VANISHING THEOREMS FOR LINEARLY OBSTRUCTED DIVISORS

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Abstract. We study divisors on the blow-up of $\mathbb{P}^n$ at points in general position that are non-special with respect to the notion of linear speciality introduced in [6]. We describe the cohomology groups of their strict transforms via the blow-up of the space along their linear base locus. We extend the result to non-effective divisors that sit in a small region outside the effective cone. As an application, we describe linear systems of divisors in $\mathbb{P}^n$ blown-up at points in star configuration and their strict transforms via the blow-up of the linear base locus.

1. Introduction

The motivation for studying vanishing theorems of divisors comes from Birational Geometry (Mori's Minimal Model Program, see [18]) and Commutative Algebra (higher order embeddings of projective varieties, see [2]). In particular, vanishing theorems have applications to positivity properties of divisors such as global generation, very ampleness and in general k-very ampleness properties.

We denote by $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a collection of s points in general position with multiplicities at least $m_1, \ldots, m_s \geq 0$ respectively. The *(affine) virtual dimension* of $\mathcal{L}$ is denoted by

$$\operatorname{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n}$$

and the *expected dimension* of $\mathcal{L}$ is defined to be $\operatorname{edim}(\mathcal{L}) = \max(\operatorname{vdim}(\mathcal{L}), 0)$.

If D is the strict transform of a general divisor in $\mathcal{L}$ in the blow-up X of $\mathbb{P}^n$ in the s points,

$$D := dH - \sum_{i=1}^{s} m_i E_i, \tag{1.1}$$

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then $\text{vdim}(D) := \text{vdim}(\mathcal{L})$ equals $\chi(X, \mathcal{O}_X(D))$, the Euler characteristic of the sheaf on X associated with D, while $\dim(\mathcal{L})$ is the number of global section of $\mathcal{O}_X(D)$, namely the dimension of the space $H^0(X, \mathcal{O}_X(D))$.

The inequality $\dim(\mathcal{L}) \geq \text{edim}(\mathcal{L})$ is always satisfied. However, if the conditions imposed by the assigned multiple points are not linearly independent, then the actual dimension of $\mathcal{L}$ is strictly greater that the expected one: in that case we say that $\mathcal{L}$ (or D) is *special*. Otherwise, whenever the actual and the expected dimension coincide we say that $\mathcal{L}$ is *non-special*.

It is obvious that knowing the dimension of $\mathcal{L}$ is equivalent to classifying the speciality of linear systems. In the last century the problem of classifying linear systems was studied with different techniques by many people. We will briefly mention a few important results. In the planar case, the Segre-Harbourne-Gimigliano-Hirschowitz conjectures describes all special linear systems. On the negative side a conjecture related to the vanishing theorems of a linear system is Nagata's conjecture that predicts the nef cone of linear systems in the blown-up plane at general points. The degeneration technique introduced by Ciliberto and Miranda (see e.g. [9, 10]) is a successful method in the study of interpolation problems. However, in spite of many partial results, both conjectures are still open in general. In the case of $\mathbb{P}^3$, there is an analogous conjectural classification of special linear systems formulated by Laface and Ugaglia (see e.g. [20]).

Due to its complexity and mysterious geometry, the simple question of predicting and computing dimensions of such vector spaces is not even conjectured when n is four or higher. In the case of $\mathbb{P}^n$ general results are rare and few things are known. The well-known Alexander-Hirschowitz theorem states that a linear system in $\mathbb{P}^n$ with arbitrary number of double points is non-special besides a list of exceptional cases in small degree (see e.g. [1, 5, 22] for more details). For higher multiplicities, the only general result known so far is a complete cohomological classification of the speciality for *only linearly obstructed* effective divisors, proved by Brambilla, Dumitrescu and Postinghel in [6] (see also [8]). One of the goals of this paper is to extend such a classification [6] to the non-effective case.

In order to classify the special divisors, one has to understand first what are the *obstructions*, namely what are the varieties that whenever contained with multiplicity in the base locus of a given divisor force $\mathcal{L}$ to be special. In [3, 4] these obstructions are named *special effect varieties*. Few examples of obstructions in small dimension were classified before [6]. The only other examples known were (-1)-curves in $\mathbb{P}^2$ and $\mathbb{P}^3$ (see [20]) and the list of exceptions from Alexander-Hirschowitz Theorem. Theorem 1.4 below (that was proved in [6, Theorem 4.6]) and Corollary 5.2 show that, for any effective divisor, *linear cycles* of arbitrary dimension are always obstructions.

In the direction of extending the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture to $\mathbb{P}^n$ and possibly to other rational projective varieties, we pose the same natural and general question as in [6].

Question 1.1 ([6, Question 1.1])**.** *Consider any non-empty linear system $\mathcal{L}$ in $\mathbb{P}^n$. Let $\widetilde{X}$ be the smooth composition of blow-ups of $\mathbb{P}^n$ along the (strict transforms of the) cycles of the base locus of $\mathcal{L}$, ordered in increasing dimension. Denote by $\widetilde{\mathcal{D}}$ the strict transform of the general divisor of $\mathcal{L}$ in $\widetilde{X}$. Does $h^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{\mathcal{D}}))$ vanish for all $i \geq 1$?*

We remark that $\widetilde{\mathcal{D}}$ is obtained blowing-up the whole base locus of D, in particular the fixed hypersurfaces. Precisely, since these divisorial components of the base locus split off the system, taking proper transform under blow-up is equivalent to the contraction of these divisorial components.

For linear divisorial components more details are presented in Section 2.

In general Question 1.1 is difficult since it requires first to describe the base locus of a linear system and second to understand the contribution given by each component of the base locus to the speciality of the linear system. An affirmative answer to Question 1.1 implies that $\dim(\mathcal{L}) = \chi(X, \mathcal{O}_X(\widetilde{\mathcal{D}}))$, translating the classical dimensionality problem for linear systems into a Riemann-Roch formula for divisors living in further blown-up spaces. We denote by $\tilde{D}$ the strict transform after the blow-up along the linear cycles of the base locus, while $\widetilde{\mathcal{D}}$ denotes the strict transform after the blow-up of all -linear and non linear- cycles of the base locus, ordered by increasing dimension. Due to the combinatorial and geometrical complexity of this problem so far we only understand properties of divisors $\tilde{D}$ and we will present them in detail.

We mention that for $s \leq n+2$ the divisors $\tilde{D}$ obtained by further blow-up of the linear base locus (described in details in Sections 5 and 6) are divisors in $\overline{\mathcal{M}}_{0,n+3}$, the moduli space of stable rational curves with $n+3$ marked points. Therefore understanding their cohomological description can be used in the study of positivity properties such as the effective cone and the ample cone of $\overline{\mathcal{M}}_{0,n+3}$ (see also [6, Section 6.3]).

In the article [6], the authors introduced a new notion of expected dimension for linear systems, that takes into account the linear obstructions and extends the notion of virtual dimension, namely *linear virtual dimension*. We mention that in this paper, we will use $\operatorname{ldim}(\mathcal{L})$ to denote the (affine) linear virtual dimension, instead of the (projective) expected linear dimension as used in [6]. Given two linear systems $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ and $\mathcal{L}_{n,d}(m'_1, \ldots, m'_s)$ with the same degree, we write $\mathcal{L} \prec_s \mathcal{L}'$ if $m_i \geq m'_i$ for all $i \in \{1, \ldots, s\}$.

Definition 1.2 ([6, Definition 3.2]). *Given a linear system, for any integer $-1 \leq r \leq s-1$ and for any multi-index $I(r) = \{i_1, \ldots, i_{r+1}\} \subseteq \{1, \ldots, s\}$, define the integer*

$$k_{I(r)} := \max(m_{i_1} + \cdots + m_{i_{r+1}} - rd, 0). \tag{1.2}$$

The (affine) linear virtual dimension *of $\mathcal{L}$ (or of D), denoted by* $\operatorname{ldim}(\mathcal{L})$, *is the number*

$$\sum_{r=-1}^{s-1} \sum_{I(r) \subseteq \{1,\ldots,s\}} (-1)^{r+1} \binom{n + k_{I(r)} - r - 1}{n}, \tag{1.3}$$

where we set $I(-1) = \emptyset$. The (affine) linear expected dimension *of $\mathcal{L}$ is defined as follows: it is 0 if $\mathcal{L} \prec_s \mathcal{L}'$ and $\operatorname{ldim}(\mathcal{L}') \leq 0$, otherwise it is the maximum between $\operatorname{ldim}(\mathcal{L})$ and 0.*

We remark that this notion is well-defined not only for all effective divisors but also for non-effective ones provided that $m_i \leq d+1$. We will study this type of divisors in Sections 5 and 6.

In this light, asking whether the dimension of a given linear system equals its linear expected dimension can be thought as a refinement of the classical question of asking whether the dimension equals the expected dimension. If the answer to this question is affirmative, then $\mathcal{L}$ (or D) is said to be a *only linearly obstructed*. Obviously, non-special linear systems are always only linearly obstructed.

There exist linear systems that are *linearly obstructed* without being *only linearly obstructed*. For instance $\mathcal{L}_{4,10}(6^7)$ contains all lines $L_{ij},\ i,j \in \{1,\ldots,7\}$ with multiplicity two in its base locus as well as the rational normal curve through the seven points, see [6, Example 6.2] for more details.

Connections between $\mathcal{L}$ being only linearly obstructed and the Fröberg-Iarrobino Conjecture (see [8]), describing the Hilbert series of an ideal generated by s forms, can be found in [6, Section 6]. This reveals the importance of the notion of linear speciality, that was achieved and developed independently from both the geometric and the algebraic setting.

Linear systems with an arbitrary number of points of bounded sum of multiplicities were classified in [6], for $n \geq 1$, $d \geq 2$, by proving that they are only linearly obstructed.

Theorem 1.3 ([6, Theorem 5.3])**.** *All non-empty linear systems with $s \leq n+2$ base points are only linearly obstructed. Moreover, if $s \geq n+3$ and*

$$\sum_{i=1}^{s} m_i \leq nd + \min(n - s(d), s-n-2), \quad 1 \leq m_i \leq d, \tag{4.1}$$

where $s(d) \geq 0$ is the number of points of multiplicity d, then $\mathcal{L}$ is non-empty and only linearly obstructed.

The new perspective introduced in [6] is built upon the cohomological study of the strict transforms of effective and only linearly obstructed divisors. More precisely, the strict transforms are taken after blowing-up their linear base locus. Moreover, in [6] a complete classification was given for effective divisors interpolating $s \leq n+2$ general points, in which range the effective cone was known (see for example [7]).

For every effective divisor D, let $D_{(r)}$ denote the strict transform of D in the space $X^n_{(r)}$ obtained as a sequence of blow-ups of $\mathbb{P}^n$ along the linear base locus of D up to dimension r, with $r \leq n-1$ (we refer to Section 2 for details about this construction):

$$D_{(r)} := D - \sum_{\rho=1}^{r} \sum_{I(\rho)\subseteq\{1,\ldots,s\}} k_{I(\rho)} E_{I(\rho)}, \tag{1.4}$$

where $E_{I(\rho)}$ denotes the (strict transform of the) exceptional divisor of the linear subspace of $\mathbb{P}^n$ of dimension ρ spanned by the points parametrized by the multi-index $I(\rho)$. Let $\bar{r}$ be the maximum dimension of the linear base locus (see Section 4.1); we will set

$$\tilde{D} := D_{(\bar{r})}. \tag{1.5}$$

To simplify notation here and throughout the paper we will also abbreviate $h^i(X^n_{(r)}, \mathcal{O}_{X^n_{(r)}}(D_{(r)}))$ by $h^i(D_{(r)})$.

Theorem 1.4 ([6, Theorem 4.6])**.** *If $s \leq n+2$ and D is effective, the following statements hold.*

(a) $h^0(D) = \mathrm{ldim}(D)$ *and* $h^i(\tilde{D}) = 0$ *for every* $i \geq 1$.

(b) *For any $0 \leq r \leq n-1$, $h^i(D_{(r)}) = 0$ for every $i \geq 1$ and $i \neq r+1$ while*

$$h^{r+1}(D_{(r)}) = \sum_{\rho=r+1}^{s-1} \sum_{I(\rho)\subseteq\{1,\ldots,s\}} (-1)^{\rho-r-1}\binom{n+k_{I(\rho)}-\rho-1}{n}.$$

The goal of this paper is to show that the same type of results as in Theorem 1.4 holds for larger classes of divisors, such as effective divisors with arbitrary number of points and non-effective divisors. The definition of strict transform after blowing-up the linear base locus is formally extended to the non-effective case in Section 5.

We also generalize the formula in Theorem 1.4 for any effective divisor, not necessarily only linearly obstructed, interpolating an arbitrary collection of general multiple points.

Theorem 1.5. *If D is any effective divisor then for any $0 \leq r \leq n-1$ we have*

$$h^{r+1}(D_{(r)}) = \sum_{\rho=r+1}^{s-1} \sum_{I(\rho)\subseteq\{1,\ldots,s\}} (-1)^{\rho-r-1}\binom{n+k_{I(\rho)}-\rho-1}{n} + \sum_{\rho=r+1}^{n} (-1)^{\rho-r-1} h^\rho(\tilde{D}).$$

In particular,

$$h^0(D) = \operatorname{ldim}(D) + \sum_{\rho=1}^{n} (-1)^\rho h^\rho(\tilde{D}).$$

Moreover, if $h^i(\tilde{D}) = 0$, for all $i \geq 1$, then $h^i(D_{(r)}) = 0$ for all $i \neq r+1$.

This result is part of Theorem 5.1 that will be proved in Section 5 in a more general setting. The geometric interpretation is that for any effective divisor D, every ρ-dimensional linear cycle $L_{I(\rho)}$, for which $k_{I(\rho)} \geq 1$ and $\rho \geq r+1$, gives a contribution with alternating sign, $(-1)^{\rho+1}$, equal to

$$\binom{n+k_{I(\rho)}-\rho-1}{n} \tag{1.6}$$

to $h^{r+1}(D_{(r)})$ and to the formula for $\operatorname{ldim}(D)$ (cfr. Theorem 5.1 and Corollary 5.2). Moreover, such a contribution is zero when $k_{I(\rho)} \leq \rho$.

The main result of this paper is a complete cohomological description of $D_{(r)}$ in the following cases, where we set

$$b := b(D) = \sum_{i=1}^{s} m_i - nd. \tag{1.7}$$

Theorem 1.6. *Statements (a) and (b) of Theorem 1.4 holds for all divisors with $m_i \leq d+1$ under the following hypothesis: $s \leq n+1$ and $b \leq n$, or $s \geq n+2$ and $b \leq \max(1, s-n-2)$.*

Moreover, if $s \leq n+1$ then $h^i(\tilde{D}) = 0$, for all $i \leq n-1$, and $h^n(\tilde{D}) = \binom{b-1}{n}$ for $b \geq n+1$ and zero otherwise.

The theorem summarizes the results contained in Theorem 4.1, Theorem 5.3 (2), Theorem 6.1 and Corollary 6.2. This result shows that it makes sense to extend

Question 1.1 to non-effective divisors in a small region outside the effective cone with a correct definition of $D_{(r)}$. In order to study classical interpolation problems, a crucial step is the study of non-effective divisors. More precisely, whenever a linear cycle is contained in the base locus of a divisor D, its normal bundle is a non-effective divisor and the cohomology groups of its multiples produce the contributions to the speciality, as in (1.6) for the linear case.

From Theorems 1.4 and 1.6, for any effective divisor with $s \leq n+2$ and any only linearly obstructed divisor satisfying the bound (4.1), one obtains $\chi(\tilde{D}) = \mathrm{ldim}(D)$. This gives a strong interpretation of the notion of linear expected dimension that, not only represents a dimension count for the linear system $\mathcal{L}$, but also computes the Euler characteristic of the sheaf $\mathcal{O}(\tilde{D})$. As a corollary of Theorem 1.6 we extend this Riemann-Roch formula to larger classes of divisors obtaining interesting combinatorial identities. In particular, toric divisors sitting on the facets of the effective cone have Euler characteristic equal to one.

This paper is organized as follows. In Section 2 we introduce the general construction and notations.

In Section 3 we provide a cohomological classification of a class of interesting divisors, namely integer multiples of *standard Cremona transformations* of the hyperplane classes.

In Section 4 we first give an explicit description of the linear base locus of any divisor D, Proposition 4.2.

In Theorem 4.1 we show that linear cycles are the only obstructions for divisors with $s \leq n+2$ or satisfying (4.1). This result is generalized in Corollary 6.2 to non-effective divisors with $m_i \leq d+1$.

In Section 5 we study the cohomology groups of toric divisors with multiplicities bounded above by $d+1$, since in this range ldim is well-defined.

In Section 6 we study vanishing theorems for non-effective divisors with $s = n+2$ points in a small region outside the effective cone, Theorem 6.1. In Corollary 6.2 we extend the result to the case of non-effective divisors with $s \geq n+3$ points with multiplicities satisfying the bound (4.1).

In Section 7 we use the vanishing theorems from Section 5 to study linear systems with points in special position. Theorem 7.3 computes the dimensions of a class of linear systems in $\mathbb{P}^n$ interpolating *star configurations* of points with higher multiplicities.

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2. Blowing-up: construction and notation

In this section we recall the main construction that was partially presented in [6, Sect. 4.1].

Let $\mathcal{I}$ be a set of subsets of $\{1, \ldots, s\}$. For every integer $0 \leq r \leq \min(n, s) - 1$ we denote by $I(r) = \{i_1, \ldots, i_{r+1}\} \in \mathcal{I}$ a multi-index of length $|I(r)| = r+1$. Let us also introduce the notation

$$\mathcal{I}(r) := \{I(\rho) \in \mathcal{I} : 0 \leq \rho \leq r\},$$
$$\mathcal{I}(r)_j := \{I(\rho) \in \mathcal{I}(r) \setminus \mathcal{I}(0) : j \in I(\rho)\}. \tag{2.1}$$

Let $p_1, \ldots, p_s$ be general points in $\mathbb{P}^n$ and, for every $I(r) \in \mathcal{I}$, let $L_{I(r)} \cong \mathbb{P}^r$ denote the r-dimensional linear subspace spanned by the points $\{p_j,\ j \in I(r)\}$, which we will refer to as a *linear r-cycle*. Notice that $L_{I(0)} = p_j$ is a point. An arbitrary multi-index will be denoted by I without specifying its cardinality.

We will assume that $\mathcal{I}$ satisfies the following properties:

(I) $\{j\} \in \mathcal{I}$, for all $j \in \{1, \ldots, s\}$;
(II) if $I \subset J$ and $J \in \mathcal{I}$, then $I \in \mathcal{I}$.

Let $\Lambda = \Lambda(\mathcal{I}) \subset \mathbb{P}^n$ be the subspace arrangement corresponding to $\mathcal{I}$, i.e. the (finite) union of the linear cycles L_I with $I \in \mathcal{I}$. Let $\bar{r}$ be the largest dimension of a linear cycle in Λ, i.e. $\bar{r} = \max_{I \in \mathcal{I}}(|I|) - 1$. Write $\Lambda = \Lambda_{(1)} + \cdots + \Lambda_{(\bar{r})}$, where $\Lambda_{(r)} = \cup_{I(r) \in \mathcal{I}} L_{I(r)}$.

Assume moreover that $\mathcal{I}$ satisfies the following condition

(III) if $I, J \in \mathcal{I}$, then $L_I \cap L_J = L_{I \cap J}$.

Notice that condition (III) is obviously satisfied when $s \leq n+1$.

We denote by $\pi_{(0)} : X^n_{(0)} \to \mathbb{P}^n$ the blow-up of $\mathbb{P}^n$ at $p_1, \ldots, p_s$, with $E_1, \ldots, E_s$ exceptional divisors. Let us also consider the sequence of blow-up maps

$$X^n_{(n-1)} \xrightarrow{\pi_{(n-1)}} \cdots \xrightarrow{\pi_{(2)}} X^n_{(1)} \xrightarrow{\pi_{(1)}} X^n_{(0)},$$

where $X^n_{(r)} \xrightarrow{\pi_{(r)}} X^n_{(r-1)}$ is the blow-up of $X^n_{(r-1)}$ along the strict transform of $\Lambda_{(r)} \subset \mathbb{P}^n$, via $\pi_{(r-1)} \circ \cdots \circ \pi_{(0)}$. For any $I(r) \in \mathcal{I}$, we denote by $E_{I(r)}$ the exceptional divisor of the cycle $L_{I(r)}$ in $X^n_{(r)}$. Notice that conditions (I), (II) and (III) ensures that, for every r, the sum of the exceptional divisors of $\pi_{(r)}$ is simple normal crossing. We will write, abusing notation, H for the strict transform in $X^n_{(r)}$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ and $E_{I(\rho)}$, for $I(\rho) \in \mathcal{I}(r-1)$, for the strict transform in $X^n_{(r)}$ of the exceptional divisor $E_{I(\rho)}$ in $X^n_{(\rho)}$, respectively.

Remark 2.1. Notice that the map and $\pi_{(n-1)} : X^n_{(n-1)} \longrightarrow X^n_{(n-2)}$ is a contraction of (strict transforms of) hyperplanes. In particular, $\operatorname{Pic}(X^n_{(n-1)}) \cong (\pi_{(n-1)})^* \operatorname{Pic}(X^n_{(n-2)})$. Thus, in our notation, for every $I(n-1) \in \mathcal{I}$ we have

$$E_{I(n-1)} = H - \sum_{I(\rho) \in \mathcal{I}(n-2)} E_{I(\rho)}. \tag{2.2}$$

The Picard group of $X^n_{(r)}$ is

$$\operatorname{Pic}(X^n_{(r)}) = \langle H, E_{I(\rho)} : I(\rho) \in \mathcal{I}(r-1) \rangle.$$

Remark 2.2. For $r = 1, \ldots, n-1$ and F a divisor on $X^n_{(r-1)}$, for any $i \geq 0$, we have

$$h^i(X^n_{(r)}, (\pi_{(r)})^* F) = h^i(X^n_{(r-1)}, F).$$

It follows from Zariski connectedness theorem and by the projection formula (see for instance [17] or [21, Lemma 1.3] for a more detailed proof.).

2.1. **The geometry of the divisor E_j in $X^n_{(r)}$.** Let $r \geq 1$. Consider the composition of blow-ups $\pi_{(r,0)} := \pi_{(r)} \circ \cdots \circ \pi_{(1)} : X^n_{(r)} \to X^n_{(0)}$. By abuse of notation we will denote by E_j the strict transform $(\pi_{(r,0)})^* E_j \in \mathrm{Pic}(X^n_{(r)})$ of the exceptional divisor $E_j \in \mathrm{Pic}(X^n_{(0)})$ of the point $p_j \in \mathbb{P}^n$.

For every multi-index $I(\rho) \in \mathcal{I}(r)_j$, let $E_{I(\rho)} \in \mathrm{Pic}(X^n_{(r)})$ be the strict transform of the exceptional divisor in $X^n_{(\rho)}$ of $L_{I(\rho)}$, and set $e_{I(\rho)|j} := E_{I(\rho)}|_{E_j}$. Moreover, let h be the hyperplane class of E_j.

Lemma 2.3. *In the above notation, a basis for the Picard group of E_j is given by h and $e_{I(\rho)|j}$, for all $I(\rho) \in \mathcal{I}(r)_j$.*

In particular, we have the isomorphism $Pic(E_j) \cong Pic(X^{n-1}_{(r-1)})$.

Rephrasing Lemma 2.3, the exceptional divisor $E_j \in \mathrm{Pic}(X^n_{(r)})$ is isomorphic to a blown-up $\mathbb{P}^{n-1}$ along linear $(\rho-1)$-cycles, $\rho \leq r$, spanned by subsets of a collection of $s-1$ general points. These $s-1$ points correspond to the lines $L_{I(1)}$, with $I(1) = \{j, l\}$, for all indices $l \in \{1, \ldots, \hat{j}, \ldots, s\}$. Similarly, the linear $(\rho-1)$-cycles blown-up in E_j correspond to the linear ρ-cycles $L_{I(\rho)}$ of $\mathbb{P}^n$ satisfying the condition that $j \in I(\rho)$.

2.2. **The geometry of the divisor $E_{I(\rho)}$ in $X^n_{(r)}$.** Let $I = I(\rho) \in \mathcal{I}(r)$ be any multi-index. Notice that if $\rho = 0$, $E_{I(\rho)}$ is the exceptional divisor of a point that was already described in Section 2.1. Consider the composition of blow-ups $\pi_{(r,\rho)} := \pi_{(r)} \circ \cdots \circ \pi_{(\rho+1)} : X^n_{(r)} \to X^n_{(\rho)}$. By abuse of notation we will denote by E_I the strict transform via $\pi_{(r,\rho)}$ in $X^n_{(r)}$ of the exceptional divisor $E_I \in \mathrm{Pic}(X^n_{(\rho)})$ of a linear ρ-cycle $L_I \subset \mathbb{P}^n$: $E_I \in \mathrm{Pic}(X^n_{(r)})$ is a Cartesian product that we are now going to describe.

Consider first the case $\rho = r$. We have the following isomorphism: $E_I \cong X^r_{(r-2)} \times \mathbb{P}^{n-r-1}$; we refer to [6, Section 4] for details. The Picard group of the first factor is generated by $\langle h, e_{I(t)} : I(t) \in \mathcal{I}(r-2) \rangle$, where $E_{I(t)}|_{E_I} =: e_{I(t)} \boxtimes 0$, while the Picard group of the second factor is generated by the hyperplane class.

Assume now that $0 \leq \rho < r$. Notice first of all that the restriction $E_{I(t)}|_{E_I}$ of $X^n_{(r)}$ is zero on both factors unless one of the following containment relations is satisfied: $I \subset I(t)$ or $I(t) \subset I$. We denote by h_b and h_f the hyperplane classes of the two factors respectively. Moreover we introduce divisors $e_{I(t)}$ on the first factor and $e_{I(t)|I}$ on the second factor according to the following intersection table:

$$\begin{aligned} H|_{E_I} &=: h_b \boxtimes 0; \\ E_{I(t)}|_{E_I} &=: e_{I(t)} \boxtimes 0, \text{ for all } I(t) \subset I,\ t \geq 0; \\ E_{I(t)}|_{E_I} &=: 0 \boxtimes e_{I(t)|I}, \text{ for all } I \subset I(t),\ t \leq r. \end{aligned} \tag{2.3}$$

Notice that if $\rho = 0$, the first factor of $E_{I(\rho)}$ is a point and we have $h_f = h$ in the notation of Section 2.1.

Remark 2.4. If $t = \rho - 1$, and $I(t) \subset I$, i.e. $L_{I(\rho-1)} \subset \mathbb{P}^n$ is a hyperplane of $L_I \subset \mathbb{P}^n$, using (2.2) we obtain the following equality:

$$E_{I(\rho-1)}|_{E_I} = \Big(H - \sum E_{I(\tau)}\Big)|_{E_I} = \Big(h - \sum e_{I(\tau)}\Big) \boxtimes 0,$$

where the sums range over the multi-indices $I(\tau) \subset I$, $I(\tau) \in \mathcal{I}(\rho-2)$. Accordingly, $e_{I(\rho-1)} = h_b - \sum e_{I(\tau)}$. A similar argument holds for divisors on the second factor, when $t = n-1$ and $I \subset I(t)$.

Lemma 2.5. *In the above notation, assume $\mathcal{I}$ is a set of multi-indices satisfying conditions* (I),(II) *and* (III). *We have*

$$E_I \cong X^{\rho}_{(\rho-2)} \times X^{n-\rho-1}_{(r-\rho-1)}.$$

Moreover, bases for the Picard groups of the two factors of the product E_I, for $I \in \mathcal{I}$ of length $\rho+1$. are given respectively by

$$\langle h_b,\, e_{I(t)} : I(t) \subset I, I(t) \in \mathcal{I}(\rho-2)\rangle;$$
$$\langle h_f,\, e_{I(t)|I(\rho)} : I \subset I(t),\ I(t) \in \mathcal{I} \setminus \mathcal{I}(\rho)\rangle.$$

The element $e_{I(t)}$ represents the exceptional divisor in $X^{\rho}_{(\rho-2)}$ of a linear t-cycle and $e_{I(t)|I(\rho)}$ represents the exceptional divisor in $X^{n-\rho-1}_{(r-\rho-1)}$ of a linear $(t-\rho-1)$-cycle which is spanned by $|I(t) \setminus I(\rho)|$ points.

We now give a characterization of the normal bundle of the exceptional divisor E_I in the space $X^n_{(r)}$. To this purpose, we introduce the following divisor on $X^{\rho}_{(\rho-2)}$:

$$\mathrm{Cr}_{\rho}(h_b) = \rho h_b - \sum_{\substack{I(t)\subset I \\ I(t)\in\mathcal{I}(\rho-2)}} (\rho - t - 1) e_{I(t)}. \tag{2.4}$$

We will give a detailed cohomological description of such a divisor in Section 3.

Lemma 2.6. *In the notation above, we have*

$$-E_I|_{E_I} = \mathrm{Cr}_{\rho}(h_b) \boxtimes h_f.$$

Proof. The proof follows from the computation of the conormal bundle of the first factor of E_I:

$$N^*_{X^{\rho}_{(\rho-2)}|X^n_{(r)}} = \mathcal{O}_{X^{\rho}_{(\rho-2)}}(\mathrm{Cr}_{\rho}(h_b)).$$

See [6, Lemma 4.3] for details. □

3. Standard Cremona transformations of hyperplane classes

We recall that the *standard Cremona transformation* of $\mathbb{P}^n$, based at the $n+1$ coordinate points, is the birational transformation defined by the following rational map:

$$\mathrm{Cr} : (x_0 : \cdots : x_n) \to (x_0^{-1} : \ldots, x_n^{-1}),$$

where $x_0, \ldots, x_n$ are homogeneous coordinates of $\mathbb{P}^n$. This map induces an action on the Picard group of $\mathbb{P}^n$ blown-up at s points, $X^n_{(0)}$. Without loss of generality we may assume that an effective divisor D of the form (1.1) is based on the $n+1$ coordinate points and other general points of the projective space and we label their corresponding exceptional divisors by $E_1, \ldots, E_{n+1}, E_{n+2}, \ldots, E_s$. The Cremona action on the divisor D is described by the following rule (see e.g. [13]). Set

$$D = dH - \sum_{i=1}^{s} m_i E_i, \quad c := (n-1)d - \sum_{i=1}^{n+1} m_i.$$

Then

$$\mathrm{Cr}(D) = (d+c)H - \sum_{i=1}^{n+1}(c+m_i)E_i - \sum_{i=n+2}^{s} m_i E_i.$$

In the case $n=3$, Cr is often called the *cubo-cubic* Cremona transformation, see for instance [20].

The divisors (2.4) that naturally arise in the blowing-up construction are the strict transforms in $X^n_{(n-1)}$ of the standard Cremona transformations of the strict transform H of the hyperplane class $\mathcal{O}_{\mathbb{P}^n}(1)$, where we abbreviate the notation for the strict transform in $X^n_{(n-1)}$ of $\mathrm{Cr}(H)$ by

$$\mathrm{Cr}_n(H) = nH - \sum_{I(\rho)\in\mathcal{I}(n-2)} (n-\rho-1)E_{I(\rho)}. \tag{3.1}$$

This is the divisor (2.4) obtained by replacing ρ by n, t by ρ, I by $\{1,\dots,n+1\}$ and h_b by H.

In this section we compute all cohomologies of any multiple of Cremona transformations of hyperplane classes. In particular we show that they have the same cohomological behavior as the same multiples of the hyperplane classes.

Let $\mathcal{I}$ be the set of all subsets of $\{1,\dots,n+1\}$. Notice that $\mathcal{I}$ satisfies conditions (I), (II) and (III) of Section 2. Recall that the canonical divisor of the blown-up projective space $X^n_{(n-1)}$ is

$$K_{X^n_{(n-1)}} = -(n+1)H + \sum_{I(\rho)\in\mathcal{I}(n-2)} (n-\rho-1)E_{I(\rho)}$$

and notice that

$$K_{X^n_{(n-1)}} + \mathrm{Cr}_n(H) = -H.$$

Theorem 3.1. *For any integer a, we have that $h^i(X^n_{(n-1)}, \mathcal{O}(a\mathrm{Cr}_n(H))) = h^i(\mathbb{P}^n, \mathcal{O}(a))$.*

In particular, we have that $h^i(X^n_{(n-1)}, \mathcal{O}(-\alpha\mathrm{Cr}_n(H))) = 0$, for $1 \le \alpha \le n$, $i \ge 0$.

Proof. If $a=0$ the statement is obvious.

Assume that $a \ge 1$. It is known that (see for example [14, Theorem 3]) $h^0(a\mathrm{Cr}_n(H)) = h^0(aH)$. Moreover, the effective divisor $a\mathrm{Cr}_n(H)$ on $X^n_{(n-1)}$ is not obstructed, that is $h^i(a\mathrm{Cr}_n(H)) = 0$, for all $i \ge 1$, see [6, Theorem 4.6].

Assume now that $a \le -1$ and denote $a = -\alpha$, where α is a positive integer.

Case $\alpha \ge n+1$. Recall that $h^i(\mathbb{P}^n, \mathcal{O}(-\alpha)) = 0$ for $i \ne n$ and that, by Serre duality, $h^n(\mathbb{P}^n, \mathcal{O}(-\alpha)) = h^0(\mathbb{P}^n, \mathcal{O}(\alpha-n-1)) > 0$. Note that

$$\begin{aligned} h^i(\mathcal{O}(-\alpha\mathrm{Cr}_n(H))) &= h^{n-i}((\alpha-1)\mathrm{Cr}_n(H) - H) \\ &= h^{n-i}\left([(\alpha-1)n-1]H - (\alpha-1)\sum_{I(\rho)\in\mathcal{I}(n-2)}(n-\rho-1)E_{I(\rho)}\right), \end{aligned}$$

where the first equality follows from Serre duality on $X^n_{(n-1)}$ and the second equality is just the expanded form of $(\alpha-1)\mathrm{Cr}_n(H) - H$. We claim that the cohomologies vanish for all $i \ne n$. To show this, we notice that the divisor

$$((\alpha-1)n-1)H - (\alpha-1)(n-1)\sum_{i=1}^{n+1} E_i$$

on $X^n_{(0)}$ is effective and that each cycle $L_{I(\rho)}$ is contained in the base locus with multiplicity $k_{I(\rho)} = (\alpha-1)(n-\rho-1)+\rho$, by [6, Lemma 2.1]. Each integer $k_{I(\rho)}$ differs from the coefficient of $E_{I(\rho)}$ in the above expression by ρ. Hence the vanishing of the ith cohomology group, for all $i \neq n$, follows by [6, Theorem 4.6]. If $i = n$, notice that

$$h^n(X^n_{(n-1)}, \mathcal{O}(-\alpha \mathrm{Cr}_n(H))) = h^0\left(X^n_{(0)}, ((\alpha-1)n-1)H - (\alpha-1)\sum_{j=0}^{n+1}(n-1)E_j\right).$$

We compute the number of global sections by preforming a standard Cremona transformation in $\mathbb{P}^n$, which preserves that number:

$$\begin{aligned} h^n(X^n_{(n-1)}, \mathcal{O}(a\mathrm{Cr}_n(H))) &= h^0(X^n_{(0)}, (\alpha-n-1))H + (n-1)\sum_{j=0}^{n+1} E_j) \\ &= h^0(X^n_{(0)}, (\alpha-n-1)H). \end{aligned}$$

Case $1 \leq \alpha \leq n$. We prove the statement by induction on n and α. The base steps $n = 2$, $\alpha = 1, 2$ are easily verified by means of Serre duality. Indeed, as the canonical divisor of $X^2_{(0)}$ is $K = -2H + E_1 + E_2 + E_3$, we have $H^i(-\mathrm{Cr}_2(H)) = H^{2-i}(-H) = 0$ and $H^i(-2\mathrm{Cr}_2(H)) = H^{2-i}(H - E_1 - E_2 - E_3) = 0$. Fix $n \geq 3$. Using (2.2), we compute the following equality

$$\begin{aligned} -\sum_{1 \in I(n-1)} E_{I(n-1)} &= -nH + \sum_{I(\rho) \in \mathcal{I}(n-2)_1} (n-\rho)E_{I(\rho)} \\ &\quad + \sum_{I(\rho) \in \mathcal{I}(n-2) \setminus \mathcal{I}(n-2)_1} (n-\rho-1)E_{I(\rho)} \\ &= -\mathrm{Cr}_n(H) + E_1 + \sum_{I(\rho) \in \mathcal{I}(n-2)_1} E_{I(\rho)}. \end{aligned}$$

For $1 \leq r \leq n-1$, and $I = I(r)$ we recall from (2.3) that $E_{I(\rho)}|_{E_I} = e_{I(\rho)} \boxtimes 0$ if $I(\rho) \subset I$, $E_{I(\rho)}|_{E_I} = 0 \boxtimes *$ if $I(\rho) \not\subseteq I$, where we use $*$ to denote the appropriate divisor, as we are only interested in the first factor. Hence, the above computation and Remark 2.4 show that

$$-\sum_{1 \in I(r-1)} E_{I(r-1)}|_{E_I} = \left(-\mathrm{Cr}_r(h) + e_1 + \sum_{I(\rho) \in \mathcal{I}(r-2)_1} e_{I(\rho)}\right) \boxtimes *.$$

Therefore,

$$-E_1 - \sum_{I(\rho) \in \mathcal{I}(r-1)_1} E_{I(\rho)}|_{E_I} = -\mathrm{Cr}_r(h) \boxtimes * \tag{3.2}$$

Recall from (2.1) that $\mathcal{I}(n-1)_1$ denotes the set of subsets of $\{1, \ldots, s\}$ of cardinality at most n containing $\{1\}$ as a proper set. For all $0 \leq r \leq n-2$, let $\{I(r)_0, \ldots, I(r)_{s_r}\} \subset \mathcal{I}(n-1)_1$ be the ordered set of all multi-indices of length $r+1$ that contain $\{1\}$, with $s_r + 1$ being its cardinality. Let $\prec$ be the lexicographical order on $\mathcal{I}(n-1)_1$ defined as follows: $I(r')_{j'} \prec I(r)_j$ if and only if $r' < r$ or $r' = r$ and $j' < j$.

In the space $X^n_{(n-1)}$ we consider the divisors $F(r, j)$ defined by recursion starting from $F = 0$ as follows:

$$F(0,0)=F-E_1,$$
$$F(r,0)=F(r-1,s_{r-1})-E_{I(r)_0},\ 1\leq r\leq n-1,$$
$$F(r,j)=F(r,j-1)-E_{I(r)_j},\ 1\leq j\leq s_r.$$

Notice that the last divisor is $F(n-1,s_{n-1})=-\mathrm{Cr}_n(H)$. We consider the following exact sequences of sheaves, performed following the order $\prec$:

$$0\to F(0,0)\to F\to F|_{E_1}\to 0,$$
$$0\to F(r,0)\to F(r-1,s_{r-1})\to F(r-1,s_{r-1})|_{E_{I(r)_1}}\to 0,\quad 1\leq r\leq n-1$$
$$0\to F(r,j)\to F(r,j-1)\to F(r,j-1)|_{E_{I(r)_j}}\to 0,\quad 1\leq j\leq s_r.$$

The divisor $-E_1$ has vanishing cohomologies, for all $n\geq 1$. Moreover, in all sequences the restricted divisor is of the form (3.2) and has therefore vanishing cohomologies, by induction on n, using Kunneth formula for the cohomology of factors. This implies that $H^i(-\mathrm{Cr}_n(H))=0$, $i\geq 0$, concluding the proof of the statement in the case $\alpha=1$.

We are left to prove the vanishing for $-\alpha\mathrm{Cr}_n(H)$, $2\leq\alpha\leq n$. For $\alpha\geq 2$, we assume the statement true for $\alpha-1$. We apply the recursive restriction procedure as above. Setting $F:=-(\alpha-1)\mathrm{Cr}_n(H)$, we get $F(n-1,s_{n-1})=-\alpha\mathrm{Cr}_n(H)$.

We first notice that $\mathrm{Cr}_n(H)|_{E_1}=-\mathrm{Cr}_{n-1}(h)$ on $E_1\cong X^{n-1}_{(n-3)}$. Indeed,

$$-nH+\alpha(n-\rho-1)\sum_{I(\rho)\in\mathcal{I}(n-2)}E_{I(\rho)}|_{E_1}=-(n-1)h+(n-\rho-1)\sum_{I(\rho)\in\mathcal{I}(n-2)\setminus\mathcal{I}(1)}e_{I(\rho)|1}.$$

Therefore, the restricted divisor of the first sequence has vanishing cohomologies by induction on n, for all $\alpha\leq n$. Moreover, a computation similar to that preceding (3.2) shows that, for all $I=I(r)_j$, $-\mathrm{Cr}_n(H)|_{E_I}=0\boxtimes *$. For every pair (r,j) the restricted divisor is of the form (3.2) and has therefore vanishing cohomologies, by induction on n. This concludes the proof. □

4. Vanishing theorems for effective only linearly obstructed divisors

Let $\mathcal{L}=\mathcal{L}_{n,d}(m_1,\ldots,m_s)$ be a non-empty linear system in $\mathbb{P}^n$. Elements of $\mathcal{L}$ are in bijection with divisors on $X^n_{(0)}$ of the form (1.1),

$$D:=dH-\sum_{i=1}^s m_iE_i.$$

Assume that the following bound is satisfied:

$$\sum_{i=1}^s m_i-nd\leq\min(n-s(d),s-n-2),\quad 1\leq m_i\leq d. \tag{4.1}$$

In [6, Section 5] the dimensions of all linear systems $\mathcal{L}$ in $\mathbb{P}^n$ (equivalently the number of global sections of the line bundles associated to the divisors D) that satisfy the bound (4.1) are given, see also Theorem 1.3. In this section we compute the dimension of all cohomology groups.

Theorem 4.1. *Statements (a) and (b) of Theorem 1.4 hold for divisors satisfying condition (4.1).*

This result implies that such linear systems are only linearly obstructed. Moreover this (partially) answers Question 1.1 for divisors satisfying (4.1).

In Section 4.1 we prove a base locus lemma, Proposition 4.2, that computes the exact multiplicity of containment of a linear cycle. In Section 4.2 we compute the dimension of the cohomology groups of the strict transforms of D after subsequently blowing-up linear cycles of the base locus in increasing dimension.

4.1. **Linear base locus lemma.** For all $I(r)=\{i_1,\ldots,i_{r+1}\}\subseteq\{1,\ldots,s\}$ with $0\le r\le \min(n,s)-1$, we introduce the integers (cfr. Definition (1.2)):

$$K_{I(r)}=K_{i_1,\ldots,i_{r+1}}:=\sum_{i\in I(r)} m_i - rd,$$
$$k_{I(r)}=k_{i_1,\ldots,i_{r+1}}:=\max(K_{I(r)},0),$$

and

$$\bar{r}=\bar{r}(\mathcal{L}):=\max(\rho|K_{I(\rho)}\ge 0).$$

Moreover, set

$$\mathcal{I}=\{I(r)\in\{1,\ldots,s\}:0\le r\le n-1, K_{I(r)}\ge 0\}. \tag{4.2}$$

Proposition 4.2. *Let $\mathcal{L}=\mathcal{L}_{n,d}(m_1,...,m_s)$ be a non-empty linear system. Let $I(r)\in\mathcal{I}$. The linear cycle $L_{I(r)}$ spanned by the points parametrized by $I(r)$ is contained with multiplicity $k_{I(r)}$ in the base locus of $\mathcal{L}$.*

Proof. Let $I(r)$ be a multi-index such that $k_{I(r)}=K_{I(r)}\ge 0$ and $\tilde{k}_{I(r)}\ge 0$ be the multiplicity with which $L_{I(r)}$ is contained in the base locus of $\mathcal{L}$. By Bézout's theorem one has $\tilde{k}_{I(r)}\ge k_{I(r)}$, see [6, Lemma 2.1] for details. We introduce the following notation:

$$R=R(\mathcal{L},I(r)):=\max(\rho|K_{I(\rho)}\ge 0, I(r)\subseteq I(\rho)).$$

If $r=n-1(=R)$ then the claim is true by [7, Lemma 4.4]. Next, for $r\le n-2$ we consider separately the following cases:

(i) $r=R=\bar{r}$,
(ii) $r=R<\bar{r}$,
(iii) $r<R$.

Case (i). We prove the statement by backward induction on R. Precisely, given $R\le n-2$, we assume that for every non-empty linear system $\mathcal{M}$ in $\mathbb{P}^n$ such that $R(\mathcal{M},I(r))=R+1$ and for every multi-index $I(R+1)$ with $K_{I(R+1)}\ge 0$, the cycle $L_{I(R+1)}$ is contained in the base locus of $\mathcal{M}$ with multiplicity $K_{I(R+1)}$, and we prove the statement for $\mathcal{L}$ and $I(r)$ with $R(\mathcal{L},I(r))=R$. Let $I(R)=\{i_1,\ldots,i_{R+1}\}$ be a multi-index with $K_{I(R)}\ge 0$, and consider the inclusions $I(R)\subset J=\{i_1,\ldots,i_{R+1},i_{R+2}\}$ and $J\subset\bar{J}=\{i_1,\ldots,i_{R+1},i_{R+2},\ldots,i_n\}$. Because $K_J\le 0$, then by induction the cycle L_J is not in the base locus of D. Let $H_{\bar{J}}:=H-\sum_{i\in\bar{J}}E_i$ denote the strict transform of the hyperplane spanned by the points indexed by elements of $\bar{J}$. We introduce the following divisor:

$$D':=D+(-K_J)H_{\bar{J}}.$$

D' is an effective divisor on $X^n_{(0)}$ and we will denote by m'_i and d' its coefficients: $d' := d - K_J$, $m'_i := m_i - K_J$ for $i \in \bar{J}$ and $m'_i := m_i$ for $i \notin \bar{J}$. By construction,

$$K'_J := \sum_{i \in J} m'_i - (R+1)d' = 0.$$

Moreover, $K'_{I(R)} = K_{I(R)} + (-K_J) \geq K_{I(R)} \geq 0$, so that $L_{I(R)}$ is contained in the base locus of D' with multiplicity at least $K'_{I(R)}$.

Assume that $\tilde{k}_{I(R)} \geq K_{I(R)} + 1$. The multiplicity of $L_{I(R)}$ in $\mathrm{Bs}(D')$ is

$$\tilde{k}'_{I(R)} \geq \tilde{k}_{I(R)} + (-K_J) \geq K_{(I(R))} + 1 + (-K_J) = K'_{I(R)} + 1.$$

For any point $p \in L_{I(R)}$ we compute the multiplicity $\tilde{k}'_{p,p_{R+2}}$ of the line spanned by the points p and $p_{i_{R+2}}$:

$$\tilde{k}'_{p,p_{R+2}} \geq \tilde{k}'_{I(R)} + m'_{R+2} - d' \geq K'_{I(R)} + 1 + m'_{R+2} - d' = K'_{I(R+1)} + 1 = 1.$$

This shows that the line spanned by p and $p_{i_{R+2}}$ is in the base locus of D' for any $p \in L_{I(R)}$. Letting p vary in $L_{I(R)}$, we obtain that L_J is in the base locus of D'. This gives a contradiction.

Case (ii). We know by the previous case that for all cycles $L_{I(\bar{r})}$, the multiplicity of containment is given by $k_{I(\bar{r})}$. For smaller cycles, $I(R)$, with $1 \leq r = R < \bar{r}$ and $K_{I(R)} \geq 0$ we run induction on $\bar{r} - R$ and the same argument from Case (ii) applies.

Case (iii). We assume that the statement holds for $I(R)$, with $I(r) \subseteq I(R)$ and such that $K_{I(R)} \geq 0$ and we show it for $I(r)$. Namely, assuming $\tilde{k}_{I(R)} = k_{I(R)}$, for all $I(R)$, we prove that $\tilde{k}_{I(r)} = k_{I(r)}$. Notice that $0 \leq K_{I(R)} \leq K_{I(r)}$, since $m_i \leq d$. Therefore, all linear subspaces $L_{I(\rho)}$ of $L_{I(R)}$ are contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity at least $K_{I(\rho)} \geq 0$, by [6, Lemma 2.1]. In particular, $L_{I(r)}$ is contained at least $K_{I(r)}$ times.

Assume now by contradiction that $L_{I(r)}$ is contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity at least $1 + K_{I(r)}$. We know that the linear cycle $L_{I(R)\setminus I(r)}$ is contained in the base locus with multiplicity at least $K_{I(R)\setminus I(r)} \geq 0$. For any point p in the cycle $L_{I(r)}$ and p' in $L_{I(R)\setminus I(r)}$ we obtain that the line spanned by p and p' is contained in the base locus with multiplicity at least $1 + K_{I(r)} + K_{I(R)\setminus I(r)} - d = 1 + K_{I(R)}$. This gives a contradiction. □

Remark 4.3. We must mention that the first part of the proof of Proposition 4.2, Case (i), was established in [6, Proposition 2.5], but we decided to include it here also for the sake of completeness.

4.2. **Vanishing theorems.** In this section we prove Theorem 4.1. The proof will be based on induction on $n \geq 1$ and s. The case $n = 1$ is trivial, the case, $s \leq n+2$ is solved in [6, Ch. 4].

Let us first recall the standard notations and definitions. Let $\mathcal{I}$ be as in Section 4.1 and let $\mathcal{I}(r) \subset \mathcal{I}$, for all $1 \leq r \leq n-1$, be the set of multi-indices of $\mathcal{I}$ of length at most $r+1$, as in Section 2.

Lemma 4.4. *In the above notation, the set $\mathcal{I}$ satisfies conditions* (I), (II) *and* (III) *of Section 2.*

Proof. Condition (I) follows by the definition.

Since $m_i \leq d$, for all i, then $K_I \geq K_J$ for $I \subset J$. Hence (II) is satisfied.

If $|I \cup J| \leq n+1$ then (III) follows easily, because one may assume that L_I and L_J are coordinate subspaces. Assume that $I \cap J = \emptyset$ and $|I \cup J| = n+2$. The inequalities (4.1) and $m_i \geq 1$ for all i, imply that $K_I + K_J = \sum_{i \in I \cup J} m_i - nd \leq 0$, hence at most one among I and J is in $\mathcal{I}$. This proves (III) in this case. If $I \cap J \neq \emptyset$ and $|I \cup J| = n+2$, we conclude by just noticing that $K_I + K_J \leq K_I + K_{J \setminus I} \leq 0$. We leave it to the reader to verify that condition (III) holds also for $|I \cup J| \geq n+3$. □

In the notation of the previous sections, let $X^n_{(r)}$ be the blow-up of $X^n_{(r-1)}$ along the union of the strict transforms of the r-cycles $L_{I(r)}$, $I(r) \in \mathcal{I}$. The total transform of $D_{(r-1)} \subset X^n_{(r-1)}$ is

$$(\pi_{(r)})^* D_{(r-1)} = dH - \sum_{I(\rho) \in \mathcal{I}(r-1)} k_{I(\rho)} E_{I(\rho)}, \tag{4.3}$$

while the strict transform of $D_{(r-1)}$ is

$$\begin{aligned} D_{(r)} &= dH - \sum_{I(\rho) \in \mathcal{I}(r-1)} k_{I(\rho)} E_{I(\rho)} - \sum_{I(r) \in \mathcal{I}} k_{I(r)} E_{I(r)} \\ &= dH - \sum_{I(\rho) \in \mathcal{I}(r)} k_{I(\rho)} E_{I(\rho)}. \end{aligned} \tag{4.4}$$

4.2.1. *Induction on the sum of the multiplicities.* Let D be as in (1.1). Modulo reordering the indices $\{1, \ldots, s\}$ if necessary, we may assume $m_1 \geq 1$. We introduce the following divisors on $X^n_{(0)}$

$$D' := D + E_1 = dH - (m_1 - 1)E_1 - \sum_{i=2}^{s} m_i E_i.$$

It corresponds to the linear system of hypersurfaces of $\mathbb{P}^n$ denoted by $\mathcal{L}' := \mathcal{L}_{n,d}(m_1 - 1, m_2, \ldots, m_s)$.

The following provides the induction step on the integer $b = \sum_{i=1}^{s} m_i - nd$, that was defined in (1.7).

Lemma 4.5. *If D satisfies (4.1), then also D' does.*

Proof. It is a trivial computation that $b(D') = b(D) - 1$. □

For all $I \in \mathcal{I}(r)_1$ (see definition (2.1)), set $k'_I := k_I(D') = \max(K_I - 1, 0)$. We have

$$D'_{(r)} = dH - (m_1 - 1)E_1 - \sum_{i=2}^{s} m_i E_i - \sum_{I \in \mathcal{I}(r)_1} k'_I E_I - \sum_{I \in \mathcal{I}(r) \setminus \mathcal{I}(r)_1} k_I E_I. \tag{4.5}$$

Notice that the linear base locus of D' is contained in that of D. Precisely, the cycles L_I with $I \notin \mathcal{I}(n-1)_1$ have the same multiplicity of containment in both base loci, while the cycles L_I with $I \in \mathcal{I}(n-1)_1$ are contained with multiplicity one more in D, see Proposition 4.2. Using the notation

$$\mathcal{I}(r)_1^{>} := \{I \in \mathcal{I}(r)_1 : K_I > 0\},$$

we can write

$$D'_{(r)} = D_{(r)} + \sum_{I \in \mathcal{I}(r)_1^{>}} E_I.$$

4.2.2. *Induction on n.* In order to employ induction on n we want to consider restricted divisors on the strict transforms of the exceptional divisor E_1 that also satisfy condition (4.1).

In the space $X^n_{(r)}$, $1 \leq r \leq n-1$, we will use the following exact sequence of sheaves, that we will refer to as *sequence of type* (A).

$$0 \longrightarrow D'_{(r)} - E_1 \longrightarrow D'_{(r)} \longrightarrow D'_{(r)}|_{E_1} \longrightarrow 0, \tag{A}$$

of which now we give a detailed description. By abuse of notation, write E_1 for the strict transform $\pi^*_{(r,0)}E_1$ in $X^n_{(1)}$ of the exceptional divisor in $X^n_{(0)}$ of the point p_1. Recall from Section 2.1 that $E_1 \cong X^{n-1}_{(r-1)}$ has Picard group generated by the hyperplane class h and by the exceptional classes $e_{I(\rho)|1} = E_{I(\rho)}|_{E_1}$, for $I(\rho) \in \mathcal{I}(r)$. The divisor $D'_{(r)}$ restricts to E_1 as

$$D'_{(r)}|_{E_1} = (m_1 - 1)h - \sum_{I \in \mathcal{I}(r)_1} k'_I e_{I|1}$$

where $k'_I := \max(K_I - 1, 0)$.

The following provides the induction step on n; in fact $D'_{(1)}|_{E_1}$ is a divisor of the form (1.1) in a blown-up $\mathbb{P}^{n-1}$ in points in general position.

Lemma 4.6. *If D satisfies* (4.1)*, then also $D'_{(1)}|_{E_1}$ does.*

Even though the same argument appeared in the proof of [6, Lemma 5.7], we include it here for the sake of completeness.

Proof. Let us assume, without loss of generality, that $d \geq m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$. Set $\bar{s} := \#\{I \in \mathcal{I}(1)_1 : k'_I > 0\}$ and consider the divisor

$$D'_{(1)}|_{E_1} = (m_1 - 1)h - \sum_{i=2}^{\bar{s}} k'_{1i} e_{1i|1}$$

in $X^{n-1}_{(0)}$. Notice that $k'_{1i} = m_1 + m_i - d - 1 \leq m_1 - 1$, as $m_i \leq d$, for all $i = 2, \ldots, \bar{s}$. If $\bar{s} \leq n-1$ the first inequality of (4.1) is trivially satisfied by $D'_{(1)}|_{E_1}$. When $\bar{s} \geq n$, we conclude by computing

$$\begin{aligned}
\sum_{i=2}^{\bar{s}} k'_{1i} &= \sum_{i=2}^{\bar{s}} m_i + \bar{s}(m_1 - d - 1) \\
&\leq \sum_{i=1}^{s} m_i - m_1 - (s - \bar{s} - 1) + \bar{s}(m_1 - d - 1) \\
&\leq nd + s - n - m_1 - (s-1) + \bar{s}(m_1 - d) \\
&\leq (n-1)(m_1 - 1).
\end{aligned}$$

□

4.2.3. *Global sections in the exact sequence of type* (A). Let us denote by

$$l(D, r) := \sum_{I(\rho) \in \mathcal{I} \setminus \mathcal{I}(r-1)} (-1)^{\rho - r - 1} \binom{n + k_{I(\rho)} - \rho - 1}{n}, \tag{4.6}$$

the number that appears in Theorem 1.4 (b). Notice that $l(D, 0) = \mathrm{ldim}(\mathcal{L})$, see Definition 1.2.

Lemma 4.7. *In the above notation, the following equality holds:*

$$l(D,r) = l(D',r) - l(D'_{(1)}|_{E_1}, r-1).$$

Proof. It follows from the equality of Newton binomials $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$, for $a, b \geq 1$. □

Proposition 4.8. *Assume that conditions (4.1) are satisfied and fix $1 \leq r \leq n-1$. If $h^1(D'_{(r)}) = 0$, then the following sequence of global sections is exact:*

$$0 \longrightarrow H^0(D'_{(r)} - E_1) \longrightarrow H^0(D'_{(r)}) \longrightarrow H^0(D'_{(r)}|_{E_1}) \longrightarrow 0.$$

Proof. It is enough to show the statement for $r = 1$, because the number of global sections of $D_{(1)}$ and $D'_{(1)}$ (and of $D_{(1)}|_{E_1}$) are preserved when taking strict transforms in the spaces $X^n_{(r)}$ ($E_1 \cong X^{n-1}_{(r-1)}$ respectively).

Notice that the kernel divisor satisfies $h^0(D'_{(1)} - E_1) = h^0(D' - E_1)$. Since $D' - E_1 = D$ by definition, we conclude that $h^0(D'_{(1)} - E_1) = h^0(D)$.

Moreover, notice that $D_{(1)}|_{E_1}$ is a divisor on $E_1 \cong X^{n-1}_{(0)}$, the $(n-1)$-dimensional projective space blown-up at points.

One can verify that $\operatorname{ldim}(D) = \operatorname{ldim}(D') - \operatorname{ldim}(D'_{(1)}|_{E_1})$, using Lemma 4.7. Furthermore, by Theorem 1.3 all three divisors are only linearly obstructed, as they satisfy (4.1) by Lemma 4.5 and Lemma 4.6. Hence $h^0(D) = h^0(D') - h^0(D'_{(1)}|_{E_1})$. □

4.2.4. *The cohomologies of the kernel of the sequences of type (A).* We now compute all cohomology groups of the divisors in the sequences of type (A), and in particular we obtain that all cohomology groups but the $(r+1)$st vanish.

Proposition 4.9. *Assume that conditions (4.1) are satisfied and fix $1 \leq r \leq n-1$. Assume that statement (b) of Theorem 1.4 holds for D' and $D'_{(1)}|_{E_1}$. Then statement (b) of Theorem 1.4 holds for $D'_{(r)} - E_1$.*

Proof. The proof is by induction on b and n. If $n = 1$ the statement is trivially true, as well as if $b = -nd$, i.e. if $m_i = 0$ for all $i = 1, \ldots, s$.

For the pair (n, b), with $n \geq 2$, and $m_1 \geq 1$, we will assume the statement to be true for $(n-1, b)$ and $(n, b-1)$.

For r such that $1 \leq r \leq \min(\bar{r}, n-1)$, in the space $X^n_{(r)}$ we consider the sequence of tyoe (A). By induction we have

- $h^i(D'_{(r)}|_{E_1}) = 0$, for all $i \neq 0, r$ and $h^r(D'_{(r)}|_{E_1}) = l(D'_{(r)}|_{E_1}, r-1)$;
- $h^{i+1}(D'_{(r)}) = 0$, for all $i \neq 0, r$ and $h^{r+1}(D'_{(r)}) = l(D', r)$.

Therefore, the long exact sequence in cohomology associated with the sequence of type (A) splits into the fundamental sequences

$$0 \to H^0(D'_{(r)} - E_1) \to H^0(D'_{(r)}) \to H^0(D'_{(r)}|_{E_1}) \to H^1(D'_{(r)} - E_1) \to 0, \tag{4.7}$$

and

$$0 \to H^r(D'_{(r)}|_{E_1}) \to H^{r+1}(D'_{(r)} - E_1) \to H^{r+1}(D'_{(r)}) \to 0. \tag{4.8}$$

Using Proposition 4.8, we obtain $h^1(D'_{(r)} - E_1) = 0$ from sequence (4.7). Moreover, using (4.8) we obtain

$$h^{r+1}(D'_{(r)} - E_1) = h^r(D'_{(r)}|_{E_1}) + h^{r+1}(D'_{(r)}) = l(D, r),$$

by Lemma 4.7. In particular, $h^i(D'_{(r)}-E_1)=0$, for all $i\neq 0, r+1$. □

4.2.5. *The cohomology of $D_{(r)}$.* We now deduce, from the above results, the cohomologies of $D_{(r)}$ for all $r\leq n-1$.

Proposition 4.10. *Assume D is effective. For all $0\leq r\leq n-1$ and $i\geq 0$ we have*

$$h^i(D'_{(r)}-E_1)=h^i(D_{(r)}).$$

Proof. If $r=0$ the statement is obvious.

Fix $1\leq\rho\leq r\leq n-1$. For every ρ, let $\{I(\rho)_0,\ldots,I(\rho)_{s_\rho}\}$ be the set of multi-indices of length $\rho+1$ that contain 1. We use the notation $\mathcal{I}(\rho)_1^{>}=\{I\in\mathcal{I}(\rho)_1: K_I>0\}$ of Section 4.2.1.

Let $\prec$ be the total order on $\mathcal{I}(\rho)_1^{>}$ inherited by the lexicographical order on $\mathcal{I}(r)_1$, that was introduced in the proof of Theorem 3.1. Precisely $I(\rho')_{j'}\prec I(\rho)_j$ if and only if $\rho'<\rho$ or $\rho'=\rho$ and $j'<j$.

In the space $X^n_{(r)}$, for every pair (ρ,j), $1\leq\rho\leq r$ and $0\leq j\leq s_\rho$, for which $I(\rho)_j\in\mathcal{I}(\rho)_1^{>}$, we consider the divisor

$$F(\rho,j)=D'_{(r)}-E_1-\sum_{\substack{I(\rho')_{j'}\prec I(\rho)_j\\ I(\rho')_{j'}\in\mathcal{I}(\rho)_1^{>}}}E_{I(\rho')_{j'}}-E_{I(\rho)_j}, \tag{4.9}$$

that we recursively define, using the same idea as in the proof of Theorem 3.1, by the following rule:

$$\begin{aligned}
F(0,0)&=D'_{(r)}-E_1,\\
F(\rho,0)&=F(\rho-1,s_{\rho-1})-E_{I(\rho)_0},\ 0\leq\rho\leq r,\\
F(\rho,j)&=F(\rho,j-1)-E_{I(\rho)_j},\ 1\leq j\leq s_\rho.
\end{aligned}$$

Notice that the last divisor obtained is $F(r,s_r)=D_{(r)}$.

We claim that, for all pairs (ρ,j) such that $I(\rho)_j\in\mathcal{I}(\rho)_1^{>}$, the first factor of the restriction of $F(\rho,j-1)$ to $E_{I(\rho)_j}$ is $-\mathrm{Cr}_\rho(h)$ (cfr. (2.4) for a definition of the divisor class $\mathrm{Cr}_\rho(h)$). The latter has vanishing cohomologies by Theorem 3.1. Hence, by the Kunneth formula, we conclude that the restriction itself has vanishing cohomologies.

In order to prove the claim, recall that the first factor of $E_{I(\rho)_j}$ is isomorphic to $X^\rho_{(\rho-1)}$. Let us expand the expression of $F(\rho,j-1)$:

$$\begin{aligned}
F(\rho,j-1)=dH-\sum_{I(t)\in\mathcal{I}(\rho-1)}k_{I(t)}E_{I(t)}-\sum_{1\leq l\leq j-1}k_{I(\rho)_l}E_{I(\rho)_l}-(k_{I(\rho)_j}-1)E_{I(\rho)_j}\\
-\sum_{j<l\leq s_\rho}(k_{I(\rho)_l}-1)E_{I(\rho)_l}-\sum_{I(t)\in\mathcal{I}\setminus\mathcal{I}(\rho)}(k_{I(t)}-1)E_{I(t)}.
\end{aligned}$$

Using the intersection table (2.3), we see that the second line above restricts to 0 on the second factor of $E_{I(\rho)_j}$. Therefore, the restriction of $F(\rho,j-1)+(k_{I(\rho)_j}-1)E_{I(\rho)_j}$ to $E_{I(\rho)_j}$ is the following divisor

$$*\boxtimes\left(dh-\sum_{\substack{I(t)\subset I(\rho)_j\\ I(t)\in\mathcal{I}(\rho-1)}}k_{I(t)}e_{I(t)}\right)=*\boxtimes\left(-k_{I(\rho)_j}\mathrm{Cr}_\rho(h)\right),$$

where $*$ denotes the appropriate divisor. The equality in the above expression is a trivial computation of which we leave the details to the reader. Finally, the self-intersection $E_{I(\rho)_j}|_{E(\rho)_j}$ is $-\mathrm{Cr}_\rho(h)$ on the first factor, see Lemma 2.6. This concludes the proof. □

Proof of Theorem 4.1. We first prove part (2); part (1) follows. The proof is by induction on n and b. If $n = 1$ the statement is trivially true, as well as if $b = -nd$, i.e. if $m_i = 0$ for all $i = 1, \ldots, s$. Fix the pair (n, b) and assume by induction that the statement is true for all divisor in the $(n-1)$-dimensional space satisfying the bound (4.1), and for all divisors with the values $(n, b-1)$. In particular, the statement is true for the divisors $D' = D + E_1$ and $D'_{(1)}|_{E_1}$. For all $1 \leq r \leq n-1$, the conclusion follows from Propositions 4.8, 4.9 and 4.10. □

5. Vanishing theorems for non-effective toric divisors

If $s \leq n+1$, the blow-up $X_{(0)}$ of $\mathbb{P}^n$ at s points in general position, that we can think of as coordinate points of $\mathbb{P}^n$, is a toric variety. We will call *toric divisor* a divisor D on $X_{(0)}$ of the form (1.1), with $s \leq n+1$:

$$D = dH - \sum_{i=1}^{n+1} m_i E_i.$$

In this section we study the cohomology of the strict transforms in $X_{(r)}$ of non-effective toric divisors with $m_i \leq d+1$ and $d, m_i \geq 0$.

Recall the notation (4.2), $\mathcal{I} = \{I(r) : K_{I(r)} \geq 0, r \leq n-1\}$. Set $\bar{r}$ to be the maximal dimension of the linear cycles $I(r)$ with positive $K_{I(r)}$.

The strict transform after blowing-up the whole linear base locus is denoted by $\tilde{D} = D_{(\bar{r})}$. For a non-effective divisor D we will call *strict transform* the divisor $D_{(r)}$ (and $\tilde{D} = D_{(\bar{r})}$) defined as the formal sum of divisors given by (4.4). Moreover, for a non-effective divisor whenever $k_{I(r)} \geq 1$, by abuse of notation, we will say that the cycle $L_{I(r)}$ is in the *base locus* of D.

We will prove the following formula for the dimension of the $(r+1)$st cohomology group of $D_{(r)}$ in terms of the speciality of $\tilde{D}$. Recall from (4.6) that $l(D, r+1)$ denotes the alternating sum of the contributions to the speciality of D given by the multiple base cycles of dimension at least $r+1$.

Theorem 5.1. *Let D in $X^n_{(0)}$ be an effective divisor interpolating an arbitrary number of points or a non-effective divisor with $m_i \leq d+1$ and $\mathcal{I}$ satisfying conditions* (I) *and* (III) *from Section 2. Then the following holds.*

(1) For any $0 \leq r \leq n-1$, we have

$$h^{r+1}(D_{(r)}) = l(D, r+1) + \sum_{\rho=r+1}^{n} (-1)^{\rho-r-1} h^\rho(\tilde{D}).$$

Moreover, if $h^i(\tilde{D}) = 0$, for all $i \geq 1$, then $h^i(D_{(r)}) = 0$ for all $i \neq r+1$.

(2) For any integer $l_{I(r)}$ with $0 \leq l_{I(r)} \leq \min(r, k_{I(r)})$, we have

$$h^i(D_{(r)}) = h^i(D_{(r)} + \sum_{I(r)} l_{I(r)} E_{I(r)}), \quad \text{for } i \geq 0.$$

Observe first that for $r=-1$ the binomial sum $l(D,0)$ defined by (4.6) becomes $\mathrm{ldim}(D)$, the linear virtual dimension of D introduced in Definition 1.2. Moreover, setting $D_{(-1)}:=D_{(0)}=D$ for $r=-1$ the theorem reads

Corollary 5.2. *For any D in $X^n_{(0)}$ effective divisor or non-effective divisor with $m_i\leq d+1$ and $\mathcal{I}$ satisfying conditions* (I) *and* (III) *from Section 2, then*

$$h^0(D)=\mathrm{ldim}(D)+\sum_{\rho=1}^{n}(-1)^\rho h^\rho(\tilde{D}).$$

Let us recall the definition (1.7):

$$b:=b(D)=\sum_{i=1}^{s}m_i-nd.$$

Theorem 5.3. *If D is any toric divisor with $m_i\leq d+1$ then the following statements hold.*

(1) $h^0(D)=\mathrm{ldim}(D)$, $h^n(\tilde{D})=\binom{b-1}{n}$ and $h^i(\tilde{D})=0$ for every $1\leq i\leq n-1$.
(2) Statement (*b*) *of Theorem 1.4 holds for every $1\leq r\leq n-1$.*

We will prove Theorems 5.1 and 5.3 in Section 5.4. The proof will be based on induction and the procedure is worked out in Sections 5.2 and 5.3.

Remark 5.4. If D is an effective toric divisor (then $m_i\leq d$ and $b(D)\leq 0$), it is $h^n(\tilde{D})=0$. In this case it is obvious that Theorem 5.3 (1) is just a particular case of [6, Theorem 4.6]. However, for the all non-effective toric divisors with $m_i\leq d+1$ this result is new. Theorem 5.3 (1) suggests that the *virtual* n-dimensional cycle should be considered in the dimension count. More precisely, this virtual cycle is detected by the $n-$th cohomology group, its contribution depends on its virtual multiplicity $k_{1,\dots,n+1}$ that becomes nothing else than $b(D)$. Moreover, for non-effective toric divisors with $m_1=d+1$ or $b(D)\leq 0$, Theorem 5.3 implies that $\mathrm{ldim}(\mathcal{L})=0$ where r runs from -1 and reaches n (see also Corollary 6.4).

5.1. **Non-effective chambers.** We recall that the effective cone $\mathrm{Eff}(X)$ of $X_{(0)}$ for $s\leq n+1$ is given by the inequalities

$$d\geq 0,\quad b\leq 0,\quad m_i\leq d,\quad \forall i\in\{1,\dots,s\}. \tag{5.1}$$

See e.g. [6, Lemma 2.2]. Moreover, for any effective toric divisor, the strict transform $\tilde{D}$ has vanishing cohomologies, see Theorem 1.4.

The following inequalities define chambers of $\mathrm{N}^1(X)\setminus\mathrm{Eff}(X)$ for which $\tilde{D}$ has vanishing theorems, by Theorem 1.6. :

$$s\leq n+1:\quad 1\leq b\leq n,\quad m_j\leq d+1; \tag{5.2a}$$

$$s\leq n+1:\quad b\leq 0,\quad m_1=d+1,\ m_j\leq d+1. \tag{5.2b}$$

Notice that under the assumptions (5.2) we have $K_{I(n-1)}\leq n-1$, for all $I(n-1)\in\mathcal{I}$. Therefore, all contributions of hyperplanes parametrized by multi-indices $I(n-1)$ that are contained with multiplicity $k_{I(n-1)}$ are zero in formula (1.3). Therefore, also see Corollary 6.4, one has

$$\chi(\tilde{D})=\mathrm{ldim}(D)\geq 0.$$

However, if we allow $b = b(D) > n$, then the above equality is no more satisfied. More precisely, (cfr. Theorem 5.3 (1)), for such divisors we have $\operatorname{ldim}(D) = 0$, while

$$\chi(\tilde{D}) = (-1)^n \binom{b-1}{n},$$

with $h^i(\tilde{D}) = 0$ for all $i < n$ and $h^n(\tilde{D}) = \binom{b-1}{n}$. We remark that $\chi(\tilde{D}) \neq \operatorname{ldim}(D)$ when $b \geq n+1$.

If one moves further away from the effective cone, by choosing for instance $m_1 = d+2$ or higher, then the strict transform after blowing-up all linear subspaces contained in the base locus may still be linearly obstructed. We illustrate some instances where this happens.

Example 5.5. *Let $D = 3H - 5E_1 - mE_2$ in $X^3_{(0)}$ be a divisor with $h^0(D) = 0$.*

- *If $m = 3$, we have $\chi(D_{(1)}) = -5$. Hence $h^1(D_{(1)}) \geq 5$.*
- *If $m = 4$, we have $\chi(D_{(1)}) = 0$. Is $h^1(D_{(1)}) = h^2(D_{(1)}) = 0$?*
- *If $m = 5$, we have $\chi(D_{(1)}) = 6$. Hence $h^2(D_{(1)}) \geq 6$.*

5.2. **Case $r = n-1$: the strict transform $D_{(n-1)}$.** In this section we give a description of the strict transforms $\tilde{D} = D_{(n-1)}$ of toric divisors D containing hyperplanes H_I with positive k_I. In this case $\tilde{D}$ is obtained from $D_{(n-2)}$ by contraction of the strict transforms of such H_I's.

Proposition 5.6. *Let D be a divisor with $s = n+1$, $d \geq 1$, $0 \leq m_i \leq d+1$.*

(1) *If $0 \leq m_i \leq d$ and $b = 0$, then $D_{(n-1)} = 0$.*
(2) *If either $b \geq n-1$ or $m_i \leq d$ for all i and $b \geq 1$, then $D_{(n-1)} = -b\mathrm{Cr}_n(H)$.*

Proof. In the first case, if $s = n$, then $m_i = d$, for all i. The strict transform is obviously $D_{(n-1)} = 0$, because it is obtained by contracting the strict transform of the hyperplane through the n points.

If $s = n+1$ with $b = 0$ and $m_j \leq d$, for all j, then all hyperplanes spanned by sets of n points are contained in the base locus of D with (exact) multiplicity

$$k_{1\ldots,\check{j},\ldots,n+1} = \sum_{i=1,\ i\neq j}^{n+1} m_i - (n-1)d = d - m_j \geq 0.$$

If $s = n+1$ and $m_j \leq d+1$, for all j, the assumption $b \geq 1$ implies that all hyperplanes spanned by sets of n points are contained in the base locus of D with (exact) multiplicity

$$k_{1\ldots,\check{j},\ldots,n+1} = \sum_{i=1,\ \neq j}^{n+1} m_i - (n-1)d = b + d - m_j \geq 0.$$

Moreover, if $m_i \leq d$ and $k_{I(r)} \geq 0$, then for any subset $I(\rho) \subset I(r)$ we have $k_{I(\rho)} \geq 0$. The same holds for $m_i \leq d+1$ and $b \geq n-1$. Therefore, in both above cases, the strict transform $D_{(n-1)}$ of D is

$$dH - \sum_{I(\rho)\in\mathcal{I}(n-2)} k_{I(\rho)} E_{I(\rho)} - \sum_{i=1}^{n+1} k_{I(n-1)} E_{I(n-1)},$$

where $E_{I(n-1)}$ is the strict transform in $X^n_{(n-1)}$ of the hyperplane of $\mathbb{P}^n$ passing through the n points parametrized by $I(n-1)$, see (2.2). One can verify that the coefficients of the hyperplane class H and the coefficient of the exceptional divisors $E_{I(\rho)}$ in the expression for $D_{(n-1)}$ are respectively

$$d-\sum_i(b-m_i+d)=-nb,$$
$$k_{I(\rho)}-\sum_{i\notin I(\rho)}(b-m_i+d)=-(n-\rho-1)b.$$

This concludes the proof. □

Corollary 5.7. *Assume that D satisfies the same assumptions of Proposition 5.6. Then statement (1) and statement (2) with $r=n-1$ of Theorem 5.3 hold.*

Proof. It follows from Theorem 3.1. □

Remark 5.8. Proposition 5.6 provides another proof that the effective cone of $X^n_{(0)}$, the space blow-up at $s\leq n+1$ points, is described by the inequalities (5.1).

5.3. **Case $r\leq n-2$: how to use induction.** In this section we will build the induction procedure, on n and b, that we will apply in order to prove the main results, Theorem 5.1 and Theorem 5.3

Remark 5.9. Since $m_i\leq d+1$, the divisor $D'_{(1)}|_{E_1}$ in $E_1\cong X^{n-1}_{(0)}$ (introduced in (4.5)) has $k'_{1i}:=\max(K_{1i}-1,0)\leq m_1$, for all $2\leq j\leq s$, namely the multiplicities of the points $e_{1i|1}$ do not exceed the multiplicity m_1-1 by more than one.

We now prove that if D satisfies conditions (5.2), then also $D'_{(1)}|_{E_1}$ does.

Lemma 5.10. *Assume D satisfies (5.2). Then $b(D'_{(1)}|_{E_1})=b(D')=b-1$.*

Proof. Assume first that $m_1=d+1$. Notice that $k'_{1i}=m_1+m_i-d-1=m_i\geq 1$, for all $1\leq i\leq s$. Therefore, the restricted divisor $D'_{(1)}|_{E_1}$ is of the form

$$dh-\sum_{i=2}^s m_ie_{1i|1},$$

and one computes

$$b(D'|_{E_1})=\sum_{i=2}^s m_i-(n-1)d=\sum_{i=1}^s m_i-nd-m_1+d=b-1.$$

Assume that we are in the case (5.2a) and moreover $m_i\leq d$, for all $1\leq i\leq s$. Fix an index $i\geq 2$ and write

$$b-1=\sum_{j=2,\ j\neq i}^s m_j-(n-1)d+(m_1+m_i-d-1).$$

As $b\geq 1$, $k'_{1i}=m_1+m_i-d-1\geq 0$. Therefore, the divisor $D'|_{E_1}$ is of the form

$$(m_1-1)h-\sum_{i=2}^s k'_{1i}e_{1i|1},$$

with possibly some of the k'_{1i} being zero. One computes

$$b(D'|_{E_1})=\sum_{i=2}^s(m_1+m_i-d-1)-(n-1)(m_1-1)=b-1.$$

□

It is clear that the map

$$H^0(D'_{(r)}) \to H^0(D'_{(r)}|_{E_1})$$

between global sections is injective, as the kernel has no non-zero sections. We prove in the next proposition that it is in fact an isomorphism.

Proposition 5.11. *Assume D satisfies* (5.2). *Then* $h^0(D'_{(1)}) = h^0(D'_{(1)}|_{E_1})$.

Proof. Notice that $h^0(D') = h^0(D'_{(1)})$. We first consider the case (5.2a) with $m_i \leq d$. Notice that $D'_{(1)}|_{E_1}$ has base points with multiplicity bounded by the degree, that is $k'_{1i} \leq m_1 - 1$, for all $2 \leq i \leq s$. If $2 \leq b \leq n$, then, using Lemma 5.10, we get that $b(D') = b(D'_{(1)}|_{E_1}) \geq 1$, therefore both divisors D' and $D'_{(1)}|_{E_1}$ have no non-zero global sections. Indeed, they both lie outside the effective cones of the respective spaces, described in (5.1). If $b = 1$, then, using Lemma 5.10, we get that $b(D') = b(D'_{(1)}|_{E_1}) = 0$, so the two divisors are effective. Using Proposition 5.6, we obtain that they both have only one non-zero global section, namely $h^0 = 1$.

We now consider the case in which $m_1 = d+1$ and one of the following conditions is satisfied: (5.2a) or (5.2b). Notice that $k'_{1i} = m_i$, for $2 \leq i \leq s$. If also $m_2 = d+1$, both D' and $D'_{(1)}|_{E_1}$ fall in the same group of conditions. Indeed, D' has a point, E_2, with multiplicity larger than the degree, that is $m_2 = d+1$ and $D'_{(1)}|_{E_1}$ has a point, $e_{12|1}$, with multiplicity larger than the degree, that is $k'_{12} = d+1$. Therefore, non of them has non-trivial global sections. If, instead, $m_i \leq d$, for all $2 \leq i \leq s$, then both D' and $D'_{(1)}|_{E_1}$ have points of multiplicity bounded by the degree. By the trivial observation that D' has a point of multiplicity d and $s-1$ points of multiplicity respectively $m_2, \ldots, m_s$ and $D'_{(1)}|_{E_1}$ has $s-1$ points of multiplicity respectively $m_2, \ldots, m_s$, we conclude that their spaces of global section have the same dimension, see for instance [6, Lemma 5.1]. □

Proposition 5.12. *Assume that conditions* (5.2) *are satisfied and fix* $1 \leq r \leq n-2$. *Assume that statements* (*a*) *and* (*b*) *of Theorem 1.4 hold for* D' *and* $D'_{(1)}|_{E_1}$. *Then the same statements hold for* $D_{(r)}$.

Using induction on n and b we obtain the following.

Corollary 5.13. *Theorem 5.3 part (2) holds if* $1 \leq r \leq n-2$.

Proof of Proposition 5.12. We prove the statement in two steps. Following the idea of the previous section, we first show that the statement holds for $D'_{(r)} - E_1$ and then that $h^i(D_{(r)}) = h^i(D'_{(r)} - E_1)$, for all $i \geq 0$.

To prove the first part, we consider the sequences in cohomology associated with the short exact sequence of type (*A*), as in the proof of Proposition 4.8. Since $h^0(D'_{(r)} - E_1) = 0$, using (4.7), Proposition 5.11 implies that $h^1(D'_{(r)} - E_1) = 0$. Moreover, we can compute the other cohomologies of $D'_{(r)} - E_1$ exploiting those of $D'_{(r)}$ and $D'_{(r)}|_{E_1}$, using (4.8).

To prove the second part, we argue as in the proof of Proposition 4.10. If $r = 0$ the statement is obvious. Fix $1 \leq \rho \leq r \leq n-2$. Recall from Section 4.2.1 the definition of $\mathcal{I}(\rho)_1^{>}$, the set of multi-indices that parametrize cycles through p_1 that are in the base locus of D, and denote

$$\mathcal{I}(\rho)_I^{\leq} := \{J \in \mathcal{I}(\rho)_1 | K_J \leq 0, J \subset I\}$$

the set of multi-indices contained in I that parametrize cycles through p_1 that are not in the base locus of D.

In the space $X^n_{(r)}$, for every pair (ρ, j), $1 \leq \rho \leq r$ and $0 \leq j \leq s_\rho$, for which $I(\rho)_j \in \mathcal{I}(\rho)^{>}_1$, we consider the divisor $F(\rho, j)$ defined in (4.9). We claim that for any pair (ρ, j) such that $I(\rho)_j \in \mathcal{I}(\rho)^{>}_1$, the divisor $F(\rho, j)$ restricts, on the first factor of $E_{I(\rho)_j}$, which is isomorphic to $X^\rho_{(\rho-1)}$, to

$$-\mathrm{Cr}_\rho(h)\left(\mathcal{I}(\rho)^{\leq}_{I(\rho)_j}\right) := -\mathrm{Cr}_\rho(h) - \sum_{I\in\mathcal{I}(\rho)^{\leq}_{I(\rho)_j}} (\rho - |I|)E_I.$$

Moreover, we claim that the latter has vanishing cohomology groups. Then, by means of the Kunneth formula, we can conclude that the restriction itself has vanishing cohomologies.

In order to prove the first claim, we argue as in the proof of Proposition 4.10. Notice that if $m_i \leq d$, for all $i = 1, \ldots, s$, then if $k_I > 0$, for some $I = I(\rho)$, then $k_J > 0$, for all $J \subset I$. In particular, $\mathcal{I}(\rho)^{\leq}_I = \emptyset$; in this case $\mathrm{Cr}_\rho(h)\left(\mathcal{I}(\rho)^{\leq}_{I(\rho)_j}\right) = \mathrm{Cr}_\rho(h)$. If, instead, some of the m_i equals $d+1$, then it is possible that $k_I > 0$, for $I = I(\rho)$, but $K_J \leq 0$, for some $J \subset I$. In the full expansion of the first factor of the restriction $F(\rho, j-1)|_{E_{I(\rho)_j}}$ computed in the proof of Proposition 4.10, notice that if $K_{I(t)} < 0$, then $k_{I(t)} = 0$ by definition so that the term $E_{I(t)}$ does not appear. Only the terms corresponding to index sets $I(t) \in \mathcal{I}(\rho)^{>}_1$ appear. Moreover, if $k_{I(\rho)} \geq 1$, then $k_{I(\rho)_j\setminus\{l\}} \geq 0$, being $m_l \leq d+1$. Therefore, one can compute that $\left(F(\rho, j-1) + (k_{I(\rho)_j} - 1)E_{I(\rho)_j}\right)|_{E_{I(\rho)_j}}$ restricts to $-k_{I(\rho)_j}\mathrm{Cr}_\rho(h)\left(\mathcal{I}(\rho)^{\leq}_{I(\rho)_j}\right)$ on the first factor of $E_{I(\rho)_j}$. Moreover, the self-intersection $E_{I(\rho)_j}|_{E(\rho)_j}$ is $-\mathrm{Cr}_\rho(h)\left(\mathcal{I}(\rho)^{\leq}_{I(\rho)_j}\right)$ and this concludes the proof of the first claim.

The second claim simply follows from Serre duality in the ρ-dimensional projective space subsequently blown-up along the cycles parametrized by $\mathcal{I} \setminus \mathcal{I}(\rho)^{\leq}_{I(\rho)_j}$, ordered by increasing dimension. Namely, the ith cohomology group of $-\mathrm{Cr}_\rho(h)\left(\mathcal{I}(\rho)^{\leq}_{I(\rho)_j}\right)$ has the same dimension as the $(\rho - i)$th cohomology group of $-h$, that is zero for all $0 \leq i \leq \rho$. □

Remark 5.14. For any divisor D as in Proposition 5.12 we can furthermore assume $m_s = m_{s+1} = \ldots = m_l = 0$ for an arbitrary collection of $l - s + 1$ extra points in general position, denoting by $\ddot{D}$ the corresponding divisor. We will say in this case that D is the *support divisor* for $\ddot{D}$. Denote by $\ddot{\mathcal{I}}$ the set of multi-indices containing at least one of these extra points of multiplicity zero. If $\ddot{I} \in \ddot{\mathcal{I}}$ is a subset with $2 \leq |\ddot{I}| \leq n$ then $K_{\ddot{I}} \leq |\ddot{I}| - 1$. The strict transform for $\ddot{D}$ is

$$\tilde{\ddot{D}} := D_{(n-2)} - \sum_{k_{\ddot{I}}>0} k_{\ddot{I}} E_{\ddot{I}}.$$

To construct $\tilde{\ddot{D}}$ by restriction sequences we remove from $\tilde{D}$ the "extra" base locus, namely the one formed by those cycles $E_{\ddot{I}}$ with $\ddot{I} \in \ddot{\mathcal{I}}$ of dimension at most $n-1$. In order to achieve this, we will restrict to all cycles $E_{\ddot{I}}$ (containing at least one of the $l - s + 1$ zero points) with positive $K_{\ddot{I}}$, starting with $|\ddot{I}| = 2$ and then gradually increasing the cardinality of $|\ddot{I}|$ by 1. Because $K_{\ddot{I}} \leq |\ddot{I}| - 1$, for all $\ddot{I} \in \ddot{\mathcal{I}}$, knowing the vanishing theorems of $D_{(n-2)}$, for divisors with $s \leq n-1$ points, and applying Theorem 5.3, we obtain inductively the vanishing theorems of the strict transform for $\tilde{\ddot{D}}$. We therefore conclude that for all $0 \leq i \leq n$, we have

$$H^i(\tilde{\ddot{D}}) = H^i(\ddot{D}_{(n-1)}) = H^i(\ddot{D}_{(n-2)}) = H^i(D_{(n-2)}) = 0.$$

5.4. **Proof of the main results: Theorem 5.1 and Theorem 5.3.** We now complete the cohomology description of toric divisors with multiplicities bounded by $d+1$, by giving all cohomology groups of their strict transforms $\tilde{D} = D_{(n-1)}$.

Proof of Theorem 5.3 (1). We split the proof in two independent parts:

(1) $b(D) \geq n-1$ or $b \geq 1$ and $m_i \leq d$ for all i.
(2) $b(D) \leq n-2$.

Case (1). By Proposition 5.6, we conclude that $\tilde{D} = -b\mathrm{Cr}(H)$. In Theorem 3.1 we proved that for all $i \neq n$ then $h^i(\tilde{D}) = h^i(\mathcal{O}(-b\mathrm{Cr}_n(H))) = 0$, while

$$h^n(\tilde{D}) = h^n(\mathcal{O}(-b\mathrm{Cr}_n(H))) = h^n(\mathbb{P}^n, \mathcal{O}(-b)) = \binom{b-(n+1)+n}{n}.$$

This concludes the proof of this case.

Case (2). It is enough to assume $m_1 = d+1$ and $b(D) \leq n-2$ to prove inductively the vanishing theorems of all cohomology groups. First if $b(D) \leq -1$, then $k_{1,\ldots,\hat{j},\ldots n+1} \leq 0$, so $\tilde{D} = D_{(n-2)}$. The claim now follows by Corollary 5.13. Furthermore, if $b(D) = 0$ and $m_1 = d+1$ then, since $k_{1,\ldots,\hat{j},\ldots n+1} \leq 1$, the claim follows from Proposition 5.12.

We reduced to the case $1 \leq b(D) \leq n-2$ and $m_1 = d+1$. Notice that in the planar case ($n=2$) since $\tilde{D} = -b\mathrm{Cr}(H)$, by Proposition 5.6 and Theorem 3.1 the claim is trivial. We use this case as the first case of induction.

For $n \geq 3$ it is easy to see that

$$\tilde{D} = \tilde{\ddot{F}},$$

where $\ddot{F}$ is a toric divisor in $\mathbb{P}^n$ with degree and multiplicities and $m_i \leq d+1$, based at $s \leq n-1$ points and extra points of multiplicity zero, as in Remark 5.14. Let F denote the support divisor of $\ddot{F}$ with $s \leq n-1$ base points. We prove, inducing on n, that $h^i(\tilde{D}) = h^i(\tilde{\ddot{F}})$. Recall that for every $\ddot{I} \in \ddot{\mathcal{I}}$ one has $k_{\ddot{I}} \leq |\ddot{I}| - 1$. Using Corollary 5.13 and the induction hypothesis as in Remark 5.14, it is easy to see that, for all $i \geq 0$, the following identities hold:

$$h^i(\tilde{D}) = h^i(\ddot{F}_{(n-1)}) = h^i(\ddot{F}_{(n-2)}) = h^i(F_{(n-2)}) = 0.$$

□

Proof of Theorem 5.1. We first recall that for any effective divisor D, *complementary* cycles, L_I and L_J, can not be simultaneously contained in the base locus since $K_I + K_J \leq 0$ for any arbitrary disjoint sets with $|I| + |J| = n$ (cfr. Lemma 4.4). Moreover, the same statement holds for non-effective divisors, D, provided that $\mathcal{I}$ satisfies (I) and (III). This observation allows us to extend the proof of Theorem 5.3 (1) to non-effective divisors.

Indeed, assuming that $k_{I(r)} \geq 1$ for some linear cycle of dimension r, $L_{I(r)}$, spanned by a subset $I(r)$ on $r+1$ points, then in our hypothesis the restriction to the exceptional divisor $E_{I(r)}$, $D_{(r-1)}|_{E_{I(r)}}$, is of the form

$$D_{(r-1)}|_{E_{I(r)}} = \tilde{G} \boxtimes 0,$$

for some $\tilde{G}$. Because $K_{I(r)^c} \leq 0$, we conclude that G is always a toric divisor in $\mathbb{P}^r$ with $k_{I(r)} = b(G)$ and $m_i \leq d+1$ of the form

$$G = dh - \sum_{i \in I(r)} m_i E_i.$$

By applying Theorem 5.3 (1) to the toric divisor G in $X^r_{(0)}$ with $m_i \leq d+1$ and $b(G) = k_{I(r)}$, we obtain

$$h^i(E_{I(r)}, D_{(r-1)}|_{E_{I(r)}}) = h^i(X^r_{(0)}, \tilde{G}) = h^i(X^r_{(0)}, -k_{I(r)}h).$$

We furthermore recall from [6, Proposition 4.10] that

$$h^{r+1}(D_{(r)}) = h^{r+1}(D_{(r+1)}) - h^{r+2}(D_{(r+1)}) + \sum_{k_{I(r)\geq 1}} \binom{n + k_{I(r)} - (r+1) - 1}{n,}$$

and that $h^i(D_{(r+1)}) = h^i(D_{(r+2)})$, for all $i \leq r+1$.

Notice if $r = n-1$, i.e. I is a subset with $|I| = n$, the same computations hold for the divisor $E_I = H_I$, a blown-up hyperplane along the cycles spanned by the subsets of I in $X^{n-1}_{(n-3)}$. Recall that for each multi-index I of cardinality n and H_I hyperplane spanned by the points parametrized by I, with $k_I \geq 1$, then

$$\tilde{D} = D_{(n-2)} - \sum_{k_I \geq 1} k_I H_I.$$

By iterating the above formulae for $\rho \geq r+1$ we conclude the proof of part (1).

To prove part (2), observe that $D_{(r)}$ is obtained from $D_{(r-1)}$ by restricting $k_{I(r)}$ times to the exceptional divisors $E_{I(r)}$, for all $I(r)$. By the above argument we obtain that the restricted divisor, for $l = 0, \ldots, k_{I(r)} - 1$, satisfies

$$h^i(E_{I(r)}, D_{(r-1)} - (k_{I(r)} - l)E_{I(r)}|_{E_{I(r)}}) = h^i(\mathbb{P}^r, -lh).$$

The right-hand side cohomology group is zero, for all $0 \leq l \leq \min(r, k_{I(r)})$. □

We now complete the proof of the main result of this section, Theorem 5.3

Proof of Theorem 5.3 (2). The case $r \leq n-2$ was already established in Corollary 5.13. The case $r = n-1$ follows from Theorem 5.3 (1). □

6. Vanishing theorem for non-effective divisors with $s \geq n+2$ points

We recall that the effective cone $\mathrm{Eff}(X)$ of $X_{(0)}$ for $s = n+2$ is given by the inequalities

$$d \geq 0, \quad b \leq 0, \quad m_i \leq d, \quad b \leq m_i, \quad \forall i \in \{1, \ldots, n+2\}, \tag{6.1}$$

where $b := b(D) = \sum_{i=1}^{n+2} m_i - nd$, as defined in (1.7). See e.g. [6, Lemma 2.2]. Moreover, for any effective toric divisor, the strict transform $\tilde{D}$ has vanishing cohomologies, see Theorem 1.4.

In this section we give vanishing theorems for the strict transforms of divisors sitting in a "small" region outside the effective cone of $X := X^n_{(0)}$, the blow-up of $\mathbb{P}^n$ in $n+2$ points in general position.

For a divisor D of the form (1.1), we consider the following chambers of $\mathrm{N}^1(X)\setminus \mathrm{Eff}(X)$:

(6.2a) $$s = n+2: \quad b = 1, \quad m_j \le d+1;$$

(6.2b) $$s = n+2: \quad b \le 0, \quad m_1 = d+1, \ m_j \le d+1.$$

The main goal of this section is to extend Theorem 5.3 by proving that any divisor not necessarily toric, satisfying conditions (6.2), has strict transform $D_{(n-1)}$, in the space consecutively blown-up along all linear cycles up to codimension 1, with vanishing cohomologies.

Theorem 6.1. *Statements (a) and (b) of Theorem 1.4 hold for non-effective divisors with $s = n+2$ points satisfying the inequalities (6.2).*

We will also generalize Theorem 4.1 to non-effective divisors with an arbitrary number of points in a small region around the effective cone, namely for $m_i \le d+1$.

Corollary 6.2. *Statements (a) and (b) of Theorem 1.4 hold for non-effective divisors with $s \ge n+3$ points such that $m_i \le d+1$ and $b \le s-n-2$.*

Remark 6.3. We say that a divisor of the form (1.4) has *non-negative coefficients* if and only if the degree d and the multiplicities m_i and $k_{I(r)}$ are non-negative. We notice that for any effective divisor D, the strict transform $\tilde{D}$ has non-negative coefficients. On the other hand, non-effective divisors behave differently. More precisely, if D is non-effective, the strict transform $\tilde{D}$ may have both non-negative and positive coefficients. For instance if we consider the divisor $D = 3H - 4E_1 - 2E_2 - E_3$ on the blow-up of $\mathbb{P}^3$ in three points, we obtain $\tilde{D} = 2H - 3E_1 - E_2 - 2E_{12} - E_{13} + E_{23}$, after contracting the strict transform on $X_{(1)}$ of the plane spanned by the three points.

For any s, if D satisfies the assumptions of Theorem 6.1 or Corollary 6.2, then $h^0(D) = h^0(\tilde{D}) = 0$. Therefore, the Euler characteristic of $\tilde{D}$ is zero. Moreover, if D is toric (i.e. $s \le n+1$) and sits on the face $b = 0$ of $\mathrm{Eff}(X)$, then $h^0(D) = 1$. In the next Corollary we compare $\mathrm{ldim}(D)$ and $\chi(\tilde{D})$ for such divisors.

Corollary 6.4. *Let D be a divisor with $m_i \le d+1$. The following holds.*

(1) If D is toric and effective with $b = 0$, then

$$\chi(\tilde{D}) = \mathrm{ldim}(D) = 1.$$

(2) If D is toric and non-effective, then $h^n(\tilde{D}) = \binom{b-1}{n}$, if $b \ge n+1$, and zero otherwise. Moreover,

$$\mathrm{ldim}(D) = 0, \quad \chi(\tilde{D}) = \mathrm{ldim}(D) + (-1)^n h^n(\tilde{D}).$$

(3) If D is non-effective such that $s \le n+1$ and $b \le n$, or $s = n+2$ and $b \le 1$, or $s \ge n+3$ and $b \le s-n-2$, then

$$\chi(\tilde{D}) = \mathrm{ldim}(D) = 0.$$

Proof. Part (1) follows from Proposition 5.6 and [6, Theorem 4.6]. Moreover, part (2) and (3) follow from Theorem 5.3 if $s \le n+1$, Theorem 6.1 if $s = n+2$ and Corollary 6.2 for $s \ge n+3$, in fact $h^0(D) = \mathrm{ldim}(D) = 0$. □

6.1. **The planar case.** The proof of Theorem 6.1 will be by induction on the dimension, n, on the degree, d and on the multiplicities, m_i. We first show the result for the planar case, $n = 2$.

Proposition 6.5. *Statements (a) and (b) of Theorem 1.4 holds for non-effective divisors in $\mathbb{P}^2$ with $s \leq 4$ points satisfying the inequalities (6.2).*

Proof. For the non-effective divisors in $\mathbb{P}^2$ based in at most four points we reduce the proof to the vanishing theorems for $\tilde{D}$. Indeed, for any line L_{ij} through two points p_i and p_j with corresponding $k_{ij} \geq 1$, we have $D|_{L_{ij}} = \mathcal{O}_{\mathbb{P}^1}(-k_{ij})$. So by Theorem 5.3 (1) we have

$$h^1(D) = \sum_{i,j} \binom{k_{ij}}{2} + h^1(\tilde{D}) - h^2(\tilde{D}),$$
$$h^0(D) = \mathrm{ldim}(D) - h^1(\tilde{D}) + h^2(\tilde{D}).$$

Therefore, to prove the statement for such divisors D it is enough to prove the vanishing theorems for $\tilde{D}$.

For $b \leq 1$, $s \leq 4$ and $m_i \leq d+1$ the vanishing of the cohomology groups $H^1(\tilde{D})$ follows from the vanishing of the cohomology groups $H^i(-H + E_1)$, for any $i \geq 0$. We leave the details to the reader. □

6.2. **Proof of Theorem 6.1.** In this section we give the proof of the vanishing theorems for divisors satisfying (6.2) by induction on n, based on the planar case discussed in Section 6.1. Without loss of generality, we order the multiplicities of the divisor D in a decreasing order $m_1 \geq m_2 \geq \cdots \geq m_s$. We recall that $\tilde{D}$ is defined in (1.5) by $D - \sum_{J:k_J>0} k_J E_J$ where J is any multi-index with $j := |J| - 1$. We introduce

$$K_J := \sum_{i \in J} m_i - jd. \tag{6.3}$$

Notice that $k_J = \max(0, K_J)$ as in (1.2).

Remark 6.6. If $b > \max(1, s - n - 2)$, then *complementary* cycles, namely cycles parametrized by I and $I^c := \{i_1, \ldots, i_{n+2}\} \setminus I$, may have $K_I, K_{I^c} \geq 1$, since $K_I + K_{I^c} = b > 1$ and condition (III) of Section 2 would not be satisfied.

Proof of Theorem 6.1. We split the proof in two parts depending on the positivity of K_J, where J is a multi-index of cardinality n. Notice first that (6.2) implies $b = K_I + K_{I^c} \leq 1$, for any index set I. Hence the set of multi-indices $\mathcal{I}$ of cycles with $K_I \geq 0$ satisfies conditions (I) and (III) of Section 2, so the hypotheses of Theorem 5.1 are satisfied (cfr. Remark 6.6).

(1) Case $K_J \leq 0$, for all multi-indices J with $|J| = n$.

In the same way as the techniques used in [6] for effective divisors with $s \leq n+2$ points were generalized to the case of effective divisors with arbitrary s in the proof of Theorem 5.1, the same techniques will be generalizes here to the case of non-effective divisors satisfying the bounds set in conditions (6.2).

We call *sequence of type (B)* an exact sequence obtained by restricting a divisor G on a blown-up projective space, $X^n_{(r)}$, to the divisor E_I, when E_I is the (strict transform of the) exceptional divisor of a linear cycle contained in the locus of G. In practice, we will use sequences of type (B) to "get rid of" the base locus of G.

We remark that whenever $|I| = n$, E_I is a fixed hyperplane passing through all points of I having a positive K_I (see Remark 2.1).

$$(B) \qquad 0 \to G - E_I \to G \to G|_{E_I} \to 0.$$

We call *sequence of type* (C) an exact sequence obtained by restricting to the strict transform of a hyperplane with negative K_J:

$$(C) \qquad 0 \to \tilde{D} - H_{1,\dots,n} \to \tilde{D} \to \tilde{D}|_{H_{1,\dots,n}} \to 0.$$

We briefly present the idea of the proof. Starting with a divisor, that is of form $\tilde{D}$ defined in (1.5), using sequences of type (C) we decrease by one degree and n multiplicities by passing from $\tilde{D}$ to the divisor in the kernel of the sequence. However, in general, the kernel divisor is not a strict transform of the form (1.4), since it can acquire simple linear base locus. We further use sequences of type (C) to eliminate the simple base locus and take the strict transform of the kernel divisor.

The proof is based on induction on the dimension, n, in the restricted divisor, and on induction on the degree d and the multiplicities m_i, in the kernel divisor.

Before proceeding with the proof, we will recall two main important ideas that help us generalize the proof of a statement like Theorem 5.1 from effective divisors to non-effective divisors.

- We first recall an important consequence of the description with inequalities of the effective cone of divisors with at most $n+2$ points, see (6.1). Complementary cycles can not be simultaneously contained in the base locus of an effective divisor (cfr. Remark 6.6). This follows from the definition of K_I (6.3) and the inequality $b = \sum_{i=1}^{n+2} m_i - nd \leq 0$ (6.1). In particular, condition (III) of Section 2 is always satisfied. This condition is crucial in the intersection theory of Section 2, that can be shortly formulated in the fact that the restriction of a divisor to an exceptional divisor, $D_{(r)}|_{E_I}$ is always toric.

  This fact generalizes to a larger class of divisors having the same intersection table of Section 2. More precisely, since under our assumptions, complementary cycles are not contained simultaneously in the base locus of a divisor D that is possibly non-effective (see Remark 6.6), results of Section 2, and in particular the restricted divisor of sequences of type (B) and (C) together with their cohomological computation, are the same as described in [6, Theorem 4.6]. Therefore, the techniques extend.
- Whenever a linear cycle of dimension $r+1$, L_I, is contained in the base locus of D with multiplicity $k_I > 0$, for constructing the strict transform $D_{(r+1)}$ starting from $D_{(r)}$, we iterate k_I times the short exact sequences of type (B) applied to the divisor $D_{(r)} - lE_I$ and we analyse the restricted divisor $D_{(r)} - lE_I|_{E_I}$, for every $0 \leq l < k_I$.

  For each such sequence, the cohomological information of the restriction $D_{(r)} - lE_I|_{E_I}$ will depend on the cohomology of multiples of the normal bundle of E_I, that was computed in Theorem 3.1. Subsequent iterations of sequences of type (B), for each such i, will compute the cohomological contribution of L_I to $h^{r+1}(D_{(r)})$, that is given by the restricted divisor and encoded in a Newton binomial of the form $\binom{n+k_I-r-2}{n}$.

Moreover, vanishing theorems for $\tilde{D}$ under the hypotheses (6.2) will prove that statements (a) and (b) of Theorem 1.4 hold, as a consequence of Theorem 5.1. Hence it will be enough to prove vanishing theorems for strict transforms $\tilde{D}$.

We recall that $\bar{r}$ denotes the maximum dimension of the linear base locus of D. The hypothesis $K_J \leq 0$ corresponds to $\bar{r} \leq n-2$, i.e. no hyperplane in the base locus, and it suffices to prove that $h^i(D_{(n-2)}) = 0$ for all $i \geq 0$.

We recall that $b = K_I + K_{I^c} \leq 1$ for any subset I. Therefore, K_I can be both positive or zero for $|I| = 2$. Moreover, we recall that the multiplicities have been arranged in decreasing order from the beginning. Furthermore, since $m_i \leq d+1$ and $b \leq 1$, we can have at most $n-1$ multiplicities equal to $d+1$. Let us assume, without loss of generality, that $J = \{1, \ldots, n\}$. Fix the hyperplane spanned by the points parametrized by J and let $H_{1,\ldots,n}$ denote its strict transform. We use first a sequence of type (*C*) for the divisor $\tilde{D}$:

$$0 \to G := \tilde{D} - H_{1,\ldots,n} \to \tilde{D} \to \tilde{D}|_{H_{1,\ldots,n}} \to 0.$$

An easy computation shows that the kernel G could have simple linear base locus, i.e. $K_I = 1$, for some subsets I of J. The only cycles that may appear as fixed part of G with corresponding $k_I = 1$ are of type E_I with $\{n+1, n+2\} \subset I$. For this we will use restriction sequences of type (*B*) to remove the simple base locus of dimension at most $n-1$, in increasing dimension, starting from the exceptional divisor of the line spanned by the last two points:

$$0 \to G_{(1)} := G - E_{n+1,n+2} \to G \to G|_{E_{n+1,n+2}} \to 0.$$

In every sequence of type (*B*) the trace has a constant form. If $|I| = n - \rho$ with $k_I > 0$ and H denotes the general hyperplane class in the projection on the second factor of E_I, $\mathbb{P}^\rho$, then the trace is

$$G|_{E_I} = * \boxtimes (-\mathrm{Cr}_\rho(H)(\check{\mathcal{J}})),$$

where $\check{\mathcal{J}}$ is a subset of the set of all index subsets of $J \setminus I$ of cardinality at most $\rho - 1$ contained in $J = \{1, \ldots, n\}$:

$$\mathrm{Cr}_\rho(H)(\check{\mathcal{J}}) := \mathrm{Cr}_\rho(H) + \sum_{J \in \check{\mathcal{J}}} (\rho - \#J) E_J.$$

See (3.1) for the definition of $\mathrm{Cr}_\rho(H)$. We claim that $h^i(-\mathrm{Cr}_\rho(H)(\check{\mathcal{J}})) = 0$, for all $i \geq 0$. The claim simply follows from Serre duality. In fact, the ith cohomology group of $-\mathrm{Cr}_\rho(H)(\check{\mathcal{J}})$ has the same dimension as the $(\rho - i)$th cohomology group of $-H$, that is zero for all $0 \leq i \leq \rho$.

By induction on ρ, we obtain that the restriction $G|_{E_I}$ has vanishing theorems. After multiple applications of sequence, of type (*B*), we denote the final kernel by $\tilde{G}$,

$$\tilde{G} := G - \sum_{\substack{k_I > 0, \\ \{n+1, n+2\} \in I}} E_I.$$

Notice that $b(G) = b(D) = b \leq 1$, so the kernel $\tilde{G}$ is now in the beginning assumption. The induction argument on degree and multiplicities applies to G starting from the toric case in Section 5, as the first step of induction. We conclude

$$h^i(G) = h^i(\tilde{G}) = 0.$$

We will now analyse the restricted divisor $\tilde{D}|_{H_{1,\ldots,n}}$ and prove by induction on n that it has vanishing theorems. We denote by e'_{n+1} the trace of the cycle $E_{n+1,n+2}$ on the hyperplane $H_{1,\ldots,n}$, $e'_{n+1} := E_{n+1,n+2}|_{H_{1,\ldots,n}}$. We have

$$\tilde{D}|_{H_{1,\ldots,n}} = dh - \sum_{i=1}^{n} m_i e_i - k_{n+1,n+2}e'_{n+1} - \sum_{\substack{k_I>0\\|I|\leq n-2}} k_I e_I - \sum_{\substack{k_M>0\\|M|=n-1}} k_M h_M,$$

where h_J is the restriction $E_J|_{H_{1,\ldots,n}}$ and E_J is a codimension-2 cycle in the base locus of D. Note that for a codimension-2 cycle J, h_J becomes (the strict transform of) a hyperplane passing through the points of J in dimension $n-1$, as in Remark 2.1. We denote by F the divisor

$$F := dh - \sum_{i=1}^{n} m_i e_i - k_{n+1,n+2}e'_{n+1}.$$

We leave to the reader to check that the restriction is a strict transform

$$\tilde{D}|_{H_{1,\ldots,n}} = \tilde{F}.$$

If $k_{n+1,n+2} = 0$, the trace $\tilde{D}|_{H_{1,\ldots,n}}$ is toric. In this case $b(F) = K_{1,\ldots,n} \leq K_{1,\ldots,n} + K_{n+1,n+2} = b(D) \leq 1$ and Remark 5.14 applies to the divisor F, proving the vanishing theorems for $\tilde{F}$.

If $k_{n+1,n+2} \geq 1$ the restriction $\tilde{D}|_{H_{1,\ldots,n}} = \tilde{F}$ satisfies inequalities (6.2) with $b(F) = b(D) \leq 1$. Since $\tilde{F}$ is a divisor in the blown-up $\mathbb{P}^{n-1}$ in $s = n+1$ points, by the induction hypothesis vanishing theorems hold for $\tilde{F}$.

We conclude that using induction on dimension and results from Section 5 the vanishing theorems hold for the divisor in the trace $\tilde{D}|_{H_{1,\ldots,n}}$ starting with Proposition 6.5 as the first step of induction $n = 2$. We use induction on degree and multiplicities to conclude the vanishing of $\tilde{G}$.

(2) Case $K_J > 0$ for some multi-index J with $|J| = n$.

In this case the problem we encounter is that the residual may have negative coefficients. Without loss of generality assume that $J = \{1,\ldots,n\}$. Write $D' := D - K_{1,\ldots,n}H_{1,\ldots,n}$ and denote the coefficients of D' by d' and m'_i respectively. Without loss of generality, we can assume that the multiplicities of D' are in decreasing order. We observe that the residual, D', has $d'+1 \geq m'_i$, $k'_{n+1,n+2} = 1$ and $b(D') = b(D)$. This implies $K'_{M,n+1,n+2} \leq n-1$. We distinguish two cases:

(2.a) $K'_{M,n+1,n+2} \leq 0$ for all subsets M of $\{1,\ldots,n\}$ with $|M| = n-2$,
(2.b) $K'_{M,n+1,n+2} > 0$ for some subset M of $\{1,\ldots,n\}$ with $|M| = n-2$.

Observe the two cases above correspond to the following cases:

(2.a) $K_{M,n+1,n+2} + K_{1,\ldots,n} \leq 0$ for all subsets M of $\{1,\ldots,n\}$ with $|M| = n-2$,
(2.b) $K_{M,n+1,n+2} + K_{1,\ldots,n} > 0$ for some subset M of $\{1,\ldots,n\}$ with $|M| = n-2$.

Since $K_I + K_{I^c} = b \leq 1$, the hypothesis that $K_{1,\ldots,n} \geq 1$ implies $K_{n+1,n+2} \leq 0$. This proves that the divisor $D|_{H_{1,\ldots,n}}$ is toric.

Case (2.a). For all subsets M of $\{1,\ldots,n\}$ with $|M| = n-2$ then $K_{M,n+1,n+2} + K_{1,\ldots,n} \leq 0$ and $K_{1,\ldots,n} > 0$. If the strict transform $\tilde{D}'$ has positive coefficients, then by Case (1) we conclude that $\tilde{D}'$, and therefore $\tilde{D}$, has vanishing cohomologies. If $\tilde{D}'$

has some negative coefficients, then $K_{1,\ldots,n} = \sum_{i=1}^{n} m_i - (n-1)d \geq m_n$. Consider further the toric divisor

$$\ddot{D} := D - m_n H_{1,\ldots,n}.$$

We conclude that the vanishing theorems for the cohomologies hold by Remark 5.14 applied to the toric divisors $\ddot{D}$, i.e. $h^i(\tilde{D}) = h^i(\tilde{D}') = h^i(\tilde{\ddot{D}}) = 0$.

Case $(2.b)$. The equality $k'_{n+1,n+2} = 1$ implies that $K'_{M,n+1,n+2}$ is bounded above by $n-1$, for any M subset of $\{1,\ldots,n\}$ of cardinality $n-2$. Therefore, we obtain

$$0 < K'_{M,n+1,n+2} = K'_{M,n+1,n+2} + K'_{1,\ldots,n} = K_{1,\ldots,n} + K_{M,n+1,n+2} \leq n-1.$$

Since $K_{1,\ldots,n} > 0$, we have that $K_{M,n+1,n+2} \leq n-1$. Now, if D' has positive coefficients, by the induction hypothesis on degree and multiplicities we conclude that the vanishing theorems hold for $\tilde{D}'$ as well as for $\tilde{D}$. If D' has negative coefficients, then we conclude by carrying out the same analysis as in Case $(2.a)$ □

6.3. **Case $s \geq n+3$.** In this section we prove the result for the case of arbitrary number of points satisfying $b \leq s-n-2$.

Proof of Corollary 6.2. The effective case $m_i \leq d$ was proved in Theorem 4.1, so we assume $m_1 = d+1$. Notice that the inequality $b \leq s-n-2$ implies $K_I + K_J = \sum_{i\in I\cup J} m_i - nd \leq 0$ for any disjoint sets I, J with $|I|+|J| = n+2$. Therefore, $\mathcal{I}$, the set of all multi-indices in $\{1,\ldots,s\}$, satisfies conditions (I) and (III) of Section 2.

By Theorem 5.1 and Corollary 5.2 it is enough to prove vanishing theorems for $\tilde{D}$. We prove it by induction on s, using as base step the cases $s \leq n+2$.

Assume first that $\tilde{D}$ has positive coefficients. Set $D' := dH - \sum_{i=1}^{n+2} m_i E_i$. Notice that $h^i(\tilde{D}) = h^i(\tilde{D}')$. Indeed, $\tilde{D}$ can be obtained from $\tilde{D}'$, by restricting m_i times to each exceptional divisor E_i, $i \geq n+3$, and then each time removing the simple base locus by means of exact sequences of type (C). The argument is the same as in the proofs of Propositions 4.9 and 4.10, and here details are left to the reader. Because $b(D') \leq 0$, by Theorem 6.1 we have $h^i(\tilde{D}') = 0$ and this proves the statement.

Assume now that $\tilde{D}$ has some negative coefficients. Notice that $K_{2,\ldots,n+1} = b(D) - m_1 - \sum_{i\geq n+2} m_i + d \leq s-n-2-(d+1)-(s-n-1)+d \leq 0$, the first inequality following from $b \leq s-n-2$ and $m_i \geq 1$. This implies that the hyperplane spanned by $\{2,\ldots,n+2\}$ is not contained in the fixed part D. Furthermore, no hyperplane spanned by $I(n-1) \subset \{2,\ldots,s\}$ is.

As in the proof of Theorem 5.3 we can write $\tilde{D} = \tilde{\ddot{F}}$, for an appropriate divisor $\ddot{F}$ with positive coefficients and at most s base points, some of them having multiplicity zero, such that $b(\ddot{F}) \leq 1$. By Remark 5.14 we have $h^i(\tilde{D}) = h^i(\tilde{\ddot{F}}) = h^i(\tilde{F})$, for all $i \geq 0$, with F, the support divisor of $\ddot{F}$, having at most $s-1$ base points. By the induction hypothesis we have $h^i(\tilde{F}) = 0$ and this concludes the proof. □

7. Vanishing theorems for points in star configuration in $\mathbb{P}^n$

As an application of the results proved in the previous sections, we compute the number of global sections and prove vanishing theorems for the cohomology groups of the strict transforms along the linear base locus of some families of divisors interpolating points in *star configuration* in $\mathbb{P}^n$.

A star configuration of points is a collection of points satisfying some particular geometric relation. They have been object of study in many papers lately, see [15, 16] and references therein.

Given l hyperplanes in $\mathbb{P}^n$ that meet properly, i.e. not three of them intersecting along a $\mathbb{P}^{n-2}$, not four of them intersecting along a $\mathbb{P}^{n-3}$ etc, a *star configuration* of *dimension r subspaces* is the set given by the $\binom{l}{n-r}$ linear subspaces of dimension r in $\mathbb{P}^n$ formed by taking all possible intersections of $n-r$ among the l hyperplanes.

In [16, Theorem 3.2], the authors compute the Hilbert function of the ideals of star configurations of dimension r subspaces of multiplicity two. This provides a complete classification of linear systems in $\mathbb{P}^n$ interpolating such a scheme. In Theorem 7.3 we compute the number of global sections of a class of linear systems in $\mathbb{P}^n$ interpolating star configurations of points obtained by $l = n+2$ hyperplanes with higher multiplicities.

Remark 7.1. When $l = n+2$, star configurations of dimension r subspaces in $\mathbb{P}^n$ are obtained as follows. Let us embed $\mathbb{P}^n \hookrightarrow H \subset \mathbb{P}^{n+1}$ and denote by $p_1, \dots, p_{n+2}$ a general collection of points of $\mathbb{P}^{n+1}$, that we may think of as the coordinate points, with respect to which H is a general hyperplane. The family of points $q_{ij} := L_{ij} \cap H \in H$, for all lines $L_{ij} = \langle p_i, p_j \rangle \subset \mathbb{P}^{n+1}$, forms a star configuration of points in H. Indeed, denoting by H_l the hyperplane spanned by all p_i's with $i \neq l$ for all $1 \leq l \leq n+2$, we can write $q_{ij} = H \cap \bigcap_{l \neq i,j} H_l$. Similarly, the family of r-linear subspaces $\lambda_I := L_I \cap H$, for all multi-indices $I = \{i_1, \dots, i_{r+2}\} \subset \{1, \dots, n+2\}$, is a star configuration of dimension r subspaces in H.

We now study effective divisors in $\mathbb{P}^n$ interpolating star configurations of points. Set $Y_{(0)} = Y^n_{(0)}$ to be the blow-up of $\mathbb{P}^n$ in the star configuration of points given as intersections of $\binom{n+2}{n}$ hyperplanes. Adopting the same notation of Remark 7.1, we call q_{ij}, $1 \leq i < j \leq n+2$ such points. Let us denote by h the hyperplane class and by e_{ij} the exceptional divisors.

Remark 7.2. As in Remark 7.1, let us embed $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ and, as in Section 2, let $X_{(1)} = X^{n+1}_{(1)}$ denote the blow-up of $\mathbb{P}^{n+1}$ at general points $p_1, \dots, p_{n+2}$ and, subsequently, along the lines spanned by those points, with exceptional divisors E_{ij}, $1 \leq i < j \leq n+1$. By writing $h = H|_H$ and $e_{ij} = E_{ij}|_H$, we obtain the following isomorphism $Y_{(0)} \cong X_{(1)}|_H$.

Given integers $d \geq m_1, \dots, m_{n+2} \geq 0$, we use the notation $k_{ij} = \max(m_i + m_j - d, 0)$. Assume

$$\sum_{i=1}^{n+2} m_i \leq (n+1)d, \tag{7.1}$$

and consider the following divisor on $Y_{(0)}$:

$$\Delta := dh - \sum_{1 \leq i < j \leq n+2} k_{ij} e_{ij}.$$

We prove that it is only linearly obstructed, with linear base locus supported on the star configurations of linear subspaces λ_I defined above (Remark 7.1) and that its subsequent strict transforms after blowing-up the dimension r star configuration have vanishing theorems.

Let $\mathcal{I}$ be the set of all multi-indices in $\{1,\dots,n+2\}$, and for each multi-index $I(r)\subset\mathcal{I}$ of cardinality $r+1$, we use the notation (1.2). For increasing r, let $Y_{(r)}=Y^n_{(r)}$ denote the blow-up of $Y_{(r-1)}$ along the strict transform of the star configuration of r-subspaces $\lambda_{I(r)}$ in H and let $\Delta_{(r)}$ be the strict transform of Δ.

Theorem 7.3. *In the above notation, we have*

$$h^0(\Delta)=\binom{n+d}{n}+\sum_{I(r)\in\mathcal{I},r\geq 1}(-1)^r\binom{n+k_{I(r)}-r}{n}.$$

Moreover, for all $1\leq r\leq n-1$, $h^i(\Delta_{(r)})=0$, *for all* $i\neq 0,r+1$, *and*

$$h^{r+1}(\Delta_{(r)})=\sum_{I(\rho)\in\mathcal{I},\rho\geq r+2}(-1)^\rho\binom{n+k_{I(\rho)}-\rho}{n}.$$

In particular, $h^i(\tilde{\Delta})=0$, *for all* $i>0$.

Proof. Recall the inclusion $H\subset\mathbb{P}^{n+1}$ and let $X_{(r+1)}:=X^{n+1}_{(r+1)}$ be the blow-up of $\mathbb{P}^{n+1}$ first at the points $p_1,\dots,p_{n+2}$ and then along the linear subspaces L_I of dimension bounded above by $r+1$, in increasing dimension. We have the following inclusion $Y_{(r)}\subset X_{(r+1)}$.

Let us consider the following divisors on $X_{(0)}$

$$D=dH-\sum_{i=2}^{n+2}m_iE_i,\quad D'=D-H.$$

Notice that, since (7.1) is satisfied, then D is effective (cfr. (5.1)) and D' is either effective or satisfies condition (5.2a).

Abusing notation, denote by H the strict transform of $H\cong\mathbb{P}^n\subset\mathbb{P}^{n+1}$ and notice that the restriction $D_{(r+1)}|_H$ belongs to the linear system associated with $\Delta_{(r)}$:

$$D_{(r+1)}|_H\subset|\Delta_{(r)}|.$$

Consider the following restriction sequence

$$0\to D_{(r+1)}-H\to D_{(r+1)}\to D_{(r+1)}|_H\to 0. \tag{7.2}$$

By Theorem 1.4 we have that $h^i(D_{(r+1)})=0$, for all $i\neq 0,r+2$. We claim that the strict transforms of the linear ρ-cycles contained in base locus of $D_{(r+1)}-H$ have multiplicity bounded above by ρ, for all $\rho\leq r+1$. Hence, by Theorem 5.1 (2), they do not provide linear obstruction to such a divisor, namely $h^i(D_{(r+1)}-H)=h^i(D'_{(r+1)})=0$, for all $i\neq 0,r+2$. To prove the claim, notice that the multiplicity of containment of $L_{I(\rho)}$ in the base locus of D' is $k'_{I(\rho)}:=\max(\sum_{i\in I(\rho)}m_i-\rho(d-1),0)$. Moreover, in $D_{(r+1)}-H$ the strict transform of the exceptional divisor of $L_{I(\rho)}$ has been removed $k_{I(\rho)}:=\max(\sum_{i\in I(\rho)}m_i-\rho d,0)$ times. Finally we have $k'_{I(\rho)}-k_{I(\rho)}\leq\rho$. Therefore, we conclude that

$$h^0(\Delta_{(r)})=h^0(D)-h^0(D'),$$
$$h^{r+1}(\Delta_{(r)})=h^{r+2}(D'_{(r+1)})-h^{r+2}(D_{(r+1)}),$$

and that all higher cohomology groups vanish, by means of the long exact sequence in cohomology associated with (7.2). $\square$

Corollary 7.4. *The linear subspace λ_I is contained with multiplicity k_I in the base locus of $\Delta_{(r)}$.*

Proof. Let D be as in the proof of Theorem 7.3. The linear base locus of $\Delta_{(r)}$ is the intersection with the hyperplane H of the linear base locus of $D_{(r+1)}$, described in Proposition 4.2, and in particular it is supported at the dimension ρ star configurations, with $\rho \geq r+1$. □

An interpretation of the above corollary is that the only obstructions are the linear subspaces λ_I.

Moreover, this suggests a definition of virtual linear dimension for divisors interpolating points in star configuration in $\mathbb{P}^n$, that generalizes the notion of virtual linear dimension for divisors interpolating points in general position that was introduced in [6] and that has been extensively studied throughout this paper. While in the general case the linear obstructions are the linear subspaces spanned by the points, in this case they are given by the star configurations of linear subspaces.

Remark 7.5. One may study general hyperplane sections of effective linearly obstructed divisors D in $\mathbb{P}^{n+1}$ interpolating arbitrary numbers of points in general position with bounds (4.1) or (6.2). Using Theorem 4.1 or Theorem 1.4 one analyses the divisor D and using Corollary 6.2 or Theorem 6.1 respectively one analyses the kernel divisor. The resulting restricted divisor $\Delta = D_{(1)}|_H$ in $Y^n_{(0)}$ interpolates points in special configuration and is linearly obstructed. Moreover, a cohomological description such as the one established in Theorem 7.3 can be obtained for such divisors.

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