DENSITY-DEGREE FUNCTION FOR SUBSETS OF $\mathbb{R}^n$

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Abstract. For all subsets E of $\mathbb{R}^n$, we define a function d_E measuring the density-degree of E at the points of $\mathbb{R}^n$. We provide some results which involve d_E. In particular we prove an approximation property stating that, given a bounded open sets Ω, the following facts hold:

(1) For all $C < \mathcal{L}^n(\Omega)$ there exists a closed subset F of $\overline{\Omega}$ such that $\mathcal{L}^n(F) > C$ and $d_F = n$ almost everywhere in F;

(2) For all $C < \mathcal{L}^n(\Omega)$ and for every proper subinterval I of $(n, +\infty)$, there exists a closed subset F of $\overline{\Omega}$ and an open subset U of Ω such that $F \supset \Omega \setminus U$, $\mathcal{L}^n(U) < \mathcal{L}^n(\Omega) - C$ (hence $\mathcal{L}^n(F) > C$) and $d_F(x) \in I$ for all $x \in \Omega \setminus U$.

1. Introduction

In the series of papers [5, 6, 7, 8, 9, 10, 11] we have investigated the following notion of superdensity: If $m \geq n$ then a subset E of $\mathbb{R}^n$ is said to be m-dense at $x_0 \in \mathbb{R}^n$ if $\mathcal{L}^n(B(x_0, r) \setminus E) = o(r^m)$ as $r \to 0^+$, where $\mathcal{L}^n$ denotes the Lebesgue outer measure in $\mathbb{R}^n$ and $B(x_0, r)$ is the open ball of radius r centered at x_0. The set of all these points x_0 is denoted by $E^{(m)}$ and it is obvious that $E^{(p)} \subset E^{(m)}$ for $p > m$. The sets E such that $E \subset E^{(m)}$ (i.e. the so called "m-dense sets") form a base topology on $\mathbb{R}^n$. In the special case when $m = n + 1 + 1/(n-1)$ this topology includes the family of locally finite perimeter subsets of $\mathbb{R}^n$, compare [6, 11].

One can roughly say that m-dense sets are closer to open sets than to generic measurable sets and actually "m-density" seems to provide a good category to generalize some classical results where "openness" is required. For example in [5, 6, 8] we have proved the following results, where Ω is an open subset of $\mathbb{R}^n$, $f \in C^1(\Omega)$, $\Phi \in C^1(\Omega, \mathbb{R}^n)$ and $F := \{x \in \Omega \mid \nabla f(x) = \Phi(x)\}$:

- *The vector field* $\operatorname{curl}\Phi$ *vanishes in* $\Omega \cap F^{(n+1)}$. Such a result generalizes the classical Schwarz theorem about the equality of mixed partial derivatives;
- *The graph of* $f|_{\Omega \cap F^{(n+1)}}$ *is* C^2*-rectifiable.* This Whitney-type property extends the obvious assertion that f restricted to the interior of F is of class C^2. Its proof is

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based on an argument which combines superdensity and differentiability in the L^1 sense (according to the definition of Calderon-Zygmund, compare [4] and [17]);

- *Given $x_0 \in \Omega \cap F^{(n+1)}$, denote by Γ the quadratic form associated to $\frac{1}{2}D\Phi(x_0)$ i.e.*

$$\Gamma(\xi) := \frac{1}{2}\langle D\Phi(x_0)\xi, \xi\rangle = \frac{1}{2}\sum_{i,j=1}^{n} \xi_i \xi_j D_i \Phi_j(x_0) \qquad (\xi \in \mathbb{R}^n)$$

and consider the family of quadratic dilatations $T_\rho : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$, with $\rho \in (0,1)$, defined by

$$T_\rho(x;t) := \left(\frac{x - x_0}{\rho}; \frac{t - f(x_0) - \nabla f(x_0) \cdot (x - x_0)}{\rho^2}\right).$$

If G_f and G_Γ denote the graph of f and the graph of Γ, respectively, then one has

$$\mathcal{H}^n \llcorner T_\rho(G_f) \to \mathcal{H}^n \llcorner G_\Gamma \qquad (\textit{as } \rho \to 0^+)$$

in the weak sense of measures.* This result establishes the existence of an osculating paraboloid over the points x_0 of $\Omega \cap F^{(n+1)}$. When x_0 is in the interior of F such a paraboloid coincides with the graph of the maximal form in the second order Taylor polynomial of f at x_0.

The interest for this subject stems naturally from the G. Alberti's result [1, Theorem 1] and the first application relating the Alberti's theorem with sets of finite perimeter has been provided by J. Fu in [12, Corollary 2]. The Fu's argument is quite technical (it uses the slicing of rectifiable currents), but an easy alternative proof based on the mentioned superdensity property of finite perimeter sets is given in [7]. In this context it is worth mentioning [3, Theorem 3.1], where the Euclidean size of the characteristic set associated to a one-codimensional submanifold of the Heisenberg group $\mathbb{H}^n$ is estimated in some special cases. Applications of superdensity to functions having derivatives in the L^1 sense are also given in [9].

In the present paper we continue the study of density-related properties of subsets of $\mathbb{R}^n$ within the classical framework of Lebesgue measure. For all subsets E of $\mathbb{R}^n$, we define a function d_E measuring the density-degree of E at the points of $\mathbb{R}^n$. More precisely: $d_E(x)$ is defined as the supremum of the set $\{k \geq n \,|\, x \in E^{(k)}\}$ if $x \in E^{(n)}$, while it is set to be zero whenever $x \notin E^{(n)}$. We provide some results concerning the level and superlevel sets of the density-degree function. In particular we prove an approximation property stating that, given a bounded open set Ω, the following facts hold:

(1) For all $C < \mathcal{L}^n(\Omega)$ there exists a closed subset F of $\overline{\Omega}$ such that $\mathcal{L}^n(F) > C$ and $d_F = n$ almost everywhere in F (compare Proposition 5.4);

(2) For all $C < \mathcal{L}^n(\Omega)$ and for every proper subinterval I of $(n, +\infty)$, there exists a closed subset F of $\overline{\Omega}$ and an open subset U of Ω such that $F \supset \Omega \setminus U$, $\mathcal{L}^n(U) < \mathcal{L}^n(\Omega) - C$ (hence $\mathcal{L}^n(F) > C$) and $d_F(x) \in I$ for all $x \in \Omega \setminus U$ (compare Theorem 5.1).

2. General notation

With $\mathcal{P}(\mathbb{R}^n)$ we indicate the collection of all subsets of $\mathbb{R}^n$. The Euclidean norms (absolute value included) are denoted by $|\cdot|$. The open ball of radius r centered at $x \in \mathbb{R}^n$ is denoted by $B(x,r)$ and ω_n indicates the measure of the unit ball $B(0,1)$. For simplicity, the ball $B(0,r)$ is indicated with B_r. The constants depending only on $p, q, \dots$ are indicated by $C(p,q,\dots)$. The usual Euclidean topology in $\mathbb{R}^n$ is indicated with $\tau(\mathbb{R}^n)$. If $E \subset \mathbb{R}^n$ then E° and $\overline{E}$ denote, respectively, the interior of E and the closure of E (w.r.t. $\tau(\mathbb{R}^n)$). Moreover φ_E is the characteristic function of E. When two subsets A and B of $\mathbb{R}^n$ are equivalent in measure, namely with respect to the Lebesgue measure $\mathcal{L}^n$, we write $A \mathrel{\mathring{=}} B$. If E is a measurable subset of $\mathbb{R}^n$, recall that the essential interior of E and the essential boundary of E are defined, respectively, as

$$\mathrm{int}^M E := \left\{ x \in \mathbb{R}^n \,\middle|\, \mathcal{L}^n(B(x,r) \backslash E) = o(r^n) \text{ as } r \to 0^+ \right\}$$

and

$$\partial^M E := \mathbb{R}^n \backslash \left(\mathrm{int}^M E \cup \mathrm{int}^M(\mathbb{R}^n \backslash E) \right).$$

Recall that $\mathrm{int}^M E \mathrel{\mathring{=}} E$ and $\mathrm{int}^M(\mathbb{R}^n \backslash E) \mathrel{\mathring{=}} (\mathbb{R}^n \backslash E)$, by the Lebesgue's density theorem, hence $\mathcal{L}^n(\partial^M E) = 0$. With $P(E)$ we denote the perimeter of a $\mathcal{L}^n$-measurable subset E of $\mathbb{R}^n$ in the sense of De Giorgi, namely

$$P(E) := \sup \left\{ \int_E \mathrm{div} \varphi \, d\mathcal{L}^n \,\middle|\, \varphi \in [C_c^1(\mathbb{R}^n)]^n \right\}$$

compare [2, Section 3.3]. The Hausdorff dimension of a set E is denoted by $\dim_H(E)$.

3. Superdense sets: a miscellany of some well-known facts

Let us begin this section by recalling the definition of base operator associated to the superdensity topology (compare [11]) and that of m-density point (compare [6, 7, 8]).

Definition 3.1. *For $m \geq n$, the operator $b_m : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is defined as follows:*

$$b_m(A) := \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r \to 0^+} \frac{\mathcal{L}^n(A \cap B(x,r))}{r^m} > 0 \right\}, \quad A \in \mathcal{P}(\mathbb{R}^n).$$

Definition 3.2. *Let $m \geq n$ and $A \in \mathcal{P}(\mathbb{R}^n)$. Then $x \in \mathbb{R}^n$ is said to be a "m-density point of A" if*

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \backslash A)}{r^m} = 0.$$

The set of m-density points of A is denoted by $A^{(m)}$.

Remark 3.1. One has the equality

$$A^{(m)} = [b_m(A^c)]^c \,. \tag{3.1}$$

This simple observation leads to the idea of defining a "superdensity topology": just as we say that A is open in the Euclidean topology whenever $A \subset A^\circ$, so we'll say that A is open with respect to b_m if $A \subset [b_m(A^c)]^c$. This idea concerns a particular case of a very general and deep theory about fine topologies arising from a base operator. The most complete reference for this subject is [14], from which we recall that a *base operator* on a set X is a map $b : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $b(\emptyset) = \emptyset$ and $b(A \cup B) = b(A) \cup b(B)$, for all $A, B \in \mathcal{P}(X)$. Such an operator is monotone and it actually determines a topology τ_b whose members are the subsets A of X satisfying $A \subset [b(A^c)]^c$. Hence a subset F of X is closed with respect τ_b (or b-closed, for shortness) if and only if $b(F) \subset F$.

Remark 3.2. As one can easily verify, the following five equalities are equivalent (recall the Lebesgue density theorem, e.g [13, Section 7.49]):

- $\mathcal{L}^n(A) = 0$;
- $b_m(A) = \emptyset$ for all $m \geq n$;
- $b_n(A) = \emptyset$;
- $A^{(m)} = \emptyset$ for all $m \geq n$;
- $A^{(n)} = \emptyset$.

Also observe that

$$A^{(m)} \cap B^{(m)} = (A \cap B)^{(m)}$$

by (3.1). Moreover, from the trivial inclusion $A, B \subset A \cup B$ which implies $A^{(m)}, B^{(m)} \subset (A \cup B)^{(m)}$ (just by definition), we get

$$A^{(m)} \cup B^{(m)} \subset (A \cup B)^{(m)}. \tag{3.2}$$

As we have already observed in [11], the proper inclusion in (3.2) can actually occur. In the special case when $m = n$ and $A^{(n)} = B^{(n)} = \emptyset$ one has $\mathcal{L}^n(A \cup B) \leq \mathcal{L}^n(A) + \mathcal{L}^n(B) = 0$, hence

$$A^{(n)} \cup B^{(n)} = (A \cup B)^{(n)} = \emptyset.$$

Remark 3.3. There exist n-dimensional subsets of $\mathbb{R}^n$ having measure zero. For example, consider $A := \cup_j C_j$, where $\{C_j\}$ is a countable family of Cantor sets such that $\dim_H(C_j) \leq \dim_H(C_{j+1}) < 1$ for all j and $\dim_H(C_j) \to 1$ as $j \to +\infty$ (compare [15, Sections 4.10 and 4.12] for a construction of such a family). Then one actually has $\mathcal{L}^1(A) = 0$ and $\dim_H(A) = 1$.

Proposition 3.1 ([11], Proposition 3.1)**.** *The following facts hold:*

(1) *b_m is a base operator;*
(2) *For all $A \subset \mathbb{R}^n$, the set $b_m(A)$ is measurable with respect to $\mathcal{L}^n$. As a consequence, $A^{(m)}$ is measurable too;*
(3) *One has $A \in \tau_{b_m}$ if and only if $A \subset A^{(m)}$. In particular τ_{b_m} is finer than $\tau(\mathbb{R}^n)$;*
(4) *If $p \geq m(\geq n)$, then $b_m(A) \subset b_p(A)$, for all $A \subset \mathbb{R}^n$. In particular τ_{b_m} is finer than τ_{b_p};*
(5) *For all $A \subset \mathbb{R}^n$, one has*

$$b_m(A) \subset \overline{A}^{\tau_{b_m}}$$

where the right hand member is the closure of A with respect to τ_{b_m}. In particular, a set A is τ_{b_m}-dense in $\mathbb{R}^n$ whenever $b_m(A) = \mathbb{R}^n$.

Theorem 3.1 ([11], Proposition 3.2)**.** *Assume $m > n$ and consider $\varepsilon > 0$. The following properties hold:*

(1) *If Ω is a bounded open set in $\mathbb{R}^n$, then there exists an open subset A of Ω such that*

$$\mathcal{L}^n(A) < \varepsilon, \qquad \Omega \subset b_m(A) \subset \overline{\Omega}. \tag{3.3}$$

In the special case when $\partial\Omega$ is Lipschitz, the set A can be chosen in such a way to satisfy

$$b_m(A) = \overline{\Omega}; \tag{3.4}$$

(2) *There exists an open subset U of $\mathbb{R}^n$ such that*

$$\mathcal{L}^n(U) < \varepsilon, \qquad b_m(U) = \mathbb{R}^n.$$

The Lebesgue's density theorem states that if E is a measurable subset of $\mathbb{R}^n$ then almost every $x \in E$ is a n-density point of E. The following result establishes that Caccioppoli sets are more dense than generic measurable sets.

Theorem 3.2 ([6], Lemma 4.1)**.** *Let E be a subset of $\mathbb{R}^n$ of locally finite perimeter and*

$$m_0 := n + 1^* = n + 1 + \frac{1}{n-1}. \tag{3.5}$$

Then $E \overset{\circ}{=} E^{(m_0)}$.

Hence every set of finite perimeter has an equivalent copy (with respect to measure) in $\tau_{b_{m_0}}$.

Proposition 3.2 ([11], Proposition 4.2)**.** *Let E be a set of locally finite perimeter in $\mathbb{R}^n$ and define $E_* := E \cap E^{(m_0)}$. Then*

$$\mathcal{L}^n(E_*) = \mathcal{L}^n(E), \qquad E_* \in \tau_{b_{m_0}}.$$

Proposition 3.3 ([11], Proposition 4.1)**.** *For all $m > m_0$ there exists a closed set F_m of positive measure and finite perimeter in $\mathbb{R}^n$ such that $F_m^{(m)} = \emptyset$.*

4. The upper m-density of $\mathcal{L}^n \llcorner E$ (with $m > n$) is an almost everywhere $\{0, +\infty\}$-valued function

We begin this section by recalling a well-known property of the functions in $L^p(\mathbb{R}^n)$ with $p \in [1, +\infty)$, compare [17, Lemma 3.7.2].

Theorem 4.1. *Let $u \in L^p(\mathbb{R}^n)$, with $p \in [1, +\infty)$. Assume that there exist a measurable subset E of $\mathbb{R}^n$ and two constants $C, a > 0$ such that*

$$\sup_{x\in E}\left(\fint_{B(x,r)} |u|^p\right)^{1/p} \leq Cr^a$$

for all $r > 0$. Then

$$\left(\fint_{B(x,r)} |u|^p\right)^{1/p} = o(r^a) \quad \text{as } r \to 0^+$$

at a.e. $x \in E$.

Remark 4.1. In general, Theorem 4.1 does not hold for $a = 0$. A trivial counterexample is provided for every p by $u := \varphi_{B(0,1)}$ and $E := \mathbb{R}^n$.

Remark 4.2. Let $a > 0$, $p \in [1, +\infty)$ and $u \in L^p_{\text{loc}}(\mathbb{R}^n)$. Observe that $\mu := \mathcal{L}^n \llcorner |u|^p$ is a Radon measure on $\mathbb{R}^n$. Then, by [15, Ch. 6, Ex. 3] with $s := n + ap$, the function

$$x \mapsto \limsup_{r\to 0^+} r^{-a}\left(\fint_{B(x,r)} |u|^p\right)^{1/p}$$

is Borel.

We shall use Theorem 4.1 to prove the following result.

Theorem 4.2. *Let $a > 0$, $p \in [1, +\infty)$ and $u \in L^p_{loc}(\mathbb{R}^n)$. Then, except for x in a null set, the function*

$$x \mapsto \limsup_{r\to 0^+} r^{-a}\left(\fint_{B(x,r)} |u|^p\right)^{1/p}$$

takes values in $\{0, +\infty\}$. In other words, if

$$Z := \left\{x \in \mathbb{R}^n \;\middle|\; 0 < \limsup_{r\to 0^+} r^{-a}\left(\fint_{B(x,r)} |u|^p\right)^{1/p} < +\infty\right\},$$

then $\mathcal{L}^n(Z) = 0$.

Proof. First of all define

$$X := \left\{x \in \mathbb{R}^n \;\middle|\; \limsup_{r\to 0^+} r^{-a}\left(\fint_{B(x,r)} |u|^p\right)^{1/p} < +\infty\right\},$$

$$X_0 := \left\{x \in \mathbb{R}^n \;\middle|\; \left(\fint_{B(x,r)} |u|^p\right)^{1/p} = o(r^a) \text{ as } r \to 0^+\right\}$$

and observe that

$$X = X_0 \cup Z.$$

The proof is divided into two steps.

First step: Assume $u \in L^p(\mathbb{R}^n)$. For $k = 1, 2, \dots,$ define

$$E_k := \left\{ x \in \mathbb{R}^n \;\middle|\; \left(\fint_{B(x,r)} |u|^p \right)^{1/p} \le k r^a \text{ for all } r > 0 \right\}.$$

and observe that (for every single k)

$$\left(\fint_{B(x,r)} |u|^p \right)^{1/p} = o(r^a) \quad \text{as } r \to 0^+$$

at a.e. $x \in E_k$, by Theorem 4.1, namely

$$\mathcal{L}^n(E_k \setminus X_0) = 0. \tag{4.1}$$

If $x \in X$ and put

$$L := \limsup_{r \to 0^+} r^{-a} \left(\fint_{B(x,r)} |u|^p \right)^{1/p}$$

then there exists $r_0 > 0$ such that

$$r^{-a} \left(\fint_{B(x,r)} |u|^p \right)^{1/p} \le L + 1$$

for all $r \le r_0$. Moreover

$$r^{-a} \left(\fint_{B(x,r)} |u|^p \right)^{1/p} \le \omega_n^{-1/p} r^{-a-n/p} \|u\|_p \to 0 \quad \text{as } r \to +\infty.$$

Thus $x \in E_k$ for k large enough, namely $X \subset \cup_k E_k$. Hence the thesis follows at once by recalling (4.1):

$$\mathcal{L}^n(Z) = \mathcal{L}^n(X \setminus X_0) \le \mathcal{L}^n\Big((\cup_k E_k) \setminus X_0 \Big) \le \sum_k \mathcal{L}^n(E_k \setminus X_0) = 0.$$

Second step: Assume $u \in L^p_{\text{loc}}(\mathbb{R}^n)$. For $R > 0$, define $u_R := u \varphi_{B_R}$ and

$$Z_R := \left\{ x \in \mathbb{R}^n \;\middle|\; 0 < \limsup_{r \to 0^+} r^{-a} \left(\fint_{B(x,r)} |u_R|^p \right)^{1/p} < +\infty \right\}.$$

Observe that $Z_R \overset{\circ}{=} Z \cap B_R$, hence

$$\mathcal{L}^n(Z_R) = \mathcal{L}^n(Z \cap B_R).$$

On the other hand, since $u_R \in L^p(\mathbb{R}^n)$, one also has

$$\mathcal{L}^n(Z_R) = 0$$

by the first step. It follows that

$$\mathcal{L}^n(Z \cap B_R) = 0$$

for all $R > 0$, that is $\mathcal{L}^n(Z) = 0$. □

Corollary 4.1. *Let E be a measurable subset of $\mathbb{R}^n$ and let $m > n$. Then, except for x in a null set, the function*

$$x \mapsto \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^m}$$

takes values in $\{0,+\infty\}$. Equivalently

$$\mathcal{L}^n\left(\left\{x\in\mathbb{R}^n \,\middle|\, 0<\limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^m} <+\infty\right\}\right)=0.$$

In terms of $E^{(m)}$ and $b_m(E)$ this means that

$$E^{(m)} \overset{\circ}{=} \left\{x\in\mathbb{R}^n \,\middle|\, \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus E)}{r^m} <+\infty\right\}$$

and

$$b_m(E) \overset{\circ}{=} \left\{x\in\mathbb{R}^n \,\middle|\, \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^m} =+\infty\right\}.$$

Proof. Apply Theorem 4.2 with $u := \varphi_E$, $p := 1$ and $a := m - n$. □

5. Density-degree function

Prior to providing the definition of density-degree function, observe that if E is a subset of $\mathbb{R}^n$ and $x\in\mathbb{R}^n$, then the set $\{k\in[n,+\infty)\,|\,x\in E^{(k)}\}$ is a (possibly empty) interval.

Definition 5.1. *Let E be a subset of $\mathbb{R}^n$. Then define the "density-degree function" $d_E:\mathbb{R}^n\to[0,+\infty]$ as follows*

$$d_E(x) := \begin{cases} \sup\left\{k\geq n\,|\,x\in E^{(k)}\right\} & \text{if } x\in E^{(n)} \\ 0 & \text{if } x\notin E^{(n)}. \end{cases}$$

For $m\in[n,+\infty)$ we also define

$$\mathrm{int}^{(m)}E := \{x\in\mathbb{R}^n\,|\,d_E(x)>m\}, \qquad \mathrm{cl}^{(m)}E := \{x\in\mathbb{R}^n\,|\,d_E(x)\geq m\}$$

and

$$\partial^{(m)}E := \mathrm{cl}^{(m)}E\setminus\mathrm{int}^{(m)}E = \{x\in\mathbb{R}^n\,|\,d_E(x)=m\}.$$

When the following identity holds

$$E \overset{\circ}{=} \partial^{(m)}E = \{x\in\mathbb{R}^n\,|\,d_E(x)=m\}$$

we say that E is a "uniformly m-dense set".

Example 5.1. If E is open, then $d_E(x)=+\infty$ for all $x\in E$. Hence

$$E\subset \mathrm{int}^{(m)}E$$

for all $m\geq n$. Observe that the strict inclusion can occur, e.g. for $E:=B_r\setminus\{0\}$ (in such a case one has $\mathrm{int}^{(m)}E=B_r$).

Example 5.2. Let $m > 2$ and

$$E := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^{m-1} \leq |x_2|\}.$$

Since (as an elementary computation shows)

$$\lim_{r\to 0^+} \frac{\mathcal{L}^2(B(0,r) \setminus E)}{r^m} \in (0, +\infty)$$

holds, one has $d_E(0) = m$. Hence $0 \in \partial^{(m)}E \setminus E^{(m)}$.

This proposition collects some very simple (nevertheless interesting) facts.

Proposition 5.1. *Let E be a subset of $\mathbb{R}^n$ and $m \in [n, +\infty)$. The following properties hold:*

(1) $\partial^{(k)}E \cap \partial^{(m)}E = \emptyset$, *if $k \neq m$ ($k \geq n$);*
(2) $\mathrm{int}^{(m)}E = \bigcup_{k>m} E^{(k)}$*;*
(3) *If $m > n$ then $\mathrm{cl}^{(m)}E = \bigcap_{l\in[n,m)} E^{(l)}$, while $\mathrm{cl}^{(n)}E = E^{(n)}$;*
(4) $\mathrm{int}^{(m)}E \subset E^{(m)} \subset \mathrm{cl}^{(m)}E$*;*
(5) *E is a uniformly m-dense set (with $m \geq n$) if and only if both the following identities hold*

$$\mathrm{cl}^{(m)}E \mathrel{\mathring{=}} E, \quad \mathrm{int}^{(m)}E \mathrel{\mathring{=}} \emptyset; \tag{5.1}$$

(6) *E is a uniformly n-dense set if and only if $\mathrm{int}^{(n)}E \mathrel{\mathring{=}} \emptyset$;*
(7) $\mathrm{int}^{(m)}E$, $\mathrm{cl}^{(m)}E$ *and* $\partial^{(m)}E$ *are measurable sets;*
(8) *The density-degree function d_E is a measurable function.*

Proof. The statements (1), (2) and (3) follow at once from Definition 5.1. The properties (2) and (3) trivially yield (4) and (7).

Let us prove (5). First observe that the "if part" of the statement follows trivially from the definition of $\partial^{(m)}E$. Conversely, if we assume $\partial^{(m)}E \mathrel{\mathring{=}} E$ we get

$$\mathrm{cl}^{(m)}E \subset E^{(n)} \mathrel{\mathring{=}} E \mathrel{\mathring{=}} \partial^{(m)}E = \mathrm{cl}^{(m)}E \setminus \mathrm{int}^{(m)}E \subset \mathrm{cl}^{(m)}E$$

hence $\mathrm{cl}^{(m)}E \mathrel{\mathring{=}} E$ (i.e. the first identity in (5.1)) and $\mathrm{cl}^{(m)}E \mathrel{\mathring{=}} \mathrm{cl}^{(m)}E \setminus \mathrm{int}^{(m)}E$. Since $\mathrm{int}^{(m)}E \subset \mathrm{cl}^{(m)}E$ this last identity yields $\mathrm{int}^{(m)}E \mathrel{\mathring{=}} \emptyset$ (i.e. the second identity in (5.1)).

Now (6) follows at once from (3) and (5).

Finally, observe that for $a \in \mathbb{R}$ one has

$$\{x \in \mathbb{R}^n \mid d_E(x) \geq a\} = \begin{cases} \mathbb{R}^n & \text{if } a \leq 0 \\ E^{(n)} & \text{if } a \in (0,n) \\ \mathrm{cl}^{(a)} E & \text{if } a \geq n \end{cases}$$

by Definition 5.1. Hence (8) follows by the half-line criterion for measurability of functions, e.g. [16, Theorem 11.15]. □

From Proposition 5.1 we obtain the following result.

Proposition 5.2. *Let E be a measurable subset of $\mathbb{R}^n$. Then the set*

$$\left\{m \in (n,+\infty) \mid \mathcal{L}^n(\partial^{(m)} E) > 0\right\}$$

is at most countable.

Proof. Observe that, for all $R > 0$ and $\varepsilon > 0$, the set

$$\left\{m \in (n,+\infty) \mid \mathcal{L}^n(B(0,R) \cap \partial^{(m)} E) \geq \varepsilon\right\}$$

has to be finite, by (1) and (7) in Proposition 5.1. Hence the conclusion follows at once by this easy identity:

$$\left\{m \in (n,+\infty) \mid \mathcal{L}^n(\partial^{(m)} E) > 0\right\} = \bigcup_{k=1}^{+\infty} \left\{m \in (n,+\infty) \mid \mathcal{L}^n(B(0,k) \cap \partial^{(m)} E) \geq \frac{1}{k}\right\}.$$

□

Now we use the machinery above to state a remark and a simple proposition about sets of finite perimeter.

Remark 5.1. Proposition 3.3 shows that m_0 is the maximum order of density which is common to all sets of finite perimeter. Hence this (up to now unanswered) question arises naturally: Does there exist a set E of positive measure and finite perimeter in $\mathbb{R}^n$, such that $\mathrm{int}^{(m_0)} E = \emptyset$ (i.e. $E^{(m)} = \emptyset$ for all $m > m_0$)?

Proposition 5.3. *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then one has*

$$\mathrm{cl}^{(m_0)} E \mathrel{\overset{\circ}{=}} E$$

with $m_0 := n + 1^* = n + 1 + 1/(n-1)$.

Proof. One obviously has

$$E^{(m_0)} \subset \mathrm{cl}^{(m_0)} E \subset E^{(n)}$$

hence the conclusion follows by recalling Theorem 3.2. □

The following result states that a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary can be arbitrarily approximated from inside by closed uniformly n-dense sets.

Proposition 5.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Then for all $C < \mathcal{L}^n(\Omega)$ there exists a closed subset F of $\overline{\Omega}$ such that $\mathcal{L}^n(F) > C$ and $\mathrm{int}^{(n)}F = \emptyset$ (in particular F is uniformly n-dense, by Proposition 5.1).*

Proof. First step: If $\partial\Omega$ is Lipschitz. Let j be an arbitrary positive integer. Then, by Theorem 3.1, there exists an open subset A_j of Ω such that

$$\mathcal{L}^n(A_j) < \frac{\mathcal{L}^n(\Omega) - C}{2^j}, \qquad b_{n+\frac{1}{j}}(A_j) = \overline{\Omega}.$$

Define

$$F_j := \overline{\Omega} \cap A_j^c, \qquad F := \bigcap_{j=1}^{\infty} F_j = \overline{\Omega} \cap \left(\bigcup_{j=1}^{\infty} A_j \right)^c.$$

Then F is closed and

$$\begin{aligned}\mathcal{L}^n(F) &= \mathcal{L}^n(\overline{\Omega}) - \mathcal{L}^n\left(\cup_j A_j\right) \geq \mathcal{L}^n(\overline{\Omega}) - \sum_j \mathcal{L}^n(A_j) > \mathcal{L}^n(\overline{\Omega}) - \Big(\mathcal{L}^n(\overline{\Omega}) - C\Big) \\ &= C.\end{aligned}$$

Moreover, by (3.1), one has

$$\begin{aligned} F_j^{(n+\frac{1}{j})} &= \left[b_{n+\frac{1}{j}}(F_j^c)\right]^c = \left[b_{n+\frac{1}{j}}\Big(A_j \cup (\overline{\Omega})^c\Big)\right]^c \\ &= \left[b_{n+\frac{1}{j}}(A_j) \cup b_{n+\frac{1}{j}}\Big((\overline{\Omega})^c\Big)\right]^c = \left[\overline{\Omega} \cup \Omega^c\right]^c \\ &= \emptyset \end{aligned}$$

for all j. Thus, for each $k > n$ we can find j such that $k > n + \frac{1}{j}$, hence

$$F^{(k)} \subset F_j^{(m)} \subset F_j^{(n+\frac{1}{j})} = \emptyset$$

namely $F^{(k)} = \emptyset$. It follows that

$$\mathrm{int}^{(n)}F = \cup_{k>n} F^{(k)} = \emptyset.$$

Second step: Without assumptions on $\partial\Omega$. Let Ω_1 be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary (e.g. a ball) such that $\Omega \subset \Omega_1$. Then, by the first step, there exists a closed subset F_1 of $\overline{\Omega}_1$ such that

$$\mathcal{L}^n(F_1) > \mathcal{L}^n(\Omega_1) - \mathcal{L}^n(\Omega) + C \tag{5.2}$$

and $\mathrm{int}^{(n)}F_1 = \emptyset$. If we define

$$F := \overline{\Omega} \cap F_1$$

one has (since $F \subset F_1$)

$$\mathrm{int}^{(n)}F \subset \mathrm{int}^{(n)}F_1 = \emptyset, \text{ i.e. } \mathrm{int}^{(n)}F = \emptyset.$$

Moreover

$$\mathcal{L}^n(F) = \mathcal{L}^n(F_1) - \mathcal{L}^n(F_1 \setminus \overline{\Omega}) > \mathcal{L}^n(\Omega_1) - \mathcal{L}^n(\Omega) + C - \mathcal{L}^n(F_1 \setminus \overline{\Omega})$$

by (5.2), where

$$\mathcal{L}^n(\Omega_1) - \mathcal{L}^n(\Omega) = \mathcal{L}^n(\overline{\Omega}_1 \setminus \Omega) \geq \mathcal{L}^n(F_1 \setminus \overline{\Omega})$$

Hence $\mathcal{L}^n(F) > C$. □

Now, on the basis of Proposition 5.4, the following conjecture seems plausible: If $m > n$, then a bounded open subset Ω of $\mathbb{R}^n$ can be arbitrarily approximated from inside by closed uniformly m-dense sets. At the moment, the best we are able to do in this direction is to prove the following result.

Theorem 5.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $m > n$. Then for all $C < \mathcal{L}^n(\Omega)$ and for all $t \in (n, m)$ there exist a closed subset F of $\overline{\Omega}$ and an open subset U of Ω such that:*

(1) *The set $\mathrm{cl}^{(t)} F$ is large, namely: $\mathrm{cl}^{(t)} F \supset \Omega \setminus U$ and $\mathcal{L}^n(U) < \mathcal{L}^n(\Omega) - C$ (hence $F \supset \Omega \setminus U$ and $\mathcal{L}^n(F) > C$);*

(2) *One has $F^{(m)} = \emptyset$ (hence $\mathrm{int}^{(m)} F = \emptyset$).*

In particular, one has $t \leq d_F(x) \leq m$ for all $x \in \Omega \setminus U$.

6. The proof of Theorem 5.1

6.1. **Preliminaries from the proof of Theorem 3.1 (compare [11]).** Let R and β be positive numbers such that

$$\overline{\Omega} \subset B_R, \quad 2^n R^n \geq 1, \quad (2^n R^n + 1)^{\frac{1}{m-n}} \geq 2$$

and

$$\beta > \max\left\{ (2^n R^n + 1)^{\frac{1}{m-n}}, \left(\frac{\varepsilon}{\omega_n}\right)^{1/n} + \frac{n^{1/2}}{2} \right\}.$$

Also define (for $h = 1, 2, \dots$)

$$\rho_h := \left(\frac{\varepsilon}{\omega_n}\right)^{\frac{1}{n}} \beta^{-\frac{hm}{n}} \tag{6.1}$$

and let Λ_h denote the lattice of step β^{-h} (in $\mathbb{R}^n$), i.e. $\Lambda_h := \beta^{-h}\mathbb{Z}^n$. Then put

$$\Gamma_h := \{P \in \Lambda_h \,|\, B(P, \rho_h) \subset \Omega\}, \quad A_h := \bigcup_{P \in \Gamma_h} B(P, \rho_h), \quad A' := \bigcup_{h=1}^{+\infty} A_h.$$

and observe that

$$\#(\Gamma_h) \leq \left(\frac{2R}{\beta^{-h}}\right)^n = 2^n R^n \beta^{nh}. \tag{6.2}$$

Then one has

$$\mathcal{L}^n(A') < \varepsilon.$$

Let A'' be an open set satisfying

$$A'' \supset B_R \backslash \Omega, \qquad \mathcal{L}^n\Big(A'' \backslash [B_R \backslash \Omega]\Big) < \varepsilon - \mathcal{L}^n(A') \tag{6.3}$$

and define

$$A := A' \cup (A'' \cap \Omega) \qquad \text{(which is a subset of } \Omega). \tag{6.4}$$

One has

$$\mathcal{L}^n(A) < \varepsilon, \qquad b_m(A) = \overline{\Omega}.$$

6.2. **The proof of Theorem 5.1. First step: Under the assumption that the boundary of Ω is Lipschitz.** Assume that $\partial\Omega$ is Lipschitz and consider the construction in Section 6.1 above, with

$$\varepsilon := (\mathcal{L}^n(\Omega) - C)\left(1 + \frac{4^n R^n}{\beta^{n(m-t)/(t-n)} - 1}\right)^{-1}.$$

Let $[B_R \backslash \Omega]_\delta$ denote the open δ-neighbourhood of $[B_R \backslash \Omega]$, namely

$$[B_R \backslash \Omega]_\delta := \{x \in \mathbb{R}^n \,|\, \mathrm{dist}(x, [B_R \backslash \Omega]) < \delta\}.$$

Since $\partial\Omega$ is Lipschitz, there exists $\delta_0 > 0$ such that

$$\mathcal{L}^n\Big([B_R \backslash \Omega]_{\delta_0} \backslash [B_R \backslash \Omega]\Big) < \varepsilon - \mathcal{L}^n(A')$$

so $A'' := [B_R \backslash \Omega]_{\delta_0/2}$ satisfies (6.3) as prescribed by the proof of Theorem 3.1. The corresponding definition of A by (6.4) becomes

$$A := A' \cup ([B_R \backslash \Omega]_{\delta_0/2} \cap \Omega).$$

Let

$$F := A^c \cap \overline{\Omega} = A^c \cap \Omega.$$

One has $b_m(A) = \overline{\Omega}$ by Theorem 3.1. Hence, recalling (3.1), we get

$$F^{(m)} = [b_m(F^c)]^c = [b_m(A \cup \Omega^c)]^c = [b_m(A) \cup b_m(\Omega^c)]^c = [\overline{\Omega} \cup \Omega^c]^c = \emptyset$$

which proves (2).

Now define

$$\gamma(\tau) := m - \frac{(m-n)n}{\tau - n}, \quad \tau > n$$

and observe that $\lim_{\tau \to m} \gamma(\tau) = \gamma(m) = m - n > 0$, hence we can find $t_0 \in (n, m)$ such that $\gamma(\tau) > 0$ for all $\tau \in (t_0, m)$. We can assume $t > t_0$ (without loss of generality), so that $\gamma := \gamma(t) > 0$.

For $h = 1, 2, \ldots,$ define

$$\tilde{\rho}_h := \Big(1 + \beta^{h\gamma/n}\Big)\,\rho_h, \qquad V_h := \bigcup_{P \in \Gamma_h} B(P, \tilde{\rho}_h). \tag{6.5}$$

Also let

$$U := \Big(\Omega \cap [B_R \backslash \Omega]_{\delta_0}\Big) \cup \bigcup_{h=1}^{\infty} V_h. \tag{6.6}$$

By (6.1), (6.2), (6.5), (6.6) and recalling that $\beta > 1$, we get

$$\begin{aligned}\mathcal{L}^n(U) &\le \mathcal{L}^n\Big(\Omega \cap [B_R \backslash \Omega]_{\delta_0}\Big) + \sum_{h=1}^{\infty} \mathcal{L}^n(V_h) \\ &< \varepsilon + \omega_n \sum_{h=1}^{\infty} \#(\Gamma_h) \left(1 + \beta^{h\gamma/n}\right)^n \rho_h^n \\ &\le \varepsilon + 2^n R^n \varepsilon \sum_{h=1}^{\infty} \beta^{hn} \left(2\beta^{h\gamma/n}\right)^n \beta^{-hm} \\ &= \varepsilon \left(1 + 4^n R^n \sum_{h=1}^{\infty} \beta^{-h(m-n-\gamma)}\right).\end{aligned}$$

But

$$m - n - \gamma = \frac{n(m-t)}{t-n} > 0$$

thus

$$\mathcal{L}^n(U) < \left(1 + \frac{4^n R^n}{\beta^{n(m-t)/(t-n)} - 1}\right)\varepsilon = \mathcal{L}^n(\Omega) - C.$$

It remains to prove that

$$\mathrm{cl}^{(t)} F \supset \Omega \setminus U. \tag{6.7}$$

To this aim consider $x \in \Omega \setminus U$, $r > 0$ and observe that the set

$$H_x(r) := \{l \ge 1 \mid B(x,r) \cap A_l \ne \emptyset\}$$

includes every h large enough, so we can define the function

$$r \mapsto h_x(r) := \min H_x(r), \quad r > 0$$

which is decreasing in that $H_x(r_1) \subset H_x(r_2)$ whenever $0 < r_1 \le r_2$. Now, the sequence

$$d_k := \mathrm{dist}\left(x, \bigcup_{l=1}^{k} A_l\right) = \mathrm{dist}\left(x, \bigcup_{l=1}^{k} \overline{A_l}\right), \qquad k = 1, 2, \dots$$

is positive, decreasing and infinitesimal. Also, since $B(x, d_k) \cap A_l = \emptyset$ for all $l \le k$, one has

$$h_x(d_k) \ge k + 1.$$

Hence

$$h_x(r) \to +\infty, \text{ as } r \to 0^+.$$

For $r > 0$ one has $B(x,r) \cap A_{h_x(r)} \ne \emptyset$, thus there exists $P \in \Gamma_{h_x(r)}$ such that the ball $B(x,r)$ intersects $B(P, \rho_{h_x(r)})$. Moreover the center x of $B(x,r)$ is outside $B(P, \tilde\rho_{h_x(r)})$ (in that $x \notin U$), so that

$$r \ge \tilde\rho_{h_x(r)} - \rho_{h_x(r)} = \rho_{h_x(r)} \beta^{\frac{h_x(r)\gamma}{n}} = \left(\frac{\varepsilon}{\omega_n}\right)^{\frac{1}{n}} \beta^{-\frac{h_x(r)}{n}(m-\gamma)}$$

i.e.

$$\beta^{-h_x(r)} \le C_1 r^{\frac{n}{m-\gamma}} \tag{6.8}$$

with $C_1 = C_1(m,n,t) := (\omega_n/\varepsilon)^{1/(m-\gamma)}$.

Now observe that if $r < \delta_0/2$ then $B(x,r)$ does not intersect $A'' = [B_R \setminus \Omega]_{\delta_0/2}$, hence

$$\mathcal{L}^n(B(x,r)\cap A) = \mathcal{L}^n(B(x,r)\cap A') \le \sum_{h\ge h_x(r)} \mathcal{L}^n(B(x,r)\cap A_h). \tag{6.9}$$

One obviously has

$$\#(\Gamma_h \cap B(x,r)) \le \#(\Lambda_h \cap B(x,r)) \le C_2 \left(\frac{r}{\beta^{-h}}\right)^n$$

for a suitable $C_2 = C_2(n)$, whereby

$$\mathcal{L}^n(B(x,r)\cap A_h) \le C_2 \left(\frac{r}{\beta^{-h}}\right)^n \omega_n \rho_h^n = C_2 \varepsilon \beta^{-h(m-n)} r^n. \tag{6.10}$$

From (6.8), (6.9) and (6.10) it follows that, for all $r < \delta_0/2$ and $x \in \Omega \setminus U$, one has

$$\begin{aligned}
\mathcal{L}^n(B(x,r)\cap A) &\le C_2\, \varepsilon r^n \sum_{h\ge h_x(r)} \beta^{-h(m-n)} \\
&= \frac{C_2\, \varepsilon r^n \beta^{-h_x(r)(m-n)}}{1-\beta^{-(m-n)}} \\
&\le \frac{C_3\, \varepsilon}{1-\beta^{-(m-n)}} r^{n+\frac{n(m-n)}{m-\gamma}} \\
&= \frac{C_3\, \varepsilon\, r^t}{1-\beta^{-(m-n)}}
\end{aligned}$$

with $C_3 = C_3(m,n,t) := C_1^{m-n} C_2$. Combining this result with the identity

$$B(x,r)\setminus F = B(x,r)\cap (A \cup (\overline{\Omega})^c) = B(x,r)\cap A$$

which holds for all $x \in \Omega$ and r small enough, we finally obtain

$$\mathcal{L}^n(B(x,r)\setminus F) = o(r^s), \text{ as } r \to 0$$

for all $x \in \Omega \setminus U$ and for all $s \in [n,t)$. This proves that

$$\Omega \setminus U \subset \bigcap_{l\in[n,t)} F^{(l)}$$

namely (6.7), by (3) of Proposition 5.1.

6.3. **The proof of Theorem 5.1. Second step: Without assumptions on $\partial\Omega$.** Let Ω_1 be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary (e.g. a ball) such that $\Omega \subset \Omega_1$ and define

$$C_1 := \mathcal{L}^n(\Omega_1) - \mathcal{L}^n(\Omega) + C. \tag{6.11}$$

Then, by the first step (Section 6.2), there exist a closed subset F_1 of $\overline{\Omega}_1$ and an open subset U_1 of Ω_1 such that

$$\mathrm{cl}^{(t)} F_1 \supset \Omega_1 \setminus U_1, \qquad \mathcal{L}^n(\Omega_1 \setminus U_1) > C_1, \qquad F_1^{(m)} = \emptyset. \tag{6.12}$$

If we define

$$F = \overline{\Omega} \cap F_1, \qquad U := \Omega \cap U_1$$

then

$$\mathcal{L}^n(\Omega \setminus U) = \mathcal{L}^n(\Omega) - \mathcal{L}^n(U) \geq \mathcal{L}^n(\Omega_1) + C - C_1 - \mathcal{L}^n(U_1) > C$$

by (6.11) and the inequality in (6.12). Moreover (since $F \subset F_1$) one has

$$F^{(m)} \subset F_1^{(m)} = \emptyset, \text{ i.e. } F^{(m)} = \emptyset.$$

It remains to prove that

$$\Omega \setminus U \subset \mathrm{cl}^{(t)} F$$

which is equivalent to

$$\Omega \setminus U \subset F^{(s)}, \text{ for all } s \in [n, t) \tag{6.13}$$

by (3) of Proposition 5.1. Observe that

$$\Omega \setminus U = \Omega \cap (\Omega^c \cup U_1^c) = \Omega \cap U_1^c = [\Omega_1 \setminus (\Omega_1 \setminus \Omega)] \cap U_1^c = \Omega_1 \cap U_1^c \cap (\Omega_1 \setminus \Omega)^c$$

hence

$$\Omega \setminus U \subset F_1^{(s)} \cap (\Omega_1 \setminus \Omega)^c$$

for all $s \in [n, t)$, by the inclusion in (6.12) and recalling again (3) of Proposition 5.1. Thus, in order to prove (6.13), it is sufficient to show that

$$F_1^{(s)} \cap (\Omega_1 \setminus \Omega)^c \subset F^{(s)}, \text{ for all } s \in [n, t). \tag{6.14}$$

To this aim, consider $s \in [n, t)$ and observe that $F_1^{(s)} \cap \Omega_1^c = \emptyset$ (in that $F_1 \subset \overline{\Omega}_1$ and $\partial\Omega_1$ is Lipschitz), hence

$$F_1^{(s)} \cap (\Omega_1 \setminus \Omega)^c = F_1^{(s)} \cap (\Omega_1^c \cup \Omega) = F_1^{(s)} \cap \Omega.$$

So, for all $x \in F_1^{(s)} \cap (\Omega_1 \setminus \Omega)^c$, one has

$$\mathcal{L}^n(B(x, r) \setminus F_1) = o(r^s), \text{ as } r \to 0$$

and $B(x, r) \subset \Omega$ for r sufficiently small. It follows that

$$B(x, r) \setminus F = B(x, r) \cap [(\overline{\Omega})^c \cup F_1^c] = B(x, r) \cap F_1^c$$

for r sufficiently small, and

$$\mathcal{L}^n(B(x, r) \setminus F) = o(r^s), \text{ as } r \to 0$$

which proves (6.14).

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