

SHARP GEOMETRIC INEQUALITIES FOR CLOSED HYPERSURFACES IN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE.

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ABSTRACT. In this paper we consider complete noncompact Riemannian manifolds (M, g) with nonnegative Ricci curvature and Euclidean volume growth, of dimension $n \geq 3$. For every bounded open subset $\Omega \subset M$ with smooth boundary, we prove that

$$\int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq \text{AVR}(g) |\mathbb{S}^{n-1}|,$$

where H is the mean curvature of $\partial\Omega$ and $\text{AVR}(g)$ is the asymptotic volume ratio of (M, g) . Moreover, the equality holds true if and only if $(M \setminus \Omega, g)$ is isometric to a truncated cone over $\partial\Omega$. An optimal version of Huisken's Isoperimetric Inequality for 3-manifolds is obtained using this result. Finally, exploiting a natural extension of our techniques to the case of parabolic manifolds, we also deduce an enhanced version of Kasue's non existence result for closed minimal hypersurfaces in manifolds with nonnegative Ricci curvature.

1. INTRODUCTION AND MAIN RESULTS

The classical Willmore inequality [74] for a bounded domain Ω of \mathbb{R}^3 with smooth boundary says that

$$\int_{\partial\Omega} H^2 d\sigma \geq 16\pi, \tag{1.1}$$

where H is the mean curvature of $\partial\Omega$. Such an inequality has been extended in [19] to submanifolds of any co-dimension in \mathbb{R}^n , for $n \geq 3$. In particular, for a bounded domain Ω in \mathbb{R}^n with smooth boundary there holds

$$\int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq |\mathbb{S}^{n-1}|, \tag{1.2}$$

with equality attained if and only if Ω is a ball. Implicit in this statement is the fact that the underlying metric by which H and $d\sigma$ are computed is the Euclidean metric $g_{\mathbb{R}^n}$. Note that the above rigidity statement can be rephrased by saying that the equality in (1.1) is fulfilled if and only if $(\partial\Omega, g_{\partial\Omega})$ is homothetic to $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, where $g_{\partial\Omega}$ is the metric induced by $g_{\mathbb{R}^n}$ on the submanifold $\partial\Omega$ and $g_{\mathbb{S}^{n-1}}$ is the standard round metric.

Recently, in [4], the Willmore-type inequality (1.2) and the corresponding rigidity statement have been deduced as a consequence of suitable monotonicity-rigidity properties of the function

$$U(t) = t^{-(n-1)} \int_{\{u=t\}} |Du|^{n-1} d\sigma, \quad t \in (0, 1], \tag{1.3}$$

associated with the level set flow of the electrostatic potential u generated by the uniformly charged body Ω . In other words, u is the unique harmonic function in $\mathbb{R}^n \setminus \overline{\Omega}$ which vanishes at infinity and such that $u = 1$ on $\partial\Omega$. More precisely, what is proven in [4] is that the function U is nondecreasing and that this monotonicity is strict unless Ω is a ball. Once this fact is established, the proof of (1.2) consists of a few lines. Indeed, exploiting first the global feature

of the monotonicity (i.e. $U(1) \geq U(0^+)$) and using then the asymptotic expansion at infinity of u and $|Du|$ one gets

$$\int_{\partial\Omega} |Du|^{n-1} d\sigma = U(1) \geq \lim_{t \rightarrow 0^+} U(t) = (n-2)^{n-1} |\mathbb{S}^{n-1}|. \quad (1.4)$$

On the other hand, computing the derivative at $t = 1$, yields

$$U'(1) = (n-2) \int_{\partial\Omega} |Du|^{n-2} \left[H - \left(\frac{n-1}{n-2}\right) |Du| \right] d\sigma,$$

and using $U'(1) \geq 0$, we deduce that

$$\int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq \int_{\partial\Omega} \left| \frac{Du}{n-2} \right|^{n-1} d\sigma, \quad (1.5)$$

where we have also applied the Hölder inequality. The coupling of the latter inequality with the former (1.4) yields the desired (1.2). In this paper, we show that the strategy described above can be adapted to a much more general setting, giving rise to a new Willmore-type inequalities, holding on manifolds with nonnegative Ricci curvature of dimension greater than three.

Throughout this paper, we systematically assume that the dimension of the underlying ambient manifold M is at least 3.

Our main result reads:

Theorem 1.1 (Willmore-type inequality). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth. If $\Omega \subset M$ is a bounded and open subset with smooth boundary, then*

$$\int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq \text{AVR}(g) |\mathbb{S}^{n-1}|, \quad (1.6)$$

where $\text{AVR}(g) \in (0, 1]$ is the asymptotic volume ratio of (M, g) . Moreover, the equality holds if and only if $(M \setminus \Omega, g)$ is isometric to

$$\left([r_0, +\infty) \times \partial\Omega, dr \otimes dr + (r/r_0)^2 g_{\partial\Omega} \right), \quad \text{with } r_0 = \left(\frac{|\partial\Omega|}{\text{AVR}(g) |\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

In particular, $\partial\Omega$ is a connected totally umbilic submanifold with constant mean curvature.

Let us remark that no connectedness assumption on Ω is required in the above statement. On the other hand, if the equality holds in (1.6), then we obtain that $\partial\Omega$ is connected as a by-product of the rigidity statement combined with the fact that (M, g) has exactly one end, see Proposition 2.10. We recall that since $\text{Ric} \geq 0$ in the above statement, then, by the classical Bishop-Gromov Volume Comparison Theorem, the function

$$(0, +\infty) \ni r \mapsto \Theta(r) = \frac{n|B(p, r)|}{r^n |\mathbb{S}^{n-1}|}$$

is nonincreasing. In particular, we have that the asymptotic volume ratio

$$\text{AVR}(g) = \lim_{r \rightarrow +\infty} \Theta(r)$$

is well defined. Moreover, we have that this limit does not depend on the point $p \in M$, and that $\lim_{r \rightarrow 0^+} \Theta(r) = 1$. Hence, we have that $0 \leq \text{AVR}(g) \leq 1$. Moreover, $\text{AVR}(g) = 1$ if and only if $(M, g) = (\mathbb{R}^n, g_{\mathbb{R}^n})$. Assuming Euclidean volume growth means assuming $\text{AVR}(g) > 0$.

Observe in particular that for $n = 3$, if $\text{Ric} \equiv 0$ then (M, g) is isometric to $(\mathbb{R}^3, g_{\mathbb{R}^3})$ and consequently $\text{AVR}(g) = 1$. On the other hand, for $n \geq 4$ there exists an important class of complete noncompact Ricci flat Riemannian manifolds with $0 < \text{AVR}(g) < 1$, that is the class of Ricci flat *Asymptotically Locally Euclidean* (ALE for short) manifolds. We refer the reader to Definition 4.13 for the precise notion. For the time being, we just recall that a n -dimensional Riemannian manifold is ALE if it is asymptotic to $((\mathbb{R}^n \setminus \{0\})/\Gamma, g_{\mathbb{R}^n})$, where Γ is a finite subgroup of $\text{SO}(n)$ acting freely on $\mathbb{R}^n \setminus \{0\}$. This family of Riemannian manifolds is widely studied. In this regard, we first mention that in [7] it is proved that any Ricci flat manifold with Euclidean volume growth and strictly faster than quadratic curvature decay is actually ALE. Moreover, we point out that 4-dimensional Ricci flat ALE manifolds appear as important examples of *gravitational instantons*, that are noncompact hyperkähler 4-manifolds with decaying curvature at infinity, introduced by Hawking in [36], in the framework of his Euclidean quantum gravity theory. An explicit example is given by the famous Eguchi-Hanson metric, introduced in [31], where $n = 4$, $\text{Ric} \equiv 0$ and $\Gamma = \mathbb{Z}_2$. We remark that ALE gravitational instantons are completely classified in [45] and [46]. Concerning the general class of gravitational instantons, let us cite, after the important works of Minerbe [60, 61, 62], the recent PhD thesis [20], where gravitational instantons with strictly faster than quadratic curvature decay are also classified. We refer the reader to the latter work and to the references therein for a more complete picture on this subject. The following corollary is the direct application of Theorem 1.1 to the class of ALE manifolds with nonnegative Ricci curvature.

Corollary 1.2. *Let (M, g) be an ALE Riemannian manifold with $\text{Ric} \geq 0$. Then,*

$$\inf \left\{ \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \mid \Omega \subset M \text{ bounded and smooth} \right\} = \frac{|\mathbb{S}^{n-1}|}{\text{card}(\Gamma)}. \quad (1.7)$$

Moreover, if the infimum is attained by some Ω , then $M \setminus \Omega$ is isometric to

$$\left([r_0, +\infty) \times (\mathbb{S}^{n-1}/\Gamma), dr \otimes dr + r^2 g_{\mathbb{S}^{n-1}/\Gamma} \right), \quad \text{with } r_0 = \left(\frac{\text{card}(\Gamma) |\partial\Omega|}{|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}, \quad (1.8)$$

for some $r_0 > 0$ and some finite subgroup Γ of $\text{SO}(n)$. In particular, $(\partial\Omega, g_{\partial\Omega})$ is homothetic to $(\mathbb{S}^{n-1}/\Gamma, g_{\mathbb{S}^{n-1}/\Gamma})$.

We observe that on ALE manifolds the rigidity is much stronger, being characterized by cones whose cross sections are homothetic to \mathbb{S}^{n-1}/Γ . Notice also that, if Γ is trivial, one recovers the classical Willmore inequality (1.2). Moreover, we point out that (1.7) also says that, in every ALE manifold with nonnegative Ricci curvature, the lower bound we find for the Willmore-type functional is actually an infimum. This fact holds true for a larger class of manifolds. Indeed, as proved in Theorem 4.12, it is sufficient to assume the hypotheses of Theorem 1.1 together with a quadratic curvature decay condition. Understanding metric and topological consequences of curvature decay conditions is a very interesting and widely studied problem in geometric analysis. Dropping any attempt of being complete, we refer the interested reader to the aforementioned [7], to the seminal [17], to [66], where the case $n = 3$ is considered, and to [77] and the references therein. To make the picture more complete, let us also mention that Willmore-type inequalities are proven in [3] for asymptotically flat (AE) static metrics in the framework of General Relativity, and in [69] for integral 2-varifolds in Cartan-Hadamard manifolds.

1.1. Monotonicity formulas in nonparabolic manifolds. Theorem 1.1 will be deduced as a consequence of the monotonicity-rigidity properties of the function U defined as in (1.3), where

now u is the unique solution to the problem

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(q) \rightarrow 0 & \text{as } d(O, q) \rightarrow +\infty, \end{cases} \quad (1.9)$$

with O being a fixed point in Ω and d a distance function on M . Observe that the hypotheses of Theorem 1.1, namely the nonnegativity of the Ricci tensor and the Euclidean volume growth, guarantee the existence of the solution to problem (1.9), as explained in Section 2.

In order to make the treatment of problem (1.9) more transparent, it is better to avoid the situation in which the function u is locally constant (and equal to 1) on some connected component of its complement $M \setminus \Omega$. To this purpose, we are going to consider bounded, open, smooth subsets *with no holes, i.e., such that their complement has no compact connected components*. On this regard, notice that, if (M, g) is not a cylinder, a subset $\Omega \subset M$ of this kind is such that $\partial\Omega$ coincides with the boundary ∂E of the unique unbounded component E of $M \setminus \Omega$ (see Proposition 2.10). Moreover, by the Strong Maximum Principle, one has that $\partial\Omega$ also coincides with the level set $\{u = 1\}$ of the solution u to (1.9), provided the latter exists.

Once that the monotonicity-rigidity of U is known, the proof of Theorem 1.1 consists of exactly the same formal steps outlined in (1.4)–(1.5), the main difference being the careful computation of the limit value

$$\lim_{t \rightarrow 0^+} U(t) = \text{AVR}(g)(n-2)^{n-1} |\mathbb{S}^{n-1}|.$$

We remark that whereas in the classical Euclidean context the limit was deduced from the pointwise asymptotic expansion of u and $|Du|$ at infinity, here it will be deduced from some quite delicate integral asymptotic expansions, in the spirit of [25]. This is an important technical difference from the Euclidean case. Such integral asymptotics will be worked out in Section 4. For completeness, we state the monotonicity-rigidity result concerning U in the wider generality of Theorem 1.3 below. Indeed, the same monotonicity-rigidity properties are shared by the whole family of functions U_β , with $\beta \geq (n-2)/(n-1)$, where $U_\beta : (0, 1] \rightarrow \mathbb{R}$ is defined as

$$U_\beta(t) = t^{-\beta \left(\frac{n-1}{n-2}\right)} \int_{\{u=t\}} |Du|^{\beta+1} d\sigma. \quad (1.10)$$

Note that U_β coincides with the function U defined in (1.3) when $\beta = n-2$. Moreover, such a Monotonicity-Rigidity Theorem holds for a wider class of manifolds than the ones with Euclidean volume growth. Namely, we prove it for any complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ admitting a solution to (1.9). This class of manifolds coincides with the thoroughly studied class of nonparabolic ones, as we are going to see in Section 2.

Theorem 1.3 (Monotonicity-Rigidity Theorem for nonparabolic manifolds). *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$. Given a bounded and open subset $\Omega \subset M$ with smooth boundary and no holes, let u be the solution to problem (1.9) and let U_β be the function defined in (1.10). Then, for every $\beta \geq (n-2)/(n-1)$, the function U_β is differentiable, with derivative*

$$\frac{dU_\beta}{dt}(t) = \beta t^{-\beta \left(\frac{n-1}{n-2}\right)} \int_{\{u=t\}} |Du|^\beta \left[H - \left(\frac{n-1}{n-2}\right) |D \log u| \right] d\sigma, \quad (1.11)$$

where \mathbf{H} is the mean curvature of the level set $\{u = t\}$ computed with respect to the unit normal vector field $\nu = -Du/|Du|$. The derivative of U_β fulfils

$$\begin{aligned} \frac{dU_\beta}{dt}(t) = \frac{\beta}{t^2} \int_{\{u < t\}} u^{2-\beta \left(\frac{n-1}{n-2}\right)} |Du|^{\beta-2} \left\{ \text{Ric}(Du, Du) + \right. \\ \left. + \left[|DDu|^2 - \left(\frac{n}{n-1}\right) |D|Du||^2 \right] + \right. \\ \left. + \left(\beta - \frac{n-2}{n-1}\right) |D^T|Du||^2 + \right. \\ \left. + \left(\beta - \frac{n-2}{n-1}\right) |Du|^2 \left[\mathbf{H} - \left(\frac{n-1}{n-2}\right) |D \log u| \right]^2 \right\} d\mu. \end{aligned} \quad (1.12)$$

In particular, U_β is nondecreasing. Moreover, $(dU_\beta/dt)(t_0) = 0$ for some $t_0 \leq 1$ and some $\beta \geq (n-2)/(n-1)$ if and only if (M, g) has Euclidean volume growth and $(\{u \leq t_0\}, g)$ is isometric to

$$\left([r_0, +\infty) \times \{u = t_0\}, dr \otimes dr + \left(\frac{r}{r_0}\right)^2 g_{\{u=t_0\}} \right), \quad \text{with } r_0 = \left(\frac{|\{u = t_0\}|}{\text{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}. \quad (1.13)$$

In this case, in particular, $\{u = t_0\}$ is a connected totally umbilic submanifold with constant mean curvature.

Observe that the quantity on the right hand side of (1.12) is nonnegative because of the refined Kato's inequality for harmonic functions reading as

$$|DDu|^2 \geq \left(\frac{n}{n-1}\right) |D|Du||^2. \quad (1.14)$$

The vanishing of (1.11) can in particular be interpreted as an overdetermining condition on (1.9), forcing the ambient manifold to split as a cone and the solution u to be radially symmetric. We mention, in this context of PDE's and splitting results, the paper [33]. Although the monotonicity of the U_β 's in the above theorem holds on every nonparabolic manifold with $\text{Ric} \geq 0$, the rigidity statement seems to be particularly effective only on manifolds with Euclidean volume growth. On the other hand, in certain contexts (e.g., in the study of ALF, ALG and ALH gravitational instantons [60, 61, 20]) it is natural to consider different types of volume growths. Studying monotonicity and rigidity properties of quantities tailored on manifolds with these kinds of behaviours at infinity will be the object of future works.

Remark 1.4. *As already observed, the Willmore-type inequalities follow from Theorem 1.3 applied with $\beta = n - 2$. In particular, for this purpose, it is sufficient to prove the latter just for $\beta \geq 1$. In fact, in this paper we propose a proof in full details suited to this range of parameters. To also cover the range $(n-2)/(n-1) \leq \beta < 1$, it is sufficient to adapt the arguments recently introduced in [4]. Some guidance in this sense will be provided in the subsequent Remark 3.8 and Remark 3.10.*

As for the Euclidean case, Theorem 1.3 will actually be proved working in the manifold $(M \setminus \Omega, \tilde{g})$ where \tilde{g} is conformally related to g by

$$\tilde{g} = u^{\frac{2}{n-2}} g,$$

where u is a solution to (1.9). In this setting, integral identities and splitting techniques are employed to infer the monotonicity of (the conformal analogue of) U_β and the related rigidity. We point out that these techniques can easily produce a more general version of Theorem 1.3 for nonparabolic ends with $\text{Ric} \geq 0$ of a noncompact Riemannian manifold. This *conformal splitting*

method, introduced in [2], has proved to be fruitful in various other situations. In [3, 10] and [9] it has been applied to the relativistic setting, while in [34, 1], geometric and analytic properties of p -harmonic functions in exterior domains have been studied, obtaining as a by-product an extended version of the classical Minkowski inequality. Moreover, a different approach to exterior problems, but still relying on a conformal change of metric, has been introduced in [8].

1.2. Monotonicity formulas in parabolic manifolds. So far, we have considered *non-parabolic* Riemannian manifolds with $\text{Ric} \geq 0$. We now turn our attention to *parabolic* Riemannian manifolds with nonnegative Ricci curvature. As we will see in Section 2, for this class of manifolds, problem (1.9) does not admit a solution, *en revanche* the following problem does

$$\begin{cases} \Delta\psi = 0 & \text{in } M \setminus \bar{\Omega} \\ \psi = 0 & \text{on } \partial\Omega \\ \psi(q) \rightarrow +\infty & \text{as } d(O, q) \rightarrow +\infty, \end{cases} \quad (1.15)$$

where $\Omega \subset M$ is any open subset with smooth compact boundary and connected unbounded complement. Observe that again in light of Proposition 2.10, if (M, g) is not a Riemannian cylinder such assumption is equivalent to requiring that Ω is an open bounded subset with smooth boundary and no holes. Also notice that, by the Strong Maximum Principle, one has that $\partial\Omega$ coincides with the level set $\{\psi = 0\}$.

Inspired by the fact that problem (1.15) presents strong formal analogies with the conformal reformulation of problem (1.9) in terms of \tilde{g} (see problem (3.5) below), we also provide a Monotonicity-Rigidity Theorem for parabolic manifolds with $\text{Ric} \geq 0$ involving ψ in place of u . For $\beta \geq 0$, we define the function $\Psi_\beta : [0, \infty) \rightarrow \mathbb{R}$ as

$$\Psi_\beta(s) = \int_{\{\psi=s\}} |\text{D}\psi|^{\beta+1} \text{d}\sigma, \quad (1.16)$$

and we prove the following result:

Theorem 1.5 (Monotonicity-Rigidity Theorem for parabolic manifolds). *Let (M, g) be a parabolic manifold with $\text{Ric} \geq 0$. Let $\Omega \subset M$ be an open subset with smooth compact boundary and connected unbounded complement, and let ψ be a solution to problem (1.15). Then, for every $\beta \geq (n-2)/(n-1)$, the function Ψ_β is differentiable with derivative*

$$\frac{\text{d}\Psi_\beta}{\text{d}s}(s) = -\beta \int_{\{\psi=s\}} |\text{D}\psi|^\beta \text{H} \text{d}\sigma,$$

where H is the mean curvature of the level set $\{\psi = s\}$ computed with respect to the unit normal vector field $\nu = \text{D}\psi/|\text{D}\psi|$. Moreover, for every $s \geq 0$, the derivative fulfils

$$\frac{\text{d}\Psi_\beta}{\text{d}s}(s) = -\beta \int_{\{\psi \geq s\}} |\text{D}\psi|^{\beta-2} \left(\text{Ric}(\text{D}\psi, \text{D}\psi) + |\text{D}\text{D}\psi|^2 + (\beta-2) |\text{D}|\text{D}\psi|^2 \right) \text{d}\mu. \quad (1.17)$$

In particular, $\text{d}\Psi_\beta/\text{d}s$ is always nonpositive. Moreover, $(\text{d}\Psi_\beta/\text{d}s)(s_0) = 0$ for some $s_0 \geq 1$ and some $\beta \geq (n-2)/(n-1)$ if and only if $(\{\psi \geq s_0\}, g)$ is isometric to the Riemannian product $([s_0, +\infty) \times \{\psi = s_0\}, d\rho \otimes d\rho + g_{\{\psi=s_0\}})$. In this case, in particular, $\{\psi = s_0\}$ is a connected totally geodesic submanifold.

Observe that the right-hand side of (1.17) is nonnegative again by (1.14), that in fact holds for any harmonic function on any Riemannian manifold.

Combining Theorem 1.3 and 1.5, we obtain as a straightforward consequence an enhanced version of a theorem by Kasue, [44, Theorem C (2)], asserting that if a smooth boundary $\partial\Omega \subset M$

has mean curvature (computed with respect to the outer unit normal) $H \leq 0$, then $H \equiv 0$ on $\partial\Omega$ and $M \setminus \Omega$ is isometric to a half cylinder. Our result actually gives precise lower bounds for the supremum of H in terms of our monotone quantities and their derivatives.

Theorem 1.6 (Enhanced Kasue's Theorem). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$, and let $\Omega \subset M$ be an open subset with smooth compact boundary and connected unbounded complement. Then, for every $\beta \geq (n-2)/(n-1)$, the following assertions hold true, where the mean curvature is computed with respect to the outward unit normal.*

(i) *If (M, g) is nonparabolic, then*

$$\sup_{\partial\Omega} H \geq \frac{1}{\int_{\partial\Omega} |Du|^\beta d\sigma} \left[U_\beta(1) + \frac{1}{\beta} \frac{dU_\beta}{dt}(1) \right] > 0, \quad (1.18)$$

where U_β is defined in (1.10) and its derivative satisfies (1.12).

(ii) *If (M, g) is parabolic, then*

$$\sup_{\partial\Omega} H \geq - \frac{1}{\int_{\partial\Omega} |D\psi|^\beta d\sigma} \left[\frac{1}{\beta} \frac{d\Psi_\beta}{ds}(0) \right] \geq 0, \quad (1.19)$$

where Ψ_β is defined in (1.16) and its derivative satisfies (1.17).

Kasue's Theorem then follows as a corollary.

Corollary 1.7 (Kasue's Theorem). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and let $\Omega \subset M$ be an open subset with smooth compact boundary and connected unbounded complement. Assume that $H \leq 0$ on $\partial\Omega$, where the mean curvature is computed with respect to the outward unit normal. Then $(M \setminus \Omega, g)$ is isometric to a Riemannian product $[0, +\infty) \times \partial\Omega$, $dr \otimes dr + g_{\partial\Omega}$ and $\partial\Omega$ is a totally geodesic connected submanifold of (M, g) .*

1.3. Isoperimetry in 3-manifolds. Finally, we combine our sharp Willmore-type inequality (1.6) with curvature flow techniques along the lines of an argument presented by Huisken in [37]. We obtain a characterization of the infimum of the Willmore functional in terms of the isoperimetric ratio of 3-manifolds with nonnegative Ricci curvature and Euclidean volume growth, refining the analogous result stated in the aforementioned contribution. This is the content of the following theorem.

Theorem 1.8 (AVR(g) & Isoperimetric Constant). *Let (M, g) be a complete noncompact 3-manifold with $\text{Ric} \geq 0$ and Euclidean volume growth. Then,*

$$\inf \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} = \inf \frac{\int_{\partial\Omega} H^2 d\sigma}{16\pi} = \text{AVR}(g), \quad (1.20)$$

where the infima are taken over bounded and open subsets $\Omega \subset M$ with smooth boundary. In particular, the following isoperimetric inequality holds for any bounded and open $\Omega \subset M$ with smooth boundary

$$\frac{|\partial\Omega|^3}{|\Omega|^2} \geq 36\pi \text{AVR}(g). \quad (1.21)$$

Moreover, equality is attained in (1.21) if and only if $M = \mathbb{R}^3$ and Ω is a ball.

Beside the characterization of the isoperimetric constant in terms of the Asymptotic Volume Ratio, the novelties with respect to [37] lie in the rigidity statement and in the fact that the infimum of the Willmore functional is taken over the whole class of bounded open subsets Ω with smooth boundary, and not just over *outward minimizing* subsets. All of these improvements substantially come from our optimal Willmore-type inequality (1.6).

It is worth noticing that the above theorem can be rephrased in terms of a Sobolev inequality with optimal constant. This is the content of the following corollary.

Corollary 1.9 (AVR(g) & Sobolev Constant). *Let (M, g) be a complete noncompact 3-manifold with $\text{Ric} \geq 0$ and Euclidean volume growth. Then*

$$\inf_{f \in W_0^{1,1}(M)} \frac{\int_M |Df| \, d\sigma}{\left(\int_M |f|^{3/2} \, d\sigma \right)^{2/3}} = \sqrt[3]{36\pi \text{AVR}(g)}. \quad (1.22)$$

Once Theorem 1.8 is established, (1.22) is obtained by very standard tools. We refer the reader to [67, pages 89-90] for a complete proof of the well known equivalence between the isoperimetric and the Sobolev inequality. On this regard, it is worth observing that relations between isoperimetry and mean curvature functionals date back to Almgren [6], while a first derivation of isoperimetric inequalities through a curvature flow has been obtained by Topping in the case of curves [71]. Isoperimetric inequalities in \mathbb{R}^n and in Cartan-Hadamard manifolds through curvature flows have been established by Schulze in [68] and [69], while the application to manifolds with nonnegative Ricci curvature is suggested in the already mentioned [37]. The techniques lectured in [37] have interesting applications also in connection with the relativistic ADM mass, see [42] for the details. Actually, as pointed out in the discussion following [22, Theorem 5.13], a positive isoperimetric/Sobolev constant for complete noncompact Riemannian manifolds with $\text{Ric} \geq 0$ and Euclidean volume growth can be deduced via the techniques introduced by Croke in [29] and [30]. However, it is known that such constant is not optimal. Other strictly related issues about isoperimetry in noncompact manifolds with $\text{Ric} \geq 0$ are treated in [63] and [21].

1.4. Summary. This paper is organized as follows. In Section 2 we review, for ease of the reader, the theory of harmonic functions on Riemannian manifolds with nonnegative Ricci curvature we are going to employ along this work. In Section 3 we introduce the conformal formulation of problem (1.9). In this setting, we prove (the conformal version of) Theorem 1.3. In Section 4 we work out the integral asymptotic estimates for the electrostatic potential on manifolds with nonnegative Ricci curvature. With these estimates at hand, we prove Theorem 1.1 and Corollary 1.2. In Section 5 we prove the Monotonicity-Rigidity Theorem for parabolic manifolds and deduce Theorem 1.6 and Corollary 1.7. In Section 6, we prove Theorem 1.8. Finally, we have included an Appendix where we describe the relations between our monotonicity formulas and some of those obtained by Colding and Colding-Minicozzi in [23] and [28].

2. HARMONIC FUNCTIONS IN EXTERIOR DOMAINS

In this section we are mainly concerned with characterizing Riemannian manifolds for which problems (1.9) and (1.15) admit a solution. We are going to see that complete noncompact nonnegatively Ricci curved manifolds for which a solution to (1.9) exists are the *nonparabolic* ones, namely, manifolds admitting a positive Green's function, while those admitting a solution to (1.15) are the *parabolic* ones.

Nothing substantially new appears in this section. We are just collecting, re-arranging and applying classical results contained in [49, 50, 51, 53, 75], and [72]. The interested reader might also refer to the nice survey [35], where the relation with the Brownian motion on manifolds is also explored, or, for a more general account on the vast subject of harmonic functions on manifolds, to the lecture notes [48] and the references therein. Other important works in this field will be readily cited along the paper. Before starting, let us mention that the results gathered in this preliminary section are spread in a huge literature, and frequently they do not

appear exactly in the form that we need or with a detailed proof. For this reason, we include the most relevant ones. Important gradient bounds are discussed too.

Along this section, we denote by D the Levi-Civita connection of the Riemannian manifold considered, and by Δ the related Laplacian. For any two points $p, q \in M$, we let $d(p, q)$ be their geodesic distance. Moreover, it is understood that we are always dealing with manifolds of dimension $n \geq 3$.

2.1. Green's functions and parabolicity. Let us begin with the definition of Green's functions on Riemannian manifolds.

Definition 2.1 (Green's function). *A smooth function*

$$G : (M \times M) \setminus \text{Diag}(M) \rightarrow \mathbb{R},$$

where $\text{Diag}(M) = \{(p, p) : p \in M\}$, is said to be a Green's function for the Riemannian manifold (M, g) if the following requirements are satisfied.

- (i) $G(p, q) = G(q, p)$ for any $p, q \in M$, $p \neq q$.
- (ii) $\Delta G(p, \cdot) = 0$ on $M \setminus \{p\}$, for any $p \in M$.
- (iii) The following asymptotic expansion holds for $q \rightarrow p$:

$$G(p, q) = (1 + o(1)) d^{2-n}(p, q). \quad (2.1)$$

It is well known that on a complete noncompact Riemannian manifold there always exists a Green's function. This result has been obtained for the first time by Malgrange in [56], while a constructive proof, best suited for applications, has been given by Li-Tam in [49]. Complete noncompact Riemannian manifolds are then divided into two classes.

Definition 2.2 (Parabolicity). *Complete noncompact Riemannian manifolds which support a positive Green's function are called nonparabolic. Otherwise they are called parabolic.*

A by-product of Li-Tam's construction of Green's function gives the following very useful characterization of parabolicity, see for example [48, Theorem 2.3] for a proof.

Theorem 2.3 (Li-Tam). *Let (M, g) be a complete noncompact Riemannian manifold. Then, it is nonparabolic if and only if there exists a positive super-harmonic function f defined on the complement of a geodesic ball $B(p, R)$ such that*

$$\liminf_{d(p, q) \rightarrow +\infty} f(q) < \inf_{\partial B(p, R)} f. \quad (2.2)$$

Notice that if (M, g) is a nonparabolic Riemannian manifold then a barrier function f as in Theorem 2.3 is just the function G restricted to $M \setminus B(p, R)$. A positive Green's function G is called *minimal* if

$$G(p, q) \leq \tilde{G}(p, q)$$

for any other positive Green's function \tilde{G} . The construction of the Green's function in [49] actually provides the minimal one.

The following theorem is a fundamental characterization of parabolicity for manifolds with $\text{Ric} \geq 0$ in terms of the volume growth of geodesic balls, first provided in [72].

Theorem 2.4 (Varopoulos). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$. Then (M, g) is nonparabolic if and only if*

$$\int_1^{+\infty} \frac{r}{|B(p, r)|} dr < +\infty, \quad (2.3)$$

for any $p \in M$, where $B(p, r)$ is a geodesic ball centered at p with radius $r \geq 0$.

The above characterization roughly says that on nonparabolic manifolds volumes are growing faster than quadratically, while on the parabolic ones they grow at most quadratically. On the other hand, on a n -dimensional complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ Bishop-Gromov's Theorem and a result of Yau [76] respectively show that the growth of volumes of geodesic balls $B(p, r)$ is controlled from above by r^n and from below by r .

From now on we focus our discussion on complete noncompact manifolds with nonnegative Ricci curvature. In the following two subsections we collect some basic though fundamental facts in this context, for the ease of references.

2.2. Harmonic functions on manifolds with nonnegative Ricci curvature. A basic tool in the study of the potential theory on Riemannian manifolds is the following celebrated gradient estimate, first provided by Yau in [75] (see also the nice presentation given in [67]).

Theorem 2.5 (Yau's Gradient Estimate). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$. Let u be a positive harmonic function defined on a geodesic ball $B(p, 2R)$ of center p and radius $2R$. Then, there exists a constant $C = C(n) > 0$ such that*

$$\sup_{x \in B(p, R)} \frac{|Du|}{u} \leq \frac{C}{R}. \quad (2.4)$$

We now apply the above inequality to a harmonic function v defined in a geodesic annulus $B(p, R_1) \setminus \overline{B(p, R_0)}$. We obtain a decay estimate on the gradient of u that we will employ several times along this paper.

Proposition 2.6. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$, and let u be a positive harmonic function defined in a geodesic annulus $B(p, R_1) \setminus \overline{B(p, R_0)}$, with $R_1 > 3R_0$. Then, there exists a geometric constant $C = C(n)$ such that*

$$|Du|(q) \leq C \frac{u(q)}{d(p, q)}, \quad (2.5)$$

for any point q such that $2R_0 \leq d(p, q) < \frac{R_1 + R_0}{2}$. In particular, if u is a harmonic function defined in $M \setminus \overline{B(p, R_0)}$, then

$$|Du|(q) \leq C \frac{u(q)}{d(p, q)} \quad (2.6)$$

for any point q with $2R_0 \leq d(p, q)$.

Proof. Let q be such that $2R_0 \leq d(p, q) < \frac{R_1 + R_0}{2}$. Then the ball $B(q, d(p, q) - R_0)$ is all contained in the annulus $B(p, R_1) \setminus \overline{B(p, R_0)}$. In particular, by Yau's inequality (2.4) we have

$$|Du|(q) \leq C \frac{u(q)}{d(p, q) - R_0} \leq 2C \frac{u(q)}{d(p, q)}.$$

Letting $R_1 \rightarrow +\infty$, we get also (2.6). □

Another important application of Yau's inequality is the following compactness result for sequences of harmonic functions. A complete proof can be found in [48, Lemma 2.1].

Lemma 2.7. *Let (M, g) be a complete Riemannian manifold with $\text{Ric} \geq 0$, and let $U \subset M$ be an open and connected subset. Let $\{f_i\}$ be a sequence of positive harmonic functions defined on U , and suppose there exists a constant C such that $f_i(p) \leq C$ at some point $p \in U$ for any $i \in \mathbb{N}$. Then, there exists a subsequence $\{f_{i_j}\}$ converging to a positive harmonic function f uniformly on any compact set $K \subset U$.*

2.3. Ends of manifolds with nonnegative Ricci curvature. It is a well-known and largely exploited fact that a complete noncompact Riemannian manifold (M, g) with nonnegative Ricci curvature that is not a Riemannian cylinder has just one end. However, since a complete proof of this fact is hard to find in standard literature, we discuss the details below.

We employ the following definition of end, that is, the one used in the works by Li and Tam, see for example [50, Definition 0.4 and discussion thereafter].

Definition 2.8 (Ends of Riemannian manifolds). *An end of a Riemannian manifold (M, g) with respect to a compact subset $K \subset M$ is an unbounded connected component of $M \setminus K$. We say that (M, g) has a finite number of ends if the number of ends with respect to any compact subset $K \subset M$ is bounded by a natural number k independent of K . In this case, we say that (M, g) has k ends if it has k ends with respect to a compact subset $K \subset M$ and to any other compact subsets of M containing K .*

To state the result, we quickly recall some terminology. A *line* in (M, g) is a curve $\gamma : \mathbb{R} \rightarrow M$ which is a minimal geodesic between any two points lying on it. A *ray* is half a line. We also recall that a complete Riemannian manifold (M, g) is called a *Riemannian cylinder* if it is isometric to the Riemannian product $(\mathbb{R} \times N^{n-1}, dt \otimes dt + g_{N^{n-1}})$, where N^{n-1} is a compact manifold.

Remark 2.9. *Using the above definition and terminology it is clear that if a Riemannian manifold has at least two ends, then it contains a line.*

Proposition 2.10. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$. If (M, g) is not a Riemannian cylinder, then it has just one end.*

Proof. Since (M^n, g) has nonnegative Ricci curvature by hypothesis, then the Cheeger-Gromoll Splitting Theorem [16] implies that

$$(M, g) \text{ is isometric to } (\mathbb{R}^m \times N^{n-m}, g_{\mathbb{R}^m} + g_{N^{n-m}}), \quad (2.7)$$

for some $m \in \{0, \dots, n\}$, where the manifold $(N^{n-m}, g_{N^{n-m}})$ has nonnegative Ricci curvature and does not contain any line. The Riemannian manifold (M, g) is a Riemannian cylinder if $m = 1$ and N^{n-1} is compact. Let us then suppose that (M, g) is not a Riemannian cylinder. We consider the cases $m = 0$, $m = 1$ and $m \geq 2$.

Case $m = 0$. In this case, (M, g) does not contain any line. Then there is no more than one end, in view of Remark 2.9.

Case $m = 1$. Since (M, g) is not a cylinder, we have that N^{n-1} is a noncompact Riemannian manifold that contains no lines. Then again by Remark 2.9, it has at most one end. Thus, also $\mathbb{R} \times N^{n-1}$ has at most one end.

Case $m \geq 2$. We show that

$$M \setminus K \text{ is connected for every compact } K \subset M. \quad (2.8)$$

In view of Definition 2.8, this readily implies that M has at most one end. Now, to check (2.8) in view of (2.7), it is sufficient to check that for every compact $Q \subset \mathbb{R}^m$ and every compact $P \subset N^{n-m}$, we have that $(\mathbb{R}^k \times N^{n-m}) \setminus (Q \times P)$ is connected. Let then $(x, q), (y, p) \in (\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ and suppose for the moment that $x, y \in Q$, so that, in turn, $q, p \notin P$. Choose $z \in \mathbb{R}^m \setminus Q$ and define the curves

$$\alpha(t) = (tx + (1-t)z, q), \quad \beta(t) = (ty + (1-t)z, p),$$

$t \in [0, 1]$, connecting (x, q) to (z, q) and (y, p) to (z, p) , respectively. Note that $\alpha(t), \beta(t) \in (\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ for every $t \in [0, 1]$, because $q, p \notin P$. Now, let $\gamma(t)$ be a continuous curve in N^{n-m} connecting q and p . Then the curve $(z, \gamma(t)) \in ((\mathbb{R}^m \setminus Q) \times N^{n-m})$ connects (z, q) to (z, p) . Gluing together the curves α, β , and (z, γ) , we obtain a continuous path lying

in $(\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ and connecting (x, q) to (y, p) . Obtaining such a curve in the case where either $x \notin Q$ or $y \notin Q$ requires a similar simpler construction. We have thus proved that $(\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ is path-connected, hence connected.

We proved that (M, g) has at most one end, if it is not a cylinder. Then, it has exactly one end, because it is noncompact. \square

Let us now describe separately some aspects of harmonic functions on nonparabolic and parabolic manifolds. In particular, we are going to characterize these two classes of manifolds through a couple of existence results for solutions of suitable boundary value problems in exterior domains (see Theorems 2.13 and 2.16 below). The monotone quantities analysed in Theorems 1.3 and 1.5 are defined along the level sets of these solutions.

2.4. The exterior problem on nonparabolic manifolds. The following is a fundamental estimate proved by Li-Yau in [53].

Theorem 2.11 (Li-Yau). *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$. Then, its minimal Green's function G satisfies*

$$C^{-1} \int_{d(p,q)}^{+\infty} \frac{r}{|B(p,r)|} dr \leq G(p,q) \leq C \int_{d(p,q)}^{+\infty} \frac{r}{|B(p,r)|} dr, \quad (2.9)$$

for some $C = C(n) > 0$.

Combining (2.3) with (2.9), we get that the minimal Green's function goes to 0 at infinity, i.e. for any fixed p in M

$$\lim_{d(p,q) \rightarrow +\infty} G(p,q) = 0. \quad (2.10)$$

An easy application of Laplace Comparison Theorem then gives the following well known fact.

Lemma 2.12. *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$, and let G be its minimal Green's function. Then, for any fixed pole p we have*

$$d^{2-n}(p,q) \leq G(p,q). \quad (2.11)$$

for any $q \neq p$ in M .

Proof. Let r be the function mapping a point q in M to $d(p,q)$. By the Laplacian Comparison Theorem, we have

$$\Delta r \leq \frac{n-1}{r}$$

in the sense of distributions (see e.g. [22, Theorem 1.128]). Therefore, we have, in the sense of distributions,

$$\Delta r^{2-n} = (n-2) [(n-1)r^{-n} - r^{1-n} \Delta r] \geq 0,$$

and then the the function $r^{2-n} - G(p, \cdot)$ is sub-harmonic. By the maximum principle, for any $\varepsilon > 0$ and $R > \varepsilon$

$$\max_{B(p,R) \setminus B(p,\varepsilon)} (r^{2-n} - G(p, \cdot)) = \max_{\partial B(p,R) \cup \partial B(p,\varepsilon)} (r^{2-n} - G(p, \cdot)).$$

We conclude by passing to the limit as $\varepsilon \rightarrow 0$ and $R \rightarrow +\infty$, taking into account the asymptotic behaviour at the pole p given by (2.1) and that $G \rightarrow 0$ at infinity, as observed in (2.10). \square

Now, we characterize the existence of a solution to problem (1.9) with the nonparabolicity of the ambient manifold. Let us first set up some notation that we are going to use in the rest of the paper. With respect to a bounded open subset $\Omega \subset M$ with smooth boundary, we denote by O a generic reference point taken inside Ω .

Theorem 2.13. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$, and let $\Omega \subset M$ be a bounded open subset with smooth boundary and no holes. Then, there exists a solution to problem (1.9) if and only if (M, g) is nonparabolic.*

Proof. The proof is an easy adaptation of arguments already presented in [50]. By Theorem 2.3 the existence of a solution to problem (1.9) implies nonparabolicity of M , since the restriction of u to $M \setminus B(O, R)$ with $\Omega \subset B(O, R)$ clearly satisfies condition (2.2).

Conversely, assume that M is nonparabolic, and consider an increasing sequence of radii $\{R_i\}_{i \in \mathbb{N}}$ such that $\Omega \subset B(O, R_1)$ and $R_i \rightarrow +\infty$. Let, for any $i \in \mathbb{N}$, u_i be the solution to the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } B(O, R_i) \setminus \bar{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial B(O, R_i). \end{cases} \quad (2.12)$$

Let now G be the minimal positive Green's function, and consider the function $G(O, \cdot)$. Due to the Maximum Principle for harmonic functions and the boundary conditions in problem (2.12) we have that

$$0 \leq u_{R_i}(q) \leq \frac{G(O, q)}{\min_{\partial\Omega} G(O, \cdot)}, \quad (2.13)$$

for $q \in B(O, R_i)$. Let then K be a compact set contained in $M \setminus \bar{\Omega}$. We can clearly suppose without loss of generality that K is contained in $B(O, R_i) \setminus \bar{\Omega}$ for any i . Then, (2.13) and Lemma 2.7 give that u_i converges up to a subsequence to a harmonic function u on K . We can clearly extend by continuity u to 1 on $\partial\Omega$. Again by (2.13), and by uniform convergence,

$$0 \leq u(q) \leq \frac{G(O, q)}{\min_{\partial\Omega} G(O, \cdot)}.$$

on $M \setminus \Omega$. Since, by (2.10), $G(O, q) \rightarrow 0$ as $d(O, q) \rightarrow +\infty$, so does u , completing the proof. \square

We conclude this section with the following easy lemma, which shows that we can control function u by the minimal Green's function G .

Lemma 2.14. *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$, and let G be its minimal Green's function. Let u be a solution to (1.9) for some open and bounded set Ω with smooth boundary and no holes, and let $O \in \Omega$. Then, there exist constants $C_1 = C_1(M, \Omega) > 0$ and $C_2 = C_2(M, \Omega) > 0$ such that*

$$C_1 G(O, q) \leq u(q) \leq C_2 G(O, q) \quad (2.14)$$

on $M \setminus \Omega$. In particular

$$C_1 d(O, q)^{2-n} \leq u(q). \quad (2.15)$$

on $M \setminus \Omega$.

Proof. Just set $0 < C_1 < 1/\max_{\partial\Omega} G(O, \cdot)$, and $C_2 > 1/\min_{\partial\Omega} G(O, \cdot)$. The claim follows from the Maximum Principle and the observation that both u and G are vanishing at infinity. The inequality (2.15) is obtained combining the lower estimate on u by (2.14) with (2.11). \square

2.5. The exterior problem on parabolic manifolds. The following inequalities, proved in [51, Theorem 2.6], can be interpreted as a version for parabolic manifolds of the Li-Yau inequalities recalled in Theorem 2.11. We point out that we are always dealing with Green's functions obtained by the Li-Tam's construction.

Theorem 2.15. *Let (M, g) be a parabolic manifold with $\text{Ric} \geq 0$, and let $p \in M$. Let G be a Green's function. Then, for any fixed $r_0 > 0$ and for any q with $d(p, q) > 2r_0$ there holds*

$$-G(p, q) \leq C_1 \int_{r_0}^{d(p, q)} \frac{r}{|B(p, r)|} dr + C_2, \quad (2.16)$$

for some constants C_1 and C_2 depending only on n , r_0 and the choice of G . Moreover, for any $R > r_0$, there holds

$$C_3 \int_{r_0}^R \frac{r}{|B(p, r)|} dr + C_4 \leq \sup_{\partial B(p, R)} (-G(p, \cdot)), \quad (2.17)$$

for some constants C_3 and C_4 depending only on n , r_0 and the choice of G .

When (M, g) is parabolic, Li-Tam proved in [50, Lemma 1.2] that the exterior problem (1.15) admits a solution. The construction of such a solution ψ , combined with Yau's inequality and Theorem 2.15, readily implies a uniform gradient bound on ψ .

Theorem 2.16. *Let (M, g) be a parabolic Riemannian manifold with $\text{Ric} \geq 0$, and let $\Omega \subset M$ be an open subset with smooth compact boundary and connected unbounded complement. Then, there exists a solution to problem (1.15). Moreover, $|\text{D}\psi|$ is uniformly bounded in $M \setminus \Omega$.*

Proof of Theorem 2.16. Let $O \in \Omega$, let $U \subset \Omega$ be an open neighborhood of O and let $D \Subset \Omega$ be a bounded open subset of Ω such that $U \Subset D$ and $\partial\Omega \subset \partial D$. Let finally K be the compact set defined by $K = \overline{D} \setminus U$. Consider, for a sequence $B(O, R_i)$ of geodesic balls with increasing radii containing Ω , a corresponding sequence of positive Green's functions $G_i(O, \cdot)$ of $B(O, R_i)$ with pole in O such that $G_i(O, x) = 0$ for $x \in \partial B(O, R_i)$. We then consider the sequence of functions defined in $B(O, R_i) \setminus \{O\}$ by

$$f_i(q) = \sup_{x \in K} G_i(O, x) - G_i(O, q).$$

The construction in [49] implies that there exists a Green's function G on M such that f_i converges to $-G(O, \cdot)$ uniformly on compact subsets of $M \setminus \{O\}$ (compare with the discussions around Lemma 1.2 in [50]). Observing that $f_i = \sup_K G_i(O, \cdot)$ on $\partial B(O, R_i)$, we set

$$a_i = \sup_{x \in K} G_i(O, x)$$

and consider the solution ψ_i to the problem

$$\begin{cases} \Delta\psi = 0 & \text{in } B(O, R_i) \setminus \overline{\Omega} \\ \psi = 0 & \text{on } \partial\Omega \\ \psi = a_i & \text{on } \partial B(O, R_i) \cap (M \setminus \overline{\Omega}). \end{cases}$$

Since $\sup_K G_i(O, \cdot) \geq \sup_{\partial\Omega} G_i(O, \cdot)$, the Maximum Principle immediately gives

$$f_i - \sup_{\partial\Omega} f_i \leq \psi_i \leq f_i \quad (2.18)$$

on $B(O, R_i) \setminus \Omega$. Since the sequence f_i is converging (uniformly on compact sets) to the Li-Tam Green's function, the second inequality in (2.18) combined with Lemma 2.7 shows that ψ_i converges uniformly on the compact subsets of $M \setminus \overline{\Omega}$ to an harmonic function ψ , that we can clearly extend to 0 on $\partial\Omega$. Moreover, since for every $q \in M \setminus \{O\}$ the sequence $f_i(q)$ converges to $-G(O, q)$ and since by (2.17) we have that $-G(O, q_j) \rightarrow +\infty$ along a sequence of points q_j such that $d(O, q_j) \rightarrow +\infty$, we use the first inequality in (2.18), to deduce that $\psi(q_j) \rightarrow +\infty$, as

$j \rightarrow +\infty$. In particular, since by [25, Lemma 3.40] ψ must admit a limit at infinity, we infer that $\psi(q) \rightarrow +\infty$, as $d(O, q) \rightarrow +\infty$. Therefore, ψ is a solution to problem (1.15).

Observe that, again by (2.18), $\psi \leq -G(O, \cdot)$. Inequality (2.6) then yields

$$|D\psi|(q) \leq C \frac{\psi(q)}{d(O, q)} \leq C \frac{-G(O, q)}{d(O, q)} \quad (2.19)$$

for some constant C and any q outside some big geodesic ball $B(O, r_0)$. Combining now (2.16) with Yau's lower bound on the growth of geodesic balls, saying that $|B(O, r)| \geq Cr$ for any $r \geq 1$ and for some constant C , we also have

$$\frac{-G(O, q)}{d(O, q)} \leq C_1 \frac{d(O, q) + C_2}{d(O, q)}$$

for q with $d(O, q) > 2r_0$ and constants C_1 and C_2 . Plugging it in (2.19), this shows that $|D\psi|$ is uniformly bounded, as claimed. \square

3. PROOF OF THE MONOTONICITY-RIGIDITY THEOREM FOR NONPARABOLIC MANIFOLDS

3.1. The conformal setting. Let (M, g) be a nonparabolic Riemannian manifold with non-negative Ricci curvature. Let $\Omega \subset M$ be a bounded and open set with smooth boundary and no holes, and let u be the solution to problem (1.9). Let us recall that the no holes assumption, combined with the parabolicity, implies that $M \setminus \Omega$ consists of a unique unbounded connected component, in virtue of Proposition 2.10. We introduce, in $M \setminus \Omega$ the metric

$$\tilde{g} = u^{\frac{2}{n-2}} g. \quad (3.1)$$

The expression for \tilde{g} is formally the same as in [2] and [4]. Let us explain why such a conformal change of metric is natural also in the current setting. Our model geometry is that of a truncated metric cone

$$(M \setminus \Omega, g) \cong \left([r_0, +\infty) \times \partial\Omega, dr \otimes dr + Cr^2 g_{\partial\Omega} \right), \quad (3.2)$$

for some positive constant r_0 and C , and where $g_{\partial\Omega}$ is the metric induced by g on $\partial\Omega$. We also assume that $\partial\Omega$ is a smooth closed sub-manifold with $\text{Ric}_{\partial\Omega} \geq (n-2)g_{\partial\Omega}$. Such a curvature assumption on $\partial\Omega$ is equivalent to supposing that the cone in (3.2) has nonnegative Ricci curvature. In this model setting, up to a suitable choice of C in (3.2), the solution to problem (1.9) is $u(r) = r^{2-n}$. With this specific u , the metric \tilde{g} becomes

$$\tilde{g} = d\rho \otimes d\rho + g_{\partial\Omega},$$

where $\rho = \log r$. In other words \tilde{g} is a (half) Riemannian cylinder over $(\partial\Omega, g_{\partial\Omega})$. In parallel, as the rigidity statement in Theorem 1.3 gives a characterization of the truncated cone metrics (3.2), so its conformal version in Theorem 3.2 characterizes truncated cylindrical metrics.

Having this in mind, we are now going to describe the general features of $(M \setminus \Omega, \tilde{g})$ in more details. Letting

$$\varphi = -\log u, \quad (3.3)$$

we have that $\tilde{g} = e^{-\frac{2\varphi}{n-2}} g$.

As before, D is the Levi-Civita connection of (M, g) . Moreover, we denote by DD the Hessian. We denote by ∇ , the Levi-Civita connection of the metric \tilde{g} , by $\nabla\nabla$ its Hessian, and we put the subscript \tilde{g} on any other quantity induced by \tilde{g} . We have, for a smooth function w

$$\nabla_\alpha \nabla_\beta w = D_\alpha D_\beta w + \frac{1}{n-2} \left(\partial_\alpha w \partial_\beta \varphi + \partial_\beta w \partial_\alpha \varphi - \langle Dw, D\varphi \rangle g_{\alpha\beta} \right),$$

where by $\langle \cdot, \cdot \rangle$ we denote the scalar product induced by g . In particular,

$$\Delta_{\tilde{g}} \varphi = 0. \quad (3.4)$$

Moreover, the Ricci tensor $\text{Ric}_{\tilde{g}}$ of \tilde{g} and the Ricci tensor Ric of g satisfy

$$\text{Ric}_{\tilde{g}} = \text{Ric} + \nabla_{\alpha} \nabla_{\beta} \varphi - \frac{d\varphi \otimes d\varphi}{n-2} + \frac{|\nabla \varphi|_{\tilde{g}}^2}{n-2} \tilde{g}. \quad (3.5)$$

Finally, by (3.4) and (3.5) problem (1.9) becomes

$$\left\{ \begin{array}{ll} \Delta_{\tilde{g}} \varphi = 0 & \text{in } M \setminus \bar{\Omega} \\ \text{Ric}_{\tilde{g}} - \nabla \nabla \varphi + \frac{d\varphi \otimes d\varphi}{n-2} = \frac{|\nabla \varphi|_{\tilde{g}}^2}{n-2} \tilde{g} + \text{Ric} & \text{in } M \setminus \bar{\Omega} \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi(q) \rightarrow +\infty & \text{as } d(O, q) \rightarrow +\infty. \end{array} \right. \quad (3.6)$$

The classical Bochner identity applied to φ in $(M \setminus \Omega, \tilde{g})$, combined with the first two equations of the above system, immediately yields the following identity

$$\Delta_{\tilde{g}} |\nabla \varphi|_{\tilde{g}}^2 - \langle \nabla |\nabla \varphi|_{\tilde{g}}^2, \nabla \varphi \rangle_{\tilde{g}} = 2 \left[\text{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla \nabla \varphi|_{\tilde{g}}^2 \right], \quad (3.7)$$

where Ric is the Ricci tensor of the background metric g . Such a relation is at the heart of this work. As a first application, we have the following fundamental correspondence between the splitting of $(M \setminus \Omega, \tilde{g})$ as a cylinder and the splitting of $(M \setminus \Omega, g)$ as a cone.

Lemma 3.1. *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$, let $\Omega \subset M$ be a bounded and open subset with smooth boundary and no holes, and let \tilde{g} and φ be defined by (3.1) and (3.3). Assume that $\nabla |\nabla \varphi|_{\tilde{g}} = 0$ on $\{\varphi \geq s_0\}$ for some $s_0 \in [0, +\infty)$.*

(i) *Then the Riemannian manifold $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to the Riemannian product*

$$([s_0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + \tilde{g}_{\{\varphi=s_0\}}).$$

In particular, $\{\varphi = s_0\}$ is a connected totally geodesic submanifold inside $(M \setminus \Omega, \tilde{g})$.

(ii) *Accordingly, for $t_0 = e^{-s_0}$, the Riemannian manifold $(\{u \leq t_0\}, g)$ has Euclidean volume growth and it is isometric to the truncated cone*

$$\left([r_0, +\infty) \times \{u = t_0\}, dr \otimes dr + \left(\frac{r}{r_0} \right)^2 g_{\{u=t_0\}} \right), \quad \text{with } r_0 = \left(\frac{|\{u = t_0\}|}{\text{AVR}(g) |\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

In particular, $\{u = t_0\}$ is a connected totally umbilic submanifold inside $(M \setminus \Omega, g)$ with constant mean curvature.

Proof. Let us first observe that plugging $\nabla |\nabla \varphi|_{\tilde{g}} = 0$ in (3.7) readily implies, since $\text{Ric} \geq 0$, that

$$\nabla \nabla \varphi \equiv 0 \quad \text{in } \{\varphi \geq s_0\} = \{u \leq t_0\}. \quad (3.8)$$

for $s_0 = -\log t_0$. The isometry of $(\{\varphi \geq s_0\}, \tilde{g})$ with a Riemannian product is then a well known fact, that was employed for example in the conclusion of the proof of the celebrated Cheeger-Gromoll's Splitting Theorem [16] (see also [2, Theorem 4.2 (i)] for a simple direct argument). Observe now that $\{\varphi = s_0\}$ is connected as a consequence of Proposition 2.10. Recalling the formula

$$\nabla_{\alpha} \nabla_{\beta} w = D_{\alpha} D_{\beta} w + \frac{1}{n-2} \left(\partial_{\alpha} w \partial_{\beta} \varphi + \partial_{\beta} w \partial_{\alpha} \varphi - \langle Dw, D\varphi \rangle g_{\alpha\beta} \right),$$

that holds for every C^2 function w , we have in particular that condition (3.8) translates into

$$0 = \nabla_{\alpha} \nabla_{\beta} \varphi = -\frac{D_{\alpha} D_{\beta} u}{u} + \left(\frac{n}{n-2} \right) \frac{D_{\alpha} u D_{\beta} u}{u^2} - \left(\frac{1}{n-2} \right) \left| \frac{Du}{u} \right|^2 g_{\alpha\beta}. \quad (3.9)$$

Observing now that

$$D_\alpha D_\beta (u^{-\frac{2}{n-2}}) = \left(\frac{2}{n-2}\right) u^{-\frac{2}{n-2}} \left[-\frac{D_\alpha D_\beta u}{u} + \left(\frac{n}{n-2}\right) \frac{D_\alpha u D_\beta u}{u^2} \right]$$

we deduce from (3.9) that

$$D_\alpha D_\beta \left(u^{-\frac{2}{n-2}} \right) = \frac{2}{(n-2)^2} \left| \frac{Du}{u} \right|^2 u^{-\frac{2}{n-2}} g_{\alpha\beta}.$$

In particular, we have that $DD(u^{-2/(n-2)})$ is proportional to the metric. By a standard result in Riemannian geometry (see e.g. [14, Theorem 1.1] for a complete proof, or [15, Section 1]) this fact implies that the potential $f = u^{-2/(n-2)}$, and thus u , depends only on the (signed) distance r from $\{u = t_0\}$ and that, up to a multiplicative factor in front of f , the metric g can be expressed as

$$g = dr \otimes dr + (f'(r))^2 g_\Sigma, \quad \Sigma = \{u = t_0\}.$$

Moreover, the associated Ricci tensor is given by

$$\text{Ric} = -(n-1) \frac{f'''}{f'} dr \otimes dr + \text{Ric}_\Sigma - ((n-2)(f'')^2 + f' f''')^2 g_\Sigma. \quad (3.10)$$

The second information obtained by plugging $\nabla|\nabla\varphi|_{\tilde{g}} = 0$ into the Bochner identity (3.7) is

$$\text{Ric}(\nabla\varphi, \nabla\varphi) = 0 \quad \text{in} \quad \{\varphi \geq s_0\} = \{u \leq t_0\}.$$

In particular this implies that $\text{Ric}(Du, Du) = 0$ in the same set. Using now expression (3.10) and recalling that Du is orthogonal to Σ , we obtain that $f'''(r) = 0$. This yields

$$g = dr \otimes dr + \left(\frac{r}{r_0}\right)^2 g_\Sigma,$$

where $r_0 > 0$ is such that $\Sigma = \{r = r_0\}$. Finally, if O is the tip of the above cone, since

$$|\partial B(O, R)| = \int_{\partial B(O, R)} d\sigma = \int_{\partial\Sigma} \left(\frac{R}{r_0}\right)^{n-1} \sqrt{\det g_{ij}^\Sigma} d\vartheta^1 \dots d\vartheta^{n-1} = \left(\frac{R}{r_0}\right)^{n-1} |\Sigma|,$$

we can explicitly express r_0 in terms of $\text{AVR}(g)$ as in (1.13) by

$$\text{AVR}(g) = \lim_{R \rightarrow +\infty} \frac{|\partial B(O, R)|}{R^{n-1} |\mathbb{S}^{n-1}|} = \frac{|\Sigma|}{r_0^{n-1} |\mathbb{S}^{n-1}|}.$$

The first identity is a well known characterization of the $\text{AVR}(g)$ in terms of areas of geodesic balls instead of volumes, and follows easily by the standard proof of the Bishop-Gromov Theorem (see e.g. [64]). The proof of the second claim is completed. \square

We now briefly record some of the main relations among geometric quantities induced by the two metrics. We omit the computations, since they are straightforward and completely analogous to those carried out in [2] and [4]. First, observe that

$$|\nabla\varphi|_{\tilde{g}} = \frac{|Du|}{u^{\frac{n-1}{n-2}}}.$$

Let H and $H_{\tilde{g}}$ be the mean curvatures of the level sets of u , that coincide with those of φ , respectively in the Riemannian manifold $(M \setminus \Omega, g)$ and in $(M \setminus \Omega, \tilde{g})$. They are computed using

the unit normal vectors $-Du/|Du|$ and $\nabla\varphi/|\nabla\varphi|_{\tilde{g}}$, respectively. Exploiting the g -harmonicity and \tilde{g} -harmonicity of u and φ , we obtain that

$$H = \frac{DDu(Du, Du)}{|Du|^3}, \quad H_{\tilde{g}} = -\frac{\nabla\nabla\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|_{\tilde{g}}^3}. \quad (3.11)$$

These quantities are related as follows

$$H_{\tilde{g}} = u^{-\frac{1}{n-2}} \left[H - \left(\frac{n-1}{n-2} \right) \frac{|Du|}{u} \right].$$

Letting $d\sigma_{\tilde{g}}$ and $d\mu_{\tilde{g}}$ denote respectively the surface and the volume measure naturally induced by \tilde{g} on $M \setminus \Omega$, we have

$$d\sigma_{\tilde{g}} = u^{\frac{n-1}{n-2}} d\sigma, \quad d\mu_{\tilde{g}} = u^{\frac{n}{n-2}} d\mu.$$

Finally, for every $\beta \geq 0$, define the conformal analogue of the U_β as the function $\Phi_\beta : [0, +\infty) \rightarrow \mathbb{R}$ mapping

$$\Phi_\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^{\beta+1} d\sigma_{\tilde{g}}. \quad (3.12)$$

The functions U_β and Φ_β , and their derivatives are related to each other as follows

$$\begin{aligned} U_\beta(t) &= \Phi_\beta(-\log t), \\ -tU'_\beta(t) &= \Phi'_\beta(-\log t), \end{aligned} \quad (3.13)$$

for $0 < t \leq 1$. The following theorem is the conformal version of the Monotonicity-Rigidity Theorem 1.3.

Theorem 3.2. *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded and open subset with smooth boundary and no holes, and let \tilde{g}, φ and Φ_β be defined respectively as in (3.1), (3.26) and (3.12). Then, for every $\beta \geq (n-2)/(n-1)$, the function Φ_β is differentiable with derivative*

$$\frac{d\Phi_\beta}{ds}(s) = -\beta \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^\beta H_{\tilde{g}} d\sigma_{\tilde{g}}, \quad (3.14)$$

where $H_{\tilde{g}}$ is the mean curvature of the level set $\{\varphi = s\}$ computed with respect to the unit normal vector field $\nu_{\tilde{g}} = \nabla\varphi/|\nabla\varphi|_{\tilde{g}}$. Moreover, for every $s \geq 0$, the derivative fulfils

$$\frac{d\Phi_\beta}{ds}(s) = -\beta e^s \int_{\{\varphi \geq s\}} \frac{|\nabla\varphi|_{\tilde{g}}^{\beta-2} \left(\text{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|_{\tilde{g}}^2 + (\beta-2) |\nabla|\nabla\varphi|_{\tilde{g}}|_{\tilde{g}}^2 \right)}{e^\varphi} d\mu_{\tilde{g}}. \quad (3.15)$$

In particular, $d\Phi_\beta/ds$ is always nonpositive. Moreover, $(d\Phi_\beta/ds)(s_0) = 0$ for some $s_0 \geq 1$ and some $\beta \geq (n-2)/(n-1)$ if and only if $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to the Riemannian product $([s_0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + \tilde{g}_{\{\varphi=s_0\}})$. In particular, $\{\varphi = s_0\}$ is a connected totally geodesic submanifold.

Observe that the right hand side in (3.15) is nonpositive because of the refined Kato's inequality for harmonic functions

$$|\nabla\nabla\varphi|_{\tilde{g}}^2 \geq \left(\frac{n}{n-1} \right) |\nabla|\nabla\varphi|_{\tilde{g}}|_{\tilde{g}}^2. \quad (3.16)$$

Notice also the striking analogy between the above statement and Theorem 1.5.

We are now going to show how to recover the Monotonicity-Rigidity Theorem 1.3 for non-parabolic manifolds from its conformal version.

Proof of Theorem 1.3 after Theorem 3.2. Deducing formulas (1.11) and (1.12) from formulas (3.14) and (3.15) is just a matter of lengthy but straightforward computations carried out using the relations between g and \tilde{g} recalled above. We sketch the main steps. First, compute

$$|\nabla\nabla\varphi|_{\tilde{g}}^2 = u^{-\frac{4}{n-2}} \left\{ \left| \frac{DDu}{u} \right|^2 + \frac{n(n-1)}{(n-2)^2} \left| \frac{Du}{u} \right|^4 - \frac{2n}{n-2} \left| \frac{Du}{u} \right|^3 H \right\}, \quad (3.17)$$

where H is defined as in (3.11), and

$$|\nabla|\nabla\varphi|_{\tilde{g}}|^2 = u^{-\frac{4}{n-2}} \left\{ \left| \frac{D|Du|}{u} \right|^2 + \left(\frac{n-1}{n-2} \right)^2 \left| \frac{Du}{u} \right|^4 - 2 \left(\frac{n-1}{n-2} \right) \left| \frac{Du}{u} \right|^3 H \right\}. \quad (3.18)$$

By (3.17) and (3.18), we can write

$$\begin{aligned} |\nabla\nabla\varphi|_{\tilde{g}}^2 + (\beta-2)|\nabla|\nabla\varphi|_{\tilde{g}}|^2 &= u^{-\frac{4}{n-2}} \left\{ \frac{|DDu|^2 - (\frac{n}{n-1})|D|Du||^2}{u^2} + \right. \\ &\quad \left. + \left(\beta - \frac{n-2}{n-1} \right) \left[\left| \frac{D|Du|}{u} \right|^2 + \left(\frac{n-1}{n-2} \right)^2 \left| \frac{Du}{u} \right|^4 - 2 \left(\frac{n-1}{n-2} \right) \left| \frac{Du}{u} \right|^3 H \right] \right\}. \end{aligned} \quad (3.19)$$

Now, considering a orthonormal frame as $\{e_1, \dots, e_{n-1}, Du/|Du|\}$, where the first $n-1$ vectors are tangent to the level sets of u , we can decompose

$$\left| \frac{D|Du|}{u} \right|^2 = \left| \frac{Du}{u} \right|^2 H^2 + \sum_{j=1}^{n-1} \left\langle \frac{D|Du|}{u}, e_j \right\rangle^2.$$

Plugging the above decomposition into (3.19), we obtain, with the aid of some algebra,

$$\begin{aligned} |\nabla\nabla\varphi|_{\tilde{g}}^2 + (\beta-2)|\nabla|\nabla\varphi|_{\tilde{g}}|^2 &= u^{-\frac{2n}{n-2}} \left\{ \left[|DDu|^2 - \left(\frac{n}{n-1} \right) |D|Du||^2 \right] \right. \\ &\quad \left. + \left(\beta - \frac{n-2}{n-1} \right) |D^T|Du||^2 \right. \\ &\quad \left. + \left(\beta - \frac{n-2}{n-1} \right) |Du|^2 \left[H - \left(\frac{n-1}{n-2} \right) |D \log u| \right]^2 \right\}, \end{aligned}$$

where

$$|D^T|Du||^2 = \sum_{j=1}^{n-1} \left\langle \frac{D|Du|}{u}, e_j \right\rangle^2.$$

The monotonicity formula (1.12) now follows easily. On the other hand, (1.11) follows from (3.14) by a direct computation.

Assume now that $U'_\beta(t_0) = 0$ for some $t_0 \in (0, 1]$ and some $\beta \geq (n-2)/(n-1)$. Then, by (3.13), $\Phi'_\beta(-\log t_0) = 0$. The rigidity statement in Theorem 1.3 then follows from the analogous statement in Theorem 3.2, simply recalling the discussion at the beginning of Subsection 3.1. \square

3.2. Proof of Theorem 3.2. As already pointed out in Remark 1.4, we restrict ourselves to the range $\beta \geq 1$, since it is sufficient for the main geometric corollaries of Theorem 1.3. The arguments are considerably easier in this regime, however in Remarks 3.8 and 3.10 we will comment on the main difficulties that arise in the case $(n-2)/(n-1) \leq \beta < 1$ and about how they can be overtaken, following the strategy described in [4].

In what follows, we are always referring to a background nonparabolic Riemannian manifold (M, g) with $\text{Ric} \geq 0$. Outside a bounded and open subset $\Omega \subset M$ with smooth boundary and

no holes, we define the conformal metric \tilde{g} as in (3.1). In particular, with the same notations as in the previous subsection, $(M \setminus \Omega, \tilde{g}, \varphi)$ is a solution to (3.6).

A first fundamental ingredient for the proof of the (conformal) Monotonicity-Rigidity Theorem 3.2 is the upper bound for the function Φ_β 's. We are going to prove it by first showing that $|\nabla\varphi|_{\tilde{g}}$ is uniformly bounded in $M \setminus \Omega$. Although not necessary for our aim, we actually provide a sharp upper bound for such a function, together with a rigidity statement when this bound is attained. The proof is substantially the one proposed in [2, Lemma 3.5] for the Euclidean case.

Proposition 3.3 (Sharp Gradient Estimate). *The inequality*

$$|\nabla\varphi|_{\tilde{g}}(q) \leq \sup_{\partial\Omega} |\nabla\varphi|_{\tilde{g}} \quad (3.20)$$

holds for every $q \in M \setminus \Omega$. Moreover, if the equality is attained for some $q \in M \setminus \Omega$, then the Riemannian manifold $(M \setminus \Omega, \tilde{g})$ is isometric to $([0, +\infty) \times \partial\Omega, d\rho \otimes d\rho + \tilde{g}_{\partial\Omega})$.

Proof. We first show that there exists a constant $C = C(M, \Omega) > 0$ such that $|\nabla\varphi|_{\tilde{g}} \leq C$ in $M \setminus \Omega$. Fixing a reference point O inside Ω , and letting $d(O, \cdot)$ be the distance from this point with respect to the metric g , we have, by (2.6), that

$$|Du|(q) \leq C \frac{u(q)}{d(O, q)}$$

outside some ball containing Ω . Then,

$$|\nabla\varphi|_{\tilde{g}}(q) = \frac{|Du|}{u^{\frac{n-1}{n-2}}}(q) \leq C \frac{u^{-\frac{1}{n-2}}(q)}{d(O, q)} \leq C C_1^{-\frac{1}{n-2}},$$

where in the last inequality we used (2.15). Consider now, for a given constant $\alpha > 0$, the auxiliary function

$$w_\alpha = |\nabla\varphi|_{\tilde{g}}^2 e^{-\alpha\varphi}.$$

Observe that by the just proved upper bound on $|\nabla\varphi|_{\tilde{g}}$ we have $w_\alpha(q) \rightarrow 0$ as $d(O, q) \rightarrow +\infty$ for any $\alpha > 0$. Moreover, a direct computation combined with (3.7) shows that w_α satisfies the identity

$$\Delta_{\tilde{g}} w_\alpha - (1 - 2\alpha) \langle \nabla w_\alpha, \nabla\varphi \rangle_{\tilde{g}} - \alpha(1 - \alpha) |\nabla\varphi|_{\tilde{g}}^2 w_\alpha = 2e^{-\alpha\varphi} \left[\text{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|_{\tilde{g}}^2 \right].$$

In particular, using the Maximum Principle, one has that for every $q \in M \setminus \Omega$ it holds

$$w_\alpha(q) \leq \sup_{\partial\Omega} |\nabla\varphi|_{\tilde{g}}^2.$$

Letting $\alpha \rightarrow 0$ in the above inequality yields (3.20) for every $q \in M \setminus \Omega$. Assume now that the maximum value of $|\nabla\varphi|_{\tilde{g}}$ is attained at some interior point $q \in M \setminus \Omega$. Then, by the Strong Maximum Principle $|\nabla\varphi|_{\tilde{g}}^2$ must be constant, since it satisfies

$$\Delta_{\tilde{g}} |\nabla\varphi|_{\tilde{g}}^2 - \langle \nabla |\nabla\varphi|_{\tilde{g}}^2, \nabla\varphi \rangle_{\tilde{g}} \geq 0.$$

Lemma 3.1 then yields the desired conclusion. \square

Remark 3.4. *The above results, in terms of $(M \setminus \Omega, g)$ and u , says that for every $q \in M \setminus \Omega$*

$$\frac{|Du|}{u^{\frac{n-1}{n-2}}}(q) \leq \sup_{\partial\Omega} \frac{|Du|}{u^{\frac{n-1}{n-2}}},$$

with equality attained only if $(M \setminus \Omega, g)$ is isometric to a truncated cone. This is exactly the content of Colding's [23, Theorem 3.2].

A direct consequence is that the functions Φ_β 's are bounded.

Corollary 3.5. *For every $\beta \geq 0$ the function $\Phi_\beta : [0, +\infty) \rightarrow \mathbb{R}$ defined in (3.12) is bounded as*

$$\Phi_\beta(s) \leq \sup_{\partial\Omega} |\nabla\varphi|_{\tilde{g}}^\beta \int_{\partial\Omega} |\nabla\varphi|_{\tilde{g}} d\sigma_{\tilde{g}}.$$

for every $s \in [0, +\infty)$.

Proof. Just observe that a simple application of the Divergence Theorem combined with the \tilde{g} -harmonicity of φ gives the constancy in s of the function

$$s \mapsto \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}} d\sigma_{\tilde{g}}.$$

Combining this observation with the bound (3.20), we have

$$\Phi_\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^\beta |\nabla\varphi|_{\tilde{g}} d\sigma_{\tilde{g}} \leq \sup_{\partial\Omega} |\nabla\varphi|_{\tilde{g}}^\beta \int_{\partial\Omega} |\nabla\varphi|_{\tilde{g}} d\sigma_{\tilde{g}},$$

for every $s \in [0, +\infty)$, as claimed. \square

A fundamental tool in the forthcoming computations, leading to the expression of the derivative of Φ_β in terms of a nonpositive integral (as in (3.15)), is the following Bochner-type identity.

Lemma 3.6 (Bochner-type identity). *At every point where $|\nabla\varphi|_{\tilde{g}} \neq 0$, the following identity holds for every $\beta \geq 0$*

$$\Delta_{\tilde{g}} |\nabla\varphi|_{\tilde{g}}^\beta - \left\langle \nabla |\nabla\varphi|_{\tilde{g}}^\beta, \nabla\varphi \right\rangle_{\tilde{g}} = \beta |\nabla\varphi|_{\tilde{g}}^{\beta-2} \left[\text{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|_{\tilde{g}}^2 + (\beta - 2) |\nabla |\nabla\varphi|_{\tilde{g}}|^2_{\tilde{g}} \right], \quad (3.21)$$

where the Ric is the Ricci tensor of the background metric g .

Proof. By a direct computation one gets

$$\Delta_{\tilde{g}} |\nabla\varphi|_{\tilde{g}}^\beta = |\nabla\varphi|_{\tilde{g}}^{\beta-2} \left[\frac{\beta}{2} \Delta_{\tilde{g}} |\nabla\varphi|_{\tilde{g}}^2 + \beta(\beta - 2) |\nabla |\nabla\varphi|_{\tilde{g}}|^2_{\tilde{g}} \right],$$

that, combined with (3.7), leads to (3.21). \square

We prove now an integral identity that will enable us to link the derivative of Φ_β to the Bochner-type formula above. This is where the assumption $\beta \geq 1$ enters the game, simplifying the proofs (see Remarks 3.8 and 3.10 for the remaining cases).

Lemma 3.7 (Fundamental Integral Identity). *Let $0 \leq s < S < +\infty$, and let $\beta \geq 1$. Then the following identity holds*

$$\begin{aligned} \int_{\{\varphi=S\}} \frac{\left\langle \nabla |\nabla\varphi|_{\tilde{g}}^\beta, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}} \right\rangle}{e^S} d\sigma_{\tilde{g}} - \int_{\{\varphi=s\}} \frac{\left\langle \nabla |\nabla\varphi|_{\tilde{g}}^\beta, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}} \right\rangle}{e^s} d\sigma_{\tilde{g}} &= \\ &= \beta \int_{\{s \leq \varphi \leq S\}} \frac{|\nabla\varphi|_{\tilde{g}}^{\beta-2} \left[\text{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|_{\tilde{g}}^2 + (\beta - 2) |\nabla |\nabla\varphi|_{\tilde{g}}|^2_{\tilde{g}} \right]}{e^\varphi} d\mu_{\tilde{g}}, \end{aligned} \quad (3.22)$$

where the Ricci tensor is referred to the background metric g .

Proof. Since all the quantities that appear in this proof are referred to the metric \tilde{g} , except for Ric, which is the Ricci tensor of the background metric g , we shorten the notation, dropping the subscript \tilde{g} . We consider the vector field

$$X = \frac{\nabla|\nabla\varphi|^\beta}{e^\varphi},$$

that is well defined at every point where $|\nabla\varphi| \neq 0$, and thus $\sigma_{\tilde{g}}$ -almost everywhere, in virtue of the regularity theory for elliptic equations recalled above. Moreover, by the Bochner-type identity (3.21), we have that wherever $|\nabla\varphi| \neq 0$ it holds

$$\operatorname{div}X = \frac{\Delta|\nabla\varphi|^\beta - \langle \nabla|\nabla\varphi|^\beta, \nabla\varphi \rangle}{e^\varphi} = \beta \frac{|\nabla\varphi|^{\beta-2} \left[\operatorname{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|^2 + (\beta-2)|\nabla|\nabla\varphi|^2 \right]}{e^\varphi}. \quad (3.23)$$

Let us consider, in $\{s \leq \varphi \leq S\}$, a tubular neighborhood of the critical set

$$B_\varepsilon[\operatorname{Crit}(\varphi)] = \bigcup_{x \in \operatorname{Crit}(\varphi)} B_\varepsilon(x),$$

It is well known that, for almost every $\varepsilon > 0$, the boundary of $B_\varepsilon[\operatorname{Crit}(\varphi)]$ is $(n-1)$ -Lipschitz rectifiable with respect to $\sigma_{\tilde{g}}$ (see e.g. [5, Theorem 2.5] for a much more general result). Then, the Divergence Theorem applies and yields

$$\begin{aligned} \int_{\{s \leq \varphi \leq S\} \setminus B_{\varepsilon_j}[\operatorname{Crit}(\varphi)]} \operatorname{div}X \, d\mu &= \int_{\{\varphi=S\} \setminus B_{\varepsilon_j}(\operatorname{Crit}(\varphi))} \frac{\left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle}{e^S} \, d\sigma - \int_{\{\varphi=s\} \setminus B_{\varepsilon_j}[\operatorname{Crit}(\varphi)]} \frac{\left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle}{e^s} \, d\sigma \\ &\quad + \int_{\partial B_{\varepsilon_j}[\operatorname{Crit}(\varphi)]} \frac{\left\langle \nabla|\nabla\varphi|^\beta, \nu_{\varepsilon_j} \right\rangle}{e^\varphi} \, d\sigma, \end{aligned}$$

for a sequence $\varepsilon_j \rightarrow 0^+$, where ν_{ε_j} is the exterior normal to $\partial B_{\varepsilon_j}[\operatorname{Crit}(\varphi)]$, that is well defined almost everywhere on such a set due to rectifiability. By Kato's inequality

$$|\nabla\nabla\varphi|^2 \geq |\nabla|\nabla\varphi||^2,$$

and identity (3.23), $\operatorname{div}X$ is nonnegative. Thus, by the Monotone Convergence Theorem, we have

$$\int_{\{s \leq \varphi \leq S\}} \operatorname{div}X \, d\mu = \lim_{\varepsilon \rightarrow 0^+} \int_{\{s \leq \varphi \leq S\} \setminus B_\varepsilon[\operatorname{Crit}(\varphi)]} \operatorname{div}X \, d\mu.$$

Moreover, observe that the length of the vector field $\nabla|\nabla\varphi|^\beta$ is bounded in $\{s \leq \varphi \leq S\}$ for $\beta \geq 1$ again by Kato's inequality

$$|\nabla|\nabla\varphi|^\beta| = \beta|\nabla\varphi|^{\beta-1}|\nabla|\nabla\varphi|| \leq \beta|\nabla\varphi|^{\beta-1}|\nabla\nabla\varphi|,$$

and thus the Dominated Convergence Theorem gives

$$\lim_{\varepsilon_j \rightarrow 0^+} \int_{\{\varphi=s\} \setminus B_{\varepsilon_j}[\operatorname{Crit}(\varphi)]} \frac{\left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle}{e^s} \, d\sigma = \int_{\{\varphi=s\}} \frac{\left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle}{e^s} \, d\sigma,$$

and analogously for $\{\varphi = S\}$. We are then left to show that

$$\lim_{\varepsilon_j \rightarrow 0^+} \int_{\partial B_{\varepsilon_j}[\text{Crit}(\varphi)]} \frac{\langle \nabla |\nabla \varphi|^\beta, \nu_\varepsilon \rangle}{e^\varphi} d\sigma = 0. \quad (3.24)$$

To see this, we observe that the integrand is bounded and we use the estimate on the tubular neighbourhoods of the critical set given in [18, Theorem 1.17]. Namely, for every $\eta > 0$ and every $\varepsilon > 0$ there exists a constant $C_\eta > 0$ such that

$$|B_\varepsilon(\text{Crit}(\varphi))| \leq C_\eta \varepsilon^{2-\eta}. \quad (3.25)$$

Let ρ be the distance function from the closed set $\text{Crit}(\varphi)$. Then, $|\nabla \rho| = 1$ almost everywhere outside $\text{Crit}(\varphi)$ (see [58, Theorem 3.1] for much deeper insights on distance functions). Therefore, by the co-area formula and (3.25), we get

$$\int_0^\varepsilon d\rho \int_{\partial B_\rho[\text{Crit}(\varphi)]} d\sigma \leq C_\eta \varepsilon^{2-\eta}.$$

By the Mean Value Theorem, we can find a sequence $\varepsilon_j \rightarrow 0^+$ such that

$$\int_{\partial B_{\varepsilon_j}[\text{Crit}(\varphi)]} d\sigma \leq C_\eta \varepsilon_j^{1-\eta},$$

and then, up to relabeling, the limit in (3.24) holds true, completing the proof. \square

Remark 3.8. *As already observed, the refined Kato inequality for harmonic function (3.16) implies that, whenever $\nabla \varphi \neq 0$, the integrand on the right hand side of (3.22) is nonnegative also for $(n-2)/(n-1) \leq \beta < 1$. However, for β in this range, the vector field $X = e^{-\varphi} \nabla |\nabla \varphi|_{\bar{g}}^\beta$ is no longer uniformly bounded in a neighborhood of the critical set of φ , and thus the above argument is not sufficient to prove (3.24). This difficulty has been recently overcome in [4], introducing a new family of tubular neighborhoods $U_\varepsilon = \{|\nabla \varphi|_{\bar{g}}^2 \leq \varepsilon\}$ of $\text{Crit}(\varphi)$, in place of the standard ones, and then modifying the vector field $X = e^{-\varphi} \nabla |\nabla \varphi|^\beta$ through a suitable cut-off function that vanishes inside U_ε . The key point is that the divergence of the modified vector field always remains nonnegative. This scheme is shown to converge when $\varepsilon \rightarrow 0$, leading to the validity of Lemma 3.7 for every $(n-2)/(n-1) \leq \beta < 1$. We refer the reader to [4] for the details of this construction.*

Building on the Fundamental Integral Identity proved in Lemma 3.7, we are now ready to deduce the following monotonicity result. Again, the case $(n-2)/(n-1) \leq \beta < 1$ can be treated adapting the argument presented in [4].

Theorem 3.9 (Monotonicity of $e^{-s} \Phi'_\beta(s)$). *Let $\beta \geq 1$. The function Φ_β defined in (3.12) is differentiable for any $s \geq 0$, and its derivative satisfies*

$$\Phi'_\beta(s) = \int_{\{\varphi=s\}} \left\langle \nabla |\nabla \varphi|_{\bar{g}}^\beta, \frac{\nabla \varphi}{|\nabla \varphi|_{\bar{g}}} \right\rangle d\sigma_{\bar{g}} = -\beta \int_{\{\varphi=s\}} |\nabla \varphi|_{\bar{g}}^\beta H_{\bar{g}} d\sigma_{\bar{g}}. \quad (3.26)$$

In particular, for any $0 \leq s < S < \infty$, we have

$$e^{-S} \Phi'_\beta(S) - e^{-s} \Phi'_\beta(s) = \beta \int_{\{s \leq \varphi \leq S\}} |\nabla \varphi|_{\bar{g}}^{\beta-2} \left[\text{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla \nabla \varphi|_{\bar{g}}^2 + (\beta-2) \left| \nabla |\nabla \varphi|_{\bar{g}} \right|_{\bar{g}}^2 \right] e^{-\varphi} d\mu_{\bar{g}}, \quad (3.27)$$

where the Ricci tensor is referred to the background metric g .

Proof. Let us drop the subscript \tilde{g} . Let $s_0 \in [0, +\infty)$, and $s \geq s_0$. Consider

$$\Phi_\beta(s) - \Phi_\beta(s_0) = \int_{\{\varphi=s\}} \left\langle |\nabla\varphi|^\beta \nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma - \int_{\{\varphi=s_0\}} \left\langle |\nabla\varphi|^\beta \nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma.$$

Ruling out the critical set of φ as shown in the proof of the above Lemma 3.7, we can apply the Divergence Theorem to get

$$\Phi_\beta(s) - \Phi_\beta(s_0) = \int_{\{s_0 \leq \varphi \leq s\}} \operatorname{div}(|\nabla\varphi|^\beta \nabla\varphi) d\mu = \int_{\{s_0 \leq \varphi \leq s\}} \left\langle \nabla|\nabla\varphi|^\beta, \nabla\varphi \right\rangle d\mu,$$

where the last identity is due to harmonicity of φ . By co-area formula, the above quantity can also be written as

$$\Phi_\beta(s) - \Phi_\beta(s_0) = \int_{s_0}^s d\tau \int_{\{\varphi=\tau\}} \left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma. \quad (3.28)$$

By the Fundamental Theorem of Calculus, the continuity of the mapping

$$\tau \mapsto I(\tau) = \int_{\{\varphi=\tau\}} \left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma$$

implies the differentiability of Φ_β , together with the first identity in (3.26). The second one follows from the first just by a direct computation involving (3.11). In order to see the continuity of I at a time τ_0 , consider, for $\tau > \tau_0$, the difference

$$e^{-\tau} I(\tau) - e^{-\tau_0} I(\tau_0) = e^{-\tau} \int_{\{\varphi=\tau\}} \left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma - e^{-\tau_0} \int_{\{\varphi=\tau_0\}} \left\langle \nabla|\nabla\varphi|^\beta, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma.$$

Then, by (3.22), we have

$$e^{-\tau} I(\tau) - e^{-\tau_0} I(\tau_0) = \beta \int_{\{\tau_0 \leq \varphi \leq \tau\}} \frac{|\nabla\varphi|^{\beta-2} \left[\operatorname{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|^2 + (\beta-2)|\nabla|\nabla\varphi|^2 \right]}{e^\varphi} d\mu.$$

In particular, by the Dominated Convergence Theorem the above integral vanishes as $\tau \rightarrow \tau_0^+$, yielding the right continuity of $e^{-\tau} I(\tau)$ at τ_0 . By a completely analogous argument with $\tau < \tau_0$ we get that $e^{-\tau} I(\tau)$ is actually continuous at τ_0 , and clearly so is I , ending the proof. \square

We are now in a position to complete the proof of Theorem 3.2.

Conclusion of the proof of Theorem 3.2. We are going to pass to the limit as $S \rightarrow +\infty$ in (3.27). The following argument is due to Colding and Minicozzi, [28]. We first prove that the derivative of Φ_β has a sign. By (3.27) we have that, for every $\beta \geq 1$ and every $0 \leq s < S < +\infty$,

$$\Phi'_\beta(S) \geq e^{(S-s)} \Phi'_\beta(s).$$

Integrating the above differential inequality, we get

$$\Phi_\beta(S) \geq \left(e^{(S-s)} - 1 \right) \Phi'_\beta(s) + \Phi_\beta(s). \quad (3.29)$$

for every $0 \leq s < S < +\infty$. Assume now, by contradiction, that $\Phi'_\beta(s) > 0$ for some $s \in [0, +\infty)$. Then, passing to the limit as $S \rightarrow +\infty$ in (3.29), we would get $\Phi_\beta(S) \rightarrow +\infty$, against the boundedness of Φ_β provided in Corollary 3.5. Thus, $\Phi'_\beta(s) \leq 0$ for every $s \in [0, +\infty)$. Therefore Φ_β is a nonincreasing, differentiable bounded function on $[0, +\infty)$, and in particular,

$\liminf_{S \rightarrow +\infty} \Phi'_\beta(S) = 0$. Passing to the inferior limit as $S \rightarrow +\infty$ in (3.27) finally gives the monotonicity formula (3.15).

For the rigidity statement, assume now that $\Phi'_\beta(s_0) = 0$ for some $s_0 \in [0, +\infty)$. Then, by (3.15), and Kato's inequality, we get $|\nabla|\nabla\varphi|| = 0$. We then conclude by Lemma 3.1. \square

Remark 3.10. *For what it concerns the rigidity statement when $\Phi'_\beta(s_0) = 0$ for some $s_0 \geq 0$ and $(n-2)/(n-1) \leq \beta < 1$, observe that the only case to be discussed differently from $\beta \geq 1$ is $\beta = (n-2)/(n-1)$. In fact, for this specific parameter, we just obtain from (3.15) that $|\nabla\nabla\varphi|_{\bar{g}} = n/(n-1)|\nabla|\nabla\varphi||_{\bar{g}}$. However, in this case, the isometry with a Riemannian cylinder substantially follows from [11, Proposition 5.1]. We refer the reader to [4] for the details of this limit case.*

4. PROOF OF THE WILLMORE-TYPE INEQUALITIES

In this section, we are going to prove the Willmore-type inequality on manifolds with nonnegative Ricci curvature and Euclidean volume growth, using the geometric features of the capacity potential u of a given bounded domain with smooth boundary Ω . For the reader's convenience we recall that u is a solution to the following problem

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \bar{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(q) \rightarrow 0 & \text{as } d(O, q) \rightarrow +\infty, \end{cases}$$

whose existence in the present context follows from Theorem 2.13. As sketched in the Introduction, the proof makes use of the global features of the Monotonicity-Rigidity Theorem, that is, it compares the behaviour of U_β in the large with $U_\beta(1)$. The well known asymptotics of u and of its gradient in the Euclidean case were the crucial tool to compute the limits of U_β in [2], that consequently gave the sharp lower bound on the Willmore energy. On manifolds with nonnegative Ricci curvature and Euclidean volume growth, the work of Colding-Minicozzi [25] actually implies that the asymptotic behaviour of the potential is completely analogous to that in \mathbb{R}^n . However, as we will clarify in Remark 4.7, there is no hope to get an Euclidean-like pointwise behaviour of Du in the general case. Nevertheless, using techniques developed in the celebrated [15], we are able to achieve asymptotic integral estimates for the gradient that in turn will let us conclude the proof of the Willmore-type inequalities (1.6).

As in Section 2, we denote by D the Levi-Civita connection on the manifold considered, by DD the Hessian and by Δ the Laplacian. Moreover, we let $r(x) = d(O, x)$, where $O \in \Omega$. To keep the notation shorter it is also useful to recall the following characterization of the electrostatic capacity of Ω

$$\text{Cap}(\Omega) = \frac{\int_{\partial\Omega} |Du| \, d\sigma}{(n-2)|\mathbb{S}^{n-1}|}, \quad (4.1)$$

in terms of the capacity potential u of Ω .

4.1. Manifolds with Euclidean volume growth. With respect to the value assumed by $\text{AVR}(g)$, we define manifolds with Euclidean volume growth and sub-Euclidean volume growth as follows.

Definition 4.1 (Volume growth). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$. Then we say that it has Euclidean volume growth if $\text{AVR}(g) > 0$, sub-Euclidean volume growth otherwise.*

It follows at once that manifolds with Euclidean volume growth satisfy (2.3), and thus, from Varopoulos' characterization, *manifolds with Euclidean volume growth are nonparabolic*. Let us also recall the well known fact that in a complete noncompact Riemannian manifold (M, g) with $\text{Ric} \geq 0$ also the function

$$(0, +\infty) \ni r \mapsto \vartheta(r) = \frac{|\partial B(O, r)|}{r^{n-1}|\mathbb{S}^{n-1}|}$$

is monotone nonincreasing for any $O \in M$, and the asymptotic volume ratio satisfies

$$\text{AVR}(g) = \lim_{r \rightarrow +\infty} \frac{|\partial B(O, r)|}{r^{n-1}|\mathbb{S}^{n-1}|}. \quad (4.2)$$

In the remainder of this section we are repeatedly going to use both the area and the volume formulations of the Bishop-Gromov Theorem without always mentioning them.

Let us now derive some rough estimates for the electrostatic potential u and its gradient Du holding on manifolds with nonnegative Ricci curvature and Euclidean volume growth, to be refined later.

Proposition 4.2. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth. Then, it is nonparabolic and the solution u to problem (1.9) for some bounded and open Ω with smooth boundary and no holes satisfies*

$$C_1 r^{2-n}(x) \leq u(x) \leq C_2 r^{2-n}(x) \quad (4.3)$$

on $M \setminus \Omega$ for some positive constants C_1 and C_2 depending on M and Ω . Moreover, if $\Omega \subset B(O, R)$, it holds

$$|Du|(x) \leq C_3 r^{1-n}(x), \quad (4.4)$$

on $M \setminus B(O, 2R)$ with $C_3 = C_3(M, \Omega) > 0$.

Proof. The first inequality in (4.3) is just (2.15). To obtain the second one, observe first that the monotonicity in the Bishop-Gromov Theorem implies, for any $O \in M$ and any $r \in (0, +\infty)$, that

$$|B(O, r)| \geq (n|\mathbb{S}^{n-1}|\text{AVR}(g)) r^n.$$

Then, the second inequality in the Li-Yau estimate (2.11), combined with the second inequality in (2.14) completes the proof of (4.3). Finally, inequality (4.4) is achieved just by plugging the upper estimate on u given by (4.3) into (2.5). \square

4.2. Asymptotics for u . The behaviour at infinity of u can be deduced along the path indicated in [25]. However, we prefer to present here a simplified version of that proof, taking advantage of some of the refinements provided in [52]. Let us first recall the following asymptotic behaviour of G (see [25, Theorem 0.1], or [52, Theorem 1.1] for a completely different proof),

$$\lim_{r(x) \rightarrow +\infty} \frac{G(O, x)}{r(x)^{2-n}} = \frac{1}{\text{AVR}(g)}. \quad (4.5)$$

Building on this fact, we deduce precise asymptotics for the capacity potential of Ω .

Lemma 4.3. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth, and let u be a solution to problem (1.9). Then*

$$\lim_{r(x) \rightarrow +\infty} \frac{u(x)}{r(x)^{2-n}} = \frac{\text{Cap}(\Omega)}{\text{AVR}(g)}. \quad (4.6)$$

Proof. By [52, Theorem 1.2], that slightly extends [25, Theorem 0.3], we have that outside some large ball $B(O, R)$ containing Ω

$$u = - \frac{G}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial B(O,R)} \frac{\partial u}{\partial \nu} d\sigma + v, \quad (4.7)$$

where G is the Green's function with pole in O , ν is the exterior unit normal to the boundary of $B(O, R)$ and v is an harmonic function defined in $M \setminus B(O, R)$ satisfying

$$|v| \leq C \frac{G}{r} \quad (4.8)$$

for some constant $C > 0$. We point out that the Green's function considered in [52] is, in our notation, $G/((n-2)|\mathbb{S}^{n-1}|)$. By the Divergence Theorem and the harmonicity of u , we infer that

$$\int_{\partial B(O,R)} \frac{\partial u}{\partial \nu} d\sigma + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} d\sigma = \int_{B(O,R) \setminus \Omega} \Delta u d\mu = 0,$$

where we denote by ν the exterior unit normal to the boundary of $B(O, R) \setminus \bar{\Omega}$. Since, on $\partial \Omega$, $\nu = Du/|Du|$, we get, by the above identity, that

$$- \frac{\int_{\partial B(O,R)} \frac{\partial u}{\partial \nu} d\sigma}{(n-2)|\mathbb{S}^{n-1}|} = \frac{\int_{\partial \Omega} |Du| d\sigma}{(n-2)|\mathbb{S}^{n-1}|} = \text{Cap}(\Omega). \quad (4.9)$$

Dividing both sides of (4.7) by r^{2-n} and passing to the limit as $r \rightarrow +\infty$ taking into account (4.5), (4.9) and (4.8), we get the claim. \square

As a straightforward corollary of the above lemma, we compute the rescaled area of large geodesic spheres in M .

Corollary 4.4. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth, and let u be a solution to (1.9). Then*

$$\lim_{R_i \rightarrow +\infty} \int_{\partial B(O, R_i)} u^{\frac{n-1}{n-2}} d\sigma = |\mathbb{S}^{n-1}| \text{AVR}(g) \left(\frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \right)^{\frac{n-1}{n-2}}$$

Proof. It follows from Lemma 4.3 and (4.2). \square

Building on the above formula (4.6) for the asymptotics of u , we now derive integral asymptotics for Du . They are achieved through an adaptation of the methods used in [15, Section 4]. Similar estimates have been widely considered in literature (see e.g. [23] [25], [24], [27]). We are providing here a complete and self-contained proof.

In the computations below, we are going to consider first and second derivatives of the distance function $x \mapsto d(O, x)$. This can be readily justified by approximating the distance function by convolution, by performing the computations below for the approximating sequence and finally passing to the limit. A scheme like this, often implicit in modern literature, has been carried out in details for example in [52]. However, in order to keep the presentation as transparent as possible, we are going to work directly with the distance function.

Proposition 4.5 (Integral Asymptotics for Du). *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth, and let u be a solution to problem (1.9).*

Then, for every $k > 1$, it holds

$$\lim_{R \rightarrow +\infty} \frac{R^{2n-2}}{|A_{R,kR}|} \int_{A_{R,kR}} \left| Du - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} Dr^{2-n} \right|^2 d\mu = 0, \quad (4.10)$$

where, for $R > 0$, we set $A_{R,kR} = B(O, kR) \setminus \overline{B(O, R)}$.

Proof. A simple integration by parts combined with the harmonicity of u leads to the following identity

$$\begin{aligned} \int_{A_{R,kR}} \left| Du - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} Dr^{2-n} \right|^2 d\mu &= - \int_{A_{R,kR}} \left(\frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \Delta r^{2-n} \right) \left(u - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{2-n} \right) d\mu + \\ &+ \int_{\partial A_{R,kR}} \left(u - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{2-n} \right) \left\langle Du - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} Dr^{2-n}, \nu \right\rangle d\sigma. \end{aligned} \quad (4.11)$$

Let us estimate separately the integrals on the right hand side of the above identity. Let $\varepsilon > 0$. Then, by (4.6),

$$\left| \frac{u}{r^{2-n}} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \right| < \varepsilon.$$

for r big enough. We have, for R big enough

$$\begin{aligned} \left| \int_{A_{R,kR}} \left(\frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \Delta r^{2-n} \right) \left(u - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{2-n} \right) d\mu \right| &\leq \int_{A_{R,kR}} \left| \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \Delta r^{2-n} \right| \left| \frac{u}{r^{2-n}} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \right| r^{2-n} d\mu \\ &\leq \varepsilon R^{2-n} \int_{A_{R,kR}} \left| \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \Delta r^{2-n} \right| d\mu \\ &= \varepsilon R^{2-n} \int_{A_{R,kR}} \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \Delta r^{2-n} d\mu, \end{aligned} \quad (4.12)$$

where in the last identity we used $\Delta r^{2-n} \geq 0$ in the sense of distributions, as already shown along the proof of Lemma 2.12. Integrating by parts Δr^{2-n} we obtain

$$\int_{A_{R,kR}} \Delta r^{2-n} d\mu = (2-n) [(kR)^{1-n} |\{r = kR\}| - R^{1-n} |\{r = R\}|].$$

In particular, by the assumption on the Euclidean volume growth, the above quantity is uniformly bounded in R . We have thus proved that the first summand on the right hand side of (4.11), for R large enough, is bounded as follows

$$\left| \int_{A_{R,kR}} \left(\frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \Delta r^{2-n} \right) \left(u - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{2-n} \right) d\mu \right| \leq C_1 \varepsilon R^{2-n}. \quad (4.13)$$

Let us turn our attention to the second integral in the right hand side of (4.11). Proceeding as in (4.12), we have

$$\begin{aligned} \int_{\partial A_{R,kR}} \left| \left(u - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{2-n} \right) \left\langle \text{Du} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \text{Dr}^{2-n}, \nu \right\rangle \right| d\sigma &\leq \\ &\leq \varepsilon R^{2-n} \int_{\partial A_{R,kR}} \left[|\text{Du}| + (n-2) \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{1-n} \right] d\sigma. \end{aligned} \quad (4.14)$$

Recall now that in Proposition 4.2 we proved that

$$|\text{Du}| \leq C_3 r^{1-n},$$

for some positive constant C_3 independent of r . Thus, by the Euclidean volume growth, the integral on the right hand side of (4.14) is uniformly bounded in R and so

$$\left| \int_{\partial A_{R,kR}} \left(u - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{2-n} \right) \left\langle \text{Du} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \text{Dr}^{2-n}, \nu \right\rangle d\sigma \right| \leq C_3 \varepsilon R^{2-n} \quad (4.15)$$

for some C_3 independent of R . Finally, by (4.11), (4.13) and (4.15), we obtain, for R big enough, the estimate

$$\frac{1}{|A_{R,kR}|} \int_{A_{R,kR}} \left| \text{Du} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \text{Dr}^{2-n} \right|^2 d\mu \leq C_4 \varepsilon \frac{R^{2-n}}{|A_{R,kR}|} \leq C_5 \varepsilon R^{2-2n},$$

for some positive constants C_4 and C_5 independent of R . In the last inequality we used the Euclidean volume growth assumption. Our claim (4.10) is thus proved. \square

From the above proposition we easily deduce the integral asymptotic behaviour of $|\text{Du}|$ on the level sets of u .

Corollary 4.6. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth, and let u be a solution to problem (1.9). Then, there exists a sequence of positive real numbers $\{t_j\}_{j \in \mathbb{N}}$ with $t_j \rightarrow 0$ as $j \rightarrow +\infty$ such that*

$$\lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} \left| |\text{Du}| - (n-2) \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} r^{1-n} \right| d\sigma = 0. \quad (4.16)$$

Proof. Let us first observe that, by means of Hölder inequality, we can deduce from the L^2 asymptotics (4.10) an analogous L^1 behaviour. Namely, for any $\varepsilon > 0$ we have

$$\frac{\int_{A_{R,kR}} \left| \text{Du} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \text{Dr}^{2-n} \right| d\mu}{R^{1-n} |A_{R,kR}|} \leq \left(\frac{\int_{A_{R,kR}} \left| \text{Du} - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \text{Dr}^{2-n} \right|^2 d\mu}{R^{2-2n} |A_{R,kR}|} \right)^{1/2} \leq \varepsilon \quad (4.17)$$

for any R large enough. Let us also employ the analogous in our setting of Colding's function b (compare with the Appendix) defined by

$$b = \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} u \right)^{-\frac{1}{n-2}}.$$

Then, as a consequence of (4.6), we have

$$\left| \frac{b}{r} - 1 \right| \leq \frac{1}{4}$$

outside some large enough ball $B(O, \bar{R})$, centered at some point $O \in M$. Building on this, we define for every $R > \bar{R}$ the annulus E_R as

$$E_R = \left\{ \frac{5}{4}R \leq b \leq \frac{6}{4}R \right\}.$$

It is immediately checked that $E_R \subseteq A_{R,2R}$. Then, setting $k = 2$ in (4.17), we have that

$$\frac{\int_{E_R} \left| Du - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} Dr^{2-n} \right| d\mu}{R^{1-n} |A_{R,2R}|} \leq \varepsilon,$$

provided R is large enough. By the Coarea Formula, the above estimate is equivalent to

$$\frac{\int_{\frac{5}{4}R}^{\frac{6}{4}R} ds \int_{\{b=s\}} \left| Du - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} Dr^{2-n} \right| \frac{1}{|Db|} d\sigma}{R^{1-n} |A_{R,2R}|} \leq \varepsilon.$$

Observing that, up to a multiplicative constant, $|Db|$ coincides with $|\nabla\varphi|_{\bar{g}} = |Du|u^{-(n-1)/(n-2)}$, one can easily deduce from Proposition 3.3 that $|Db|$ is uniformly bounded by $\sup_{\partial\Omega} |Db|$. This fact, combined with the Mean Value Theorem, yields the existence of $s_R \in (5R/4, 6R/4)$ such that

$$\int_{\{b=s_R\}} \left| Du - \frac{\text{Cap}(\Omega)}{\text{AVR}(g)} Dr^{2-n} \right| d\sigma \leq C_1 R^{-n} |A_{R,2R}| \varepsilon \leq C_2 \varepsilon,$$

for constants C_1 and C_2 independent of R . The last inequality is due to Bishop-Gromov's comparison for geodesic annuli. The above estimate clearly implies the existence of a sequence t_j as in the statement. \square

Remark 4.7. *The integral asymptotic for $|Du|$ given by Corollary 4.6 cannot, in general, be improved to a pointwise asymptotic expansion at infinity on a manifold with nonnegative Ricci curvature and Euclidean volume growth. Indeed, the validity of such a formula would imply $|Du| \neq 0$ outside some big ball $B(O, R)$, and, in turn, $M \setminus B(O, R)$ would be diffeomorphic to $\partial B(O, R) \times [0, +\infty)$. This would imply that M has finite topological type. However, Menguy provided in [59] examples of manifolds of dimension $n \geq 4$ with $\text{Ric} \geq 0$ and $\text{AVR}(g) > 0$ with infinite topological type.*

4.3. Final steps of the proof. Recalling that

$$U_\beta(t) = t^{-\beta\left(\frac{n-1}{n-2}\right)} \int_{\{u=t\}} |Du|^{1+\beta} d\sigma,$$

it is easy to realise that we have now at hand all the elements to compute the limit as $t \rightarrow 0^+$.

Proposition 4.8. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth. Then, for every $\beta \geq 0$, we have that*

$$\lim_{t \rightarrow 0^+} U_\beta(t) = \text{Cap}(\Omega)^{1-\frac{\beta}{n-2}} \text{AVR}(g)^{\frac{\beta}{n-2}} (n-2)^{\beta+1} |\mathbb{S}^{n-1}|. \quad (4.18)$$

Proof. We multiply and divide inside the integral in (4.16) by $u^{(n-1)/(n-2)}$. Using the asymptotics of u obtained in Lemma 4.3, we obtain

$$\lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} \left| \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right| - (n-2) \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1}{n-2}} \right| u^{\frac{n-1}{n-2}} d\sigma = 0. \quad (4.19)$$

As an immediate consequence of the above limit we have

$$\lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} |Du| d\sigma = (n-2) \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1}{n-2}} \lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} u^{\frac{n-1}{n-2}} d\sigma. \quad (4.20)$$

On the other hand, since u is harmonic, the Divergence Theorem yields

$$\int_{\{u=t\}} |Du| d\sigma = \int_{\partial\Omega} |Du| d\sigma$$

for any $t \in (0, 1]$. Hence, taking into account the well known characterization of the electrostatic capacity (4.1), one has that (4.20) readily implies

$$\lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} u^{\frac{n-1}{n-2}} d\sigma = |\mathbb{S}^{n-1}| \text{AVR}(g) \left(\frac{\text{Cap}(\Omega)}{\text{AVR}(g)} \right)^{\frac{n-1}{n-2}}. \quad (4.21)$$

Observe now that clearly

$$\begin{aligned} \int_{\{u=t_j\}} \left| \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right| - (n-2) \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1}{n-2}} \right|^{1+\beta} u^{\frac{n-1}{n-2}} d\sigma &\leq \\ &\leq \int_{\{u=t_j\}} \left| \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right| - (n-2) \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1}{n-2}} \right| u^{\frac{n-1}{n-2}} d\sigma \times \\ &\quad \times \sup_{\{u=t_j\}} \left| \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right| - (n-2) \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1}{n-2}} \right|^\beta. \end{aligned} \quad (4.22)$$

Thanks to the uniform bounds on $|Du|/u^{(n-1)/(n-2)} = |\nabla\varphi|_{\tilde{g}}$ given in (3.20), the second factor on the right hand side of (4.22) is uniformly bounded with respect to t_j . Thus, by (4.22) and (4.19), we deduce that, for every $\beta \geq 0$, it holds

$$\lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} \left| \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right| - (n-2) \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1}{n-2}} \right|^{1+\beta} u^{\frac{n-1}{n-2}} d\sigma = 0.$$

In particular, the above limit implies that

$$\lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right|^{1+\beta} u^{\frac{n-1}{n-2}} d\sigma = (n-2)^{1+\beta} \left(\frac{\text{AVR}(g)}{\text{Cap}(\Omega)} \right)^{\frac{1+\beta}{n-2}} \lim_{j \rightarrow +\infty} \int_{\{u=t_j\}} u^{\frac{n-1}{n-2}} d\sigma,$$

which, combined with (4.21), establishes the validity of our claim for some sequence $t_j \rightarrow 0^+$. However, by the boundedness and monotonicity of U_β the whole limit as $t \rightarrow 0^+$ exists and coincides with the one that we have just computed. \square

Let us briefly discuss the case of sub-Euclidean volume growth in the following remark.

Remark 4.9. *If (M, g) has sub-Euclidean volume growth, that is, if*

$$\lim_{r \rightarrow +\infty} \frac{|B(p, r)|}{r^n} = 0$$

for any $p \in M$, it is easy to realize that $\lim_{t \rightarrow 0^+} U_\beta = 0$ for any $\beta \geq 0$. Indeed, by (2.14), (2.11) and (2.6), one easily obtains

$$\frac{|Du|}{u^{\frac{n-1}{n-2}}} \leq C \left[\frac{|B(p, r)|}{r^n} \right]^{\frac{1}{n-2}}$$

outside some ball containing Ω for some $C = C(M, \Omega)$. This implies that

$$\lim_{t \rightarrow 0^+} U_\beta(t) = \lim_{t \rightarrow 0^+} \int_{\{u=t\}} \left| \frac{Du}{u^{\frac{n-1}{n-2}}} \right|^\beta |Du| d\sigma = 0,$$

where we used the constancy of $t \mapsto \int_{\{u=t\}} |Du| d\sigma$. This computation clearly shows that U_β cannot be employed to deduce a Willmore-type inequality on manifolds with sub-Euclidean volume growth, and it supports the perception that the infimum of the Willmore-type functional is zero on these manifolds. This actually happens for example on noncompact Riemannian manifolds with a metric which is asymptotic to a warped product metric of the following type

$$g = d\rho \otimes d\rho + C\rho^{2\alpha} g|_\Sigma,$$

where Σ is a compact hypersurface, $C > 0$ and $0 < \alpha < 1$. Indeed, this is readily checked by computing the Willmore-type functional on large level sets $\{\rho = r\}$.

Proof of Theorem 1.1. Let us first focus on the case where Ω has no holes, i.e., $M \setminus \Omega$ has no compact connected components. In this case, with Theorem 1.3 and Proposition 4.8 at hand, Theorem 1.1 follows exactly as in the Euclidean proof recalled in the Introduction. Precisely, let $\beta = n - 2$. Then, (4.18) reads

$$\lim_{t \rightarrow 0^+} U_{n-2}(t) = \text{AVR}(g) (n-2)^{n-1} |\mathbb{S}^{n-1}|.$$

Moreover, by the nonnegativity of expression (1.11) proved in Theorem 1.3, and the Hölder inequality,

$$\frac{n-1}{n-2} \int_{\partial\Omega} |Du|^{n-1} d\sigma \leq \int_{\partial\Omega} |Du|^{n-2} H d\sigma \leq \left(\int_{\partial\Omega} |Du|^{n-1} d\sigma \right)^{(n-2)/(n-1)} \left(\int_{\partial\Omega} H^{n-1} d\sigma \right)^{1/(n-1)},$$

that gives

$$\int_{\partial\Omega} |Du|^{n-1} d\sigma \leq (n-2)^{n-1} \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma.$$

Finally, one has

$$\begin{aligned} \text{AVR}(g) (n-2)^{n-1} |\mathbb{S}^{n-1}| &= \lim_{t \rightarrow 0^+} U_{n-2}(t) \leq U_{n-2}(1) = \int_{\partial\Omega} |Du|^{n-1} d\sigma \leq \\ &\leq (n-2)^{n-1} \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma. \end{aligned}$$

This completes the proof of the Willmore-type inequality, when Ω has no holes. Notice that in the penultimate passage we took advantage of the no holes assumption to deduce that $\partial\Omega$ coincides with the level set $\{u = 1\}$, via the Strong Maximum Principle.

We now discuss how to remove the no hole assumption and prove the general statement. As already observed, under our assumptions, $M \setminus \Omega$ has exactly one unbounded connected component. Let us denote by E such a connected component and by D its complement, i.e.,

$D = M \setminus E$. Observe that D is by construction a bounded open set with smooth boundary and no holes. Hence, by the previous discussion, we have that

$$\int_{\partial D} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq \text{AVR}(g) |\mathbb{S}^{n-1}|.$$

The validity of (1.6) now follows from the fact that $\partial D \subseteq \partial\Omega$. For the rigidity part, let us suppose that

$$\text{AVR}(g) |\mathbb{S}^{n-1}| = \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma = \int_{\partial D} \left| \frac{H}{n-1} \right|^{n-1} d\sigma + \int_{\partial\Omega \setminus \partial D} \left| \frac{H}{n-1} \right|^{n-1} d\sigma.$$

By the previous inequality we have that the last summand in the rightmost hand side must be zero. Since $\partial\Omega \setminus \partial D$ is only given by a finite number of smooth compact hypersurfaces, we can invoke the Kasue's Theorem [44, Theorem C (2)] to deduce that $\partial\Omega \setminus \partial D$ is empty. In particular Ω has no holes and the thesis follows from the rigidity part of Theorem 1.3 \square

4.4. Application to ALE manifolds. We can improve our Willmore-type inequality if (M, g) satisfies a *quadratic curvature decay* condition, showing that, in this case, the lower bound $\text{AVR}(g) |\mathbb{S}^{n-1}|$ on the Willmore functional is actually an infimum. Let us recall the following well known definition.

Definition 4.10 (Quadratic curvature decay). *A complete noncompact Riemannian manifold (M, g) has quadratic curvature decay if there exists a point $p \in M$ and a constant $C = C(M, p)$ such that*

$$|\text{Riem}|(q) \leq C d(p, q)^{-2},$$

where by Riem we denote the Riemann curvature tensor of (M, g) .

When this assumption is added on a Riemannian manifold with $\text{Ric} \geq 0$ and Euclidean volume growth (M, g) , [25, Proposition 4.1] gives the following asymptotic behaviour of the gradient and the Hessian of the minimal Green's function G .

Theorem 4.11. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$, Euclidean volume growth and quadratic curvature decay, and let G be its minimal Green's function. Then*

$$\lim_{d(p,x) \rightarrow +\infty} \frac{|D_x G|(p, x)}{d^{1-n}(p, x)} = \frac{(n-2)}{\text{AVR}(g)}, \quad (4.23)$$

$$\lim_{d(p,x) \rightarrow +\infty} \left| \text{DD}_x \left(G^{2/(2-n)} \right) (p, x) - 2 \left(\frac{1}{\text{AVR}(g)} \right)^{\frac{2}{2-n}} g(x) \right| = 0. \quad (4.24)$$

Observe that arguing as in Remark 4.7, one realizes that (4.23) actually implies that Riemannian manifolds satisfying the assumptions of the above Theorem have finite topological type.

Theorem 4.11 enables us to prove that the Willmore-type functional of large level sets of the Green's function approaches $\text{AVR}(g) |\mathbb{S}^{n-1}|$. This fact, combined with our Willmore-type inequality (1.6), easily yields the following refinement.

Theorem 4.12. *Let (M, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$, Euclidean volume growth and quadratic curvature decay. Then,*

$$\inf \left\{ \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \mid \Omega \subset M \text{ bounded and smooth} \right\} = \text{AVR}(g) |\mathbb{S}^{n-1}|. \quad (4.25)$$

Moreover, the infimum is attained at some bounded and smooth $\Omega \subset M$ if and only if $(M \setminus \Omega, g)$ is isometric to

$$\left([r_0, +\infty) \times \partial\Omega, dr \otimes dr + (r/r_0)^2 g_{\partial\Omega} \right), \quad \text{with } r_0 = \left(\frac{|\partial\Omega|}{\text{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

Proof. Let $p \in M$ be fixed, and let G be the minimal Green's function of (M, g) . Let us denote again by G the function $q \mapsto G(p, q)$. In light of Theorem 1.1, it suffices to prove that

$$\lim_{t \rightarrow +\infty} \int_{\{G^{2/(2-n)}=t\}} \left| \frac{H}{n-1} \right|^{n-1} d\sigma = \text{AVR}(g)|\mathbb{S}^{n-1}|. \quad (4.26)$$

To see this, consider, at a point x , orthonormal vectors $\{e_1, \dots, e_{n-1}\}$ tangent to the level set $\{G = G(x)\}$. Then, letting $r(x) = d(p, x)$, we have, by (4.24),

$$\lim_{r(x) \rightarrow +\infty} \sum_{i=1}^{n-1} \text{DD} \left(G^{\frac{2}{2-n}} \right) (e_i, e_i)(x) = 2(n-1) \left(\frac{1}{\text{AVR}(g)} \right)^{\frac{1}{2-n}}. \quad (4.27)$$

The mean curvature of the level sets of $G^{2/(2-n)}$ is clearly computed as

$$H_{G^{2/(2-n)}} = \frac{\sum_{i=1}^{n-1} \text{DD} \left(G^{\frac{2}{2-n}} \right) (e_i, e_i)}{|DG^{2/(2-n)}|},$$

that, combined with (4.27), (4.23) and (4.5) gives

$$\lim_{r(x) \rightarrow +\infty} r(x) H_{G^{2/(2-n)}}(x) = (n-1).$$

Combining it again with (4.5) and Euclidean volume growth gives (4.26), in fact completing the proof. \square

We now particularize Theorem 4.12 to *Asymptotically Locally Euclidean* (ALE) manifolds, proving Corollary 1.2. We adopt the following definition, that is a sort of extension of the one considered in the celebrated [7] – where striking relations between curvature decay conditions and behaviour at infinity of manifolds are drawn – and sensibly weaker than the one used by Joyce in the classical reference [43].

Definition 4.13 (ALE manifolds). *We say that a complete noncompact Riemannian manifold (M, g) is ALE (of order τ) if there exist a compact set $K \subset M$, a ball $B \subset \mathbb{R}^n$, a diffeomorphism $\Psi : M \setminus K \rightarrow \mathbb{R}^n \setminus B$, a subgroup $\Gamma < SO(n)$ acting freely on $\mathbb{R}^n \setminus B$ and a number $\tau > 0$ such that*

$$(\Psi^{-1} \circ \pi)^* g(z) = g_{\mathbb{R}^n} + O(|z|)^{-\tau} \quad (4.28)$$

$$\left| \partial_i [(\Psi^{-1} \circ \pi)^* g] \right| (z) = O(|z|)^{-\tau-1} \quad (4.29)$$

$$\left| \partial_i \partial_j [(\Psi^{-1} \circ \pi)^* g] \right| (z) = O(|z|)^{-\tau-2}, \quad (4.30)$$

where π is the natural projection $\mathbb{R}^n \rightarrow \mathbb{R}^n/\Gamma$, $z \in \mathbb{R}^n \setminus B$ and $i, j = 1, \dots, n$.

Proof of Corollary 1.2. Conditions (4.28), (4.29), (4.30) readily imply that ALE manifolds have Euclidean volume growth and quadratic curvature decay. Moreover, condition (4.28) and a direct computation give that

$$\text{AVR}(g) = \frac{|\mathbb{S}^{n-1}/\Gamma|}{|\mathbb{S}^{n-1}|} = \frac{1}{\text{card}(\Gamma)}. \quad (4.31)$$

The characterization (1.7) then follows immediately from (4.25).

Assume now that the infimum of the Willmore functional is attained at some $\Omega \subset M$. Then, by the rigidity part in Theorem 4.12, $M \setminus \Omega$ is isometric to a truncated cone over $\partial\Omega$. However, by (4.28), (M, g) is also C^0 -close at infinity to a metric cone with link \mathbb{S}^{n-1}/Γ . Since the cross sections of a cone are all homothetic to each other, $\partial\Omega$ is homothetic to \mathbb{S}^{n-1}/Γ , that is, they are diffeomorphic and $g_{\partial\Omega} = \lambda^2 g_{\mathbb{S}^{n-1}/\Gamma}$ for some positive constant λ . This fact, together with (4.31) in the rigidity part of Theorem 1.1 implies that

$$\left([r_0, +\infty) \times \partial\Omega, dr \otimes dr + (\lambda r/r_0)^2 g_{\mathbb{S}^{n-1}/\Gamma} \right), \quad \text{with } r_0 = \left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}/\Gamma|} \right)^{\frac{1}{n-1}}.$$

In particular, one has

$$|\partial B(O, R)| = \left(\frac{R\lambda}{r_0} \right)^{n-1} |\mathbb{S}^{n-1}/\Gamma|.$$

Combining this with (4.31) we conclude that $\lambda = r_0$, proving the isometry with (1.8) and completing the proof. \square

5. PROOF OF THE ENHANCED KASUE'S THEOREM

The proof of Theorem 1.5 is completely analogous to the proof of Theorem 3.2, for this reason most of the details can be easily adapted from the previous section and will be left to the interested reader. For $\beta \geq 0$, we recall the definition of the function $\Psi_\beta : [0, +\infty) \rightarrow \mathbb{R}$ satisfying

$$\Psi_\beta(s) = \int_{\{\psi=s\}} |D\psi|^{\beta+1} d\sigma,$$

where ψ is a solution to problem (1.15) for some open subset $\Omega \subset M$ with smooth compact boundary and connected unbounded complement. Combining the uniform bound on $|D\psi|$ given in Theorem 2.16 with the constancy of Ψ_0 , we obtain, as in Corollary 3.5, that Ψ_β is uniformly bounded in s for any $\beta \geq 0$. We record this fact for future reference.

Lemma 5.1. *Let (M, g) be a parabolic manifold with $\text{Ric} \geq 0$, and let ψ be a solution to problem (1.15). Then, the function Ψ_β defined in (1.16) is uniformly bounded for every $\beta \geq 0$.*

We are now in a position to prove the Monotonicity-Rigidity Theorem for parabolic manifolds with nonnegative Ricci curvature.

Proof of Theorem 1.5. As for Theorem 3.2, we only prove Theorem 1.5 for $\beta \geq 1$. To include the remaining cases it is sufficient to follow the strategy suggested in Remark 1.4. Let us begin with the computation of a Bochner-type identity for $|D\psi|^\beta$, in the same spirit as in Lemma 3.6

$$\Delta |D\psi|^\beta = \beta |D\psi|^{\beta-2} \left[|DD\psi|^2 + (\beta-2) |D|D\psi||^2 + \text{Ric}(D\psi, D\psi) \right]. \quad (5.1)$$

Observe that the right hand side of (5.1) is nonnegative if $\beta \geq (n-2)/(n-1)$, by means of the refined Kato's inequality (3.16) for harmonic functions. Applying the Divergence Theorem and the co-area formula in the same fashion as in the proof of (3.28), we get

$$\Psi_\beta(s) - \Psi_\beta(s_0) = \int_{s_0}^s d\tau \int_{\{\psi=\tau\}} \left\langle D|D\psi|^\beta, \frac{D\psi}{|D\psi|} \right\rangle d\sigma. \quad (5.2)$$

Exactly as in Lemma 3.7, we can use the Divergence Theorem and (5.1) to obtain that for every $0 \leq \tau_0 < \tau$ it holds

$$\begin{aligned} \int_{\{\psi=\tau\}} \left\langle D|\mathbf{D}\psi|^\beta, \frac{D\psi}{|\mathbf{D}\psi|} \right\rangle d\sigma - \int_{\{\psi=\tau_0\}} \left\langle D|\mathbf{D}\psi|^\beta, \frac{D\psi}{|\mathbf{D}\psi|} \right\rangle d\sigma &= \\ &= \beta \int_{\{\tau_0 \leq \psi \leq \tau\}} |\mathbf{D}\psi|^{\beta-2} \left[|\mathbf{D}\mathbf{D}\psi|^2 + (\beta-2)|\mathbf{D}|\mathbf{D}\psi||^2 + \text{Ric}(D\psi, D\psi) \right] d\mu. \end{aligned}$$

Combining this integral identity with (5.2), we deduce, as in the proof of Theorem 3.9, that Ψ_β is differentiable, that

$$\Psi'_\beta(s) = \int_{\{\psi=s\}} \left\langle D|\mathbf{D}\psi|^\beta, \frac{D\psi}{|\mathbf{D}\psi|} \right\rangle d\sigma = -\beta \int_{\{\psi=s\}} |\mathbf{D}\psi|^\beta H d\sigma,$$

and that, for every $S \geq s \geq 0$, it holds

$$\Psi'_\beta(S) - \Psi'_\beta(s) = \beta \int_{\{s \leq \psi \leq S\}} |\mathbf{D}\psi|^{\beta-2} \left(\text{Ric}(D\psi, D\psi) + |\mathbf{D}\mathbf{D}\psi|^2 + (\beta-2)|\mathbf{D}|\mathbf{D}\psi||^2 \right) d\mu \geq 0. \quad (5.3)$$

Since Ψ_β is bounded by Lemma 5.1, we can argue as in the conclusion of the proof of Theorem 3.2 to pass to the limit as $S \rightarrow +\infty$ in (5.3) and obtain the monotonicity formula (1.17). The rigidity part of the statement is obtained exactly as for that of Theorem 3.2. \square

The Enhanced Kasue's Theorem 1.6 now follows at once.

Proof of Theorem 1.6 and Corollary 1.7. Assume first that (M, g) is nonparabolic. Then, by Proposition 2.10 the assumptions in the statement of Theorem 1.6 are equivalent to those in Theorem 1.3. Using (1.11) and (1.12) at $t = 1$ we get

$$\left(\frac{n-1}{n-2} \right) U_\beta(1) + \frac{1}{\beta} \frac{dU_\beta}{dt}(1) = \int_{\partial\Omega} H |\mathbf{D}u|^\beta d\sigma \leq \sup_{\partial\Omega} H_{\partial\Omega} \int_{\partial\Omega} |\mathbf{D}u|^\beta d\sigma,$$

that is (1.18). If (M, g) is parabolic, one can prove inequality (1.19) in a completely analogous fashion, using Theorem 1.5 together with $\partial\Omega = \{\psi = 0\}$, as guaranteed under our assumptions by the Strong Maximum Principle. Since $U_\beta > 0$, it is easy to deduce from (1.18) and (1.19) that $H \leq 0$ on $\partial\Omega$ if and only if (M, g) is parabolic and $\Psi'_\beta(0) = 0$. The rigidity statement in Theorem 1.5 gives then Corollary 1.7. \square

6. THE ISOPERIMETRIC INEQUALITY FOR 3-MANIFOLDS

As already discussed in the Introduction, we show here how to use our Willmore inequality (1.6) to improve a result stated by Huisken in [37], in which the infimum of the Willmore energy is characterized in terms of the infimum of the isoperimetric ratio on 3-manifolds with nonnegative Ricci curvature. In concrete, we are going to prove Theorem 1.8, whose statement is recalled hereafter for the reader's convenience.

Theorem 6.1. *Let (M, g) be a complete noncompact 3-manifold with $\text{Ric} \geq 0$ and Euclidean volume growth. Then,*

$$\inf \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} = \inf \frac{\int_{\partial\Omega} H^2 d\sigma}{16\pi} = \text{AVR}(g),$$

where the infima are taken over bounded and open subsets $\Omega \subset M$ with smooth boundary. In particular, the following isoperimetric inequality holds for any bounded and open $\Omega \subset M$ with smooth boundary

$$\frac{|\partial\Omega|^3}{|\Omega|^2} \geq 36\pi \text{AVR}(g).$$

Moreover, equality is attained in (1.21) if and only if $M = \mathbb{R}^3$ and Ω is a ball.

Remark 6.2. We point out that in the 3-dimensional case Theorem 1.8 extends Theorem 4.12 to any complete noncompact manifold with $\text{Ric} \geq 0$ and Euclidean volume growth, with no curvature assumptions at infinity. Such enhancement is implicitly due to the fact that in dimension $n = 3$ the topology of nonnegatively Ricci curved manifolds is completely understood (see [54]).

6.1. Huisken's argument. Let us briefly present Huisken's heuristic argument to deduce an isoperimetric inequality from Willmore's through the Mean Curvature Flow. We first recall that a sequence of orientable hypersurfaces $F_t(p) : \Sigma \rightarrow M$ immersed in a Riemann manifold (M, g) , evolves through the Mean Curvature Flow if

$$\frac{d}{dt}F_t(p) = -H(p, t)\nu(p, t),$$

where H is the mean curvature of $\Sigma_t = F_t(\Sigma)$ and ν is its (exterior, in the case where Σ_t is the boundary of a domain) unit normal. Accordingly, we say that $\{\Omega_t\}$ is a mean curvature flow if the boundaries are evolving through mean curvature flow in the sense explained above. Let then Ω be an open bounded set with smooth boundary and let $\{\Omega_t\}$, with $t \in [0, T)$, be a smooth Mean Curvature Flow starting at Ω . Suppose, in addition, that

$$\lim_{t \rightarrow T^-} |\Omega_t| = 0. \quad (6.1)$$

Consider, for some constant $C > 0$ to be defined later, the Isoperimetric Difference

$$D(t) = |\partial\Omega_t|^{3/2} - C|\Omega_t|. \quad (6.2)$$

Taking derivatives in t , and using standard formulas (see for example [41, Theorem 3.2]), one finds

$$\frac{d}{dt}D(t) = -\frac{3}{2}|\partial\Omega_t|^{1/2} \int_{\partial\Omega_t} H^2 d\sigma + C \int_{\partial\Omega_t} H d\sigma,$$

that, through Hölder inequality, gives

$$\frac{d}{dt}D(t) \leq \left(|\partial\Omega_t| \int_{\partial\Omega_t} H^2 d\sigma \right)^{1/2} \left[C - \frac{3}{2} \left(\int_{\partial\Omega_t} H^2 d\sigma \right)^{1/2} \right].$$

Thus, if we choose C such that

$$C \leq \frac{3}{2} \left(\int_{\partial\Omega} H^2 d\sigma \right)^{1/2} \quad (6.3)$$

for any bounded and smooth $\Omega \subset M$, $t \mapsto D(t)$ is nonincreasing. This implies that

$$D(0) = |\partial\Omega|^{3/2} - C|\Omega| \geq \lim_{t \rightarrow T^-} D(t) \geq 0,$$

where we have also used (6.1). The above comparison in particular gives the (possibly non sharp) isoperimetric inequality

$$\frac{|\partial\Omega|^{3/2}}{|\Omega|} \geq C.$$

In [37], the constant C is chosen to be the infimum of the right hand side of (6.3), when Ω varies in the class of outward minimizing domains.

6.2. Tools from the Mean Curvature Flow of mean-convex domains. We are first concerned with the accurate justification of the above computations. This will be accomplished with the help of a couple of important results due to Schulze and White, respectively. In the first part of our treatment we assume that the boundary $\partial\Omega$ of the bounded set Ω is smooth and mean-convex, that we understand as $H > 0$. We will see later how to deal with the general cases.

Since the Mean Curvature Flow (MCF for short) is likely to develop singularities, one needs to consider an appropriate weak notion in order to state the following useful results. In particular, we consider the Weak Mean Curvature Flow in the sense defined in [32]. A special case of the regularity theorem [73, Theorem 1.1] gives

Theorem 6.3 (White's Regularity Theorem). *Let (M, g) be a 3-dimensional Riemannian manifold, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t \in [0, T]}$ be its Weak Mean Curvature Flow. Then, the boundary of Ω_t is smooth for almost every $t \in [0, T]$.*

We point out that the maximal time T might *a priori* be infinite on a general Riemannian manifold. We are going to combine the above regularity result with the following special case of [68, Proposition 7.2], that is a weak version of the monotonicity of the isoperimetric ratio. It can be checked, indeed, that the computations performed to obtain such a result do not involve the geometry of the underlying manifold.

Theorem 6.4 (Schulze). *Let (M, g) be a 3-dimensional Riemannian manifold, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t \in [0, T]}$ be its Weak Mean Curvature Flow. Assume there exists a universal constant $C \geq 0$ such that*

$$C \leq \frac{3}{2} \left(\int_{\partial\Omega_t} H^2 d\sigma \right)^{1/2}$$

for almost every $t \in [0, T]$. Then, the Isoperimetric Difference $t \mapsto D(t)$ defined as in (6.2) using the constant C , is nonincreasing for every $t \in [0, T]$.

Remark 6.5. *A different tool that might be used to deal with the singularities would be the theory of the Mean Curvature Flow with surgery, recently developed by Brendle and Huisken in [12] and [13]. On this regard, one should first make clear whether the monotonicity of the Isoperimetric Difference survives the surgeries.*

The following theorem provides a complete description of the long time behaviour of the Weak MCF of a surface moving inside a 3-dimensional Riemannian manifolds, and it can be readily deduced from [73, Theorem 11.1].

Theorem 6.6 (Long time behaviour of MCF). *Let (M, g) be a 3-dimensional Riemannian manifold, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t \in [0, T]}$ be its Weak Mean Curvature Flow. If $|\Omega_t|$ and $|\partial\Omega_t|$ do not vanish at finite time as $t \rightarrow T^-$, then Ω_t converges smoothly to a subset K , and the boundary of any connected component of K is a stable minimal submanifold.*

As a consequence, if (M, g) contains no bounded subsets with minimal boundary, the Weak MCF of a bounded set with mean-convex boundary is going to vanish. In particular, combining Corollary 1.7 with Theorem 6.6, one gets the following corollary.

Corollary 6.7. *Let (M, g) be a complete, noncompact, 3-dimensional Riemannian manifold with $\text{Ric} \geq 0$ and no cylindrical ends, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t \in [0, T]}$ be its Weak Mean Curvature Flow. Then, T is finite and $|\Omega_t|$ and $|\partial\Omega_t|$ tend to 0 as $t \rightarrow T^-$.*

So far we have at hand all the ingredients that allow to completely justify the computations of Subsection 6.1 and in turns to prove the Isoperimetric Inequality for mean-convex domains.

6.3. Proof of Theorem 1.8. In order to prove the isoperimetric inequality for any possibly non mean-convex domain Ω , we are going to consider the minimizing hull Ω^* (see [39, Section 1]). By [70], $\partial\Omega^*$ enjoys $C^{1,1}$ regularity, and by the minimizing property, its weak mean curvature $H_{\partial\Omega^*}$ is nonnegative. We will actually flow Ω^* by mean curvature. To this aim, we will invoke [40, Lemma 2.6], where the authors show that C^1 bounded hypersurfaces with nonnegative variational mean curvature can be approximated in $C^{1,\beta} \cap W^{2,p}$, for any $\beta \in (0, 1)$ and $p \in [1, \infty)$ by smooth submanifolds with strictly positive mean-curvature. Interestingly, the approximating sequence is built through an appropriate notion of mean curvature flow starting from such a C^1 hypersurface. Although presented in \mathbb{R}^n , the proof given in [40] can be easily adapted to go through the case of a general ambient Riemannian manifold, and we refer the reader to [78, Lemma 4.4] for the details of this extension. Such a result has also been pointed out in [47, Lemma 4.2] in the ambient setting of a Kottler-Schwarzschild manifold, and used in the proof of [68, Corollary 1.2], where the ambient manifold was a Cartan-Hadamard 3-manifold. For our aim, where we are interested in approximating $C^{1,1}$ hypersurfaces, it actually suffices to argue as in [39, Lemma 5.6]. We include here the precise and general statement.

Lemma 6.8 (Huisken-Ilmanen's approximation lemma). *Let (M, g) be a Riemannian manifold, and let $F : \Sigma \hookrightarrow M$ be a C^1 closed immersed hypersurface. Assume Σ has nonnegative weak mean curvature, that is, there exists a nonnegative function H defined almost everywhere on Σ such that*

$$\int_{\Sigma} \text{div}_{\Sigma} X \, d\sigma = \int_{\Sigma} H \langle X, \nu \rangle \, d\sigma$$

for any compactly supported vector field X of M . Assume also that Σ is not minimal, that is, there exists a subset $K \subset \Sigma$ of positive measure such that $H > 0$ on K . Then, F is of class $C^{1,\beta} \cap W^{2,p}$ and there exists a family of smooth immersions $F(\cdot, \varepsilon) : \Sigma \hookrightarrow M$, with $\varepsilon \in (0, \varepsilon_0]$ such that

$$\frac{d}{d\varepsilon} F(p, \varepsilon) = -H_{\Sigma_{\varepsilon}}(p, \varepsilon) \nu(p, \varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_0]$, where $H_{\Sigma_{\varepsilon}}$ is the mean curvature of Σ_{ε} and ν its outer unit normal, and

$$\lim_{\varepsilon \rightarrow 0^+} F(p, \varepsilon) = F(p)$$

locally uniformly in $C^{1,\beta} \cap W^{2,p}$. Moreover, $H_{\Sigma_{\varepsilon}} > 0$ for any $\varepsilon \in (0, \varepsilon_0]$.

Remark 6.9. *Observe that if Σ is a non minimal C^2 hypersurface with $H_{\Sigma} \geq 0$, then the approximation of Σ by means of a family of smooth mean-convex hypersurfaces $\{\Sigma_{\varepsilon}\}_{\varepsilon > 0}$ is a straightforward procedure. Indeed, it is sufficient to run the MCF starting at Σ for short time (see [57] for an account about the classical existence theory), say until some time $\varepsilon_0 > 0$. This provides a family of hypersurfaces $\{\Sigma_{\varepsilon}\}_{\varepsilon \in (0, \varepsilon_0]}$, whose mean curvatures satisfy (see e.g. [41, Theorem 3.2]) the following reaction-diffusion equation,*

$$\frac{\partial}{\partial \varepsilon} H = \Delta H + H \left(|h|^2 + \text{Ric}(\nu, \nu) \right),$$

where h is the second fundamental form of the evolving hypersurface and Ric is the Ricci tensor of the ambient manifold. Then, a standard maximum principle for parabolic equations (see e.g.

Theorem 7 and subsequent remarks in [65]) shows that the mean curvature of Σ_ε is strictly positive for every $\varepsilon \in (0, \varepsilon_0]$, unless Σ were minimal.

We can finally prove Theorem 1.8.

Proof of Theorem 1.8. We argue as in [68, proof of Corollary 1.2]. Let us first suppose that the boundary of Ω is strictly mean-convex, that is, $H_{\partial\Omega} > 0$. Let $\{\Omega_t\}_{t \in [0, T]}$ be a mean curvature flow starting from Ω . Then, by Theorem 6.3, for almost any $t \in [0, T)$ the boundary $\partial\Omega_t$ is a smooth submanifold. Let C be defined as

$$C = \inf \left\{ \frac{3}{2} \left(\int_{\partial\Omega} H^2 d\sigma \right)^{1/2} \mid \Omega \subset M \text{ bounded set with smooth boundary} \right\}. \quad (6.4)$$

Then, Theorem 6.4 guarantees that, with the above choice of C , the Isoperimetric Difference $t \mapsto D(t)$ defined in (6.2) is nonincreasing for $t \in [0, T)$. Moreover, since the Euclidean volume growth prevents the presence of cylindrical ends, by Corollary 6.7 $D(t)$ tends to 0 as $t \rightarrow T^-$. This implies the inequality

$$\frac{|\partial\Omega|^{3/2}}{|\Omega|} \geq C$$

for any Ω with smooth mean-convex boundary. If this is not the case, take the minimizing hull Ω^* of Ω (see [39, Section 1] for details). By [70] (compare also with the comprehensive [39, Theorem 1.3]) $\partial\Omega^*$ is a $C^{1,1}$ hypersurface. Observe that, by the minimizing property, $|\partial\Omega^*| \leq |\partial\Omega|$, while trivially $|\Omega^*| \geq |\Omega|$. Hence, proving a lower bound on the isoperimetric ratio for Ω^* readily implies that the same lower bound holds for Ω . Moreover, again by the minimizing property, we have that $H_{\partial\Omega^*} \geq 0$ (see also [39, (1.15)]). Also notice that $\partial\Omega^*$ cannot be minimal, for otherwise would be smooth and thus (M, g) would have a cylindrical end, in virtue of Corollary 1.7, against the Euclidean volume growth assumption. By Lemma 6.8, we find a sequence of smooth hypersurfaces Σ_ε with $H_{\Sigma_\varepsilon} > 0$ approximating $\partial\Omega^*$ locally uniformly in $C^{1,\beta}$ for any $\beta \in (0, 1)$. Arguing as above, we thus obtain the isoperimetric inequality

$$\frac{|\partial\Sigma_\varepsilon|^{3/2}}{|\Sigma_\varepsilon|} \geq C,$$

that, through letting $\varepsilon \rightarrow 0^+$, gives the isoperimetric inequality for Ω^* , and, in turn, for any bounded Ω with smooth boundary. Combining it with our Willmore inequality (1.6), we get

$$\inf \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} \geq \inf \frac{\left(\int_{\partial\Omega} H^2 d\sigma \right)}{16\pi} \geq \text{AVR}(g), \quad (6.5)$$

where the infima are taken over any bounded Ω with smooth boundary. We now want to prove that the equality sign holds in both the above inequalities, as stated in (1.20). To do so, we fix a point $O \in M$ and we observe that by the Bishop-Gromov Theorem, we can find, for every $\delta > 0$, a radius R_δ such that

$$\frac{|\partial B(O, R_\delta)|^3}{36\pi|B(O, R_\delta)|^2} \leq \text{AVR}(g) + \delta.$$

Observe that we can suppose $\partial B(O, R_\delta)$ to be smooth. Otherwise, it suffices to consider in place of $B(O, R_\delta)$ a smooth set whose perimeter and volume approximate $|\partial B(O, R_\delta)|$ and $|B(O, R_\delta)|$, respectively (this can be done by standard tools, see e.g. [55, Remark 13.2]). This proves that

$$\inf \left\{ \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} \mid \Omega \subset M \text{ bounded and smooth} \right\} \leq \text{AVR}(g).$$

Combining the above inequality with (6.5), gives (1.20).

To prove the rigidity statement, we assume now that (1.21) holds with the equality sign for some smooth and bounded $\Omega \subset M$. In virtue of (6.4) and of (1.20), we have that

$$C = \sqrt{36\pi\text{AVR}(g)}.$$

By the minimizing property, Ω^* satisfies the same equality (recall that we actually proved the isoperimetric inequality for minimizing hulls). We claim that the equality sign also holds for the region enclosed by any approximating Σ_ε as above. Indeed, by Lemma 6.8, this family is a smooth Mean Curvature Flow, and then, by the monotonicity of the Isoperimetric Difference, for any fixed $\varepsilon_1 \in (0, \varepsilon_0]$ we have

$$D(\varepsilon) \geq D(\varepsilon_1) \geq 0$$

for every $\varepsilon \in (0, \varepsilon_1)$. Since Σ_ε converges to $\partial\Omega^*$ as $\varepsilon \rightarrow 0^+$, and on Ω^* the isoperimetric difference is 0, $D(\varepsilon) \rightarrow 0^+$ as $\varepsilon \rightarrow 0^+$, and thus $D(\varepsilon_1) = 0$ as well. Since ε_1 was arbitrarily chosen, the region enclosed by any of the Σ_ε with $\varepsilon \in (0, \varepsilon_0]$ satisfies the equality in the isoperimetric inequality, as claimed. In particular, from the same computations as in Subsection 6.1, for any fixed $\varepsilon \in (0, \varepsilon_0]$, we have that

$$0 = \frac{dD}{d\varepsilon}(\varepsilon) \leq \left(|\Sigma_\varepsilon| \int_{\Sigma_\varepsilon} \mathbf{H}^2 d\sigma \right)^{1/2} \left[\sqrt{36\pi\text{AVR}(g)} - \frac{3}{2} \left(\int_{\Sigma_\varepsilon} \mathbf{H}^2 d\sigma \right)^{1/2} \right] \leq 0.$$

This implies that the equality sign holds in the Willmore inequality for Σ_ε , and thus, by the rigidity statement in Theorem 1.1, the exterior of Σ_ε in (M, g) is isometric to a cone. More precisely, if we call Ω_ε the (open and bounded) region enclosed by Σ_ε , we have that $(M \setminus \Omega_\varepsilon)$ is isometric to

$$\left([r_\varepsilon, +\infty) \times \Sigma_\varepsilon, dr \otimes dr + (r/r_\varepsilon)^2 g_{\Sigma_\varepsilon} \right), \quad \text{where } r_\varepsilon = \left(\frac{|\Sigma_\varepsilon|}{4\pi\text{AVR}(g)} \right)^{1/2}.$$

Hence, it is easily seen that the MCF $\{\Sigma_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ is given by totally umbilic hypersurfaces coinciding with the cross sections of the cone

$$(M \setminus \Omega_\varepsilon, g) \cong \left([r_\varepsilon, +\infty) \times \partial\Omega^*, dr \otimes dr + \frac{4\pi\text{AVR}(g)}{|\partial\Omega^*|} r^2 g_{\partial\Omega^*} \right). \quad (6.6)$$

We now claim that the MCF $\{\Sigma_\varepsilon\}_{\varepsilon > 0}$ does not develop singularities before the extinction time ε^* . Letting $(0, \varepsilon_*)$ be the maximal interval of existence of the smooth MCF, we claim that $\varepsilon_* = \varepsilon^*$. In fact, from (6.6) one can easily see that the mean curvature of Σ_ε is given by $(n-1)/r_\varepsilon$, and in turn the squared norm of its second fundamental form is equal to $(n-1)/r_\varepsilon^2$. It follows then by [38, Theorem 7.1] that ε_* is such that $r_{\varepsilon_*} = 0$, and thus coincides with the extinction time of the flow, i.e., $\varepsilon^* = \varepsilon_*$. We have hence deduced the isometry

$$(M \setminus \{O\}, g) \cong \left((0, +\infty) \times \partial\Omega^*, dr \otimes dr + \frac{4\pi\text{AVR}(g)}{|\partial\Omega^*|} r^2 g_{\partial\Omega^*} \right),$$

for some $O \in M$. In particular, the surface area of the geodesic balls centered at O decays as $4\pi r^2 \text{AVR}(g)$, and, since g is smooth at O , this implies that $\text{AVR}(g) = 1$. By Bishop-Gromov, we infer that (M, g) is isometric to $(\mathbb{R}^3, g_{\mathbb{R}^3})$ and $\partial\Omega^*$ is isometric to a sphere. This implies that $\Omega = \Omega^*$, since, otherwise, the mean curvature of $\partial\Omega^*$ would be null on the points not belonging to $\partial\Omega$ (compare with [39, (1.15)]), leading to a contradiction. We have thus shown that Ω is a ball, completing the proof. \square

APPENDIX: COMPARISON WITH THE MONOTONICITY FORMULAS BY COLDING AND MINICOZZI

In this section, we provide a comparison between our monotonicity formulas and those obtained by Colding and by Colding-Minicozzi in [23] and [28], respectively. To start with, let u be a solution of (1.9) in a nonparabolic Riemannian manifold (M, g) with $\text{Ric} \geq 0$, for a bounded subset $\Omega \subset M$ with smooth boundary and no holes, and set

$$b = u^{-\frac{1}{n-2}}. \quad (6.7)$$

Note that $b = 1$ on $\partial\Omega$ and that $b \rightarrow +\infty$ at infinity. Associated with the level sets of b , consider the family of functions $\{A_\beta\}$, where $A_\beta : [1, +\infty) \rightarrow [0, +\infty)$ is defined for every $\beta \geq 0$ as

$$A_\beta(r) = \frac{1}{r^{n-1}} \int_{\{b=r\}} |Db|^{\beta+1} d\sigma.$$

Now, replacing u by a minimal Green's function $G(O, \cdot)$, for some pole $O \in M$, the above defined A_β is exactly the quantity considered in [26, formula (1.1)]. Note that in Colding's setting, the level sets $\{b = r\}$ are considered for every $r > 0$, since $b(q) \rightarrow 0$ as $d(O, q) \rightarrow 0$.

Our aim is to see how the monotonicity of our family of functions $\{\Phi_\beta\}$ translates in terms of the family $\{A_\beta\}$. First of all, it is straightforward from (3.1)–(3.3) and (6.7) that

$$b = e^{\frac{\varphi}{n-2}}, \quad |Db| = \frac{|\nabla\varphi|_{\tilde{g}}}{n-2}, \quad d\sigma = b^{n-1} d\sigma_{\tilde{g}}, \quad (6.8)$$

and in turn that

$$\Phi_\beta(s) = (n-2)^{\beta+1} A_\beta(e^{\frac{s}{n-2}}), \quad \text{for every } s \geq 0. \quad (6.9)$$

We look at the derivative (3.14) of Φ_β and at its equivalent expression (3.15). In particular, the volume integral (3.15) contains the following terms.

$$\text{Ric}(\nabla\varphi, \nabla\varphi) = \left(\frac{n-2}{2}\right)^2 \text{Ric}(Db^2, Db^2), \quad (6.10)$$

and

$$\begin{aligned} |\nabla\nabla\varphi|_{\tilde{g}}^2 + (\beta-2)|\nabla|\nabla\varphi|_{\tilde{g}}|_{\tilde{g}}^2 &= \left(\frac{n-2}{2}\right)^2 \left\{ \left| \text{DD}b^2 - \frac{\Delta b^2}{n} g \right|^2 \right. \\ &\quad \left. + (\beta-2) |D^T|Db|^2 \right. \\ &\quad \left. + (\beta-2) |Db^2|^2 \left[\text{H} - (n-1) |D \log b| \right]^2 \right\} \end{aligned} \quad (6.11)$$

which have been expressed in terms of the function b and of the metric g via some computations (compare with the proof of (1.12)). Differentiating both sides of (6.9) and writing expression (3.15) in terms of b and g through formulas (6.8), (6.10) and (6.11), we obtain

$$\begin{aligned} \frac{dA_\beta}{dr}(r) &= \frac{(n-2)^{-\beta}}{r} \frac{d\Phi_\beta}{ds}((n-2) \log r) \\ &= -\frac{\beta}{4} r^{n-3} \int_{\{b>r\}} |Db|^{\beta-2} \left\{ \text{Ric}(Db^2, Db^2) + \left| \text{DD}b^2 - \frac{\Delta b^2}{n} g \right|^2 \right. \\ &\quad \left. + (\beta-2) |D^T|Db|^2 \right. \\ &\quad \left. + (\beta-2) |Db^2|^2 \left[\text{H} - (n-1) |D \log b| \right]^2 \right\} b^{2-2n} d\mu \leq 0. \end{aligned} \quad (6.12)$$

Setting $b = 2$ in the above formula, we obtain exactly the integrand of the right hand side of [23, (2.106)], that, arguing as in the conclusion of the present Theorem 3.2, leads to the monotonicity

of A_2 . For a general $\beta \geq (n-2)/(n-1)$, in [28, Theorem 1.3] the monotonicity of A_β is inferred grouping the terms in (6.12) in a different way. Observe indeed that for $\beta < 2$ the volume integral in (6.12) does not evidently carry a sign. On the other hand, (6.11) combined with Kato's inequality immediately show the nonnegativity of the expression.

We close this appendix by showing how our methods can be applied also to obtain the Monotonicity-Rigidity Theorem for the Green's function, obtaining a new (conformal) proof of Colding-Minicozzi's [28, Theorem 1.3]. Indeed, let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$, let G be its minimal Green's function and consider the new metric on $M \setminus \{O\}$

$$\tilde{g} = G(O, \cdot)^{\frac{2}{n-2}} g,$$

for some point $O \in M$. Set

$$\varphi = -\log G(O, \cdot).$$

Then, we have that the triple M, \tilde{g}, φ satisfies the system

$$\left\{ \begin{array}{ll} \Delta_{\tilde{g}} \varphi = 0 & \text{in } M \setminus \{O\} \\ \text{Ric}_{\tilde{g}} - \nabla \nabla \varphi + \frac{d\varphi \otimes d\varphi}{n-2} = \frac{|\nabla \varphi|_{\tilde{g}}^2}{n-2} \tilde{g} + \text{Ric} & \text{in } M \setminus \{O\} \\ \varphi(q) \rightarrow +\infty & \text{as } d(O, q) \rightarrow +\infty \\ \varphi(q) \rightarrow -\infty & \text{as } d(O, q) \rightarrow 0. \end{array} \right.$$

We denote by d the distance with respect to g . Define the function $\Phi_\beta : \mathbb{R} \mapsto \mathbb{R}$ given by

$$\Phi_\beta(s) = \int_{\{\varphi=s\}} |\nabla \varphi|_{\tilde{g}}^{\beta+1} d\sigma_{\tilde{g}}.$$

All the theory developed in Section 3 holds with trivial modification for Φ_β as above, and immediately yields a conformal Monotonicity-Rigidity Theorem for the Green's function.

Theorem 6.10. *Let (M, g) be a nonparabolic Riemannian manifold with $\text{Ric} \geq 0$. Let G be its minimal Green's function. Then, with the notations above, we have*

$$\frac{d\Phi_\beta}{ds}(s) = -\beta e^s \int_{\{\varphi \geq s\}} \frac{|\nabla \varphi|_{\tilde{g}}^{\beta-2} \left(\text{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla \nabla \varphi|_{\tilde{g}}^2 + (\beta-2) |\nabla |\nabla \varphi|_{\tilde{g}}|^2 \right)}{e^\varphi} d\mu_{\tilde{g}}$$

In particular, Φ'_β is always nonpositive. Moreover, $(d\Phi_\beta/ds)(s_0) = 0$ for some $s_0 \in \mathbb{R}$ and some $\beta \geq (n-2)/(n-1)$ if and only if $\{\varphi \geq s_0\}$ is isometric to the Riemannian product $([s_0, \infty) \times \{\varphi = s_0\}, dp \otimes dp + \tilde{g}|_{\{\varphi=s_0\}})$.

The above Theorem clearly translates in terms of (M, g) and G , exactly as Theorem 1.3 was deduced from Theorem 3.2.

Acknowledgements. *The authors are grateful to C. Arezzo, A. Carlotto, A. Farina, G. Huisken, L. Mari, D. Peralta-Salas, F. Schulze and P. Topping for useful comments and discussions during the preparation of the paper. They also express their gratitude to the anonymous referees for many constructive comments and remarks, that helped improving the quality of the paper. The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), which is part of the Istituto Nazionale di Alta Matematica (INdAM) and are partially funded by the GNAMPA project "Aspetti geometrici in teoria del potenziale lineare e nonlineare". The paper was partially completed during the authors' attendance to the program "Geometry and Relativity" organized by the Erwin Schrödinger International Institute for Mathematics and Physics (ESI).*

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