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A PARTIAL STRATIFICATION OF SECANT VARIETIES OF VERONESE VARIETIES VIA CURVILINEAR SUBSCHEMES

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ABSTRACT. We give a partial "quasi-stratification" of the secant varieties of the order d Veronese variety $X_{m,d}$ of $\mathbb{P}^m$. It covers the set $\sigma_t(X_{m,d})^\dagger$ of all points lying on the linear span of curvilinear subschemes of $X_{m,d}$, but two "quasi-strata" may overlap. For low border rank two different "quasi-strata" are disjoint and we compute the symmetric rank of their elements. Our tool is the Hilbert schemes of curvilinear subschemes of Veronese varieties. To get a stratification we attach to each $P \in \sigma_t(X_{m,d})^\dagger$ the minimal label of a quasi-stratum containing it.

INTRODUCTION

Let $\nu_d : \mathbb{P}^m \hookrightarrow \mathbb{P}^{\binom{m+d}{m}-1}$ be the order d Veronese embedding with $d \geq 3$. We write $X_{m,d} := \nu_d(\mathbb{P}^m)$. An element of $X_{m,d}$ can be described both as the projective class of a d-th power of a homogeneous linear form in $m+1$ variables and as the projective class of a completely decomposable symmetric d-mode tensor. In many applications like Chemometrics (see e.g. [27]), Signal Processing (see e.g. [23]), Data Analysis (see e.g. [5]), Neuroimaging (see e.g. [17]), Biology (see e.g. [25]) and many others, the knowledge of the minimal decomposition of a tensor in terms of completely decomposable tensors turns out to be extremely useful. This kind of decomposition is strictly related to the concept of secant varieties of varieties parameterizing tensors (if the tensor is symmetric one has to deal with secant varieties of Veronese varieties).

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Let $Y \subseteq \mathbb{P}^N$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ of characteristic zero.

For any point $P \in \mathbb{P}^N$ the *Y-rank $r_Y(P)$ of P* is the minimal cardinality of a finite set of points $S \subset Y$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denotes the linear span:

$$r_Y(P) := \min\{s \in \mathbb{N} \mid \exists\, S \subset Y, \sharp(S) = s, \text{ with } P \in \langle S \rangle\}. \tag{1}$$

If Y is the Veronese variety $X_{m,d}$ the Y-rank is also called the "symmetric tensor rank". The minimal set of points $S \subset X_{m,d}$ that realizes the symmetric tensor rank of a point $P \in X_{m,d}$ is also said to be the set that realizes either the "CANDECOMP/PARAFAC decomposition" or the "canonical decomposition" of P.

Set $X := X_{m,d}$. The natural geometric object that one has to study in order to compute the symmetric tensor rank either of a symmetric tensor or of a homogeneous polynomial is the set that parameterizes points in $\mathbb{P}^N$ having X-rank smaller or equal than a fixed value $t \in \mathbb{N}$. For each integer $t \geq 1$ let the t-th secant variety $\sigma_t(X) \subseteq \mathbb{P}^N$ of a variety $X \subset \mathbb{P}^N$ be the Zariski closure in $\mathbb{P}^N$ of the union of all $(t-1)$-dimensional linear subspaces spanned by t points of $X \subset \mathbb{P}^N$:

$$\sigma_t(X) := \overline{\bigcup_{P_1,\dots,P_t \in X} \langle P_1, \dots, P_t \rangle} \tag{2}$$

For each $P \in \mathbb{P}^N$ the *border rank $b_X(P)$ of P* is the minimal integer t such that $P \in \sigma_t(X)$:

$$b_X(P) := \min\{t \in \mathbb{N} \mid P \in \sigma_t(X)\}. \tag{3}$$

We denote by $\sigma_t^0(X)$ the set of the elements belonging to $\sigma_t(X)$ of fixed X-rank t:

$$\sigma_t^0(X) := \{P \in \sigma_t(X) \mid r_X(P) = t\} \tag{4}$$

Observe that if $\sigma_{t-1}(X) \neq \mathbb{P}^N$, then $\sigma_t^0(X)$ contains a non-empty open subset of $\sigma_t(X)$.

Some of the recent papers on algorithms that are able to compute the symmetric tensor rank of a symmetric tensor (see [9], [7], [10]) use the idea of giving a stratification of the t-th secant variety of the Veronese variety via the symmetric tensor rank. In fact, since $\sigma_t(X) = \overline{\sigma_t^0(X)}$, the elements belonging to $\sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ have X-rank strictly bigger than t. What some of the known algorithms for computing the symmetric rank of a symmetric tensor T do is firstly to test the equations of the secant varieties of the Veronese varieties (when known) in order to find the X-border rank of T, and secondly to use (when available) a stratification via the symmetric tensor rank of $\sigma_t(X)$. For the state of the art on the computation of the symmetric rank of a symmetric tensor see [16], [10], [22] Theorem 5.1, [9],

§3, for the case of rational normal curves, [9] for the case $t = 2, 3$, [7] for $t = 4$.

Moreover, the recent paper [12], has shown the importance of the study of the smoothable 0-dimensional schemes in order to understand the structure of the points belonging to secant varieties to Veronese varieties.

We propose here the computation of the symmetric tensor rank of a particular class of the symmetric tensors whose symmetric border rank is strictly less than its symmetric rank. We will focus on those symmetric tensors that belong to the linear span of a 0-dimensional curvilinear sub-scheme of the Veronese variety. We will indicate in Notation 6 this set as $\sigma_t(X)^\dagger$. We use a well-known stratification of the subset of the Hilbert scheme $\mathrm{Hilb}^t(\mathbb{P}^m)_c$ of curvilinear zero-dimensional subschemes of $\mathbb{P}^m$ with degree t. Taking the unions of all $\langle \nu_d(A)\rangle$, $A \in \mathrm{Hilb}(\mathbb{P}^m)_c$, we get $\sigma_t(X)^\dagger$. From each stratum U of $\mathrm{Hilb}^t(\mathbb{P}^m)_c$ we get a quasi-stratum $\cup_{A\in U}\langle A\rangle$ of $\sigma_t(X)$. In this way we do not obtain a stratification of $\sigma_t(X)^\dagger$, because a point of $\sigma_t(X)^\dagger$ may be in the intersection of the linear spans of elements of two different strata of $\mathrm{Hilb}^t(\mathbb{P}^m)_c$. We may get a true stratification of $\sigma_t(X)^\dagger$ taking a total ordering of the set of all strata of $\mathrm{Hilb}^t(\mathbb{P}^m)_c$ and assigning to any $P \in \sigma_t(X)^\dagger$ only the stratum of $\mathrm{Hilb}^t(\mathbb{P}^m)_c$ with minimal label among the strata with P in their image. The strata of $\mathrm{Hilb}^t(\mathbb{P}^m)_c$ have a natural partial ordering with maximal element $(1, \dots, 1)$ corresponding to $\sigma_t^0(X)$ and the next maximal one $(2, \dots, 1)$ (Notation 4 and Lemma 1). Hence $\sigma_t(X)^\dagger\backslash\sigma_t^0(X)$ has a unique maximal quasi-stratum and we may speak about the general element of the unique component of maximal dimension of $\sigma_t(X)^\dagger\backslash\sigma_t^0(X)$. If $t \leq (d+1)/2$, then our quasi-stratification of $\sigma_t(X)^\dagger$ is a true stratification, because the images of two different strata of $\mathrm{Hilb}(\mathbb{P}^m)_c$ are disjoint (Theorem 1). We may give the lexicographic ordering to the labels of $\mathrm{Hilb}^t(\mathbb{P}^m)_c$ to get a total ordering and hence a true stratification of $\sigma_t(X)^\dagger$, but it is rather artificial: there is no reason to say that the quasi-stratum $(3, 1, \dots, 1)$ comes before the quasi-stratum $(2, 2, 1, \dots, 1)$.

For very low t (i.e. $t \leq \lfloor (d-1)/2\rfloor$), we will describe the structure of $\sigma_t(X)^\dagger$: we will give its dimension, its codimension in $\sigma_t(X)$ and the dimension of each stratum (see Theorem 1). Moreover in the same theorem we will show that for such values of t, the symmetric border rank of the projective class of a homogeneous polynomial $[F] \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ is computed by a unique 0-dimensional subscheme $W_F \subset X$ and that the generic $[F] \in \sigma_t(X)^\dagger$ is of the form $F = L^{d-1}M + L_1^d + \cdots + L_{t-2}^d$ with $L, L_1, \dots, L_{t-2}, M$ linear forms. To compute the dimension of the 3 largest strata of our stratification we will use Terracini's lemma (see Propositions 1, 2 and 3).

We will also prove several results on the symmetric ranks of points $P \in \mathbb{P}^N$ whose border rank is computed by a scheme related to our stratification (see Proposition 5 and Theorem 2). In all cases that we will be able to compute, we will have $b_X(P) + r_X(P) \leq 3d - 2$, but we will need also additional conditions on the scheme computing $b_X(P)$ when $b_X(P) + r_X(P) \geq 2d + 2$.

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1. The quasi-stratification

For any scheme T let T_{red} denote its reduction. We begin this section by recalling the well known stratification of the curvilinear 0-dimensional subschemes of any smooth connected projective variety $Y \subset \mathbb{P}^r$. Expert readers can skip this section and refer to it only for Notation.

Notation 1. For any integral projective variety $Y \subset \mathbb{P}^r$ let $\beta(Y)$ be the maximal positive integer such that every 0-dimensional scheme $Z \subset Y$ with $\deg(Z) \leq \beta(Y)$ is linearly independent, i.e. $\dim(\langle Z \rangle) = \deg(Z) - 1$ (see [13], Lemma 2.1.5, or [7], Remark 1, for the Veronese varieties).

Remark 1. Let $Z \subset \mathbb{P}^m$ be any 0-dimensional scheme. If $\deg(Z) \leq d + 1$, then $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$. If Z is the union of $d + 2$ collinear points, then $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 1$. Therefore $\beta(X_{m,d}) = d + 1$.

Notation 2. Fix an integer $t \geq 1$. Let $A(t)$ (resp. $A'(t)$) be the set of all non-increasing sequences $t_1 \geq t_2 \geq \cdots \geq t_s \geq 0$ (resp. $t_1 \geq \cdots \geq t_s > 0$) such that $\sum_{i=1}^{s} t_i = t$.
For each $\underline{t} = (t_1, \dots, t_s) \in A(t)$ let $l(\underline{t})$ be the number of the non zero t_i's, for $i = 1, \dots, s$.
Set $B(t) := A(t) \setminus \{(1, \dots, 1)\}$ in which the string $(1, \dots, 1)$ has t entries. Set $B'(t) = B(t) \cap A'(t)$.

$A(t)$ is the set of all partitions of the integer t. The integer $l(\underline{t})$ is the length of the partition $\underline{t}$.

Definition 1. *Let $Y \subset \mathbb{P}^r$ be a smooth and connected projective variety of dimension m. For every positive integer t let $Hilb^t(Y)$ denote the Hilbert scheme of all degree t 0-dimensional subschemes of Y.*

If $m \leq 2$, then $\mathrm{Hilb}^t(Y)$ is smooth and irreducible ([19], Propositions 2.3 and 2.4, [20], page 4).

We now introduce some subsets of $\mathrm{Hilb}^t(Y)$ that will give the claimed stratification.

Notation and Remark 1. Let $Y \subset \mathbb{P}^r$ be a smooth connected projective variety of dimension m.

- For every positive integer t let $\mathrm{Hilb}^t(Y)_0$ be the set of all disjoint unions of t distinct points of Y.
  Observe that $\mathrm{Hilb}^t(Y)_0$ is a smooth and irreducible quasi-projective variety of dimension mt. If $m \le 2$, then $\mathrm{Hilb}^t(Y)_0$ is dense in $\mathrm{Hilb}^t(Y)$ (see [19], [20], page 4). For arbitrary $m = \dim(Y)$ the irreducible scheme $\mathrm{Hilb}^t(Y)_0$ is always open in $\mathrm{Hilb}^t(Y)$.
- Let $\mathrm{Hilb}^t(Y)_+$ be the closure of $\mathrm{Hilb}^t(Y)_0$ in reduction $\mathrm{Hilb}^t(Y)_{red}$ of the scheme $\mathrm{Hilb}^t(Y)$. The elements of $\mathrm{Hilb}^t(Y)_+$ are called the *smoothable* degree t subschemes of Y.
  If $t \gg m \ge 3$, then there are non-smoothable degree t subschemes of Y ([21], [20], page 6).
- An element $Z \in \mathrm{Hilb}^t(Y)$ is called *curvilinear* if at each point $P \in Z_{red}$ the Zariski tangent space of Z has dimension ≤ 1 (equivalently, Z is contained in a smooth subcurve of Y). Let $\mathrm{Hilb}^t(Y)_c$ denote the set of all degree t *curvilinear* subschemes of Y. $\mathrm{Hilb}^t(Y)_c$ is a smooth open subscheme of $\mathrm{Hilb}^t(Y)_+$ ([26], bottom of page 86). It contains $\mathrm{Hilb}^t(Y)_0$.

Fix now $O \in Y$ with $Y \subset \mathbb{P}^r$ being a smooth connected projective variety of dimension m. Following [20], page 3, we state the corresponding result for the punctual Hilbert scheme of $\mathcal{O}_{Y,O}$, i.e. the scheme parametrizing all degree t zero-dimensional schemes $Z \subset Y$ such that $Z_{red} = \{O\}$ (here instead of “ curvilinear ” several references use the word “ collinear ”) .

Remark 2. For each integer $t > 0$ the subset of the punctual Hilbert scheme parametrizing the degree t curvilinear subschemes of Y with P as its reduction is smooth, connected and of dimension $(t-1)(m-1)$.

Notation 3. Fix an integer $s > 0$ and a non-increasing sequence of integers $t_1 \ge \cdots \ge t_s > 0$ such that $t_1 + \cdots + t_s = t$ and $\underline{t} = (t_1, \dots, t_s)$. Let $\mathrm{Hilb}^t(Y)_c[t_1, \dots, t_s]$ denote the subset of $\mathrm{Hilb}^t(Y)_c$ parametrizing all elements of $\mathrm{Hilb}^t(Y)_c$ with s connected components of degree $t_1, \dots, t_s$ respectively. We also write it as $\mathrm{Hilb}^t(Y)_c[\underline{t}]$.

Remark 3. Since the support of each component $\mathrm{Hilb}^t(Y)_c[\underline{t}]$ varies in the m-dimensional variety $Y \subset \mathbb{P}^r$, the theorem on the punctual Hilbert scheme quoted in Remark 2 says that $\mathrm{Hilb}^t(Y)_c[t_1, \dots, t_s]$ is an irreducible algebraic set of dimension $ms + \sum_{i=1}^{s}(t_i - 1)(m-1) = mt + s - t$, i.e. of codimension $t - s$ in $\mathrm{Hilb}^t(Y)_c$. Each stratum $\mathrm{Hilb}^t(Y)_c[\underline{t}]$ is non-empty, irreducible and different elements of $A'(t)$ give disjoint strata, because any curvilinear subscheme has a unique type $\underline{t}$.

Hence if $t \ge 2$ we have:

$$\mathrm{Hilb}^t(Y)_c = \sqcup_{\underline{t} \in A'(t)} \mathrm{Hilb}^t(Y)_c[\underline{t}] = \mathrm{Hilb}^t(Y)_0 \bigsqcup \sqcup_{\underline{t} \in B'(t)} \mathrm{Hilb}^t(Y)_c[\underline{t}].$$

Different strata may have the same codimension, but there is a unique stratum of codimension 1: it is the stratum with label $(2,1,\dots,1)$. This stratum parametrizes the disjoint unions of a tangent vector to Y and $t-2$ disjoint points of Y.

Notation 4. Take now a partial ordering $\preceq$ on $A(t)$ writing $(a_1,\dots,a_x) \preceq (b_1,\dots,b_y)$ if and only if $\sum_{j=1}^{i} a_j \le \sum_{j=1}^{i} b_j$ for all integers $i \ge 1$, in which we use the convention $a_j := 0$ for all $j > x$ and $b_j = 0$ for all $j > y$. In the theory of partitions the partial ordering $\preceq$ is called the *dominance partial ordering*.

The next lemma is certainly well-known, but we were unable to find a reference.

Lemma 1. *Fix $(t_1,\dots,t_s) \in B(t)$.*

(a) *The stratum $Hilb^t(Y)_c[t_1,\dots,t_s]$ is in the closure of the stratum $Hilb^t(Y)_c[2,1,\dots,1]$.*
(b) *If $t_1 \ge 3$, then the stratum $Hilb^t(Y)_c[t_1,\dots,t_s]$ is in the closure of the stratum $Hilb^t(Y)_c[3,1,\dots,1]$.*
(c) *if $t_2 \ge 2$, then the stratum $Hilb^t(Y)_c[t_1,\dots,t_s]$ is in the closure of the stratum $Hilb^t(Y)_c[2,2,1,\dots,1]$.*

Proof. We only check part (c), because the proofs of parts (a) and (b) are similar. Fix $Z \in \mathrm{Hilb}^t(Y)_c[t_1,\dots,t_s]$. Take a smooth curve $C \subseteq Y$ containing Z and write $Z = \sum_{i=1}^{s} t_i P_i$ as a divisor of C, with $P_i \neq P_j$ for all $i \neq j$. Since $t_1 \ge 2$, the effective divisor $t_1 P_1$ is a flat degeneration of a family of divisors $\{Z_\lambda\}$ of C in which each Z_λ is the disjoint union of a connected degree 2 divisor and $t_1 - 2$ distinct points. Similarly, the divisor $t_2 P_2$ is a flat degeneration of a family of divisors $\{Z'_\lambda\}$ of C in which each Z'_λ is the disjoint union of a connected degree 2 divisor and $t_2 - 2$ distinct points. Obviously for each $i \ge 3$ the divisor $t_i P_i$ is smoothable inside C, i.e. it is a flat degeneration of flat family of t_i distinct points. The product of these parameter spaces is a parameter space for a deformation of Z to a flat family of elements of $\mathrm{Hilb}^t(C)_c[2,2,1,\dots,1]$. Since $C \subseteq Y$, we have $\mathrm{Hilb}^t(C)_c[2,2,1,\dots,1] \subseteq \mathrm{Hilb}^t(Y)_c[2,2,1,\dots,1]$ and the inclusion is a morphism. Hence (c) is true. $\square$

We recall the following lemma ([13], Lemma 2.1.5, [9], Proposition 11, [6], Remark 1).

Lemma 2. *Let $Y \subset \mathbb{P}^r$ be a smooth and connected subvariety. Fix an integer k such that $k \le \beta(Y)$, where $\beta(Y)$ is defined in Notation 1, and $P \in \mathbb{P}^r$. Then $P \in \sigma_k(Y)$ if and only if there exists a smoothable 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) = k$ and $P \in \langle Z \rangle$.*

The following lemma shows a very special property of the curvilinear subschemes.

Lemma 3. *Let $Y \subset \mathbb{P}^r$ be a smooth and connected subvariety. Let $W \subset Y$ be a linearly independent curvilinear subscheme of Y. Fix a general $P \in \langle W \rangle$. Then $P \notin \langle W' \rangle$ for any $W' \subsetneq W$.*

Proof. A curvilinear subscheme of a smooth variety is locally a complete intersection. Hence it is Gorenstein. Hence the lemma is a particular case of [13], Lemma 2.4.4. It may be also proved in the following elementary way, which in addition gives a description of $\langle W \rangle \setminus (\cup_{W' \subsetneq W} \langle W' \rangle)$. Fix any $W' \subsetneq W$. Since $\deg(W') < \deg(W) \leq \beta(Y)$, we have $\dim(\langle W' \rangle) = \deg(W') - 1 < \deg(W) - 1 = \dim(\langle W \rangle)$. Hence it is sufficient to show that W has only finitely many proper subschemes. Take a smooth quasi-projective curve $C \supset W$. W is an effective Cartier divisor $\sum_{i=1}^{s} b_i P_i$ with $P_i \in C$, $b_i > 0$ for all i and $\sum_{i=1}^{s} b_i = \deg(W)$. Any $W' \subseteq W$ is of the form $\sum_{i=1}^{s} a_i P_i$ for some integers a_i such that $0 \leq a_i \leq b_i$ for all i. □

We introduce the following Notation.

Notation 5. For each integral variety $Y \subset \mathbb{P}^r$ and each $Q \in Y_{reg}$ let $[2Q, Y]$ denote the first infinitesimal neighborhood of Q in Y, i.e. the closed subscheme of Y with $(\mathcal{I}_{Q,Y})^2$ as its ideal sheaf. We call any $[2Q, Y]$, with $Q \in Y_{reg}$, a double point of Y.

Remark 4. Observe that $[2Q, Y]_{red} = \{Q\}$ and $\deg([2Q, Y]) = \dim(Y) + 1$.

The following observation shows that Lemma 3 fails for some non-curvilinear subscheme.

Remark 5. Fix an integral variety $Y \subset \mathbb{P}^r$ and a smooth point Q of Y. The linear space $\langle [2Q, Y] \rangle$ has dimension $\dim(Y)$ (it is usually called the Zariski tangent space or embedded Zariski tangent space of Y at Q). Fix any $P \in \langle [2Q, Y] \rangle \setminus \{Q\}$. The line $R := \langle \{P, Q\} \rangle$ is spanned by the degree 2 effective divisor $[2P, R]$. Since $P \in \langle [2Q, Y] \rangle$, we have $[2P, R] \subset Y$.

Notation 6. For any integer $t > 0$ let $\sigma_t(X)^\dagger$ denote the set of all $P \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ such that there is a curvilinear degree t subscheme $Z \subset X_{reg}$ such that $P \in \langle Z \rangle$.

Remark 6. Let $X \subset \mathbb{P}^N$ be the Veronese variety $X_{m,d}$ with $N = \binom{n+d}{d} - 1$. Fix integer $t > 0$, $s > 0$ and $t_1 \geq \cdots \geq t_s > 0$ such that $t_1 + \cdots + t_s = t$. Let $\sigma_t(X)_c[t_1, \ldots, t_s]$ denote the set of all $P \in \sigma_t(X)_c$ such that $P \in \langle Z \rangle$ for some curvilinear scheme Z with s connected components of degree $t_1, \ldots, t_s$. If $P \in \sigma_t(X)^\dagger$, then $P \in \sigma_t(X)_c[t_1, \ldots, t_s]$ for some $s, t_1, \ldots, t_s$ with $t_1 \geq 2$. The point $P \in \sigma_t(X)^\dagger$ may be contained in different sets $\sigma_t(X)_c[t_1, \ldots, t_s]$,

$\sigma_t(X)_c[w_1,\dots,w_k]$, except under very restrictive conditions (see e.g. Theorem 1 for a sufficient condition). To get a stratification one could attach to each $P \in \sigma_t(X_{m,d})^\dagger$ the minimal label of a quasi-stratum containing it.

We recall the following definition ([1]).

Definition 2. *Let $X_1,\dots,X_t \subset \mathbb{P}^r$ be integral and non-degenerate subvarieties (repetitions are allowed). The join $J(X_1,\dots,X_t)$ of $X_1,\dots,X_t$ is the closure in $\mathbb{P}^r$ of the union of all $(t-1)$-dimensional vector spaces spanned by t linearly independent points $P_1,\dots,P_t$ with $P_i \in X_i$ for all i.*

From Definition 2 we obviously get that $\sigma_t(X_1) = J(\underbrace{X_1,\dots,X_1}_{t})$.

Definition 3. *Let $S(X_1,\dots,X_t) \subset X_1 \times \cdots \times X_t \times \mathbb{P}^r$ be the closure of the set of all $(P_1,P_2,\dots,P_t,P)$ such that $P \in \langle\{P_1,\dots,P_t\}\rangle$ and $P_i \in X_i$ for all i. We call $S(X_1,\dots,X_t)$ the abstract join of the subvarieties $X_1,\dots,X_t$ of $\mathbb{P}^r$.*

The abstract join $S(X_1,\dots,X_t)$ is an integral projective variety and we have $\dim(S(X_1,\dots,X_t)) = t-1+\sum_{i=1}^t \dim(X_i)$. The projection of $X_1\times\cdots\times X_t \times \mathbb{P}^r \to \mathbb{P}^r$ induces a proper morphism $u_{X_1,\dots,X_t} : S(X_1,\dots,X_t) \to \mathbb{P}^r$ such that $u_{X_1,\dots,X_t}(S(X_1,\dots,X_t)) = J(X_1,\dots,X_t)$. The embedded join $J(X_1,\dots,X_t)$ has the expected dimension $t-1+\sum_{i=1}^t \dim(X_i)$ if and only if $u_{X_1,\dots,X_t}$ is generically finite.

2. Curvilinear subschemes and tangential varieties to Veronese varieties

From now on in this paper we fix integers $m \geq 2$, $d \geq 3$ and take $N := \binom{m+d}{m} - 1$ and $X := X_{m,d}$ the Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^N$.

Definition 4. *Let $\tau(X) \subseteq \mathbb{P}^N$ be the tangent developable of X, i.e. the closure in $\mathbb{P}^N$ of the union of all embedded tangent spaces T_PX, $P \in X_{reg}$:*

$$\tau(X) := \overline{\bigcup_{P\in X} T_PX}$$

Remark 7. Obviously $\tau(X) \subseteq \sigma_2(X)$ and $\tau(X)$ is integral. Since $d \geq 3$, the variety $\tau(X)$ is a divisor of $\sigma_2(X)$.

Definition 5. *For each integer $t \geq 3$ let $\tau(X,t) \subseteq \mathbb{P}^N$ be the join of $\tau(X)$ and $\sigma_{t-2}(X)$:*

$$\tau(X,t) := J(\tau(X), \sigma_{t-2}(X)).$$

We recall that $\min\{N, t(m+1)-2\}$ is the expected dimension of $\tau(X,t)$.

Here we fix integers d,t with $t \geq 2$, d not too small and look at $\tau(X,t)$ from many points of view.

Remark 8. The set $\tau(X,t)$ is nothing else than the closure inside $\sigma_t(X)$ of the largest stratum of our stratification, i.e. is the stratum given by $\mathrm{Hilb}^t(X)_c[2,1,\cdots,1]$ (Lemma 1).

For any integral projective scheme W, any effective Cartier divisor D of W and any closed subscheme Z of W the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of W with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(W)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0 \tag{5}$$

The long cohomology exact sequence of (5) gives the following well-known result, often called Castelnuovo's lemma.

Lemma 4. *Let $Y \subseteq \mathbb{P}^r$ be any integral projective variety and D an effective Cartier divisor of Y. Fix any $L \in Pic(Y)$. Then*

$$h^i(Y,\mathcal{I}_Z \otimes L) \le h^i(Y,\mathcal{I}_{Res_D(Z)} \otimes L(-D)) + h^i(D,\mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

for every $i \in \mathbb{N}$.

Notation 7. For any $Q \in \mathbb{P}^m$ and any integer $k \ge 2$ let kQ denote the $(k-1)$-infinitesimal neighborhood of Q in $\mathbb{P}^m$, i.e. the closed subscheme of $\mathbb{P}^m$ with $(\mathcal{I}_Q)^k$ as its ideal sheaf. The scheme kQ will be called a k-point of $\mathbb{P}^m$.

We give here the definition of a $(2,3)$-point as it is in [14], p. 977.

Definition 6. *Fix a line $L \subset \mathbb{P}^m$ and a point $Q \in L$. The $(2,3)$ point of $\mathbb{P}^m$ associated with (Q,L) is the closed subscheme $Z(Q,L) \subset \mathbb{P}^m$ with $(\mathcal{I}_Q)^3 + (\mathcal{I}_L)^2$ as its ideal sheaf.*

Notice that $2Q \subset Z(Q,L) \subset 3Q$.

In [8], Lemma 3.5, by using the theory of inverse systems, the authors introduced a zero-dimensional subscheme of $4Q$ and used it to compute the dimension of the secant varieties to the second osculating variety of $X_{m,d}$. Hence our computations with $4Q$ that will be done in Lemma 7 may be useful for joins of the second osculating variety of a Veronese and several copies of the Veronese.

Remark 9. Fix $Q \in \mathbb{P}^m$ and a zero-dimensional scheme $Z_1 \subset \mathbb{P}^m \setminus \{Q\}$. Since $Z(Q,L) \subset 3Q$, if $h^1(\mathbb{P}^m,\mathcal{I}_{3Q\cup Z_1}(d)) = 0$, then $h^1(\mathbb{P}^m,\mathcal{I}_{Z(Q,L)\cup Z_1}(d)) = 0$.

Lemma 5. *Fix an integer t such that $(m+1)(t-2)+2m < N = \binom{m+d}{d}-1$, general $P_0,\dots,P_{t-2} \in \mathbb{P}^m$ and a general line $L \subset \mathbb{P}^m$ such that $P_0 \in L$. Set*

$$Z := Z(P_0,L)\bigcup(\cup_{i=1}^{t-2}2P_i), \quad Z' := 3P_0\bigcup(\cup_{i=1}^{t-2}2P_i).$$

(i) *If* $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$, *then* $\dim(\tau(X,t)) = t(m+1) - 2$.
(ii) *If* $h^1(\mathbb{P}^m, \mathcal{I}_{Z'}(d)) = 0$, *then* $\dim(\tau(X,t)) = t(m+1) - 2$.

Proof. If $t = 2$, then $\tau(X,t) = \tau(X)$ and the part (i) for this case is proved in [14]. The case $t \geq 3$ of part (i) follows from the case $t = 2$ and Terracini's lemma([1], part (2) of Corollary 1.11), because $\tau(X,t)$ is the join of $\tau(X)$ and $t-2$ copies of X. Part (ii) follows from part (i) and Remark 9. □

Remark 10. Let $A \subset \mathbb{P}^m$, $m \geq 2$, be a connected curvilinear subscheme of degree 3. Up to a projective transformation there are two classes of such schemes: the collinear ones (i.e. A is contained in a line, i.e. $\nu_d(A)$ is contained in a degree d rational normal curve) and the non-collinear ones, i.e. the ones that are contained in a smooth conic of $\mathbb{P}^m$. We have $h^1(\mathbb{P}^m, \mathcal{I}_A(1)) > 0$ if and only if A is contained in a line. Thus the semicontinuity theorem for cohomology gives that the set of all A's not contained in a line forms a non-empty open subset of the corresponding stratum $(3, 0, \dots, 0)$ and, in this case, we will say that A is *not collinear.* The family of all degree 3 connected and non-collinear schemes A covers an integral variety of dimension $3m-2$. If $d \geq 5$ any non-collinear connected curvilinear scheme appears as the scheme computing the border rank of a point of $\sigma_3(X) \setminus \sigma_2(X)$ with symmetric rank $2d-1$, while the collinear ones give points with symmetric rank $d-1$ ([9], Theorem 34).

Lemma 6. *Fix integers* $m \geq 2$ *and* $d \geq 5$. *If* $m \leq 4$, *then assume* $d \geq 6$. *Set* $\alpha := \lfloor \binom{m+d-1}{m}/(m+1) \rfloor$. *Let* $Z_i \subset \mathbb{P}^m$, $i = 1, 2$, *be a general union of* i *triple points and* $\alpha - i$ *double points. Then* $h^1(\mathcal{I}_{Z_i}(d)) = 0$.

Proof. Fix a hyperplane H of $\mathbb{P}^m$ and call E_i, $i \in \{1, 2\}$, the union of i triple points of $\mathbb{P}^m$ with support on H. Hence $E_i \cap H$ is a disjoint union of i triple points of H. Since $d \geq 5$, we have $h^1(H, \mathcal{I}_{H \cap E_i}(d)) = 0$. Let $W_i \subset \mathbb{P}^m$ be a general union of $\alpha - i$ double points. Since W_i is general, we have $W_i \cap H = \emptyset$.
If we prove that $h^1(\mathcal{I}_{E_i \cup W_i}(d)) = 0$, then, by semicontinuity, we also get that $h^1(\mathcal{I}_Z(d)) = 0$ for $i \in \{1, 2\}$.
By Lemma 4 it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_H(W_i \cup E_i)}(d-1)) = 0$.
Since $W_i \cap H = \emptyset$, we have $\mathrm{Res}_H(W_i) = W_i$ and $\mathrm{Res}_H(W_i \cup E_i) = W_i \sqcup \mathrm{Res}_H(E_i)$. Hence $\mathrm{Res}_H(W_i \cup E_i)$ is a general union of α double points, with the only restriction that the reductions of two of these double points are contained in the hyperplane H. Any two points of $\mathbb{P}^m$, $m \geq 2$, are contained in some hyperplane. The group $\mathrm{Aut}(\mathbb{P}^m)$ acts transitively on the set of all hyperplanes of $\mathbb{P}^m$. The cohomology groups of projectively equivalent subschemes of $\mathbb{P}^m$ have the same dimension. Hence we may consider $W_i \sqcup \mathrm{Res}_H(E_i)$ as a general union of α double points of $\mathbb{P}^m$. Since $(m+1)\alpha \leq \lfloor \binom{m+d-1}{m}/(m+1) \rfloor$, $d-1 \geq 4$ and $d-1 \geq 5$ if $m \leq 4$, a famous theorem

of Alexander and Hirschowitz on the dimensions of all secant varieties to Veronese varieties gives $h^1(\mathcal{I}_{\mathrm{Res}_H(W_i\cup E_i)}(d-1)) = 0$ (see [2], [3], [4], [15], [11]). □

Lemma 7. *Fix integers $m \ge 2$ and $d \ge 6$. If $m \le 4$, then assume $d \ge 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1)\rfloor$. Let $Z \subset \mathbb{P}^m$ be a general union of one quadruple point and $\beta - 1$ double points. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. Fix a hyperplane H and call E a quadruple point of $\mathbb{P}^m$ with support on H. Hence $E \cap H$ is a quadruple point of H. Since $d \ge 2$, we have $h^1(H, \mathcal{I}_{H\cap E}(d)) = 0$. Let $W \subset \mathbb{P}^m$ be a general union of $\beta - 1$ double points. Since W is general, we have $W \cap H = \emptyset$.
If we prove that $h^1(\mathcal{I}_{E\cup W}(d)) = 0$ then, by semicontinuity, we also get that $h^1(\mathcal{I}_Z(d)) = 0$. By Lemma 4 it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_H(W\cup E)}(d-1)) = 0$.
Since $W\cap H = \emptyset$, we have $\mathrm{Res}_H(W) = W$ and $\mathrm{Res}_H(W\cup E) = W \sqcup \mathrm{Res}_H(E)$. Hence $\mathrm{Res}_H(W \cup E)$ is a general union of $\beta - 1$ double points and one triple point with support on H. Since $\mathrm{Aut}(\mathbb{P}^m)$ acts transitively, the scheme $\mathrm{Res}_H(W\cup E)$ may be seen as a general disjoint union of $\beta-1$ double points and one triple point. Now it is sufficient to apply the case $i = 1$ of Lemma 6 for the integer $d' := d - 1$. □

Each set $\sigma_t(X)_c[t_1, \dots, t_s]$ is constructible and its closure $\overline{\sigma_t(X)_c[t_1, \dots, t_s]}$ inside $\sigma_t(X)$ is a projective variety, perhaps a union of several irreducible components.

Definition 7. *Set $\Gamma_1 := \sigma_t(X)_c[2, 1, \dots, 1]$ and $\Gamma_2 := \sigma_t(X)_c[2, 2, 1, \dots, 1]$.*

Let Γ_3 be the set of all $P \in \sigma_t(X)_c[3, 1, \dots, 1]$ such that $P \in \langle Z\rangle$ with Z a union of $t - 3$ simple points and a connected and non-collinear degree 3 curvilinear scheme.

Remark 11. For every $P \in \Gamma_1$ there is a scheme $Z_P \subset X$ such that $P \in \langle Z_P\rangle$ and Z_P has one connected component of degree 2 and $t - 2$ simple points.

For every $P \in \Gamma_2$ there is a scheme $Z_P \subset X$ such that $P \in \langle Z_P\rangle$ and Z_P has two connected components of degree 2 and $t - 4$ simple points.

For every $P \in \Gamma_3$ there is a scheme $Z_P \subset X$ such that $P \in \langle Z_P\rangle$ and Z_P has $t - 3$ simple points and one connected component which is curvilinear, of degree 3 and non-collinear.

Proposition 1. *Set $\alpha := \lfloor \binom{m+d-1}{m}/(m+1)\rfloor$. Fix an integer $t \ge 3$ such that $t \le \alpha - 1$. Then $\Gamma_1 \ne \emptyset$, Γ_1 is irreducible and Γ_1 has pure codimension 1 in $\sigma_t(X)$.*

Proof. Lemma 6 and Terracini's lemma ([1], part (2) of Corollary 1.11) give that the join $\tau(X,t)$ (see Definition 5) has the expected dimension. This is equivalent to say that the set of all points $P \in \langle Z_1 \cup \{P_1,\dots,P_{t-2}\}\rangle$ with Z_1 a tangent vector of X has the expected dimension, i.e. codimension 1 in $\sigma_t(X)$. Obviously $\tau(X,t) \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. The set Γ_1 is irreducible, because it is an open subset of a join of irreducible subvarieties. □

The proof of Proposition 1 can be analogously repeated for the following two propositions.

Proposition 2. *Set* $\alpha := \lfloor \binom{m+d-1}{m}/(m+1)\rfloor$*. Fix an integer* $t \geq 3$ *such that* $t \leq \alpha - 2$*. Then* $\Gamma_2 \neq \emptyset$*,* Γ_2 *is irreducible and* Γ_2 *has pure codimension* 2 *in* $\sigma_t(X)$*.*

Proof. This proposition can be proved in the same way of Proposition 1 just quoting the case $i = 2$ of Lemma 6 instead of the case $i = 1$ of the same lemma. □

Proposition 3. *Set* $\beta := \lfloor \binom{m+d-2}{m}/(m+1)\rfloor$*. Fix an integer* $t \geq 3$ *such that* $t \leq \beta - 1$*. Then* $\Gamma_3 \neq \emptyset$*,* Γ_3 *is irreducible,* Γ_3 *is dense in* $\sigma_t(X)_c[3,1,\dots,1]$ *and* Γ_3 *has pure codimension* 2 *in* $\sigma_t(X)$*.*

Proof. This proposition can be proved in the same way of Proposition 1 just quoting Lemma 7 instead of Lemma 6 and using Remark 10. □

Remark 12. Observe that if we interpret the Veronese variety $X_{m,d}$ as the variety that parameterizes the projective classes of homogeneous polynomials of degree d in $m+1$ variables that can be written as d-th powers of linear forms then:

- The elements $F \in \Gamma_1$ can all be written in the following two ways:
$$F = L^{d-1}M + L_1^d + \cdots + L_{t-2}^d,$$
$$F = M_1^d + \cdots + M_d^d + L_1^d + \cdots + L_{t-2}^d.$$
- The elements $F \in \Gamma_2$ can all be written in the following two ways:
$$F = L^{d-1}M + L'^{d-1}M' + L_1^d + \cdots + L_{t-4}^d;$$
$$F = M_1^d + \cdots + M_d^d + M_1'^d + \cdots + M_d'^d + L_1^d + \cdots + L_{t-4}^d.$$
- The elements $F \in \Gamma_3$ can be written either in one of the following two ways:
$$F = L^{d-2}Q + L_1^d + \cdots + L_{t-3}^d;$$
$$F = N_1^d + \cdots + N_{2d-1}^d + L_1^d + \cdots + L_{t-3}^d;$$
or in one of the following two ways:
$$F = L^{d-1}M + L_1^d + \cdots + L_{t-3}^d,$$

$$F = M_1^d + \cdots + M_d^d + L_1^d + \cdots + L_{t-3}^d.$$

where L, $L'M$, $M'L_1, \dots, L_{t-2}$, $M_1, \dots, M_d$, $M_1', \dots, M_d'$, $N_1, \dots, N_{2d-1}$ are linear forms and Q is a quadratic form. Actually $M_1, \dots, M_d$ and $M_1', \dots, M_d'$ are binary forms (see [9], Theorem 32 and Theorem 34).

3. The ranks and border ranks of points of Γ_i

Here we compute the rank $r_X(P)$ for certain points $P \in \tau(X, t)$ when t is not too big with respect to d. The cases $t = 2, 3$ are contained in [9], Theorems 32 and 34. The case $t = 4$ is contained in [6], Theorem 1.

We first handle the border rank.

Theorem 1. *Fix an integer t such that $2 \le t \le \lfloor (d-1)/2 \rfloor$. For each $P \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ there is a unique $W_P \in Hilb^t(X)$ such that $P \in \langle W_P \rangle$.*

(a) *The constructible set $\sigma_t(X)^\dagger$ is non-empty, irreducible and of dimension $(m+1)t - 2$. For a general $P \in \sigma_t(X)^\dagger$ the associated $W \subset X$ computing $b_X(P)$ has a connected component of degree 2 (i.e. a tangent vector) and $t-2$ reduced connected components.*

(b) *We have a set-theoretic partition $\sigma_t(X)^\dagger = \sqcup_{\underline{t} \in B'(t)} \sigma(\underline{t})$, where $B'(t)$ are defined in Notation 2, in which each set $\sigma(\underline{t})$ is an irreducible and non-empty constructible subset of dimension $(m+1)t-1-t+l(\underline{t})$, where $l(\underline{t})$ is defined in Notation 2. The strata $\sigma(2, 1, \dots, 1)$ is the only stratum with dimension $t(m+1)-2$ and all the other strata are in the closure of $\sigma(2, 1, \dots, 1)$.*

(c) *$\sigma(2, 2, \dots, 1)$ and $\sigma(3, 1, \dots, 1)$ are the only strata of codimension 1 of $\sigma_t(X)^\dagger$.*

(d) *If $t_1 \ge 3$ (resp. $t_2 \ge 3$), then the stratum $\sigma(t_1, \dots, t_s)$ is in the closure of $\sigma(3, 1, \dots, 1)$ (resp. $\sigma(2, 2, \dots, 1)$).*

(e) The complement of $\sigma_t(X)^\dagger$ inside $\sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ has codimension at least 3 if $t \ge 3$, or it is empty if $t = 2$.

Proof. Fix $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$. Remark 1 gives $\beta(X) = d+1 \ge t$. Therefore Lemma 2 gives the existence of some $W \subset X$ such that $\deg(W) = t$, $P \in \langle W \rangle$ and W is smoothable. Since $2t \le d+1$, we can use [6], Lemma 1, to say that W is unique. Moreover, if $A \subset X$ is a degree t smoothable subscheme, $Q \in \langle A \rangle$ and $Q \notin \langle A' \rangle$ for any $A' \subsetneq A$, then Lemma 2 gives $Q \in \sigma_t(X) \setminus \sigma_{t-1}(X)$. If A is curvilinear, then it is smoothable and $\cup_{A' \subsetneq A} \langle A' \rangle \subsetneq \langle A \rangle$. Hence each degree t curvilinear subscheme W of X contributes a non-empty open subset U_W of the $(t-1)$-dimensional projective space $\langle W \rangle$ and $U_{W_1} \cap U_{W_2} = \emptyset$ for all curvilinear W_1, W_2 such that $W_1 \neq W_2$. Hence

$$\sigma_t(X)^\dagger = \sqcup_{\underline{t} \in A'(t)} (\sqcup_{W \in \mathrm{Hilb}^t(X)[\underline{t}]} U_W).$$

Each algebraic constructible set $\sigma(\underline{t}) := \sqcup_{W \in \mathrm{Hilb}^t(X)[\underline{t}]} U_W$ is irreducible and of dimension $t-1+tm+l(\underline{t})-t$. This partition of $\sigma_t(X)^\dagger$ into non-empty irreducible constructible subsets is the partition claimed in part (b).

Parts (b), (c) and (d) follows from Lemma 1.

Now we prove part (e). Every element of $\mathrm{Hilb}^2(X)$ is either a tangent vector or the disjoint union of two points. Hence $\mathrm{Hilb}^2(X) = \mathrm{Hilb}^2(X)_c$. Hence we may assume $t \geq 3$. Fix $P \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ such that $P \notin \sigma_t(X)^\dagger$. By Lemma 2 there is a smoothable $W \subset X$ such that $\deg(W) = t$ and $P \in \langle W \rangle$. Since $2t \leq \beta(X)$, such a scheme is unique. Hence it is sufficient to prove that the set $\mathbb{B}_t$ of all 0-dimensional smoothable schemes with degree t and not curvilinear have dimension at most $mt-3$. Call $\mathbb{B}_t(s)$ the set of all $W \in \mathbb{B}_t$ with exactly s connected components.
First we assume that W is connected. Set $\{Q\} := W_{red}$. Since in the local Hilbert scheme of $\mathcal{O}_{X,Q}$ the smoothable colength t ideals are parametrized by an integral variety of dimension $(m-1)(t-1)$ and a dense open subset of it is formed by the ideals associated to curvilinear subschemes, we have $\dim(\mathbb{B}_t(1)) \leq m + (m-1)(t-1) - 1 = mt - t = \dim(\mathrm{Hilb}^t(X)_c) - t$.
Now we assume $s \geq 2$. Let $W_1, \dots, W_s$ be the connected components of W, with at least one of them, say W_s, not curvilinear. Set $t_i = \deg(W_i)$. We have $t_1 + \cdots + t_s = t$. Since W_s is not curvilinear, we have $t_s \geq 3$ and hence $t-s \geq 2$. Each W_i is smoothable. Hence each W_i, $i < s$, depends on at most $m + (m-1)(t_i - 1) = mt_i + 1 - t_i$ parameters. We saw that $\mathbb{B}_{t_s}(1)$ depends on at most $mt_s - t_s$ parameters. Hence $\dim(\mathbb{B}_t(s)) \leq mt + s - 1 - t$. $\square$

Proposition 4. *Assume $m \geq 2$. Fix integers d, t such that $2 \leq t \leq d$. Fix a curvilinear scheme $A \subset \mathbb{P}^m$ such that $\deg(A) = t$ and $\deg(A \cap L) \leq 2$ for every line $L \subset \mathbb{P}^m$. Set $Z := \nu_d(A)$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Then $b_X(P) = t$ and Z is the only 0-dimensional scheme W such that $\deg(W) \leq t$ and $P \in \langle W \rangle$.*

Proof. Since $t \leq d+1$, Z is linearly independent. Since Z is curvilinear, Lemma 3 gives the existence of many points $P' \in \langle Z \rangle$ such that $P' \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Let $W \subset X$ be a minimal degree subscheme such that $P \in \langle W \rangle$. Set $w := \deg(W)$. The minimality of w gives $w \leq t$. If $w = t$, then we assume $W \neq Z$. Now it is sufficient to show that these conditions give a contradiction. Write $Z := \nu_d(A)$ and $W = \nu_d(B)$ with A and B subschemes of $\mathbb{P}^m$, $\deg(A) = t$ and $\deg(B) = w$. We have $P \in \langle W \rangle \cap \langle Z \rangle$, then, since $W \neq Z$, by [6], Lemma 1, the scheme $W \cup Z$ is linearly dependent. We have $\deg(B \cup A) \leq t + w \leq 2d$. Since $W \cup Z$ is linearly dependent, we have $h^1(\mathcal{I}_{B \cup A}(d)) > 0$. Hence, by [9], Lemma 34, there is a line $R \subset \mathbb{P}^m$ such that $\deg(R \cap (B \cup A)) \geq d+2$. By assumption we have $\deg(R \cap A) \leq 2$. Hence $\deg(B \cap R) \geq d$. In our set-up we get $w = d$ and $B \subset R$. Since

$P \in \langle W\rangle$, we get $P \in \langle \nu_d(R)\rangle$. That means that P belongs to the linear span of a rational normal curve. Therefore the border rank of P is computed by a curvilinear scheme which has length $\le \lfloor (d+1)/2\rfloor$ (apply Lemma 2 to [16] or [22], Theorem 4.1, or [9], Theorem 23), a contradiction. $\square$

Proposition 5. *Fix a line $L \subset \mathbb{P}^m$ and set $D := \nu_d(L)$. Fix positive integers t_1, s_1, a 0-dimensional scheme $Z_1 \subset D$ such that $\deg(Z_1) = t_1$ and $S_1 \subset X \setminus D$ such that $\sharp(S_1) = s_1$. Assume $2 \le t_1 \le d/2$, $0 \le s_1 \le d/2$, that Z_1 is not reduced and $\dim(\langle D \cup S_1\rangle) = d + s_1$. Fix $P \in \langle Z_1 \cup S_1\rangle$ such that $P \notin \langle W\rangle$ for any $W \subsetneq Z_1 \cup S_1$. We have $\sharp(\langle Z_1\rangle \cap \langle \{P\} \cup S_1\rangle) = 1$. Set $\{Q\} := \langle Z_1\rangle \cap \langle \{P\} \cup S_1\rangle$. Then $b_X(P) = t_1 + s_1$, $r_X(P) = d + 2 + s_1 - t_1$, $Z_1 \cup S_1$ is the only subscheme of X computing $b_X(P)$ and every subset of X computing $r_X(P)$ contains S_1. If $2s_1 < d$, then every subset of X computing $r_X(P)$ is of the form $A \cup S_1$ with $A \subset D$, $\sharp(A) = d+2-s_1$ and A computing $r_D(Q)$.*

Proof. Obviously $b_X(P) \le t_1+s_1$. Since $P \in \langle Z_1 \cup S_1\rangle \subset \langle D \cup S_1\rangle$, $P \notin \langle S_1\rangle$ and $\langle D\rangle$ has codimension s_1 in $\langle D \cup S_1\rangle$, the linear subspace $\langle Z_1\rangle \cap \langle \{P\} \cup S_1\rangle$ is a unique point, $\{Q\}$. Since $\deg(Z_1) \le d + 1 = \beta(X) = \beta(D)$ (Remark 1), the scheme Z_1 is linearly independent. Since $P \notin \langle W\rangle$ for any $W \subsetneq Z_1 \cup S_1$, we have $\langle Z_1\rangle \cap \langle \{P\} \cup S_1\rangle \ne \emptyset$. Since $\langle Z_1\rangle \subset \langle D\rangle$, we get $\{Q\} = \langle Z_1\rangle \cap \langle \{P\} \cup S_1\rangle$. Hence Z_1 compute $b_D(Q)$ (Lemma 2). By Lemma 2 we also have $b_X(Q) = b_D(Q) = t_1$. Since Z_1 is not reduced, we have $r_D(Q) = d+2-t_1$ ([16] or [22], theorem 4.1, or [9], §3). We have $r_X(Q) = r_D(Q)$ ([24], Proposition 3.1, or [22], subsection 3.2). Write $Z_1 = \nu_d(A_1)$ and $S_1 = \nu_d(B_1)$ with $A_1, B_1 \subset \mathbb{P}^m$. Lemma 2 gives $b_X(P) \le t_1 + s_1$. Assume $b_X(P) \le t_1+s_1-1$ and take $W = \nu_d(E)$ computing $b_X(Q)$ for a certain 0-dimensional scheme $E \subset \mathbb{P}^m$. Hence $\deg(W) \le 2t_1+2s_1-1$. Since $P \in \langle W\rangle \cap \langle Z_1 \cup S_1\rangle$, by the already quoted [6], Lemma 1, we get $h^1(\mathbb{P}^m, \mathcal{I}_{E\cup A_1\cup B_1}(d)) > 0$. Hence there is a line $R \subset \mathbb{P}^m$ such that $\deg(R \cap (E \cup Z_1 \cup S_1)) \ge d + 2$.

First assume $R = L$. Hence $L \cap (A_1 \cup B_1) = A_1$. Hence $\deg(E \cap L) \ge d + 2 - t_1$. Set $E' := E \cap L$, $E'' := E \setminus E'$, $W' := \nu_d(E')$ and $W'' := \nu_d(E'')$. Since $P \in \langle W' \cup W''\rangle$, there is $O \in \langle W'\rangle$ such that $P \in \langle \{O\} \cup W''\rangle$. Hence $b_X(P) \le b_X(O) + \deg(W'')$. Since $O \in \langle D\rangle$, we have $r_X(O) \le r_D(O) \le \lfloor (d+2)/2\rfloor < d + 2 - t_1 \le \deg(W')$, contradicting the assumption that W computes $b_X(P)$.

Now assume $R \ne L$. Since the scheme $L \cap R$ has degree 1, while the scheme $A_1 \cap L$ has degree t_1, we get $\deg(R \cap E) \ge d + 2 - s_1 > (d + 2)/2$. As above we get a contradiction.

Now assume $b_X(P) = t_1+s_1$, but that $W \ne Z_1 \cup S_1$ computes $b_X(P)$. As above we get a line R such that $\deg((W \cup Z_1 \cup S_1) \cap R) \ge d+2$. As above we get $R = L$. Since $P \in \langle Z_1 \cup S_1\rangle$, there is $U \in \langle D\rangle$ such that Z_1 computes the border D-rank of U and $P \in \langle U \cup S_1\rangle$. Take $A \subset D$ computing $r_D(U)$. By

[16] or [22], Theorem 4.1, or [9] we have $\sharp(A) = d+2-t_1$. Since $P \in \langle A \cup S_1 \rangle$ and $A \cap S_1 = \emptyset$, we have $r_X(P) \le d+2+s_1-t_1$. Assume the existence of some $S \subset X$ computing $r_X(P)$ and such that $\sharp(S) \le d+1+s_1-t_1$. Hence $\deg(S \cup S_1 \cup Z_1) \le d+1+2s_1 \le 2d+1$. Write $S = \nu_d(B)$. We proved that $Z_1 \cup S_1$ computes $b_X(P)$. By [6], Theorem 1, we have $B = B_1 \sqcup S_1$ with $B_1 = L \cap B$. Hence $\sharp(B_1) \le d+1-t_1$. Since $P \in \langle B_1 \cup S_1 \rangle$, there is $V \in \langle B_1 \rangle$ such that $P \in \langle V \cup S_1 \rangle$. Hence $r_X(P) \le r_X(V) + s_1$. Since B computes $r_X(P)$ and $V \in \langle B_1 \rangle$, we get $r_X(V) = \sharp(B_1)$ and that B_1 computes $r_X(V)$. Since $\nu_d(B_1) \subset D$, we have $V = Q$. Recall that $b_X(Q) = b_D(Q)$ and that Z_1 is the only subscheme of X computing $r_X(Q)$. We have $r_X(Q) = r_D(Q) = d+2-t_1$. Hence $\sharp(B_1) \ge d+2-t_1$, a contradiction.

If $2s_1 < d$, then the same proof works even if $\sharp(B) = d+2+s_1-t_1$ and prove that any set computing $r_X(P)$ contains S_1. □

Lemma 8. *Fix a hyperplane $M \subset \mathbb{P}^m$ and 0-dimensional schemes A, B such that B is reduced, $A \ne B$, $h^1(\mathcal{I}_A(d)) = h^1(\mathcal{I}_B(d)) = 0$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Res_M(A\cup B)}(d-1)) = 0$. Set $Z := \nu_d(A)$, $S := \nu_d(B)$. Then $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) = h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d))$ and Z and S are linearly independent. Assume the existence of a point $P \in \langle Z \rangle \cap \langle S \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. Set $F := (B \setminus (B \cap M)) \cap A$. Then $B = (B \cap M) \sqcup F$ and $A = (A \cap M) \sqcup F$.*

Proof. Since $h^1(\mathcal{I}_A(d)) = h^1(\mathcal{I}_B(d)) = 0$, both Z and S are linearly independent. Since $h^2(\mathcal{I}_{A\cup B}(d-1)) = 0$, the residual sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_M(A\cup B)}(d-1) \to \mathcal{I}_{A\cup B}(d) \to \mathcal{I}_{(A\cup B)\cap M}(d) \to 0.$$

gives $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) = h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d))$. Assume the existence of P as in the statement. Set $B_1 := (B \cap M) \cup F$.

(a) Here we prove that $B = (B \cap M) \cup F$, i.e. $B = B_1$. Since $P \notin \langle S' \rangle$ for any $S' \subsetneq S$, it is sufficient to prove $P \in \langle \nu_d(B_1) \rangle$. Since Z and S are linearly independent, Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle S \rangle) = \deg(Z \cap S) - 1 + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since $\mathrm{Res}_M(A \cup B_1) \subseteq \mathrm{Res}_M(A \cup B)$ and $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_M(A\cup B)}(d-1)) = 0$, we have that $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B_1}(d)) = h^1(M, \mathcal{I}_{(A\cup B_1)\cap M}(d))$. Since $M \cap (A \cup B_1) = M \cap (A \cup B)$, we get $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B_1}(d)) = h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since both schemes Z and $\nu_d(B)$ are linearly independent, Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle \nu_d(B) \rangle) = \deg(A \cap B) - 1 + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since both schemes Z and $\nu_d(B_1)$ are linearly independent, Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle \nu_d(B_1) \rangle) = \deg(A \cap B_1) - 1 + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since $A \cap B_1 = A \cap B$, we get $\dim(\langle Z \rangle \cap \langle S \rangle) = \dim(\langle Z \rangle \cap \langle \nu_d(B_1) \rangle$. Since $\langle Z \rangle \cap \langle \nu_d(B_1) \rangle \subseteq \langle Z \rangle \cap \langle S \rangle$, we get $\langle Z \rangle \cap \langle \nu_d(B_1) \rangle = \langle Z \rangle \cap \langle S \rangle$. Hence $P \in \langle \nu_d(B_1) \rangle$.

(b) In a very similar way we get $A = (A \cap M) \sqcup F$ (see steps (b), (c) and (d) of the proof of Theorem 1 in [6]). $\square$

Theorem 2. *Assume $m \ge 3$. Fix integers $d \ge 5$ and $3 \le t \le d$. Fix a line $L \subset \mathbb{P}^m$, a degree 2 connected subscheme $A_1 \subset L$ and a reduced set $A_2 \subset \mathbb{P}^m \setminus L$, such that $\sharp(A_2) = t-2$. Set $A := A_1 \cup A_2$, $Z_i := \nu_d(A_i)$, $i = 1, 2$, and $Z := Z_1 \cup Z_2$. Assume that A is in linearly general position in $\mathbb{P}^m$. Set $\mathcal{Z} := \{P \in \langle Z\rangle : P \notin \langle Z'\rangle$ for any $Z' \subsetneq Z\}$. Then $\mathcal{Z}$ is the complement in $\langle Z\rangle \cong \mathbb{P}^{t-1}$ of $t-1$ hyperplanes. For any $P \in \mathcal{Z}$ we have $b_X(P) = t$ and $r_X(P) = d+t-2$.*

Proof. Since $\deg(A) \le d+1$, we have $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = 0$. Hence the scheme Z is linearly independent. Hence $\mathcal{Z}$ is the complement in $\langle Z\rangle \cong \mathbb{P}^{t-1}$ of $t-1$ hyperplanes. Fix any $P \in \mathcal{Z}$. Proposition 4 gives $b_X(P) = t$. Fix a set $B \subset \mathbb{P}^m$ such that $S := \nu_d(B)$ computes $r_X(P)$. Assume $r_X(P) < d+t-2$, i.e. $\sharp(S) \le d+t-3$. Since $t \le d$, we have $r_X(P) + t \le 3d-3$.

(a) Until step (g) we assume $m = 3$. We have $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) > 0$ ([6], Lemma 1). Hence $A \cup B$ is not in linearly general position (see [18], Theorem 3.2). Hence there is a plane $M \subset \mathbb{P}^3$ such that $\deg(M \cap (A \cup B)) \ge 4$. Among all such planes we take one, say M_1, such that the integer $x_1 := \deg(M_1 \cap (A \cup B))$ is maximal. Set $E_1 := A \cup B$ and $E_2 := \mathrm{Res}_{M_1}(E_1)$. Notice that $\deg(E_2) = \deg(E_1) - x_1$. Define inductively the planes $M_i \subset \mathbb{P}^3$, $i \ge 2$, the schemes E_{i+1}, $i \ge 2$, and the integers x_i, $i \ge 2$, by the condition that M_i is one of the planes such that the integer $x_i := \deg(M_i \cap E_i)$ is maximal and then set $E_{i+1} := \mathrm{Res}_{M_i}(E_i)$. We have $E_{i+1} \subseteq E_i$ (with strict inclusion if $E_i \ne \emptyset$) for all $i \ge 1$ and $E_i = \emptyset$ for all $i \gg 0$. For all integers t and $i \ge 1$ there is the residual exact sequence

$$0 \to \mathcal{I}_{E_{i+1}}(t-1) \to \mathcal{I}_{E_i}(t) \to \mathcal{I}_{E_i \cap M_i, M_i}(t) \to 0. \tag{6}$$

Let u be the minimal positive integer i such that and $h^1(M_i, \mathcal{I}_{M_i \cap E_i}(d+1-i)) > 0$. Use at most $r_X(P) + t$ times the exact sequences (6) to prove the existence of such an integer u. Any degree 3 subscheme of $\mathbb{P}^3$ is contained in a plane. Hence for any $i \ge 1$ either $x_i \ge 3$ or $x_{i+1} = 0$. Hence $x_i \ge 3$ for all $i \le u-1$. Since $r_X(P) + t \le 3d$, we get $u \le d$.

(b) Here we assume $u = 1$. Since A is in linearly general position, we have $\deg(M_1 \cap A) \le 3$. First assume $x_1 \ge 2d+2$. Hence $\sharp(B) \ge \sharp(B \cap M_1) \ge 2d-1 > d+t-3$, a contradiction. Hence $x_1 \le 2d+1$. Since $h^1(M_1, \mathcal{I}_{M_1 \cap E_1}(d)) > 0$, there is a line $T \subset M_1$ such that $\deg(T \cap E_1) \ge d+2$ ([9], Lemma 34). Since A is in linearly general position, we have $\deg(A \cap T) \le 2$. Hence $\deg(T \cap B) \ge d$. Assume for the moment $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) > 0$. Hence $x_2 \ge d+1$. Since by hypothesis $d \ge 4$, $x_2 \le x_1$ and $x_1 + x_2 \le 3d+1$, we have $x_2 \le 2d-1$. Hence [9], Lemma 34, applied to the integer $d-1$ gives the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(E_2 \cap R) \ge d+1$.

Since A is in linearly general position, we also get $\deg(R \cap E_2) \le 2$ and hence $\deg(R \cap B \cap E_2) \ge d-1$. Hence $\sharp(S) \ge 2d-1$, a contradiction. Now assume $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) = 0$. Lemma 8 gives the existence of a set $F \subset \mathbb{P}^3 \setminus M_1$ such that $A = (A \cap M_1) \sqcup F$ and $B = (B \cap M_1) \sqcup F$. Hence $\sharp(F) = \deg(A) - \deg(A \cap M_1) \ge t-1$. Since $\sharp(B \cap M_1) \ge d$, we obtained a contradiction.

(c) Here and in steps (d), (e), and (f) we assume $m = 3$ and $u \ge 2$. We first look at the possibilities for the integer u. Since every degree 3 closed subscheme of $\mathbb{P}^3$ is contained in a plane, either $x_i \ge 3$ or $x_{i+1} = 0$. Since $r_X(P) + t \le 3d-3$, we get $x_i = 0$ for all $i > d$. Hence $u \le d$. We have $x_u \ge d+3-u$ (e.g. by [9], Lemma 34). Since the sequence x_i, $i \ge 1$, is non-increasing, we get $r_X(P) + 2 + t - 2 \le u(d+3-u)$. Since the function $s \mapsto s(d+3-s)$ is concave in the interval $[2, d+1]$, we get $u \in \{2, 3, d\}$.

(d) Here we assume $u = 2$. Since $3d+1 \ge x_1 + x_2 \ge 2x_2$, we get $x_2 \le 2(d-1)+1$. Hence there is a line $R \subset \mathbb{P}^3$ such that $\deg(E_2 \cap R) \ge d+1$. We claim that $x_1 \ge d+1$. Indeed, since $A \cup B \nsubseteq R$, there is a plane $M \subset R$ such that $\deg(M \cap (A \cup B)) > \deg((A \cup B) \cap R) \ge d+1$. The maximality property of x_1 gives $x_1 \ge d+2$. Since A is in linearly general position, we have $\deg(A \cap R) \le 2$ and $\deg(A \cap M_1) \le 3$. Hence $\deg(B \cap E_2 \cap R) \ge d-1$ and $r_X(P) \ge (x_1 - 3) + d - 1 \ge 2d-2 \ge d+t-2$, a contradiction.

(e) Here we assume $u = 3$. Since $h^1(M_3, \mathcal{I}_{M_3 \cap E_3}(d-2)) > 0$, there is a line $R \subset M_3$ such that $\deg(E_3 \cap T) \ge d$. This is absurd, because $x_1 \ge x_2 \ge x_3 \ge d$ and $x_1 + x_2 + x_3 \le r_X(P) + t \le d + 2t - 3 \le 3d-3$.

(f) Here we assume $u = d$. The condition " $h^1(\mathcal{I}_{M_d \cap E_d}(1)) > 0$ " says that either $M_d \cap E_d$ contains a scheme of length ≥ 3 contained in a line R or $x_d \ge 4$. Since $x_d \ge 3$, we have $r_X(P) + t \ge x_1 + \cdots + x_d \ge 3d$. Since $t \le d$ and $r_X(P) \le d + t - 3$, this is absurd.

(g) Here we assume $m > 3$. We make a similar proof, taking as M_i, $i \ge 1$, hyperplanes of $\mathbb{P}^m$. Any 0-dimensional scheme of degree at most m of $\mathbb{P}^m$ is contained in a hyperplane. Hence either $x_i \ge m$ or $x_{i+1} = 0$. With these modification we repeat the proof of the case $m = 3$. $\square$

Corollary 1. *Assume $m \ge 3$. Fix integers $d \ge 5$ and $3 \le t \le d$. There is a dense open subset $\mathcal{U}$ of $\sigma_t(X)^\dagger \setminus \sigma_t(X)^0$ such that $b_X(P) = t$ and $r_X(P) = d + t - 2$ for all $P \in \mathcal{U}$.*

Proof. The irreducible constructible set $\sigma(2, 1, \ldots, 1)$ is dense in $\sigma_t(X)^\dagger \setminus \sigma_t(X)^0$. Any degree 2 subscheme of $\mathbb{P}^m$ is contained in a line. Hence there is a dense open subset $\mathcal{U}$ of $\sigma_t(X)^\dagger \setminus \sigma_t(X)^0$ such that for every $P \in \mathcal{U}$ there is $Z \subset X$ such that $\deg(Z) = t$, $P \in \langle Z \rangle$, $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$, and $Z = \nu_d(A \cup S)$ with $A \subset \mathbb{P}^m$ degree 2 and connected, $S \subset \mathbb{P}^m \setminus L$, $\sharp(S) = t-2$ and $A \cup S$ in linearly general position. Apply Theorem 2. $\square$

The following example is the transposition of [7], Example 2, to our set-up.

Example 1. Fix a smooth plane conic $C \subset \mathbb{P}^m$, $m \geq 2$, and positive integers $d \geq 5$, x, y, a_i, $1 \leq i \leq x$, and b_j, $1 \leq j \leq y$, such that $\sum_{i=1}^{x} a_i + \sum_{j=1}^{y} b_j = 2d+2$. Fix $x+y$ distinct points $P_1, \dots, P_x, Q_1, \dots, Q_y$ of C. Let $A \subset C$ be the effective degree $\sum_{i=1}^{x} a_i$ divisor of C in which each P_i appears with multiplicity a_i. Let $B \subset C$ be the effective degree $\sum_{j=1}^{j} b_j$ divisor of C in which each Q_j appears with multiplicity b_j. Since C is projectively normal, $h^0(C, \mathcal{O}_C(d)) = 2d+1$ and $h^1(C, \mathcal{I}_E(d)) = 0$ for every divisor E of C with degree at most $2d+1$, the set $\langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$ is a unique point, P, $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$. Since $h^1(C, \mathcal{I}_E(d)) = 0$ for every divisor E of C with degree at most $2d+1$, it is easy to check that $b_X(P) = \min\{\deg(A), \deg(B)\}$. Thus P is contained in two different quasi-strata of $\sigma_t(X_{m,d})^\dagger$ for $t \geq \max\{\deg(A), \deg(B)\}$. If $\deg(A) = \deg(B) = d+1$, then $P \in \sigma_{d+1}(X_{m,d})^\dagger \setminus \sigma_d(X_{m,d})$ and both A and B compute the border rank of P.

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