three non collinear landmarks solves any ambiguity and it enables global observability for any possible trajectory with $v_k \neq 0$, as stated in the following.

Theorem 4: Consider a robot with kinematic (3), with output function (4), $M = 3$ (three landmarks) and moving with non-null forward velocity $v_k \neq 0$. Then the state of the system is globally observable for any trajectory.

Proof: The Cartesian coordinates (x_0, y_0) can be found using the usual trilateration technique. In our notation, starting from three landmark measurements (4)

$$\begin{aligned}
\mathcal{Y}_{i,0}^2 &= (x_0 - X_i)^2 + (y_0 - Y_i)^2 = \\
&= X_i^2 + x_0^2 + y_0^2 + Y_i^2 - 2X_ix_0 - 2Y_iy_0,
\end{aligned}$$

we can subtract $\mathcal{Y}_{2,0}^2$ and $\mathcal{Y}_{3,0}^2$ to $\mathcal{Y}_{1,0}^2$, hence generating a system of two equations in two unknowns (x_0, y_0), whose coefficient matrix $\Sigma = \left[\begin{smallmatrix} X_2 - X_1 & Y_2 - Y_1 \\ X_3 - X_1 & Y_3 - Y_1 \end{smallmatrix}\right]$ is invertible since the points are not collinear. The result is then

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{2}\Sigma^{-1}\begin{bmatrix} \mathcal{Y}_{1,0}^2 - \mathcal{Y}_{2,0}^2 - X_1^2 - Y_1^2 + X_2^2 + Y_2^2 \\ \mathcal{Y}_{1,0}^2 - \mathcal{Y}_{3,0}^2 - X_1^2 - Y_1^2 + X_3^2 + Y_3^2 \end{bmatrix}.$$

For the computation of angle θ_0, we can proceed as in the proof of Theorem 3. Considering (5) for $i = 1, 2, 3$ and $k = 0$, then subtracting $\mathcal{Y}_{2,1}^2$ and $\mathcal{Y}_{3,1}^2$ to $\mathcal{Y}_{1,1}^2$, we end up with a system of linear equations, i.e.

$$\begin{bmatrix} X_2 - X_1 & Y_2 - Y_1 \\ X_3 - X_1 & Y_3 - Y_1 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \frac{1}{2A_0}\begin{bmatrix} \mathcal{Y}_{1,1}^2 - \mathcal{Y}_{2,1}^2 - \mathcal{Y}_{1,0}^2 + \mathcal{Y}_{2,0}^2 \\ \mathcal{Y}_{1,1}^2 - \mathcal{Y}_{3,1}^2 - \mathcal{Y}_{1,0}^2 + \mathcal{Y}_{3,0}^2 \end{bmatrix},$$

that has a unique solution in (C_0, S_0) if, again, the points are not collinear. ■

IV. Mapping

Let us now consider the case in which the position of the landmarks is unknown and that the robot needs to construct a map of the environment associating some coordinates with each of the landmarks. As introduced earlier, our problem is in this case to reconstruct (X_i, Y_i) with respect to the origin of $\langle W \rangle$ placed in (x_0, y_0) and with the X_w axis oriented along θ_0. Let us start form the simplified situation in which we want to map three landmarks (X_i, Y_i), for $i = 1, 2, 3$, that remain always in sight of the robot and that the robot moves for 3 steps. Using (5) for $i = 1, 2, 3$ and for $k = 0, 1$, the following matrix equation can be derived

$$L_3B_3 = H_3, \tag{8}$$

where $B_3 = [X_1, Y_1, X_2, Y_2, X_3, Y_3]^T$, $L_3 = \text{diag}(P_{0,1}, P_{0,1}, P_{0,1})$, $P_{0,1} = \left[\begin{smallmatrix} F_0 & G_0 \\ F_1 & G_1 \end{smallmatrix}\right]$, $F_i = -2A_iC_i$, $G_i = -2A_iS_i$, $H_3 = [H_{1,1}, H_{1,2}, H_{2,1}, H_{2,2}, H_{3,1}, H_{3,2}]^T$, $H_{i,j} = \mathcal{Y}_{i,j} - \mathcal{Y}_{i,j-1} - A_{j-1}^2 - 2A_{j-1}C_{j-1}x_{j-1} -$

$2A_{j-1}S_{j-1}y_{j-1}$, that, once solved, returns the solution to the mapping problem. In order to solve the problem, let us consider the invertibility of the system matrix. Given the recursive structure of the matrix it is possible to see that

$$D = \det\left(\begin{bmatrix} F_0 & G_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_0 & G_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_0 & G_0 \\ F_1 & G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_1 & G_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_1 & G_1 \end{bmatrix}\right) = \\ = (F_0G_1 - F_1G_0) = 4A_0A_1(C_0S_1 - C_1S_0)^3$$

Using the $\mathbb{S}_k$ if and $\mathbb{T}_k$ notation and recalling (6) and (7), we have after some algebraic manipulations

$$D^{1/3} = \begin{cases} 4v_0v_1T_s^2 \sin(\theta_1 - \theta_0) = 0, & \mathbb{S}_0\mathbb{S}_1, \\ 8\frac{v_0v_1T_s}{\omega_1}\sin^2\left(\frac{\omega_1T_s}{2}\right), & \mathbb{S}_0\mathbb{T}_1, \\ 8\frac{v_1v_0T_s}{\omega_0}\sin^2\left(\frac{\omega_0T_s}{2}\right), & \mathbb{T}_0\mathbb{S}_1, \\ 8\frac{v_1v_0}{\omega_0\omega_1}\sin\left(\frac{\omega_0T_s}{2}\right)\sin\left(\frac{\omega_1T_s}{2}\right)\sin\left(\frac{(\omega_0+\omega_1)T_s}{2}\right), & \mathbb{T}_0\mathbb{T}_1. \end{cases}$$

Hence, if the robot moves with the three landmarks in sight their position can be reconstructed in two steps if the robot: a) moves one step turning (i.e., if we exclude the case $\mathbb{S}_0\mathbb{S}_1$) with an angular velocity $\omega_k \neq \frac{2\pi}{T_s}$; b) turns for two consecutive steps (i.e., $\mathbb{T}_0\mathbb{T}_1$) and $\omega_0 \neq -\omega_1$. Under one of these hypotheses, given the particular structure of the matrix, it is also possible to compute the positions of the landmarks in closed-form for the i-th landmark as

$$X_i = \frac{A_1S_1H_{i,1} - A_0S_0H_{i,2}}{\Sigma_d}, Y_i = \frac{-A_1C_1H_{i,1} + A_0C_0H_{i,2}}{\Sigma_d},$$

where $\Sigma_d = 2A_0A_1(C_0S_1 - C_1S_0)$. The idea sketched above can be generalised to an arbitrary number of landmarks relaxing the in-sight requirement along the motion, as shown in the following.

Theorem 5: Assume that there are M landmarks and that the robot makes a number of steps such that it sees each landmark at least twice in two different steps. Moreover, assume that at least one of the steps in which the robot sees each landmark is a turn. Then it is possible to reconstruct the position of the M landmarks.

Proof: Let K be the total number of steps. Let j be the index associated with the j-th landmark. Let $j_1 \in \{1, 2, \ldots, K\}$ represent the first step in which the landmark is in sight and $j_2 \in \{1, 2, \ldots, K\} \setminus \{j_1\}$ the second step. After a few re-arrangements, it is possible to study the resulting system of linear equations $L_MB_M = H_M$, which is a generalised version of (8) with M landmarks. Given the structure of the matrix, it is possible to show that:

$$\det(A) = \prod_{j=1}^{M} (F_{j_1}G_{j_2} - F_{j_2}G_{j_1}).$$

Therefore, the matrix is invertible if and only if $F_{j_1}G_{j_2} - F_{j_2}G_{j_1} \neq 0, \forall j$. Proceeding as in the discussion made above for three landmarks, we can see that the acceptable transitions are: $\mathbb{T}_{j_1}\mathbb{S}_{j_2}$, $\mathbb{T}_{j_1}\mathbb{T}_{j_2}$, $\mathbb{S}_{j_1}\mathbb{T}_{j_2}$, with the same conditions stated previously. ■

V. Extensions to 3D

The results above are easy to generalise for fixed wing aircrafts having their kinematics described by the following equation:

$$\dot{x} = v\cos\phi\cos\theta,\ \dot{y} = v\cos\phi\sin\theta, \\ \dot{z} = v\sin\phi,\ \dot{\theta} = \frac{g}{v}\tan\psi \tag{9}$$

In this model, $(x,\ y,\ z)$ are the Cartesian coordinates of the origin of a frame attached to the vehicle, v is the air-speed, g is the magnitude of the gravity at the sea level, θ is the heading angle, ϕ is the pitch angle and ψ is the roll angle. In this context, we are customarily assuming that the vehicle moves in coordinated turn condition and that the roll and pitch dynamics are much faster than the heading and altitude dynamics. Therefore, we can consider the roll and flight path angle, respectively ψ and ϕ, as command variables [13]. Assuming piece-wise constant inputs and velocity, we can write the state $s_k = [x_k, y_k, z_k, \theta_k]^T$ discrete time equivalent dynamic with sampling period T_s:

$$\begin{aligned} x_{k+1} &= \begin{cases} x_k + v_kT_s\cos\phi_k\cos\theta_k & \text{if } \psi_k = 0, \\ x_k + 2\frac{v_k}{\omega_k}\cos\phi_k\sin\left(\frac{\omega_k}{2}T_s\right)a_x & \text{otherwise,} \end{cases} \\ y_{k+1} &= \begin{cases} y_k + v_kT_s\cos\phi_k\sin\theta_k & \text{if } \psi_k = 0, \\ y_k + 2\frac{v_k}{\omega_k}\cos\phi_k\sin\left(\frac{\omega_k}{2}T_s\right)a_y & \text{otherwise,} \end{cases} \\ z_{k+1} &= z_k + v_kT_s\sin\phi_k, \\ \theta_{k+1} &= \theta_k + \omega_kT_s, \end{aligned} \tag{10}$$

where $a_x = \cos\left(\theta_k + \frac{\omega_k}{2}T_s\right)$, $a_y = \sin\left(\theta_k + \frac{\omega_k}{2}T_s\right)$ and $\omega_k = \frac{g}{v_k}\tan\psi_k$. Given the model (10), we can re-write (5) as follows:

$$\mathcal{Y}_{i,k+1}^2 = \\ = \begin{cases} \mathcal{Y}_{i,k}^2 + (v_kT_s)^2 + 2v_kT_s\cos\phi_k\cos\theta_k(x_k - X_i) + \\ +2v_kT_s\cos\phi_k\sin\theta_k(y_k - Y_i) + \\ +2v_kT_s\sin\phi_k(z_k - Z_i) \quad \text{If } \omega_k = 0 \\ \mathcal{Y}_{i,k}^2 + \left(2\frac{v_k}{\omega_k}\cos\phi_k\right)^2 + (v_kT_s\sin\phi_k)^2 + \\ +4\frac{v_k}{\omega_k}\cos\phi_k\cos\left(\theta_k + \frac{\omega_kT_s}{2}\right)(x_k - X_i) + \\ +4\frac{v_k}{\omega_k}\cos\phi_k\sin\left(\theta_k + \frac{\omega_kT_s}{2}\right)(y_k - Y_i) + \\ +2v_kT_s\sin\phi_k(z_k - Z_i) \quad \text{If } \omega_k \neq 0. \end{cases} \tag{11}$$

By using this expression, it is possible to revisit the results we found for the planar unicycle-like robot, for which we give an example in the following.

Theorem 6: Consider the case of $M = 4$ non coplanar landmarks in position $H_i = (X_i, Y_i, Z_i)$, $i = 1, \ldots, 4$. Then, the whole state s_0 can be reconstructed using two consecutive measurements $\mathcal{Y}_{0,i}$, $\mathcal{Y}_{1,i}$.

Proof: The proof follows the one presented for Theorem 4. Indeed, introduce the following notation.

$$A_k = \begin{cases} 2v_k T_s \cos\phi_k & \text{if } \omega_k = 0, \\ 4\frac{v_k}{\omega_k}\cos\phi_k & \text{if } \omega_k \neq 0, \end{cases}$$
$$B_k = \begin{cases} (v_k T_s)^2 & \text{if } \omega_k = 0, \\ \left(2\frac{v_k}{\omega_k}\cos\phi_k\right)^2 + (v_k T_s \sin\phi_k)^2 & \text{if } \omega_k \neq 0, \end{cases}$$
$$D_k = 2v_k T_s \sin\phi_k$$
$$C_k = \begin{cases} \cos\theta_k & \text{if } \omega = 0, \\ \cos(\theta_k + \omega_k T_s/2) & \text{if } \omega_k \neq 0, \end{cases}$$
$$S_k = \begin{cases} \sin\theta_k & \text{if } \omega = 0, \\ \sin(\theta_k + \omega_k T_s/2) & \text{if } \omega_k \neq 0. \end{cases}$$

It is then possible to rewrite (11) as:

$$\begin{aligned} \mathcal{Y}_{i,k+1}^2 =& \mathcal{Y}_{i,k}^2 + B_k + A_k C_k (x_k - X_i) + \\ &+ A_k S_k (y_k - Y_i) + D_k (z_k - Z_i) \end{aligned} \quad (12)$$

If we consider one step of evolution, from $k = 0$ and subtract the last three landmark equations $j = 2, 3, 4$ from the first one, we find

$$\begin{aligned} &\mathcal{Y}_{1,1}^2 - \mathcal{Y}_{j,1}^2 - (\mathcal{Y}_{1,0}^2 - \mathcal{Y}_{j,0}^2 + D_0(Z_j - Z_1)) = \\ &= A_0 C_0 (X_j - X_1) + A_0 S_0 (Y_j - Y_1). \end{aligned}$$

By defining the left hand side as Σ_j, the previous relation can be rewritten in matrix form as:

$$\frac{1}{A_0}\begin{bmatrix} \Sigma_2 \\ \Sigma_3 \\ \Sigma_4 \end{bmatrix} = \begin{bmatrix} X_2 - X_1 & Y_2 - Y_1 \\ X_3 - X_1 & Y_3 - Y_1 \\ X_4 - X_1 & Y_4 - Y_1 \end{bmatrix} \begin{bmatrix} C_0 \\ S_0 \end{bmatrix}.$$

If the four landmarks are not coplanar, then any landmark triplet randomly extracted is not collinear, and, hence, the solution of the system of linear equation in S_0 and C_0 can be solved, which leads to θ_0.

For the computation of (x_0, y_0, z_0), we can apply a rather standard trilateration technique, yielding the linear system

$$\begin{aligned} &\begin{bmatrix} X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X_3 - X_1 & Y_3 - Y_1 & Z_3 - Z_1 \\ X_4 - X_1 & Y_4 - Y_1 & Z_4 - Z_1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \\ &\frac{1}{2}\begin{bmatrix} \mathcal{Y}_{1,0}^2 - \mathcal{Y}_{2,0}^2 - X_1^2 - Y_1^2 - Z_1^2 + X_2^2 + Y_2^2 + Z_2^2 \\ \mathcal{Y}_{1,0}^2 - \mathcal{Y}_{3,0}^2 - X_1^2 - Y_1^2 - Z_1^2 + X_3^2 + Y_3^2 + Z_3^2 \\ \mathcal{Y}_{1,0}^2 - \mathcal{Y}_{4,0}^2 - X_1^2 - Y_1^2 - Z_1^2 + X_4^2 + Y_4^2 + Z_4^2 \end{bmatrix}, \end{aligned}$$

that can be solved since the selected triplet is not coplanar. ■

VI. Conclusion

In this paper we have presented and discussed conditions for the global observability of a vehicle extracting information on its current location from a number of range sensors deployed in the environment. A possible paradigm of this rather general analysis is a robot travelling across an environment instrumented with a number of UWB landmarks. The main results are the conditions that make the system state globally observable for different families of trajectories and for different deployments of the sensors. A key enabler for our analysis is the use of the discrete–time formulation induced by the sensor sampling time that simplifies the analytical form of the equations that need to be solved to reconstruct the state. The same formulation allows us to set up the complementary problem: constructing a map of the landmarks position with respect to the initial position of the robot. The analysis has been developed for systems moving in the plane, but we have shown its easy generalisation to the three dimensional case (e.g., fixed wing aerial vehicles).

We envision possible extensions of the work in different directions, such as: 1. using the observability analysis to develop effective filtering techniques based on a precise characterisation of the noise propagation, 2. using our localisation technique with noisy sensors to set up a Simultaneous Localisation and Mapping (SLAM) problem, 3.extending our results toward different types of sensors (e.g., passive RFID), 4. accounting for measurements affected by Poissonian noise (e.g., data losses due to occasional occlusions).

References

[1] V. Magnago, L. Palopoli, R. Passerone, D. Fontanelli, and D. Macii, "Effective Landmark Placement for Robot Indoor Localization With Position Uncertainty Constraints," *IEEE Trans. on Instrumentation and Measurement*, pp. 1–13, 2019, available online.

[2] G. P. Huang, N. Trawny, A. I. Mourikis, and S. I. Roumeliotis, "Observability-based consistent ekf estimators for multi-robot cooperative localization," *Autonomous Robots*, vol. 30, no. 1, pp. 99–122, 2011.

[3] G. L. Mariottini, G. Pappas, D. Prattichizzo, and K. Daniilidis, "Vision-based localization of leader-follower formations," in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 635–640.

[4] S. Cedervall and X. Hu, "Nonlinear observers for unicycle robots with range sensors," *IEEE transactions on automatic control*, vol. 52, no. 7, pp. 1325–1329, 2007.

[5] H. Sert, W. Perruquetti, A. Kokosy, X. Jin, and J. Palos, "Localizability of unicycle mobiles robots: An algebraic point of view," in *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems*. IEEE, 2012, pp. 223–228.

[6] F. A. Belo, P. Salaris, D. Fontanelli, and A. Bicchi, "A complete observability analysis of the planar bearing localization and mapping for visual servoing with known camera velocities," *International Journal of Advanced Robotic Systems*, vol. 10, no. 4, p. 197, 2013.

[7] A. Bicchi, D. Prattichizzo, A. Marigo, and A. Balestrino, "On the observability of mobile vehicle localization," in *Theory and Practice of Control and Systems*. World Scientific, 1998, pp. 142–147.

[8] A. Martinelli, "The unicycle in presence of a single disturbance: Observability properties," in *2017 Proceedings of the Conference on Control and its Applications*. SIAM, 2017, pp. 62–69.

[9] V. Magnago, L. Palopoli, A. Motroni, P. Nepa, D. Fontanelli, D. Macii, A. Buffi, and B. Tellini, "Robot Localisation based on Phase Measures of backscattered UHF-RFID Signals," in *Proc. IEEE Int. Instrumentation and Measurement Technology Conference (I2MTC)*. Auckland, New Zealand: IEEE, May 2019, to appear.

[10] H. Sert, A. Kksy, and W. Perruquetti, "A single landmark based localization algorithm for non-holonomic mobile robots," in *2011 IEEE International Conference on Robotics and Automation*, May 2011, pp. 293–298.

[11] A. Isidori, *Nonlinear control systems*. Springer Science & Business Media, 2013.

[12] F. Arrichiello, G. Antonelli, A. Aguiar, and A. Pascoal, "An observability metric for underwater vehicle localization using range measurements," *Sensors*, vol. 13, no. 12, pp. 16 191–16 215, 2013.

[13] R. W. Beard, J. Ferrin, and J. Humpherys, "Fixed wing uav path following in wind with input constraints," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 6, pp. 2103–2117, 2014.