

# SYMMETRIC TENSOR RANK WITH A TANGENT VECTOR: A GENERIC UNIQUENESS THEOREM

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ABSTRACT. Let  $X_{m,d} \subset \mathbb{P}^N$ ,  $N := \binom{m+d}{m} - 1$ , be the order  $d$  Veronese embedding of  $\mathbb{P}^m$ . Let  $\tau(X_{m,d}) \subset \mathbb{P}^N$ , be the tangent developable of  $X_{m,d}$ . For each integer  $t \geq 2$  let  $\tau(X_{m,d}, t) \subseteq \mathbb{P}^N$ , be the join of  $\tau(X_{m,d})$  and  $t-2$  copies of  $X_{m,d}$ . Here we prove that if  $m \geq 2$ ,  $d \geq 7$  and  $t \leq 1 + \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ , then for a general  $P \in \tau(X_{m,d}, t)$  there are uniquely determined  $P_1, \dots, P_{t-2} \in X_{m,d}$  and a unique tangent vector  $\nu$  of  $X_{m,d}$  such that  $P$  is in the linear span of  $\nu \cup \{P_1, \dots, P_{t-2}\}$ , i.e. a degree  $d$  linear form  $f$  (a symmetric tensor  $T$  of order  $d$ ) associated to  $P$  may be written as

$$f = L_{t-1}^{d-1} L_t + \sum_{i=1}^{t-2} L_i^d, \quad (T = v_{t-1}^{\otimes(d-1)} v_t + \sum_{i=1}^{t-2} v_i^{\otimes d})$$

with  $L_i$  linear forms on  $\mathbb{P}^m$  ( $v_i$  vectors over a vector field of dimension  $m+1$  respectively),  $1 \leq i \leq t$ , that are uniquely determined (up to a constant).

## 1. INTRODUCTION

In this paper we want to address the question of the uniqueness of a particular decomposition for certain given homogeneous polynomials. An analogous question can be rephrased in terms of uniqueness of a particular tensor decomposition of certain given symmetric tensors. In fact, given a homogeneous polynomial  $f$  of degree  $d$  in  $m+1$  variables defined over an algebraically closed field  $\mathbb{K}$ , there is an obvious way to associate a symmetric tensor  $T \in S^d(V_{\mathbb{K}})$ , with  $\dim(V_{\mathbb{K}}) = m+1$ , to the form  $f$ . We will always work over an algebraically closed field  $\mathbb{K}$  such that  $\text{char}(\mathbb{K}) = 0$ . Fix integers  $m \geq 2$  and  $d \geq 3$ . Let  $j_{m,d} : \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ ,  $N := \binom{m+d}{m} - 1$ , be the order  $d$  Veronese embedding of  $\mathbb{P}^m$  and set  $X_{m,d} := j_{m,d}(\mathbb{P}^m)$  (we often write  $X$  instead of  $X_{m,d}$ ). Let  $\mathbb{K}[x_0, \dots, x_m]_d$  be the polynomial ring of homogeneous degree  $d$  polynomials in  $m+1$  variables over  $\mathbb{K}$  and let  $V_{\mathbb{K}}^*$  be the dual space of  $V_{\mathbb{K}}$ . Since obviously  $\mathbb{P}^m \simeq \mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_1) \simeq \mathbb{P}(V_{\mathbb{K}}^*)$ , an element of the Veronese variety  $X_{m,d}$  can be interpreted either as the projective class of a  $d$ -th power of a linear form  $L \in \mathbb{K}[x_0, \dots, x_m]_1$  or as the projective class of a symmetric tensor  $T \in S^d(V_{\mathbb{K}}^*) \subset (V_{\mathbb{K}}^*)^{\otimes d}$  for which there exists  $v \in V_{\mathbb{K}}^*$  s.t.  $T = v^{\otimes d}$ .

For each integer  $t$  such that  $1 \leq t \leq N$  let  $\sigma_t(X)$  denote the closure in  $\mathbb{P}^N$  of the union of all  $(t-1)$ -dimensional linear subspaces spanned by  $t$  points of  $X$  (the  $t$ -secant variety of  $X$ ). From this definition one can understand that the generic element of  $\sigma_t(X_{m,d})$  can be interpreted either as  $[f] = [L_1^d + \dots + L_t^d] \in$

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$\mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_d)$  with  $L_1, \dots, L_t \in \mathbb{K}[x_0, \dots, x_m]_1$  or as  $[T] = [v_1^{\otimes d} + \dots + v_t^{\otimes d}] \subset \mathbb{P}(S^d(V_{\mathbb{K}}^*))$  with  $v_1, \dots, v_t \in V_{\mathbb{K}}^*$ . For a given form  $f$  (or a symmetric tensor  $T$ ), the minimum integer  $t$  for which there exists such a decomposition is called the symmetric rank of  $f$  (or of  $T$ ). Finding those  $v_i$ 's,  $i = 1, \dots, t$  such that  $T = v_1^{\otimes d} + \dots + v_t^{\otimes d}$ , with  $t$  the symmetric rank of  $T$ , is known as the Tensor Decomposition problem and it is a generalization of the Singular Value Decomposition problem for symmetric matrices (i.e. if  $T \in S^2(V_{\mathbb{K}}^*)$ ). The existence and the possible uniqueness of the decompositions of a form  $f$  as  $L_1^d + \dots + L_t^d$  with  $t$  minimal is studied in certain cases in [6], [8], [10], [11].

Let  $\tau(X) \subseteq \mathbb{P}^N$  be the tangent developable of  $X$ , i.e. the closure in  $\mathbb{P}^N$  of the union of all embedded tangent spaces  $T_P X$ ,  $P \in X$ . Obviously  $\tau(X) \subseteq \sigma_2(X)$  and  $\tau(X)$  is integral. Since  $d \geq 3$ , the variety  $\tau(X)$  is a divisor of  $\sigma_2(X)$  ([5], Proposition 3.2). An element in  $\tau(X_{m,d})$  can be described both as  $[f] \in \mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_d)$  for which there exists two linear forms  $L_1, L_2 \in \mathbb{K}[x_0, \dots, x_m]_1$  such that  $f = L_1^{d-1}L_2$ , and as  $[T] \in \mathbb{P}(S^d(V_{\mathbb{K}}^*))$  for which there exists two vectors  $v_1, v_2 \in V_{\mathbb{K}}^*$  such that  $T = v_1^{\otimes d-1}v_2$  ([5], [4]).

Fix integral positive-dimensional subvarieties  $A_1, \dots, A_s \subset \mathbb{P}^N$ ,  $s \geq 2$ . The join  $[A_1, A_2]$  is the closure in  $\mathbb{P}^N$  of the union of all lines spanned by a point of  $A_1$  and a different point of  $A_2$ . If  $s \geq 3$  define inductively the join  $[A_1, \dots, A_s]$  by the formula  $[A_1, \dots, A_s] := [[A_1, \dots, A_{s-1}], A_s]$ . The join  $[A_1, \dots, A_s]$  is an integral variety and  $\dim([A_1, \dots, A_s]) \leq \min\{N, s-1 + \sum_{i=1}^s \dim(A_i)\}$ . The integer  $\min\{N, s-1 + \sum_{i=1}^s \dim(A_i)\}$  is called the *expected dimension* of the join  $[A_1, \dots, A_s]$ . Obviously  $[A_1, \dots, A_s] = [A_{\sigma(1)}, \dots, A_{\sigma(s)}]$  for any permutation  $\sigma : \{1, \dots, s\} \rightarrow \{1, \dots, s\}$ . The secant variety  $\sigma_t(X)$ ,  $t \geq 2$ , is the join of  $t$  copies of  $X$ . For each integer  $t \geq 3$  let  $\tau(X, t) \subseteq \mathbb{P}^N$  be the join of  $\tau(X)$  and  $t-2$  copies of  $X$ . We recall that  $\min\{N, t(m+1)-2\}$  is the expected dimension of  $\tau(X, t)$ , while  $\min\{N, t(m+1)-1\}$  is the expected dimension of  $\sigma_t(X)$ . In the range of triples  $(m, d, t)$  we will meet in this paper both  $\tau(X, t)$  and  $\sigma_t(X)$  have the expected dimensions and hence  $\tau(X, t)$  is a divisor of  $\sigma_t(X)$ . An element in  $\tau(X_{m,d}, t)$  can be described both as  $[f] \in \mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_d)$  for which there exist linear forms  $L_1, \dots, L_t \in \mathbb{K}[x_0, \dots, x_m]_1$  such that  $f = L_{t-1}^{d-1}L_t + \sum_{i=1}^{t-2} L_i^d$ , and as  $[T] \in \mathbb{P}(S^d(V_{\mathbb{K}}^*))$  for which there exist  $v_1, \dots, v_t \in V_{\mathbb{K}}^*$  such that  $T = v_{t-1}^{\otimes(d-1)}v_t + \sum_{i=1}^{t-2} v_i^{\otimes d}$ .

After [3], it is natural to ask the following question.

**Question 1.** Assume  $d \geq 3$  and  $\tau(X, t) \neq \mathbb{P}^N$ . Is a general point of  $\tau(X, t)$  in the linear span of a unique set  $\{P_0, P_1, \dots, P_{t-2}\}$  with  $(P_0, P_1, \dots, P_{t-2}) \in \tau(X) \times X^{t-2}$ ?

For non weakly  $(t-1)$ -degenerate subvarieties of  $\mathbb{P}^N$  the corresponding question is true by [8], Proposition 1.5. Here we answer it for a large set of triples of integers  $(m, d, t)$  and prove the following result.

**Theorem 1.** Fix integers  $m \geq 2$  and  $d \geq 6$ . If  $m \leq 4$ , then assume  $d \geq 7$ . Set  $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ . Assume  $3 \leq t \leq \beta + 1$ . Let  $P$  be a general point of  $\tau(X, t)$ . Then there are uniquely determined points  $P_1, \dots, P_{t-2} \in X$  and  $Q \in \tau(X)$  such that  $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$ , i.e. (since  $d > 2$ ) there are uniquely determined points  $P_1, \dots, P_{t-2} \in X$  and a unique tangent vector  $\nu$  of  $X$  such that  $P \in \langle \{P_1, \dots, P_{t-2}\} \cup \nu \rangle$ .

In terms of homogeneous polynomials Theorem 1 may be rephrased in the following way.

**Theorem 2.** *Fix integers  $m \geq 2$  and  $d \geq 6$ . If  $m \leq 4$ , then assume  $d \geq 7$ . Set  $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ . Assume  $3 \leq t \leq \beta+1$ . Let  $P$  be a general point of  $\tau(X, t)$  and let  $f$  be a homogeneous degree  $d$  form in  $\mathbb{K}[x_0, \dots, x_m]$  associated to  $P$ . Then  $f$  may be written in a unique way*

$$f = L_{t-1}^{d-1} L_t + \sum_{i=1}^{t-2} L_i^d$$

with  $L_i \in \mathbb{K}[x_0, \dots, x_m]_1$ ,  $1 \leq i \leq t$ .

In the statement of Theorem 2 the form  $f$  is uniquely determined only up to a non-zero scalar, and (as usual in this topic) “uniqueness” may allow not only a permutation of the forms  $L_1, \dots, L_{t-2}$ , but also a scalar multiplication of each  $L_i$ .

In terms of symmetric tensors Theorem 1 may be rephrased in the following way.

**Theorem 3.** *Fix integers  $m \geq 2$  and  $d \geq 6$ . If  $m \leq 4$ , then assume  $d \geq 7$ . Set  $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ . Assume  $3 \leq t \leq \beta+1$ . Let  $P$  be a general point of  $\tau(X, t)$  and let  $T \in S^d(V_{\mathbb{K}}^*)$  be a symmetric tensor associated to  $P$ . Then  $T$  may be written in a unique way*

$$T = v_{t-1}^{\otimes(d-1)} v_t + \sum_{i=1}^{t-2} v_i^{\otimes d}$$

with  $v_i \in V_{\mathbb{K}}^*$ ,  $1 \leq i \leq t$ .

As above, in the statement of Theorem 3 the tensor  $T$  and the vectors  $v_i$ 's are uniquely determined only up to non-zero scalars.

To prove Theorem 1, and hence Theorems 2 and 3, we adapt the notion and the results on weakly defective varieties described in [6]. It is easy to adapt [6] to joins of different varieties instead of secant varieties of a fixed variety if a general tangent hyperplane is tangent only at one point ([7]). However, a general tangent space of  $\tau(X)$  is tangent to  $\tau(X)$  along a line, not just at the point of tangency. Hence a general hyperplane tangent to  $\tau(X, t)$ ,  $t \geq 3$ , is tangent to  $\tau(X, t)$  at least along a line. We prove the following result.

**Theorem 4.** *Fix integers  $m \geq 2$  and  $d \geq 6$ . If  $m \leq 4$ , then assume  $d \geq 7$ . Set  $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ . Assume  $t \leq \beta+1$ . Let  $P$  be a general point of  $\tau(X, t)$ . Let  $P_1, \dots, P_{t-2} \in X$  and  $Q \in \tau(X)$  be the points such that  $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$ . Let  $\nu$  be the tangent vector of  $X$  such that  $Q$  is a point of  $\langle \nu \rangle \setminus \nu_{red}$ . Let  $H \subset \mathbb{P}^N$  be a general hyperplane containing the tangent space  $T_P \tau(X, t)$  of  $\tau(X, t)$ . Then  $H$  is tangent to  $X$  only at the points  $P_1, \dots, P_{t-2}, \nu_{red}$ , the scheme  $H \cap X$  has an ordinary node at each  $P_i$ , and  $H$  is tangent to  $\tau(X) \setminus X$  only along the line  $\langle \nu \rangle$ .*

## 2. PRELIMINARIES

**Notation 1.** Let  $Y$  be an integral quasi-projective variety and  $Q \in Y_{reg}$ . Let  $\{kQ, Y\}$  denote the  $(k-1)$ -th infinitesimal neighborhood of  $Q$  in  $Y$ , i.e. the closed subscheme of  $Y$  with  $(\mathcal{I}_Q)^k$  as its ideal sheaf. If  $Y = \mathbb{P}^m$ , then we write  $kQ$  instead of  $\{kQ, \mathbb{P}^m\}$ . The scheme  $\{kQ, Y\}$  will be called a  $k$ -point of  $Y$ . We also say that a 2-point is a double point, that a 3-point is a triple point and a 4-point is a quadruple point.

We give here the definition of a  $(2, 3)$ -point as it is in [5], p. 977.

**Definition 1.** Let  $\mathfrak{q} \subset \mathbb{K}[x_0, \dots, x_m]$  be the reduced ideal of a simple point  $Q \in \mathbb{P}^m$ , and let  $l \subset \mathbb{K}[x_0, \dots, x_m]$  be the ideal of a reduced line  $L \subset \mathbb{P}^m$  through  $Q$ . We say that  $Z(Q, L)$  is a  $(2, 3)$ -point if it is the zero-dimensional scheme whose representative ideal is  $(\mathfrak{q}^3 + l^2)$ .

**Remark 1.** Notice that  $2Q \subset Z(Q, L) \subset 3Q$ .

We recall the notion of weak non-defectivity for an integral and non-degenerate projective variety  $Y \subset \mathbb{P}^r$  (see [6]). For any closed subscheme  $Z \subset \mathbb{P}^r$  set:

$$(1) \quad \mathcal{H}(-Z) := |\mathcal{I}_{Z, \mathbb{P}^r}(1)|.$$

**Notation 2.** Let  $Z \subset \mathbb{P}^r$  be a zero-dimensional scheme such that  $\{2Q, Y\} \subseteq Z$  for all  $Q \in Z_{red}$ . Fix  $H \in \mathcal{H}(-Z)$  where  $\mathcal{H}(-Z)$  is defined in (1). Let  $H_c$  be the closure in  $Y$  of the set of all  $Q \in Y_{reg}$  such that  $T_Q Y \subseteq H$ .

The contact locus  $H_Z$  of  $H$  is the union of all irreducible components of  $H_c$  containing at least one point of  $Z_{red}$ .

We use the notation  $H_Z$  only in the case  $Z_{red} \subset Y_{reg}$ .

Fix an integer  $k \geq 0$  and assume that  $\sigma_{k+1}(Y)$  doesn't fill up the ambient space  $\mathbb{P}^r$ . Fix a general  $(k+1)$ -uple of points in  $Y$  i.e.  $(P_0, \dots, P_k) \in Y^{k+1}$  and set

$$(2) \quad Z := \cup_{i=0}^k \{2P_i, Y\}.$$

The following definition of weakly  $k$ -defective variety coincides with the one given in [6].

**Definition 2.** A variety  $Y \subset \mathbb{P}^r$  is said to be *weakly  $k$ -defective* if  $\dim(H_Z) > 0$  for  $Z$  as in (2).

In [6], Theorem 1.4, it is proved that if  $Y \subset \mathbb{P}^r$  is not weakly  $k$ -defective, then  $H_Z = Z_{red}$  and that  $\text{Sing}(Y \cap H) = (\text{Sing}(Y) \cap H) \cup Z_{red}$  for a general  $Z = \cup_{i=0}^k \{2P_i, Y\}$  and a general  $H \in \mathcal{H}(-Z)$ . Notice that  $Y$  is weakly 0-defective if and only if its dual variety  $Y^* \subset \mathbb{P}^{r*}$  is not a hypersurface.

In [7] the same authors considered also the case in which  $Y$  is not irreducible and hence its joins have as irreducible components the joins of different varieties.

**Lemma 1.** Fix an integer  $y \geq 2$ , an integral projective variety  $Y$ ,  $L \in \text{Pic}(Y)$  and  $P \in Y_{reg}$ . Set  $x := \dim(Y)$ . Assume  $h^0(Y, \mathcal{I}_{(y+1)P} \otimes L) = h^0(Y, L) - \binom{x+y}{x}$ . Fix a general  $F \in |\mathcal{I}_{yP} \otimes L|$ . Then  $P$  is an isolated singular point of  $F$ .

*Proof.* Let  $u : Y' \rightarrow Y$  denote the blowing-up of  $Y$  at  $P$  and  $E := u^{-1}(P)$  the exceptional divisor. Since  $\dim(Y) = x$ , we have  $E \cong \mathbb{P}^{x-1}$ . Set  $R := u^*(L)$ . For each integer  $t \geq 0$  we have  $u_*(R(-tE)) \cong \mathcal{I}_{tP} \otimes L$ . Thus the push-forward  $u_*$  induces an isomorphism between the linear system  $|R(-tE)|$  on  $Y'$  and the linear system  $|\mathcal{I}_{tP} \otimes L|$  on  $Y$ . Set  $M := R(-yE)$ . Since  $\mathcal{O}_{Y'}(E)|_E \cong \mathcal{O}_E(-1)$  (up to the identification of  $E$  with  $\mathbb{P}^{x-1}$ ), we have  $R(-tE)|_E \cong \mathcal{O}_E(t)$  for all  $t \in \mathbb{N}$ . Consider on  $Y'$  the exact sequence:

$$(3) \quad 0 \rightarrow M(-E) \rightarrow M \rightarrow \mathcal{O}_E(y) \rightarrow 0$$

Our hypothesis implies that  $h^0(Y, \mathcal{I}_{yP} \otimes L) = h^0(Y, L) - \binom{x+y-1}{x}$ . Thus our assumption implies  $h^0(Y', M(-E)) = h^0(Y', R) - \binom{x+y}{x} = h^0(Y', R) - \binom{x+y-1}{x} - \binom{x+y-1}{x-1} = h^0(Y', M) - h^0(E, \mathcal{O}_E(y))$ . Thus (3) gives the surjectivity of the restriction map

$\rho : H^0(Y', M) \rightarrow H^0(E, M|_E)$ . Since  $y \geq 0$ , the line bundle  $M|_E$  is spanned. Thus the surjectivity of  $\rho$  implies that  $M$  is spanned at each point of  $E$ . Hence  $M$  is spanned in a neighborhood of  $E$ . Bertini's theorem implies that a general  $F' \in |M|$  is smooth in a neighborhood of  $E$ . Since  $F$  is general and  $|M| \cong |\mathcal{I}_{yP} \otimes L|$ ,  $P$  is an isolated singular point of  $F$ .  $\square$

### 3. $\tau(X, t)$ IS NOT WEAK DEFECTIVE

In this section we fix integers  $m \geq 2$ ,  $d \geq 3$  and set  $N = \binom{m+d}{m} - 1$  and  $X := X_{m,d}$ . The variety  $\tau(X)$  is 0-weakly defective, because a general tangent space of  $\tau(X)$  is tangent to  $\tau(X)$  along a line. Terracini's lemma for joins implies that a general tangent space of  $\tau(X, t)$  is tangent to  $\tau(X, t)$  at least along a line (see Remark 3). Thus  $\tau(X, t)$  is weakly 0-defective. To handle this problem and prove Theorem 1 we introduce another definition, which is tailor-made to this particular case. As in [5] we want to work with zero-dimensional schemes on  $X$ , not on  $\tau(X)$  or  $\tau(X, t)$ . We consider  $X = j_{m,d}(\mathbb{P}^m)$  and the 0-dimensional scheme  $Z \subset X$  which is the image (via  $j_{m,d}$ ) of the general disjoint union of  $t - 2$  double points and one  $(2, 3)$ -point of  $\mathbb{P}^m$ , in the case of [5] (see Definition 1). We will often work by identifying  $X$  with  $\mathbb{P}^m$ , so e.g. notice that  $\mathcal{H}(-\emptyset)$  is just  $|\mathcal{O}_{\mathbb{P}^m}(d)|$ .

**Remark 2.** Fix  $P \in X$  and  $Q \in T_P X \setminus \{P\}$ . Any two such pairs  $(P, Q)$  are projectively equivalent for the natural action of  $\text{Aut}(\mathbb{P}^m)$ . We have  $Q \in \tau(X)_{reg}$  and  $T_Q \tau(X) \supset T_P X$ . Set  $D := \langle \{P, Q\} \rangle$ . It is well-known that  $D \setminus \{P\}$  is the set of all  $O \in \tau(X)_{reg}$  such that  $T_Q \tau(X) = T_O \tau(X)$  (e.g. use that the set of all  $g \in \text{Aut}(\mathbb{P}^m)$  fixing  $P$  and the line containing  $P$  associated to the tangent vector induced by  $Q$  acts transitively on  $T_P X \setminus D$ ).

**Definition 3.** Fix a general  $(O_1, \dots, O_{t-2}, O) \in (\mathbb{P}^m)^{t-1}$  and a general line  $L \subset \mathbb{P}^m$  such that  $O \in L$ . Set  $Z := Z(O, L) \cup \bigcup_{i=1}^{t-2} 2O_i$ . We say that the variety  $\tau(X, t)$  is not *drip defective* if  $\dim(H_Z) = 0$  for a general  $H \in |\mathcal{I}_Z(d)|$ .

We are now ready for the following lemma.

**Lemma 2.** Fix an integer  $t \geq 3$  such that  $(m + 1)t < n$ . Let  $Z_1 \subset \mathbb{P}^m$  be a general union of a quadruple point and  $t - 2$  double points. Let  $Z_2$  be a general union of 2 triple points and  $t - 2$  double points. Fix a general disjoint union  $Z = Z(O, L) \cup (\bigcup_{i=1}^{t-2} 2P_i)$ , where  $Z(O, L)$  is a  $(2, 3)$ -point as in Definition 1 and  $O, L$  and  $\{P_1, \dots, P_{t-2}\} \subset \mathbb{P}^m$  are general. Assume  $h^1(\mathbb{P}^m, \mathcal{I}_{Z_1}(d)) = h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$ . Then:

- (i)  $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$ ;
- (ii)  $\tau(X, t)$  is not drip defective;
- (iii) a general  $H \in \mathcal{H}(-Z)$  has an ordinary quadratic singularity at each  $P_i$ .

*Proof.* Set  $W := 3O \cup (\bigcup_{i=1}^{t-2} 2P_i)$ . The definition of a  $(2, 3)$ -point gives that  $Z(O, L) \subset 3O$ . Thus  $Z \subset W \subset Z_2$ . Hence  $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) \leq h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$ . Hence part (i) is proven.

To prove part (ii) of the lemma we need to prove that  $\dim(H_Z) = 0$  for a general  $H \in \mathcal{H}(-Z)$ . Since  $W \subsetneq Z_1$  and  $h^1(\mathbb{P}^m, \mathcal{I}_{Z_1}(d)) = 0$ , we have  $\mathcal{H}(-W) \neq \emptyset$ . Since  $W_{red} = Z_{red}$  and  $Z \subset W$ , to prove parts (ii) and (iii) of the lemma it is sufficient to prove  $\dim((H_W)_c) = 0$  for a general  $H_W \in \mathcal{H}(-W)$ , where  $W$  is as above and  $(H_W)_c$  is as in Notation 2. Assume that this is not true, therefore:

- (1) either the contact locus  $(H_W)_c$  contains a positive-dimensional component  $J_i$  containing some of the  $P_i$ 's, for  $1 \leq i \leq t-2$ ,
- (2) or the contact locus  $(H_W)_c$  contains a positive-dimensional irreducible component  $T$  containing  $Q$ .

Set  $Z_3 := \cup_{i=1}^{t-3} 2P_i$  and  $Z' := 3O \cup Z_3$ .

(a) Here we assume the existence of a positive dimensional component  $J_i \subset (H_W)_c$  containing one of the  $P_i$ 's, say for example  $J_{t-2} \ni P_{t-2}$ . Thus a general element of  $|\mathcal{I}_W(d)|$  is singular along a positive-dimensional irreducible algebraic set containing  $P_{t-2}$ . Let  $w : M \rightarrow \mathbb{P}^m$  denote the blowing-up of  $\mathbb{P}^m$  at the points  $O, P_1, \dots, P_{t-3}$ . Set  $E_0 := w^{-1}(O)$  and  $E_i := w^{-1}(P_i)$ ,  $1 \leq i \leq t-3$ . Let  $A$  be the only point of  $M$  such that  $w(A) = P_{t-2}$ . For each integer  $y \geq 0$  we have  $w_*(\mathcal{I}_{yA} \otimes w^*(\mathcal{O}_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \dots - 2E_{t-3})) = \mathcal{I}_{Z' \cup yP_{t-2}}(d)$ . Applying Lemma 1 to the variety  $M$ , the line bundle  $w^*(\mathcal{O}_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \dots - 2E_{t-3})$ , the point  $A$  and the integer  $y = 2$  we get a contradiction.

(b) Here we prove the non-existence of a positive-dimensional  $T \subset (H_W)_c$  containing  $O$ . Let  $w_1 : M_1 \rightarrow \mathbb{P}^m$  denote the blowing-up of  $\mathbb{P}^m$  at the points  $P_1, \dots, P_{t-2}$ . Set  $E_i := w_1^{-1}(P_i)$ ,  $1 \leq i \leq t-2$ . Let  $B \in M_1$  be the only point of  $M_1$  such that  $w_1(B) = O$ . For each integer  $y \geq 0$  we have  $w_{1*}(\mathcal{I}_{yB} \otimes w_1^*(\mathcal{O}_{\mathbb{P}^m}(d))(-2E_1 - \dots - 2E_{t-2})) = \mathcal{I}_{Z' \cup yO}(d)$ . Since  $h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$  and  $|\mathcal{I}_{Z_2}(d)| \subset |\mathcal{I}_Z(d)|$ , by Lemma 1 (with  $y = 3$ ) we get a contradiction.  $\square$

In [3], Lemmas 5 and 6, we proved the following two lemmas:

**Lemma 3.** *Fix integers  $m \geq 2$  and  $d \geq 5$ . If  $m \leq 4$ , then assume  $d \geq 6$ . Set  $\alpha := \lfloor \binom{m+d-1}{m} / (m+1) \rfloor$ . Let  $Z_i \subset \mathbb{P}^m$ ,  $i = 1, 2$ , be a general union of  $i$  triple points and  $\alpha - i$  double points. Then  $h^1(\mathcal{I}_{Z_i}(d)) = 0$ .*

**Lemma 4.** *Fix integers  $m \geq 2$  and  $d \geq 6$ . If  $m \leq 4$ , then assume  $d \geq 7$ . Set  $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ . Let  $Z \subset \mathbb{P}^m$  be a general union of one quadruple point and  $\beta - 1$  double points. Then  $h^i(\mathcal{I}_Z(d)) = 0$ .*

We will use the following set-up.

**Notation 3.** Fix any  $Q \in \tau(X) \setminus X$ . For  $d \geq 3$  the point  $Q$  uniquely determines a point  $B \in X$  and (up to a non-zero scalar) a tangent vector  $\nu$  of  $X$  with  $\nu_{red} = \{B\}$ . We have  $Q \in \langle \nu \rangle \setminus \{B\}$  and  $T_Q\tau(X)$  is tangent to  $\tau(X) \setminus X$  exactly along the line  $\langle \nu \rangle = \langle \{B, Q\} \rangle$ . Let  $O \in \mathbb{P}^m$  be the only point such that  $j_{n,d}(O) = B$ . Let  $u_O : \tilde{X} \rightarrow \mathbb{P}^m$  be the blowing-up of  $O$ . Let  $E := u_O^{-1}(O)$  denote the exceptional divisor. For all integers  $x, e$  set  $\mathcal{O}_{\tilde{X}}(x, eE) := u^*(\mathcal{O}_{\mathbb{P}^m}(x))(eE)$ . Let  $\mathcal{H}$  denote the linear system  $|\mathcal{O}_{\tilde{X}}(d, -3E)|$  on  $\tilde{X}$ .

**Remark 3.** When  $d \geq 4$ , the line bundle  $\mathcal{O}_{\tilde{X}}(d, -3E)$  is very ample,  $u_*(\mathcal{O}_{\tilde{X}}(d, -3E)) = \mathcal{I}_{3O}(1)$ ,  $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(d, -3E)) = \binom{m+d}{m} - \binom{m+2}{m}$  and  $h^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(d, -3E)) = 0$  for all  $i > 0$ .

**Lemma 5.** *Fix integers  $m \geq 2$  and  $d \geq 5$ . If  $m \leq 4$ , then assume  $d \geq 6$ . Set  $\alpha := \lfloor \binom{m+d-1}{m} / (m+1) \rfloor$ . Fix an integer  $t$  such that  $3 \leq t \leq \alpha$ . The linear system  $\mathcal{H}$  on  $\tilde{X}$  is not  $(t-3)$ -weakly defective. For a general  $O_1, \dots, O_{t-2} \in \tilde{X}$  a general  $H \in |\mathcal{H}(-2O_1 - \dots - 2O_{t-2})|$  is singular only at the points  $O_1, \dots, O_{t-2}$  which are ordinary double points of  $H$ .*

*Proof.* Fix general  $O_1, \dots, O_{t-2} \in \tilde{X}$ . Fix  $j \in \{1, \dots, t-2\}$  and set  $Z' := 3O_j \cup \bigcup_{i \neq j} 2O_i$ ,  $Z'' := \bigcup_{i=1}^{t-2} 2O_i$  and  $W := 3O_j \cup \bigcup_{i \neq j} 2O_i$ . We have  $u_*(\mathcal{I}_{Z'}(d, -3E)) \cong \mathcal{I}_{W \cup 3O}(1)$ . The case  $i = 2$  of Lemma 3 gives  $h^1(\mathcal{I}_Z(d, -3E)) = 0$ . Lemma 1 applied to a blowing-up of  $\mathbb{P}^m$  at  $\{O, O_1, \dots, O_{t-2}\} \setminus \{O_j\}$  shows that a general  $H \in \mathcal{H}(-Z)$  has as an isolated singular point at  $O_j$ . Since this is true for all  $j \in \{1, \dots, t-2\}$ ,  $\mathcal{H}$  is not  $(t-3)$ -weakly defective (just by the definition of weak defectivity). The second assertion follows from the first one and [6], Theorem 1.4.  $\square$

Now we can apply Lemmas 2, 3, 4 and 5 and get the following result.

**Theorem 5.** *Fix integers  $m \geq 2$  and  $d \geq 6$ . If  $m \leq 4$ , then assume  $d \geq 7$ . Set  $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$ . Fix an integer  $t$  such that  $3 \leq t \leq \beta + 1$ . Then  $\tau(X, t)$  is not drip defective.*

*Proof.* Fix general  $P_1, \dots, P_{t-2}, O \in \mathbb{P}^m$  and a general line  $L \subset \mathbb{P}^m$  such that  $O \in L$ . Set  $Z := Z(O, L) \cup \bigcup_{i=1}^{t-2} 2P_i$ ,  $W := 3O \cup \bigcup_{i=1}^{t-2} 2P_{t-2}$ ,  $W' := 3O \cup 3O_1 \cup \bigcup_{i=2}^{t-2} 2P_{t-2}$  and  $W'' := 4O \cup \bigcup_{i=1}^{t-2} 2P_{t-2}$ . Take  $O_i \in \tilde{X}$  such that  $u_O(O_i) = P_i$ ,  $1 \leq i \leq t-2$ . Since  $u_{O_*}(\mathcal{I}_{2O_1 \cup \dots \cup 2O_{t-2}}(d, -4E)) \cong \mathcal{I}_W(d)$ , Lemma 4 gives  $h^1(\mathcal{I}_{2O_1 \cup \dots \cup 2O_{t-2}}(d, -4E)) = 0$ . Since  $Z(O, L) \subset 3O$ , the case  $y = 3$  of Lemma 1 applied to the blowing-up of  $\mathbb{P}^m$  at  $O_1, \dots, O_{t-2}$  shows that a general  $H \in |\mathcal{I}_W(d)|$  has an isolated singularity at  $O$  with multiplicity at most 3.  $\square$

Recall that  $\text{Sing}(\tau(X)) = X$  and that for each  $Q \in \tau(X) \setminus X$  there is a unique  $O \in X$  and a unique tangent vector  $\nu$  to  $X$  at  $O$  such that  $Q \in \langle \nu \rangle$  and that  $\langle \nu \rangle \setminus \{O\}$  is the contact locus of the tangent space  $T_Q\tau(X)$  with  $\tau(X) \setminus X$ .

Let  $P$  be a general point of  $\tau(X, t)$ , i.e. fix a general  $(P_1, \dots, P_{t-2}, Q) \in X^{t-2} \times \tau(X)$  and a general  $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$ .

*Proof of Theorem 1.* Fix a general  $P \in \tau(X, t)$ , say  $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$  with  $(P_1, \dots, P_{t-2}, Q)$  general in  $X^{t-2} \times \tau(X)$ . Terracini's lemma for joins ([1], Corollary 1.10) gives  $T_P\tau(X, t) = \langle T_{P_1}X \cup \dots \cup T_{P_{t-2}}X \cup T_Q\tau(X) \rangle$ . Let  $O$  be the point of  $\mathbb{P}^m$  such that  $Q \in T_{j_{m,d}(O)}X$ . Let  $\mathcal{H}'$  (resp.  $\mathcal{H}''$ ) be the set of all hyperplane  $H \subset \mathbb{P}^N$  containing  $T_Q\tau(X)$  (resp.  $T_P\tau(X, t)$ ). We may see  $\mathcal{H}'$  and  $\mathcal{H}''$  as linear systems on the blowing-up  $\tilde{X}$  of  $\mathbb{P}^m$  at  $O$ . Take  $O_i \in \tilde{X}$ ,  $1 \leq i \leq t-2$ , such that  $P_i = u(O_i)$  for all  $i$ . We have  $\mathcal{H}'' = \mathcal{H}'(-2P_1 - \dots - 2P_{t-2})$  and  $\mathcal{H} \subseteq \mathcal{H}'$ , where  $\mathcal{H}$  is defined in Notation 3. Since  $(P_1, \dots, P_{t-2})$  is general in  $X^{t-2}$  for a fixed  $Q$  and  $\mathcal{H} \subseteq \mathcal{H}'$ , Lemma 5 gives that a general  $H \in \mathcal{H}''$  intersects  $X$  in a divisor which, outside  $O$ , is singular only at  $P_1, \dots, P_{t-2}$  and with an ordinary node at each  $P_i$ . Now assume  $P \in \langle \{P'_1, \dots, P'_{t-2}, Q'\} \rangle$  for some other  $(P'_1, \dots, P'_{t-2}, Q') \in X^{t-2} \times \tau(X)$ . Since  $P$  is general in  $\tau(X, t)$  and  $\tau(X, t)$  has the expected dimension, the  $(t-1)$ -ple  $(P'_1, \dots, P'_{t-2}, Q')$  is general in  $X^{t-2} \times \tau(X)$ . Hence  $H \cap X$  is singular at each  $P'_i$ ,  $1 \leq i \leq t-2$ , and with an ordinary node at each  $P'_i$ . Since  $O$  is not an ordinary node of  $H \cap X$ , we get  $\{P_1, \dots, P_{t-2}\} = \{P'_1, \dots, P'_{t-2}\}$ . Thus  $O = O'$ . Hence  $H$  is tangent to  $\tau(X)_{reg}$  exactly along the line  $\langle \{Q, O\} \rangle \setminus \{O\}$ . Hence  $Q' \in \langle \{Q, O\} \rangle$ . Assume  $Q \neq Q'$ . Since  $P$  is general in  $\tau(X, t)$ , then  $P \notin \tau(X, t-1)$ . Hence  $Q' \notin \langle \{P_1, \dots, P_{t-2}\} \rangle$  and  $Q \notin \langle \{P_1, \dots, P_{t-2}\} \rangle$ . Thus  $\langle \{P_1, \dots, P_{t-2}, Q\} \rangle \cap \langle \{P_1, \dots, P_{t-2}, Q'\} \rangle = \langle \{P_1, \dots, P_{t-2}\} \rangle$  if  $Q \neq Q'$ . Since  $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle \cap \langle \{P_1, \dots, P_{t-2}, Q'\} \rangle$ , we got a contradiction.  $\square$

*Proof of Theorem 4.* The case  $t = 2$  is well-known and follows from the following fact: for any  $O \in X$  and any  $Q \in T_OX \setminus \{O\}$  the group  $G_O := \{g \in$

$\text{Aut}(\mathbb{P}^n) : g(O) = O$  acts on  $T_O X$  and the stabilizer  $G_{O,Q}$  of  $Q$  for this action is the line  $\langle \{O, Q\} \rangle$ , while  $T_O X \setminus \langle \{O, Q\} \rangle$  is another orbit for  $G_{O,Q}$ . Thus we may assume  $t \geq 3$ . Fix a general  $P \in \tau(X, t)$  and a general hyperplane  $H \supset T_P \tau(X, t)$ . If  $H$  is tangent to  $\tau(X)$  at a point  $Q' \in \tau(X) \setminus X$ , then it is tangent along a line containing  $Q'$ . Let  $E \in X$  be the only point such that  $Q' \in T_E X$ . We get  $T_E X \subset \tau(X, t)$  and that  $H \cap T_E X$  is larger than the double point  $2E \subset X$ . Theorem 1 gives that  $Q, Q'$  and  $E$  are collinear, i.e  $H$  is tangent only along the line  $\nu$ .  $\square$

## REFERENCES

- [1] B. Ådlandsvik, Joins and higher secant varieties. *Math. Scand.* **61** (1987), no. 2, 213–222.
- [2] E. Ballico, On the weak non-defectivity of Veronese embeddings of projective spaces. *Cent. Eur. J. Math.* **3** (2005), no. 2, 183–187.
- [3] E. Ballico, A. Bernardi, Partial stratification of secant varieties of Veronese varieties via curvilinear subschemes, arXiv: 1010.3546v1 [math.AG].
- [4] A. Bernardi, A. Gimigliano, M. Idà. Computing symmetric rank for symmetric tensors. *J. Symb. Comput.* **46** (2011) 34–53.
- [5] M. V. Catalisano, A. V. Geramita, A. Gimigliano, On the secant varieties to the tangential varieties of a Veronesean. *Proc. Amer. Math. Soc.* **130** (2002), no. 4, 975–985.
- [6] L. Chiantini, C. Ciliberto, Weakly defective varieties. *Trans. Amer. Math. Soc.* **454** (2002), no. 1, 151–178.
- [7] L. Chiantini, C. Ciliberto, On the dimension of secant varieties. *J. Eur. Math. Soc.* **12** (2010), no. 5, 1267–1291.
- [8] C. Ciliberto, F. Russo, Varieties with minimal secant degree and linear systems of maximal dimension on surfaces. *Adv. Math.* **206** (2006), no. 1, 1–50.
- [9] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
- [10] M. Mella, Singularities of linear systems and the Waring problem. *Trans. Amer. Math. Soc.* **358** (2006), no. 12, 5523–5538.
- [11] M. Mella, Base loci of linear systems and the Waring problem. *Proc. Amer. Math. Soc.* **137** (2009), no. 1, 91–98.

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