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Selected Topics in Analysis in Metric Measure Spaces

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Contents

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Declaration of Authorship

I hereby declare that this thesis, titled "Selected Topics in Analysis in Metric Measure Spaces" and submitted to the University of Trento in support of my application for the degree of Doctor of Philosophy, is the result of my own research. The work presented was carried out by myself except in the cases listed below:

- The first chapter is introductory and does not contain any new results.
- The second chapter is joint work with Professor Andrea Pinamonti (University of Trento) and Professor Gareth Speight (University of Cincinnati, US). The original paper [22] has been accepted for publishing, the version presented in this thesis is slightly modified with respect to the original. Part of the original article was done while I and A. Pinamonti were visiting the University of Cincinnati (supported by funding from the Taft Research Center and the University of Trento) and while G. Speight was visiting the University of Trento (supported by funding from the University of Trento).
- The third chapter is joint work with fellow PhD student Alberto Maione (University of Trento) and researchers Dr. Ariel Salort (University of Buenos Aires, AG) and Dr. Eugenio Vecchi (University of Milan). It was published in [21]. As for the second chapter, the version presented in this thesis has been adjusted for practical reasons. The same article was also part of the PhD thesis of Alberto Maione.
- The fourth chapter is again joint work with Professor Andrea Pinamonti (University of Trento) and Professor Gareth Speight (University of Cincinnati, US).

All the information derived from the literature have been acknowledged and a list of references is provided at the end of the thesis.

This dissertation was not previously presented for any degree of any level in this or other institutions.

Introduction

In recent years it has become clear that a large part of geometric analysis, geometric measure theory and real analysis in Euclidean spaces may be generalized to more general settings, see for example [20, 24, 39, 44, 52, 66, 73, 77].

One important area of research studies these topics in Carnot groups. Carnot groups admit a great deal of structure, including translations, dilations, path distance and a Haar measure. Hence it is natural to ask which Euclidean results have a Carnot group generalization. Those generalizations range from classical theorems like the Whitney extension theorem, Lusin approximation, Pansu's Rademacher theorem and the study of sets of finite perimeter, to more recent results, like the study of gradients flows, potential theory, geometric measure theory and PDEs [11, 36, 37, 39, 40, 46, 53, 61, 63, 65, 75, 76, 78, 86, 88, 91]. In Chapter 2 we present problems related to Lusin's approximation result in the Heisenberg group, a special case of Carnot group.

Another prolific trend seeks to find more generalized results in the theory of Sobolev spaces. Starting from the seminal work in the Euclidean setting done by Bourgain, Brezis and Mironescu [12], many authors have taken their work in several directions both in the Euclidean setting and in the Carnot one.

Generalizations came in many forms, from changing the underlying space [6], to changing the power law [9]. Among the many works we also cite [10, 26, 44, 50, 61, 64, 71, 79, 80, 87]. We devote Chapter 3 to the study of asymptotic results in the Carnot-Sobolev setting when the power law considered comes from an Orlicz function.

A third direction is the study of PI spaces, introduced for the first time in [43]. Those are metric spaces that admit a doubling measure and a Poincaré inequality. PI spaces can be very different from the usual Euclidean spaces or even from Carnot groups, nonetheless it is possible to recover a large part of classical results in geometric measure theory and its applications. Some of the relevant work in this field are [7, 24, 47, 48, 73, 82, 85]. In Chapter 4 we present results about the differentiability of Lipschitz functions in Laakso spaces, an example of PI spaces introduced for the first time in [51].

The structure of the thesis is as follows.

The first chapter is introductory and contains the relevant mathematical background that is shared through the thesis.

The second chapter corresponds to [22]. In this work we tackled the C^m Lusin approximation theorem in the Heisenberg group $\mathbb{H}^n$. The classical result [57] gives conditions under which a curve in $\mathbb{R}^n$ admits a C^m approximation outside of a set of small measure. It relies on the notion of approximate differentiability. In the Heisenberg group the more relevant curves are the horizontal ones. These are absolutely continuous curves whose tangent lies in a distinguished sub-bundle of the tangent bundle and they are also used to define the Carnot-Charathéodory distance. To obtain a result that respects the horizontal structure it is necessary to look for additional hypotheses. This resulted in two main theorems: Theorem 2.3.1 and Theorem 2.4.1. The first illustrates the C^m Lusin approximation for horizontal curves and contains new assumptions with respect to the Euclidean counterpart. The second result shows why these additional assumptions are optimal and cannot be weakened, illustrating the differences with the Euclidean case.

The third chapter corresponds to [21]. In this paper we introduced the concept of Fractional Orlicz-Sobolev spaces in the context of Carnot groups and studied their properties. In the Euclidean setting Fractional Orlicz-Sobolev spaces were introduced to generalize the classical limiting embedding theorems from Bourgain, Brezis and Mironescu [12], in the case in which behaviours more general than power laws are considered. In recent years results in the spirit of [12], but generalized to the setting of Carnot groups, have begun to appear in the literature. This, together with [9], inspired our idea for the paper. The main result we obtained is Theorem 3.2.1. This is a Bourgain-Brezis-Mironescu type formula for Orlicz-Sobolev spaces in Carnot groups.

In the fourth chapter the focus moves to PI spaces and in particular to Laakso spaces. They were first introduced in [51] as an example of a doubling metric measure space with a Poincaré inequality that cannot be bi-Lipschitz embedded in $\mathbb{R}^n$ for any n. Laakso showed the remarkable fact that his construction gives a space of Hausdorff dimension Q for any given $Q > 1$. The main characteristic that makes Laakso spaces so difficult to work with is the fact that they have fractional dimension and do not possess a group structure. Properties that are used when working in the Euclidean or in the Carnot setting, like the existence of a family of dilations and translations, do not make sense in Laakso spaces. For this reason the usual approach to many problems fails and new methodologies must be employed. With this in mind we decided to investigate the relationship between the existence of maximal directional derivatives and the differentiability of Lipschitz functions in Laakso spaces. We also studied the differentiability of the distance function. In this chapter we present our most recent research in this area.

Chapter 1

Preliminaries

In this section we will recall the basic notions needed for the main chapters. First we will recall some general facts about metric measure spaces and differentiability spaces. Then we will give three examples of differentiability space by introducing Carnot groups, the Heisenberg group and Laakso spaces. They will be the main settings for the thesis.

More specific results that are needed only for a single chapter will be listed at the beginning of each chapter.

1.1 Differentiability Spaces

Definition 1.1.1. A *metric measure space* is a complete and separable metric space (X, d) equipped with a locally finite and positive Borel measure μ. Metric measure spaces are denoted by (X, d, μ) or simply (X, μ).

The metric measure spaces studied in this thesis will have two additional assumptions: the doubling property and the Poincaré inequality.

Definition 1.1.2. A Borel measure μ on a metric space (X, d) is said to be doubling (w.r.t. the distance d) if there exists a constant $C \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

for all $r > 0$ and $x \in X$.

Roughly speaking, being doubling means that the measure of balls does not explode too quickly when the radius grows.

A classic example of a doubling measure on a metric space is the n-dimensional Lebesgue measure on $\mathbb{R}^n$ with the Euclidean distance. In this case we also know the exact value of the constant: $C = 2^n$. On the other hand it is possible to define a Borel measure on $\mathbb{R}$ that is absolutely continuous wuth respect to the Lebesgue measure but is not doubling. Take for example $[a,b] \subset \mathbb{R}$ and define a measure μ on $\mathbb{R}$ as

$$\mu(A) = \int_A \chi_{[a,b]}(x)dx$$

where $\chi_{[a,b]}$ is the characteristic function of $[a,b]$. Then it is easy to see that $\mu \ll \mathcal{L}^1$ but μ is not doubling on $\mathbb{R}$.

The classical Poincaré inequality in $\mathbb{R}^n$ is a tool to control the oscillation of a function on a ball by means of its derivatives.

Theorem 1.1.3 (Poincaré inequality)**.** *Let $B \subset \mathbb{R}^n$ be a ball and $1 \leq p < \infty$. Then there exists a constant C, depending only on n and p, such that*

$$\fint_B |u - u_B| \leq C(n,p)\,\mathrm{diam}(B)\left(\fint_B |\nabla u|^p\right)^{1/p}$$

for all $u \in W^{1,p}(B)$. Here $\fint$ denotes the average integral and $u_b := \fint_B u$.

In general metric measure space we may not have a suitable notion of derivative. One way to solve this problem is with the use of *upper gradients.*

Definition 1.1.4. A Borel function $\rho : X \to [0,\infty]$ is an *upper gradient* for a function $u : X \to \mathbb{R}$ if

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma \rho\, ds \tag{1.1.1}$$

for every rectifiable curve $\gamma : [a,b] \to X$. Here with $\int_\gamma \rho\, ds$ we mean the line integral as defined, for example, in [44, Chapter 5].

The name upper gradient comes from the fact that it was introduced to replace the standard notion of gradient in general metric measure spaces. If we chose $(X,d,\mu) = (\mathbb{R}^n, d_E, \mathcal{L}^n)$ (d_E is the Euclidean distance) it is possible to see that

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma |\nabla u|\, dx$$

for any $\gamma : [a,b] \to \mathbb{R}^n$ rectifiable, i.e. the modulus of the classical gradient is an upper gradient.

Definition 1.1.5 (Poincaré inequality for metric measure spaces)**.** We say that the metric measure space X support a p-Poincaré inequality if every ball $B \subset X$ has positive finite measure and if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\fint_B |u - u_B| d\mu \leq C\ \mathrm{diam}(B)\left(\fint_{\lambda B} \rho^p d\mu\right)^{1/p} \tag{1.1.2}$$

for every open ball B, for every integrable function $u : B \to X$ and for every ρ that is an upper gradient for u. Here λB is the ball with the same center of B but radius scaled by a factor λ.

From the Hölder inequality it follows that if a space supports a p-Poincaré inequality, then it also supports a q-Poincaré inequality for every $q \geq p$.

As shown for example in [44], $\mathbb{R}^n$ supports a 1-Poincaré inequality. On the other hand, there are spaces that does not support any p-Poincaré inequality. An example in this sense is a totally disconnected metric measure space in which the only rectifiable curves are the constant ones. Between these extremal cases, there are a plethora of examples of metric measure spaces that support a p-Poincaré inequality for some $p > 1$, see for example [62]. The interested reader will find an in depth study of upper gradients and p-Poincaré inequalities in [44].

Notation 1.1.6. A doubling metric measure space that supports a p-Poincaré inequality for some $p \geq 1$ is called a *PI space*.

There are several directions in which the study of PI spaces can branch. One is the study of differentiability of functions in metric measure spaces, see for example [3, 24, 56, 73]. To this regard, in [24] Cheeger proved that Rademacher's theorem holds in these spaces, meaning that Lipschitz functions are differentiable μ-almost everywhere. More precisely, given a metric measure space (X, d, μ), Cheeger showed the existence of a family of Borel charts $\{(U_i, \phi_i)\}_{i\in I}$ and of positive integers $\{n_i\}_{i\in I}$ with the following properties:

i) The family $\{U_i\}_{i\in I}$ covers X in the sense that $U_i \subset X$ is Borel for each i and
$$\mu\left(X \setminus \bigcup_{i\in I} U_i\right) = 0.$$

ii) $\phi_i : U_i \to \mathbb{R}^{n_i}$ is Lipschitz for each i.

iii) Every Lipschitz function $f : X \to \mathbb{R}$ is differentiable at x for μ-almost every $x \in U_i$, i.e. there exists a unique $df(x) \in \mathbb{R}^{n_i}$ such that
$$\limsup_{y\to x} \frac{|f(y) - f(x) - \langle df(x), \phi_i(y) - \phi_i(x)\rangle|}{d(x, y)} = 0.$$

Another direction is to find an inverse to Rademacher's theorem. This means, given a null set $N \subset X$, find a Lipschitz map $f : X \to \mathbb{R}$ that is differentiable at no point of N. This question lead to the notion of *universal differentiability set* and much research has been done in this regard, both in the Euclidean setting and in more general metric measure spaces: [31, 32, 33, 82, 83].

In the works presented here I decided to focus on two classes of PI spaces: Carnot groups and Laakso spaces. The remainder of this introduction will be used to briefly introduce those spaces, more detailed information will be given in the next chapters.

1.1.1 Carnot Groups

To introduce Carnot groups we need the definitions of Lie algebra and Lie group.

Definition 1.1.7. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with an alternating bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *Lie bracket*, that satisfies the Jacobi identity, i.e. such that

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

for each $x, y, z \in \mathfrak{g}$.

A *Lie group* is a finite dimensional smooth manifold with a group structure in which the group operations of multiplication and inversion are smooth maps.

An elementary example of Lie algebra is $\mathfrak{g} = \mathbb{R}^3$ with the bracket operation defined as the cross product of vectors: $[x, y] = x \times y$. An elementary example of a Lie group is $(\mathbb{R}, +)$.

Definition 1.1.8. A *Carnot group* $\mathbb{G}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification. Namely, there exist linear subspaces $V_1, \dots, V_k$ such that

i) $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$;

ii) $[V_1, V_i] := \operatorname{span}\{[X, Y] : X \in V_1, Y \in V_i\} = V_{i+1}$ for $1 \leq i \leq k-1$;

iii) $V_k \neq \{0\}$ and $[V_1, V_k] = \{0\}$.

$V_1, \dots, V_k$ are called *layers* and $k \in \mathbb{N}$ is called the *step* of the Carnot group.

A key concept when working with Carnot groups is the exponential map.

Definition 1.1.9. Let $\mathbb{G}$ be a Carnot group and $\mathfrak{g}$ its Lie algebra. The *exponential map* of $\mathbb{G}$ is the map $\exp : \mathfrak{g} \to \mathbb{G}$ defined by

$$\exp(X) = \gamma(1)$$

where $\gamma : \mathbb{R} \to \mathbb{G}$ is the unique one-parameter subgroup of $\mathbb{G}$ whose tangent vector at the identity is X. More explicitly exp is the value at the time $t = 1$ of the path $\gamma(t)$ that is the unique solution of the Cauchy problem

$$\begin{cases} \gamma'(t) = X(\gamma(t)) \\ \gamma(0) = 0. \end{cases}$$

As shown for example in [11] the exponential map is a global diffeomorphism.

Take $x \in \mathbb{G}$. Then x is the image of some $p \in \mathfrak{g}$ via the exponential map:

$$\mathbb{G} \ni x = \exp(p).$$

Since the Lie algebra of a Carnot group is a finite dimensional vector space, we can chose a basis $X_1, \dots, X_n$ for it. We can use this basis to define a map that identifies the Lie algebra and hence, via exp, $\mathbb{G}$ with $\mathbb{R}^n$. In particular we can choose a basis that is adapted to the stratification.

Definition 1.1.10. Let $\mathbb{G}$ be a Carnot group for which $\mathfrak{g} = V_1 \oplus \dots \oplus V_k$ and denote by m_i the dimension of V_i. A basis $X_1, \dots, X_n$ for $\mathfrak{g}$ is said to be *adapted to the stratification* if $X_1, \dots, X_{m_1}$ is a basis for V_1, $X_{m_1+1}, \dots, X_{m_1+m_2}$ is a basis for V_2, and so on.

Definition 1.1.11. The *exponential coordinate map* is defined as

$$\begin{aligned} \mathbb{G} &\longrightarrow \mathbb{R}^n \\ x &\mapsto (a_1, \dots, a_n) \end{aligned}$$

where $(a_1, \dots, a_n) \in \mathbb{R}^n$ are the unique real coefficients such that $p = \sum_{i=1}^n a_i X_i$ and $x = \exp(p)$.

The explicit expression of the group law of $\mathbb{G}$ can be deduced from the group law of $\mathfrak{g}$, using the fact that the exponential map is invertible (see [11, Chapter 2]). Given $x, y \in \mathbb{G}$ and denoting by $\cdot$ the operation in $\mathbb{G}$ and by $*$ the operation in $\mathfrak{g}$ we have

$$x \cdot y = \exp\left(\exp^{-1}(x) * \exp^{-1}(y)\right).$$

The group law can then be used to define a diffeomorphism, usually called *left–translation* $\tau_y : \mathbb{G} \to \mathbb{G}$ for every $y \in \mathbb{G}$, defined as

$$\tau_y(x) := y \cdot x \quad \text{for every } x \in \mathbb{G}.$$

Let $h_0 = 0$ and $h_i = m_1 + \dots + m_i$ for $1 \le i \le k$. Denoting points of $\mathbb{G}$ by $(a_1, \dots, a_n) \in \mathbb{R}^n$, the *homogeneity* of a_j is the number $d_j \in \mathbb{N}$ defined by

$$d_j = i \text{ whenever } h_{i-1} + 1 \le j \le h_i.$$

For any $\lambda > 0$ the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is defined in coordinates by

$$\delta_\lambda(a_1, \dots, a_n) = (\lambda^{d_1} a_1, \dots, \lambda^{d_n} a_n).$$

The dilation satisfies $\delta_\lambda(x \cdot y) = \delta_\lambda(x) \cdot \delta_\lambda(y)$.

Definition 1.1.12. A Borel measure μ on a Carnot group $\mathbb{G}$ such that

$$\mu(gE) = \mu(E) \text{ for each } g \in \mathbb{G} \text{ and each Borel set } E \subset \mathbb{G}$$

is called a *Haar measure on* $\mathbb{G}$.

Haar measures are unique up to scaling by positive constants, hence sets of measure zero are defined without ambiguity. If we identify $\mathbb{G}$ with $\mathbb{R}^n$ then the n-dimensional Lebesgue measure $\mathcal{L}^n$ provides a Haar measure on $\mathbb{G}$, see e.g. [11, Proposition 1.3.21].

We introduce the concept of horizontal curve. It will be used to defined a distance on $\mathbb{G}$ and it will also play a major role in Chapter 2.

Recall that a curve $\gamma : [a, b] \to \mathbb{R}^n$ is *absolutely continuous* if it is differentiable almost everywhere, $\gamma' \in L^1([a, b])$ and

$$\gamma(t) = \gamma(a) + \int_a^t \gamma'(s)ds$$

whenever $t \in [a, b]$.

From now on the dimension of the first layer will be denoted simply with m instead of m_1.

Definition 1.1.13. An absolutely continuous curve $\gamma : [a, b] \to \mathbb{G}$ is called *horizontal* it there exist a vector of m functions $g = (g_1, \dots, g_m) \in L^1([a, b])$ such that

$$\gamma'(t) = \sum_{i=1}^{m} g_i(t) X_i(\gamma(t))$$

for almost every $t \in [a, b]$.

We define the *horizontal length* of such a curve as

$$L(\gamma) = \int_a^b |g(t)|\, dt.$$

Remarkably, any two points in $\mathbb{G}$ can be connected by a horizontal curve, as stated by the Chow-Rashevskii Theorem [11, Theorem 19.1.3].

Definition 1.1.14. The *Carnot-Carathéodory distance* d_{CC} (or simply d) on a Carnot group $\mathbb{G}$ is defined, for every $x, y \in \mathbb{G}$, by

$$d(x, y) = \inf\{L(\gamma) \ : \ \gamma : [0, 1] \to \mathbb{G} \text{ is and horizontal curve joining } x \text{ to } y\}.$$

For convenience we will denote $d(x, 0)$ simply by $d(x)$ when needed.

For each $x, y, z \in \mathbb{G}$ and $\lambda > 0$, the Carnot-Carathéodory distance satisfies the following relations with the dilations and translations introduced above:

i) $d(z \cdot x, z \cdot y) = d(x, y)$

ii) $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$.

The topology induced by this distance is the same as the topology induced by the Euclidean distance, however the two distances are not bi-Lipschitz equivalent. Indeed it can be proven that for any compact $K \subset \mathbb{G}$ there exist constants $c_1, c_2 > 0$ such that

$$c_1|x - y| \leq d(x, y) \leq c_2|x - y|^{1/k} \text{ for each } x, y \in K.$$

It has been shown that a Carnot group equipped with the Lebesgue measure and the Carnot-Carathéodory distance supports a 1-Poincaré inequality. This, thanks to the Hölder inequality, implies that Carnot groups support a p-Poincaré inequality for any $p \geq 1$. For the proof of this fact we remand to [41] and the references therein.

1.1.2 The Heisenberg Group

The focus of chapter 2 will be entirely on the Heisenberg group, a particular example of a Carnot group.

Definition 1.1.15. The *Heisenberg group* $\mathbb{H}^n$ is the Carnot group whose Lie algebra $\mathfrak{h}_n$ admits a stratification of the form

$$\mathfrak{h}_n = V_1 \oplus V_2.$$

A basis for $\mathfrak{h}_n$ that is adapted to the stratification is $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ where $V_1 = \text{span}\{X_i, Y_i : i \leq 1 \leq n\}$ and $V_2 = \text{span}\{T\}$. The only non-trivial Lie brackets are

$$[X_i, Y_i] = -4T \ i \leq 1 \leq n. \tag{1.1.3}$$

$\mathbb{H}^n$ can be represented in coordinates by $\mathbb{R}^{2n+1}$, whose points we denote by (x, y, t) with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The group law is given by:

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i') \right). \tag{1.1.4}$$

The identity is $(0_n, 0_n, 0) \in \mathbb{R}^{2n+1}$ and the inverse of $(x, y, t) \in \mathbb{R}^{2n+1}$ is $(-x, -y, -t)$. The left translation by $p \in \mathbb{H}^n$ is the mapping $\tau_p : \mathbb{H}^n \to \mathbb{H}^n$ defined by $\tau_p(q) = p \cdot q$.

Similarly we can express the dilations in coordinates. For any $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ is defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

We equip $\mathbb{H}^n$ with vector fields

$$X_i = \partial_{x_i} + 2y_i \partial_t, \quad Y_i = \partial_{y_i} - 2x_i \partial_t, \quad 1 \leq i \leq n, \quad T = \partial_t.$$

Here $\partial_{x_i}, \partial_{y_i}$ and ∂_t denote the coordinate vectors in $\mathbb{R}^{2n+1}$, which may be interpreted as operators on differentiable functions.

Clearly, for the Heisenberg group $\mathbb{H}^n$ the Haar measure is given by the $(2n+1)$-dimensional Lebesgue measure. We can specialize Definition 1.1.13 to the simplest case of the Heisenberg group.

Definition 1.1.16. A vector in $\mathbb{R}^{2n+1}$ is *horizontal* at $p \in \mathbb{R}^{2n+1}$ if it is a linear combination of the vectors $X_i(p), Y_i(p), 1 \leq i \leq n$.

An absolutely continuous curve γ in the Heisenberg group is *horizontal* if, at almost every point t, the derivative $\gamma'(t)$ is horizontal at $\gamma(t)$.

We can characterize horizontal curves by the behaviour of their last coordinate, as expressed in the following lemma.

Lemma 1.1.17. *An absolutely continuous curve $\gamma\colon [a, b] \to \mathbb{R}^{2n+1}$ is an horizontal curve in the Heisenberg group if and only if, for $t \in [a, b]$:*

$$\gamma_{2n+1}(t) = \gamma_{2n+1}(a) + 2 \sum_{i=1}^{n} \int_a^t (\gamma_i' \gamma_{n+i} - \gamma_{n+i}' \gamma_i).$$

This lemma will be used repeatedly throughout the thesis.

The first Heisenberg group $\mathbb{H}^1 = \mathbb{H}$

In this thesis I will restrict to the first Heisenberg group $\mathbb{H}^1$, which is simply denoted by $\mathbb{H}$. The results in $\mathbb{H}^n$ are analogous.

In the first Heisenberg group $\mathbb{H}$, the relevant equations for an absolutely continuous curve to be horizontal simplify to

$$\gamma_3(t) = \gamma_3(a) + 2 \int_a^t (\gamma_1' \gamma_2 - \gamma_2' \gamma_1). \tag{1.1.5}$$

1.1.3 Laakso Spaces

The second class of PI spaces that we are interested in is the class of Laakso spaces, introduced by Laakso in [51]. The main idea behind Laakso's construction is to start from a Cartesian product involving some Cantor-like fractal and from there to create a set of identifications, called wormholes. Those identifications are then used to make sure that the resulting set is path-connected and that the Laakso space is a metric measure space in which we have a notion of geodesic.

To start the construction choose $1 < Q < 2$ and let t satisfy

$$\log_{1/t} 2 = Q - 1.$$

From this imposition we have that $0 < t < \frac{1}{2}$. Denote with $K \subset [0,1]$ the Cantor-like fractal satisfying the set equation $K = tK \cup (tK + 1 - t)$ and notice that it has Hausdorff dimension $Q - 1$. We now have the starting set in which we want to define the identifications: take $I = [0,1]$ and consider $I \times K$ with the metric induced by $\mathbb{R}^2$. The height of $(x_1, x_2) \in I \times K$ will be denoted by $h(x_1, x_2) := x_1$.

Remark 1.1.18. It may seem counter-intuitive to say that the first component is the vertical one, however this is the notation used by Laakso in his original work [51]. We decided to keep the same notation for practical reasons.

Choose the unique integer n such that $\frac{1}{n+1} < t \leq \frac{1}{n}$ and let $\mathbf{m} = \{m_k\}$ be a sequence of integers n or $n+1$ such that, for every i

$$\frac{n}{n+1} \prod_{j=1}^{i} m_j^{-1} \leq t^i \leq \frac{n+1}{n} \prod_{j=1}^{i} m_j^{-1}. \tag{1.1.6}$$

Once a sequence $\mathbf{m}$ has been fixed we define a family of functions $\Omega = \{\omega_k\}$, one for each positive integer k, that will be used to locate the levels at which the identifications happen. Each ω_k has k inputs $n_1, \ldots, n_k$, maps to $[0,1]$ and is defined as

$$\omega_k(n_1, \ldots, n_k) = \sum_{j=1}^{k} n_j \prod_{h=1}^{j} m_h^{-1} \tag{1.1.7}$$

where the n_j are integers such that $0 \leq n_j < m_j$ for each j and moreover $n_k \neq 0$ for $k \geq 2$.

For example in the model case in which K is the middle third Cantor set we have that $t = 1/3$. In this case it is easy to see that the choice $m_j = 3$ for each j satisfies (1.1.6). With this particular choice (1.1.7) becomes

$$\omega_k(n_1, \ldots, n_k) = \frac{n_1}{3} + \cdots + \frac{n_k}{3^k}.$$

The condition on n_k ensures that if $\omega = \omega_k(n_1, \dots, n_k)$ for certain k and $n_1, \dots, n_k$, then it is not possible to find h and $n'_1, \dots, n'_h$ such that $\omega = \omega_h(n'_1, \dots, n'_h)$. This in particular means that there is no overlapping among different levels of identification, i.e. a wormhole of level k is only a wormhole of that particular level and can not be found at previous or successive levels.

Definition 1.1.19. $l \in [0, 1]$ is said to be a *wormhole level* if $l = \omega_k(n_1, \dots, n_k)$ for an appropriate k. Notice that, since wormhole levels do not overlap, k is unique and hence we can actually speak of wormhole level of order k, or k-wormhole.

Notice that 0 is a wormhole level of order 1, however 1 is never a wormhole level. Nonetheless we can go as near as we want to 1 with wormhole levels of increasing order. We are now ready to define the identifications. The idea is that the Cantor-like fractal can be seen as formed by cells of various depth. To see that define the functions $f_0(x) = tx$ and $f_1(x) = tx + 1 - t$. The sets K_0 and K_1 are then defined as $K_0 = f_0(K) = tK$ and $K_1 = f_1(K) = tK + 1 - t$. Similarly denote with $K_{00} = f_0(K_0)$ and $K_{01} = f_0(K_1)$ the second iteration of K_0:

$$K_{00} = f_0(K_0) = t(K_0) = t^2 K$$
$$K_{01} = f_0(K_1) = t(K_1) = t^2 K + t - t^2.$$

More in general if a is a string of 0s and 1s, then the concatenations K_{a0} and K_{a1} will denote the sets $f_a(K_0)$ and $f_a(K_1)$. Here f_a denotes the composition of functions f_0 and f_1 that comes from the string a. For more details about self-similar fractals we remand to [45].

For any given $n \in \mathbb{N}$ there are 2^n cells of depth n, one for each string of length n made up by 0s and 1s. Wormholes of level k will be used to jump among cells of the same depth.

Definition 1.1.20. We define an equivalence relation $\sim$ on $I \times K$ as follows. For each $n \in \mathbb{N}$ and wormhole level $\{w(m_1, \dots, m_n)\}$ of order n, identify pairwise $\{w(m_1, \dots, m_n)\} \times K_{a0}$ and $\{w(m_1, \dots, m_n)\} \times K_{a1}$ for each binary string a of length $n-1$. More precisely, a point $(x_1, x_2) \in \{w(m_1, \dots, m_n)\} \times K_{a0}$ is identified with $(x_1, x_2 + t^{n-1}(1-t)) \in \{w(m_1, \dots, m_n)\} \times K_{a1}$. Such a point is called a wormhole of order n.

The Laakso space is the set of equivalence classes $F := (I \times K)/\sim$.

Define $s\colon I \times K \to F$ by $s(x_1, x_2) = [x_1, x_2]$, where $[x_1, x_2]$ denotes the equivalence class in F of $(x_1, x_2) \in I \times K$. We define the height $h\colon F \to I$ by $h([x_1, x_2]) = x_1$. Notice this is well defined because points identified in the construction of F have the same coordinate in I. We define a metric d on F by

$$d(x, y) = \inf\{\mathcal{H}^1(p)\colon s(p) \text{ is a path joining } x \text{ and } y\},$$

where paths p are considered in $I \times K$. The following proposition is [51, Proposition 1.1]. It describes the behaviours of geodesics in F.

Proposition 1.1.21. *Let $x, y \in F$ and assume $h(x) \geq h(y)$. Let $[a, b] \subset I$ be a smallest possible interval that contains heights of x and y and all the wormhole levels needed to connect those points with a path. Let p be any path starting from x, going downwards to height a, then upwards to height b and then again downwards to y. Then p is a geodesic connecting x and y. All the geodesics are of that form for some interval $[a', b']$ such that $b' - a' = b - a$. We will call $[a, b]$ a* minimal height interval *for x and y.*

The following Proposition is [51, Proposition 1.2]. It relates minimal height intervals to the distance among points in F.

Proposition 1.1.22. *Let $x, y \in F$ and let $[a, b]$ be a minimal height interval for the two points. Then*

$$d(x, y) = 2b - 2a - |h(x) - h(y)|.$$

It is shown in [51] that the metric measure space $(F, d, \mathcal{H}^Q)$ does support a Poincaré inequality and that it is Ahlfors Q-regular and hence doubling. Thus it is a PI space.

Chapter 2

A C^m Lusin Approximation Theorem for Horizontal Curves in the Heisenberg Group

This chapter corresponds to [22]. In this article we proved a C^m Lusin approximation theorem for horizontal curves in the Heisenberg group. The result states that, if an absolutely continuous horizontal curve has a $(m-1)$-times L^1-differentiable almost everywhere horizontal velocity, then it coincides with a C^m horizontal curve, except on a set of small measure. We also showed that the result no longer holds if L^1 differentiability is replaced by approximate differentiability, showing that our result is optimal and highlighting differences between the Heisenberg and Euclidean settings.

2.1 Motivation

In mathematical analysis it is often useful to understand when a rough map can be approximated by a smoother one. For instance, Lusin's theorem asserts that every measurable function on $\mathbb{R}^n$ is continuous after removing a set of small measure from the domain. Another useful result states that every absolutely continuous curve in $\mathbb{R}^n$ has the 1-Lusin property, which means that it coincides with a C^1 curve except for a set of small measure (Theorem 2.2.1). The position and velocity of absolutely continuous curves are related according to the Fundamental Theorem of Calculus, so these curves are important in analysis and geometry.

Concerning higher regularity, if a curve in $\mathbb{R}^n$ is approximately differentiable of order m almost everywhere (Definition 2.2.3), then it has the m-Lusin property allowing approximation by C^m curves (Theorem 2.2.5) [57]. In this chapter we will study the m-Lusin property for horizontal curves in the Heisenberg group. The key difference between the Heisenberg group and Euclidean space is that in the Heisenberg group both the initial and the approximating curve must be horizontal, which means they are constrained to move in a smaller, but still rich, family of directions.

In the Heisenberg group, the 1-Lusin property is known to be true for all absolutely continuous horizontal curves with the requirement that the approximating curve can be chosen both C^1 and horizontal. More precisely, every absolutely continuous horizontal curve can be approximated by a C^1 horizontal curve (Theorem 2.2.2) [86]. A similar result holds in step two Carnot groups [53] and more general pliable Carnot groups [46]. However the natural analogue is not true in the Engel group, a Carnot group of step three [86]. This highlights that the approximation depends on the space considered and Euclidean results do not always extend to the Carnot group setting. Proving that smooth approximations exist is closely connected to validity of a Whitney extension theorem. In Euclidean spaces, the Whitney extension theorem (Theorem 2.2.11) [8, 89] characterizes when a collection of continuous functions defined on a compact set can be extended to a C^m function on some larger set. To prove a Lusin approximation result from a Whitney extension theorem, one typically restricts to a large compact set where the original mapping satisfies the hypotheses of the Whitney extension theorem and then obtains a C^m mapping which agrees with the starting map on a large set. To apply this idea in the Heisenberg group it is important to have an analogue of the Whitney extension theorem for curves in the Heisenberg group. Such a theorem is indeed known for C^1 horizontal curves in the Heisenberg group [91] and in more general spaces [46]. Very recently it was also understood for C^m horizontal curves in the Heisenberg group [78].

There are two main results in this chapter. The first one is Theorem 2.3.1 which asserts that if $\Gamma = (f, g, h)$ is an absolutely continuous horizontal curve in $\mathbb{H}^1$ with f', g' almost everywhere $(m-1)$-times L^1 differentiable (Definition 2.2.7), then Γ has the m-Lusin property. This should be compared with the previously known analogue in Euclidean space (Theorem 2.2.5), which has the weaker hypothesis that f, g, h are m times approximately differentiable almost everywhere (Definition 2.2.3). To prove Theorem 2.3.1 the strategy is to adapt the proof of Theorem 2.2.5 from [57] to the Heisenberg group using stronger hypothesis to restrict to a compact set on which it is possible to apply the C^m Whitney extension theorem for horizontal curves in the Heisenberg group (Theorem 2.2.12) recently proved [78].

The second main result is Theorem 2.4.1, which illustrates the difference between the Heisenberg setting and the Euclidean setting and justifies the hypotheses of Theorem 2.3.1. Here we explained how to construct an absolutely continuous horizontal curve Γ in $\mathbb{H}^1$ such that f, g, h are almost everywhere twice L^p differentiable for all $p \geq 1$, f', g', h' are almost everywhere once approximately differentiable, yet Γ does not have the 2-Lusin approximation property. This shows that the Euclidean hypothesis of twice approximate differentiability is not sufficient in $\mathbb{H}^1$ and that one really needs to assume differentiability properties of the derivatives f', g', h' rather than only on f, g, h.

In the main results we restricted our attention to the first Heisenberg group $\mathbb{H} = \mathbb{H}^1$. We expect that the natural analogue of Theorem 2.3.1 is also true in higher dimensional Heisenberg groups $\mathbb{H}^n$ with similar proofs but more cumbersome notation.

2.2 Preliminaries

The relevants preliminaries about Carnot groups and, more specifically, the Heisenberg Group, can be found in Chapter 1. In the sequel we will need some classical facts about horizontal curves.

Clearly Lemma 1.1.17 implies that for any horizontal curve γ we have

$$\gamma'_{2n+1}(t) = 2\sum_{i=1}^{n}(\gamma'_i(t)\gamma_{n+i}(t) - \gamma'_{n+i}(t)\gamma_i(t)) \quad \text{for a.e. } t \in [a,b]. \tag{2.2.1}$$

If we assume that γ is C^1, this equality holds for every $t \in [a,b]$. If we further assume that γ is C^m for some $m > 1$, then, for $1 \leq k \leq m$, we may write

$$D^k\gamma_{2n+1}(t) = \sum_{i=1}^{n}\mathcal{P}^k\left(\gamma_i(t), \gamma_{n+i}(t), \gamma'_i(t), \gamma'_{n+i}(t), \ldots, D^k\gamma_i(t), D^k\gamma_{n+i}(t)\right) \tag{2.2.2}$$

for all $t \in [a,b]$ where $\mathcal{P}^k$ is a polynomial determined by the Leibniz rule. For a C^m horizontal curve γ in the first Heisenberg group $\mathbb{H}$, the equations simplify to

$$\gamma_3^k = 2\sum_{i=0}^{k-1}\binom{k-1}{i}(\gamma_1^{k-i}\gamma_2^i - \gamma_2^{k-i}\gamma_1^i) \qquad \text{for } 1 \leq k \leq m. \tag{2.2.3}$$

A classical result about the 1-Lusin property in Euclidean spaces can be found for example in [5, 35].

Theorem 2.2.1. *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be absolutely continuous. Then for every $\varepsilon > 0$ there exists a C^1 map $\Gamma\colon [a,b] \to \mathbb{R}^n$ such that*

$$\mathcal{L}^1(\{t \in [a,b] : \Gamma(t) \neq \gamma(t)\}) < \varepsilon.$$

We also recall the analogous result for the 1-Lusin property in Heisenberg groups [86].

Theorem 2.2.2. *Absolutely continuous horizontal curves in $\mathbb{H}^n$ have the 1-Lusin property.*

It is also known that absolutely continuous horizontal curves have the 1-Lusin property in step two Carnot groups [53], in pliable Carnot groups [46] and in suitable sub-Riemannian manifolds [84].

2.2.1 Approximate Differentiability and Integral Differentiability

Recall that if $f\colon \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$ and $l \in \mathbb{R}$, then $\operatorname{aplim}_{y\to x} f(x) = l$ means that for every $\varepsilon > 0$ the set

$$\{y \in \mathbb{R}^d : |f(y) - l| \leq \epsilon\}$$

has density one at x, i.e.

$$\lim_{R\to 0} \frac{\mathcal{L}^d(B(x,R) \cap \{y \in \mathbb{R}^d : |f(y) - l| \leq \epsilon\})}{\mathcal{L}^d(B(x,R))} = 1.$$

Definition 2.2.3. Given $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$, we say that a function $u\colon \mathbb{R}^d \to \mathbb{R}$ is *m times approximately differentiable at x* if there exists a polynomial $P^m_{u,x}$ of degree at most m such that

$$\operatorname*{aplim}_{y\to x} \frac{|u(y) - P^m_{u,x}(y)|}{|y-x|^m} = 0. \tag{2.2.4}$$

Remark 2.2.4. The polynomial $P^m_{u,x}$ in Definition 2.2.3 is uniquely determined and can be expressed in the form

$$P^m_{u,x}(y) = \sum_{|\alpha|\leq m} \frac{u_\alpha(x)}{|\alpha|!}(y-x)^\alpha \tag{2.2.5}$$

for some $u_\alpha(x) \in \mathbb{R}$ [57].

As a special case of a recent result from [57] we get a C^m version of the Lusin property.

Theorem 2.2.5. *Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ is measurable and approximately differentiable of order m almost everywhere. Then for every $\varepsilon > 0$ there exists a C^m map $\Gamma\colon [a,b] \to \mathbb{R}^n$ such that*

$$\mathcal{L}^1(\{x \in [a,b] : \Gamma(t) \neq \gamma(t)\}) < \varepsilon.$$

In the Heisenberg group we give the following definition.

Definition 2.2.6. An absolutely continuous horizontal curve $\Gamma\colon [a,b] \to \mathbb{H}^n$ is said to have the *Lusin property of order m* if for every $\epsilon > 0$ there exists a C^m horizontal curve $\widetilde{\Gamma}\colon [a,b] \to \mathbb{H}^n$ such that

$$\mathcal{L}^1(\{x \in [a,b] : \widetilde{\Gamma}(x) \neq \Gamma(x)\}) < \varepsilon.$$

We will also refer to the Lusin property of order m as the m-Lusin property.

Throughout this thesis we use the usual notation for integral averages

$$\fint_A f = \frac{1}{\mathcal{L}^d(A)} \int_A f$$

for any $A \subset \mathbb{R}^d$ and $f\colon A \to \mathbb{R}$ for which the expression is well defined.

Definition 2.2.7. Let $u\colon \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$, $p \in [1,\infty)$, and $m \in \mathbb{N}$.

We say that u is *m times L^p differentiable at x* if there exists a polynomial $P^m_{u,x}$ on $\mathbb{R}^d$ of degree at most m such that

$$\left[\fint_{B(x,\rho)} |u(y) - P^m_{u,x}(y)|^p \,\mathrm{d}y\right]^{1/p} = o(\rho^m). \tag{2.2.6}$$

Remark 2.2.8. As noted for instance in [1], if u is m times L^p differentiable at x then u is also m times approximately differentiable at x with the same choice of $P^m_{u,x}$.

2.2.2 Jets and Whitney Extension

Definition 2.2.9. A *jet* of order $m \in \mathbb{N}$ on a set $K \subset \mathbb{R}$ consists of a collection of $(m+1)-$continuous functions $F = (F^k)_{k=0}^m$ on K.

Given such a jet F and $a \in K$, the *Taylor polynomial of order m of F at a* is

$$T^m_a F(x) = \sum_{k=0}^{m} \frac{F^k(a)}{k!}(x-a)^k \quad \text{for all } x \in \mathbb{R}.$$

If m or a are clear from the context, we may write TF for the Taylor polynomial. We will also use the notation $F(x)$ for $F^0(x)$.

Given a jet F of order m on $K \subset \mathbb{R}$, for $a \in K$ and $0 \leq k \leq m$ we define

$$(R^m_a F)^k(x) = F^k(x) - \sum_{\ell=0}^{m-k} \frac{F^{k+\ell}(a)}{\ell!}(x-a)^\ell \quad \text{for all } x \in \mathbb{R}.$$

Definition 2.2.10. A jet F of order m on K is a *Whitney field of class C^m on K* if, for every $0 \leq k \leq m$, we have

$$(R_a^m F)^k(b) = o(|a-b|^{m-k})$$

as $|a-b| \to 0$ with $a, b \in K$.

We now recall the classical Whitney extension theorem in the special case that the domain is a subset of $\mathbb{R}$ [89].

Theorem 2.2.11 (Classical Whitney extension theorem)**.** *Let K be a closed subset of an open set $U \subset \mathbb{R}$. Then there is a continuous linear mapping W from the space of Whitney fields of class C^m on K to $C^m(U)$ such that*

$$D^k(WF)(x) = F^k(x) \quad \text{for } 0 \leq k \leq m \text{ and } x \in K,$$

and WF is C^∞ on $U \setminus K$.

We now recall the Whitney extension theorem for C^m horizontal curves in $\mathbb{H}$ from [78]. Suppose F, G, H are jets of order m on $K \subset \mathbb{R}$. For $a, b \in K$, we define the *area discrepancy*

$$\begin{aligned} A(a,b) := H(b) - H(a) - 2\int_a^b ((T_a^m F)'(T_a^m G) - (T_a^m G)'(T_a^m F)) \\ + 2F(a)(G(b) - T_a^m G(b)) - 2G(a)(F(b) - T_a^m F(b)) \end{aligned} \tag{2.2.7}$$

and the *velocity*

$$V(a,b) := (b-a)^{2m} + (b-a)^m \int_a^b (|(T_a^m F)'| + |(T_a^m G)'|)\,. \tag{2.2.8}$$

Theorem 2.2.12. *Let $K \subset \mathbb{R}$ be compact and F, G, H be jets of order m on K. Then (F, G, H) extends to a C^m horizontal curve $(f, g, h)\colon \mathbb{R} \to \mathbb{H}$ if and only if*

i) F, G, H are Whitney fields of class C^m on K,

ii) For $1 \leq k \leq m$ the following equation holds at all points of K

$$H^k = 2\sum_{i=0}^{k-1} \binom{k-1}{i} (F^{k-i}G^i - G^{k-i}F^i), \tag{2.2.9}$$

iii) $A(a,b)/V(a,b) \to 0$ uniformly as $(b-a) \to 0$ with $a, b \in K$.

Finally we state for future use the following fact about polynomials from [78].

Lemma 2.2.13. *Let P be a polynomial of degree n, $a < b$, and $\|P\|_\infty := \max_{[a,b]} |P|$. Then*

$$\frac{1}{8n^2}\|P\|_\infty \leq \fint_a^b |P| \leq \|P\|_\infty.$$

2.2.3 Facts about Approximate Derivatives and L^1 Derivatives

In this section we prove several lemmas which will be useful later.

Lemma 2.2.14. *Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous and $m \geq 2$.*

Suppose f' is $(m-1)$-times L^1 differentiable at a point $x \in (a,b)$ with L^1 derivative given by the polynomial $P^{m-1}_{f,x}$ of degree at most $m-1$. Then f is m times L^1 differentiable at x with L^1 derivative $Q^m_{f,x}$ of degree at most m defined by $Q^m_{f,x}(y) := f(x) + \int_x^y P^{m-1}_{f,x}(t)\,\mathrm{d}t$.

Proof. Denote $P = P^{m-1}_{f,x}$ and define $Q = Q^m_{f,x}$ by $Q^m_{f,x}(y) := f(x) + \int_x^y P(t)\,\mathrm{d}t$. Let $\varepsilon > 0$. From the definition of L^1 differentiability we have for all sufficiently small $\rho > 0$

$$\fint_{B(x,\rho)} |f'(t) - P(t)|\,\mathrm{d}t \leq \varepsilon\rho^{m-1}/2.$$

Absolute continuity gives for all $y \in B(x,\rho)$,

$$\begin{aligned} |f(y) - Q(y)| &= \left| f(x) + \int_x^y f'(t)\,\mathrm{d}t - \left(f(x) + \int_x^y P(t)\,\mathrm{d}t \right) \right| \\ &= \left| \int_x^y (f'(t) - P(t))\,\mathrm{d}t \right| \\ &\leq \int_{B(x,\rho)} |f'(t) - P(t)|\,\mathrm{d}t \\ &\leq \varepsilon\rho^m. \end{aligned}$$

Hence given $\varepsilon > 0$, we have for all sufficiently small $0 < \rho < 1$

$$\fint_{B(x,\rho)} |f(y) - Q(y)|\,\mathrm{d}y \leq \fint_{B(x,\rho)} \varepsilon\rho^m = \varepsilon\rho^m.$$

This proves the lemma. □

Lemma 2.2.15. *Suppose $(f,g,h)\colon [a,b] \to \mathbb{H}$ is an absolutely continuous horizontal curve in $\mathbb{H}$ and f', g' are $(m-1)$-times L^1 differentiable at a point $x \in [a,b]$ for some $m \geq 2$. Then h is m times L^1 differentiable at x. More precisely, denote*

$$R(y) := h(x) + 2\int_x^y (P'Q - Q'P),$$

where P, Q are the L^1 derivatives of order m of f, g respectively which exist by Lemma 2.2.14. Let $\widetilde{R}$ be the polynomial of degree at most m such that $R(y) - \widetilde{R}(y)$ is divisible by $(y-x)^{m+1}$. Then $\widetilde{R}$ is the L^1 derivative of h of order m at x.

Proof. Let R be defined as in the statement of the lemma. Fix $0 < \varepsilon < 1$. Then there exists $\delta > 0$ such that for all $0 < \rho < \delta$ we have

$$\fint_{B(x,\rho)} |f - P| \leq \varepsilon \rho^m, \qquad \fint_{B(x,\rho)} |f' - P'| \leq \varepsilon \rho^{m-1},$$

and

$$\fint_{B(x,\rho)} |g - Q| \leq \varepsilon \rho^m, \qquad \fint_{B(x,\rho)} |g' - Q'| \leq \varepsilon \rho^{m-1}.$$

Let $0 < \rho < \delta$ and $y \in B(x, \rho)$. We estimate as follows, using the fact (f, g, h) is a horizontal curve and (2.2.1),

$$\begin{aligned} h(y) - R(y) &= h(x) + \int_x^y h' - h(x) - 2\int_x^y (P'Q - Q'P) \\ &= 2\int_x^y ((f'g - P'Q) + (Q'P - g'f)). \end{aligned}$$

We estimate the first term as follows

$$\begin{aligned} \left| 2\int_x^y (f'g - P'Q) \right| &\leq 2\int_{B(x,\rho)} |f'g - P'Q| \\ &= 4\rho \fint_{B(x,\rho)} |f'g - P'Q|. \end{aligned}$$

Since $f'g - P'Q = (f' - P')g + P'(g - Q)$ and g, P' are continuous hence bounded on $[a, b]$, we can continue our estimate as follows

$$\begin{aligned} 4\rho \fint_{B(x,\rho)} |f'g - P'Q| &\leq 4\rho \left(\|g\|_\infty \fint_{B(x,\rho)} |f' - P'| + \|P'\|_\infty \fint_{B(x,\rho)} |g - Q| \right) \\ &\leq 4\rho \left(\|g\|_\infty \varepsilon \rho^{m-1} + \|P'\|_\infty \varepsilon \rho^m \right) \\ &\leq C\varepsilon \rho^m \end{aligned}$$

for a constant C independent of y and ρ. The estimate of $2\int_x^y (Q'P - g'f)$ is similar. Hence we obtain $|h(y) - R(y)| \leq C\varepsilon\rho^m$ for all $0 < \rho < \delta$. Consequently

$$\fint_{B(x,\rho)} |h - R| \leq C\varepsilon \rho^m.$$

To conclude we notice that if $\widetilde{R}$ is the polynomial of degree at most m defined in the statement of the lemma then for some constant C independent of $\rho < 1$ we have

$$\begin{aligned} \fint_{B(x,\rho)} |h - \widetilde{R}| &\leq \fint_{B(x,\rho)} |h - R| + \fint_{B(x,\rho)} |\widetilde{R} - R| \\ &\leq \fint_{B(x,\rho)} |h - R| + C\rho^{m+1}. \end{aligned}$$

Hence

$$\fint_{B(x,\rho)} |h - \widetilde{R}| = o(\rho^m)$$

so h is m times L^1 differentiable at x with derivative $\widetilde{R}$. □

We next prove Proposition 2.2.17 which shows that approximate differentiability almost everywhere leads to Whitney fields on large compact sets. Our argument is adapted from [57] where a similar result is proved under slightly different assumptions. As in [57] we need the following lemma, attributed to De Giorgi in [19].

Lemma 2.2.16 (De Giorgi)**.** *Let $A > 0$ be fixed and let E be a measurable subset of the ball $B(x,r)$ in $\mathbb{R}^n$ such that $\mathcal{L}^n(E) \geq Ar^n$. Then for each $m \in \mathbb{N}$ there is a positive constant C, depending only on n,m and A, such that for each polynomial p of degree at most m and for every multi-index α*

$$|D^\alpha p(x)| \leq \frac{C}{r^{n+|\alpha|}} \int_E |p(y)| dy.$$

Proposition 2.2.17. *Let $u\colon [a,b] \to \mathbb{R}$ be measurable and m times approximately differentiable almost everywhere. Let the approximate derivative at almost every point x be denoted by*

$$P^m_{u,x}(y) = \sum_{i=0}^{m} \frac{u_i(x)}{i!}(y-x)^i.$$

Then for every $\varepsilon > 0$ there exists a compact set $K \subset [a,b]$ with $\mathcal{L}^1([a,b] \setminus K) \leq \varepsilon$ such that $\Gamma = (u_i)_{i=0}^m$ is a C^m Whitney field on K.

Proof. It is proven in [57] that all the functions u_i are measurable under the given hypotheses. Let $0 < \delta < 1$ and $0 < \varepsilon < 1$ be fixed for the moment. For every $x \in [a,b]$ where u is approximately differentiable and $r > 0$, define

$$W(x,r) := \{y \in [a,b] \cap [x-r, x+r] : |u(y) - P^m_{u,x}(y)| > \delta |x-y|^m\}.$$

Each set $W(x,r)$ is measurable because all the u_i are measurable. We can write $W(x,r) = \{y \in [a,b] : (x,y) \in T(r)\}$, where

$$T(r) := \{(x,y) \in [a,b] \times [a,b] : |x-y| < r, |u(y) - P^m_{u,x}(y)| > \delta |x-y|^m\}.$$

Since T is measurable, it follows $x \mapsto \mathcal{L}^1(W(x,r))$ is a measurable function of x. For $n \in \mathbb{N}$ define the sets

$$B_n := \{x \in [a,b] : \mathcal{L}^1(W(x,r)) \leq r/4 \quad \text{for all } r \leq 1/n\}. \tag{2.2.10}$$

Since $x \mapsto \mathcal{L}^1(W(x,r))$ is a measurable function of x and $\mathcal{L}^1(W(x,r))$ is monotonic in r for each fixed x, it is easy to show that the sets B_n are measurable. Clearly $B_n \subset B_{n+1}$ for every n. Since u is m times approximately differentiable almost everywhere, it follows $\mathcal{L}^1([a,b]\setminus\bigcup_{n=1}^\infty B_n) = 0$. Consider two points $x, y \in B_n$ with $x \leq y$ and $|x-y| \leq 1/n$. Let $r = |y-x|$ and define the measurable sets

$$S(x,y) := [x,y] \setminus (W(x,r) \cup W(y,r)).$$

Then

$$\mathcal{L}^1(S(x,y)) \geq |y-x| - \mathcal{L}^1(W(x,r)) - \mathcal{L}^1(W(y,r)) \geq r/2.$$

Define the polynomial $q := P^m_{u,y} - P^m_{u,x}$. For $z \in S(x,y)$ we estimate $|q(z)|$ as follows

$$|q(z)| \leq |P^m_{u,y}(z) - u(z)| + |u(z) - P^m_{u,x}(z)| \leq \delta(|z-y|^m + |x-z|^m) \leq 2\delta r^m.$$

We apply De Giorgi's Lemma to the polynomial q with $E = S(x,y)$ and $A = 1/2$ to obtain for every k

$$|D^k q(y)| = |u_k(y) - D^k P^m_{u,x}(y)| \leq \frac{C}{r^{1+k}} \int_{S(x,y)} |q(z)| \,\mathrm{d}z \leq 2C\delta r^{m-k}.$$

Recall $\varepsilon > 0$ was fixed earlier. Since $\mathcal{L}^1([a,b] \setminus \bigcup_{n=1}^\infty B_n) = 0$ and the sets B_n are increasing, we may choose $N \in \mathbb{N}$ such that $\mathcal{L}^1([a,b] \setminus B_N) \leq \varepsilon/2$. We then choose K a compact subset of B_N with $\mathcal{L}^1([a,b] \setminus K) \leq \varepsilon$. Now we recall the dependence of K on ε, δ and denote $K = K(\varepsilon,\delta)$ and $N = N(\varepsilon,\delta)$. The set $K(\varepsilon,\delta)$ has the following two properties for a constant C depending only on m:

i) $\mathcal{L}^1([a,b] \setminus K(\varepsilon,\delta)) \leq \varepsilon$,

ii) For every $0 \leq k \leq m$ and $x, y \in K(\varepsilon,\delta)$ with $|x-y| \leq 1/N(\varepsilon,\delta)$ we have

$$|u_k(y) - D^k P^m_{u,x}(y)| \leq 4C\delta|x-y|^{m-k}.$$

We now put our compact sets together. Fix $\varepsilon > 0$ and define

$$K = \bigcap_{n=1}^\infty K(\varepsilon/2^n, 1/n).$$

Using i) for the sets $K(\varepsilon/2^n, 1/n)$, we estimate the measure of K as follows

$$\begin{aligned}\mathcal{L}^1([a,b] \setminus K) &\leq \sum_{n=1}^\infty \mathcal{L}^1([a,b] \setminus K(\varepsilon/2^n, 1/n)) \\ &\leq \sum_{n=1}^\infty \varepsilon/2^n \\ &= \varepsilon.\end{aligned}$$

Using ii) for the sets $K(\varepsilon/2^n, 1/n)$, we see that K has the following property. Whenever $0 \leq k \leq m$ and $x, y \in K$ satisfy $|x-y| \leq N(\varepsilon/2^n, 1/n)$ for some $n \in \mathbb{N}$,

$$|u_k(y) - D^k P^m_{u,x}(y)| \leq 4C|x-y|^{m-k}/n.$$

In other words, for every $0 \leq k \leq m$ we have

$$|u_k(y) - D^k P^m_{u,x}(y)| = o(|x-y|^{m-k})$$

as $|x-y| \to 0$ with $x, y \in K$. Hence $\Gamma = (u_i)_{i=0}^m$ is a C^m Whitney field on K. □

2.3 C^m Horizontal Lusin Approximation for Horizontal Curves with L^1 Differentiable Velocity

In this section we prove our first main theorem. Before giving the statement we first recall that if $f\colon [a,b] \to \mathbb{R}$ is m times L^1 differentiable at a point $x \in [a,b]$, then we denote the L^1 derivative at x by

$$P^m_{f,x}(y) = \sum_{i=0}^{m} \frac{f_i(x)}{i!}(y-x)^i,$$

where $f_i(x) \in \mathbb{R}$ for $0 \leq i \leq m$. Also, if a function $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous and f' is $(m-1)$-times L^1 differentiable at a point $x \in [a,b]$, then f is m times L^1 differentiable at x with derivative given by Lemma 2.2.14.

Theorem 2.3.1. *Let $I \subset \mathbb{R}$ be an interval and $\Gamma = (f,g,h) : I \to \mathbb{H}$ be an absolutely continuous horizontal curve such that f' and g' are $m-1$ times L^1 differentiable at almost every point of I. Then Γ has the m-Lusin property. Further, for every $\eta > 0$ there is a C^m horizontal curve $\widetilde{\Gamma} = (\widetilde{f}, \widetilde{g}, \widetilde{h})\colon I \to \mathbb{H}$ such that*

$$\mathcal{L}^1\left(\bigcup_{k=0}^{m}\{x \in I : \widetilde{f}^k(x) \neq f_k(x) \ \text{ or } \ \widetilde{g}^k(x) \neq g_k(x) \ \text{ or } \ \widetilde{h}^k(x) \neq h_k(x)\}\right) < \eta.$$

Proof. Using Lemma 2.2.14 it follows that f and g are m times L^1 differentiable almost everywhere. By Lemma 2.2.15 we also know that h is m times L^1 differentiable almost everywhere. At almost every $x \in I$ denote the L^1 derivative of f by

$$P^m_{f,x}(y) = \sum_{k=0}^{m} \frac{f_k(x)}{k!}(y-x)^k$$

where the f_k are measurable functions by [57]. Similarly define the L^1 derivatives $P^m_{g,x}$ and $P^m_{h,x}$ with coefficients $g_k(x)$ and $h_k(x)$ at almost every point x, which are measurable functions of x.

Fix $\eta > 0$. Choose a compact set $K \subset I$ satisfying $\mathcal{L}^1(I \setminus K) < \eta$ with the following properties:

i) the jets F, G, H defined on K by

$$F^k = f_k|_K, \; G^k = g_k|_K \;\text{ and }\; H^k = h_k|_K \;\text{ for } 0 \le k \le m$$

are Whitney fields of class C^m on K.

ii) For every $\varepsilon > 0$ there is $\delta > 0$ such that if $a, b \in K$ with $|b - a| < \delta$ then

$$\int_a^b |f' - (T_a^m F)'| \le \epsilon (b-a)^{m-1} \quad \text{and} \quad \int_a^b |g' - (T_a^m G)'| \le \epsilon (b-a)^{m-1}. \tag{2.3.1}$$

The first property above is possible using Proposition 2.2.17. To obtain the second property we use the almost everywhere $(m-1)$ times L^1 differentiability of f' and g', Lemma 2.2.14, elementary measure theory, and the fact that $P^m_{f,a} = T^m_a F$ and $P^m_{g,a} = T^m_a G$. We now show that the hypotheses of Theorem 2.2.12 hold for the jets F, G, H on the compact set K.

Verification of Theorem 2.2.12 i). This follows directly from the definition of K.

Verification of Theorem 2.2.12 ii). We need to check (2.2.9), which we recall states

$$H^k = 2 \sum_{i=0}^{k-1} \binom{k-1}{i} (F^{k-i} G^i - G^{k-i} F^i) \quad \text{on } K \text{ for } 1 \le k \le m.$$

Fix $a \in K$ and let $TF = T_a^m F$, $TG = T_a^m G$, $TH = T_a^m H$ for simplicity. Using Lemma 2.2.15, we know

$$(P^m_{h,a})' = 2((P^m_{f,a})'(P^m_{g,a}) - (P^m_{g,a})'(P^m_{f,a})) + S_a'(y),$$

where $S_a(y)$ is a polynomial divisible by $(y-a)^{m+1}$. Hence

$$(TH)' = 2((TF)'(TG) - (TG)'(TF)) + S_a'(y).$$

Differentiating the Taylor polynomials as was done to derive (2.2.9) yields

$$(TH)^k = 2 \sum_{i=0}^{k-1} \binom{k-1}{i} ((TF)^{k-i}(TG)^i - (TG)^{k-i}(TF)^i) + S_a^k \quad \text{on } K \text{ for } 1 \le k \le m,$$

where the polynomial $S_a^k(y)$ is divisible by $(y-a)^{m+1-k}$. In particular, $S_a^k(a)=0$ for $1\leq k\leq m$. Since the Taylor polynomials are based at a, $(TF)^i(a)=F^i(a)$ for $0\leq i\leq m$ and similarly for G and H. Hence substituting in a we obtain

$$H^k(a)=2\sum_{i=0}^{k-1}\binom{k-1}{i}(F^{k-i}(a)G^i(a)-G^{k-i}(a)F^i(a))$$

for all $a\in K$ and $1\leq k\leq m$ as required.

Verification of Theorem 2.2.12(3). Given $0<\varepsilon<1$ fixed, let $\delta>0$ be chosen as above. Fix $a,b\in K$ with $0<b-a<\delta$. For ease of notation we write $TF=T_a^mF$ and $TG=T_a^mG$. For simplicity we will consider only the case

$$F(a)=G(a)=H(a)=0.$$

Otherwise one can use the translation invariance of $A(a,b)$ and $V(a,b)$ as in [78]. Since $F(a)=G(a)=H(a)=0$, $A(a,b)$ is of the form

$$A(a,b)=H(b)-H(a)-2\int_a^b((TF)'TG-TF(TG)').$$

Since (f,g,h) is a horizontal curve, we have

$$H(b)-H(a)=h(b)-h(a)=2\int_a^b(f'g-fg').$$

We estimate $|A(a,b)|$ as follows

$$\begin{aligned}&\left|H(b)-H(a)-2\int_a^b((TF)'TG-TF(TG)')\right|\\&\qquad\leq 2\left(\int_a^b|f'g-(TF)'TG|+\int_a^b|fg'-TF(TG)'|\right).\end{aligned}$$

We will show how to estimate the first term after the inequality, the second one will follow by changing the roles of f and g. First we pass to the average

$$\int_a^b|f'g-(TF)'TG|=(b-a)\fint_a^b|f'g-(TF)'TG|$$

and then we decompose the argument as

$$f'g-(TF)'TG=(f'-(TF)')(g-TG)+(f'-(TF)')TG+(g-TG)(TF)'.$$

We then obtain

$$\begin{aligned}(b-a)\fint_a^b |f'g-(TF)'TG| \leq (b-a)\Bigg[&\left(\fint_a^b |f'-(TF)'|\right)||g-TG||_\infty \\ &+\left(\fint_a^b |f'-(TF)'|\right)||TG||_\infty \\ &+\left(\fint_a^b |g-TG|\right)||(TF)'||_\infty\Bigg].\end{aligned}$$

From (2.3.1) we obtain

$$\left(\fint_a^b |f'-(TF)'|\right) \leq \epsilon(b-a)^{m-1}.$$

Absolute continuity of g, the Fundamental Theorem of Calculus, and (2.3.1) gives

$$||g-TG||_\infty \leq (b-a)\left(\fint_a^b |g'-(TG)'|\right) \leq \epsilon(b-a)^m.$$

Using Lemma 2.2.13 we have

$$||(TF)'||_\infty \leq C\fint_a^b |(TF)'|$$

for some constant $C \geq 1$ depending only on m. Using $TG(a) = G(a) = g(a) = 0$ and again the Fundamental Theorem of Calculus, we have

$$||TG||_\infty \leq \int_a^b |(TG)'|.$$

Combining all together we get

$$\begin{aligned}\int_a^b |f'g-(TF)'TG| &\leq \epsilon^2(b-a)^{2m} + \epsilon(b-a)^m\int_a^b |(TG)'| + C\epsilon(b-a)^m\int_a^b |(TF)'|. \\ &\leq C\varepsilon V(a,b)\end{aligned}$$

By doing the same computation with f and g switched we obtain

$$|A(a,b)| \leq 4C\varepsilon V(a,b)$$

whenever $a, b \in K$ with $0 < b-a < \delta$. This yields Theorem 2.2.12(3).

Conclusion. We have shown that the jets F, G, H satisfy the hypotheses of Theorem 2.2.12 on the compact set K. Hence $\Gamma = (F,G,H)$ extends to a C^m horizontal curve $\widetilde{\Gamma} = (\widetilde{f}, \widetilde{g}, \widetilde{h})\colon I \to \mathbb{H}^1$ satisfying

$$\widetilde{f}^k|_K = F^k, \quad \widetilde{g}^k|_K = G^k, \quad \widetilde{h}^k|_K = H^k \quad \text{for } 0 \leq k \leq m.$$

From the definition of the compact set K and the jets F, G, H we have

$$\begin{aligned}
&\mathcal{L}^1\left(\bigcup_{k=0}^{m}\{x \in I : \widetilde{f}^k(x) \neq f_k(x) \text{ or } \widetilde{g}^k(x) \neq g_k(x) \text{ or } \widetilde{h}^k(x) \neq h_k(x)\}\right)\\
&\leq \mathcal{L}^1(I \setminus K)\\
&< \eta.
\end{aligned}$$

This completes the proof of the theorem. □

2.4 A Horizontal Curve with no Lusin Approximation

In this section we prove our second main theorem, which justifies the hypotheses of Theorem 2.4.1 and highlights the difference between the settings of Euclidean space and the Heisenberg group.

Theorem 2.4.1. *There exists $\Gamma = (f, g, h)\colon [0,1] \to \mathbb{H}$ which is absolutely continuous and horizontal with the following properties:*

i) Almost everywhere the maps f, g, h are twice L^p differentiable for all $p \geq 1$,

ii) Almost everywhere the maps f', g', h' are once approximately differentiable,

iii) Γ does not admit a C^2 horizontal Lusin approximation.

We use the remainder of this section to prove Theorem 2.4.1.

2.4.1 Construction of the Horizontal Curve

Parameters for the Construction

Fix decreasing sequences $h_n, \lambda_n > 0$ with

$$\sum_{n=1}^{\infty} 2^n \lambda_n < \infty, \quad h_n/\lambda_n \to 0, \quad 4^n h_n \to \infty, \quad \frac{1}{\lambda_{n+1}^2} \sum_{k=n+1}^{\infty} 2^{k-n} h_k^2 \to 0. \quad (2.4.1)$$

One possible choice is $h_n = 1/3^n$ and $\lambda_n = (2/5)^n$. A consequence of (2.4.1) is

$$\sum_{n=1}^{\infty} 2^n h_n < \infty. \quad (2.4.2)$$

We next fix a decreasing sequence $w_n > 0$ such that

$$w_n \leq 1/2^{6n}, \qquad \frac{1}{\lambda_{n+1}} \sum_{k=n+1}^{\infty} 2^{k-n} w_k \to 0, \tag{2.4.3}$$

and

$$\frac{1}{\lambda_{n+1}^{2p+1}} \sum_{k=n+1}^{\infty} 2^{k-n} w_k h_k^p \to 0 \qquad \text{for every } p \geq 1. \tag{2.4.4}$$

This is possible since w_n can be chosen very small compared to h_n and λ_n.

The Sets I_n and I

For each $n \geq 1$ we inductively define sets $I_n \subset [0,1]$, each a disjoint union of finitely many open intervals, as follows. Firstly, I_1 is the open interval with center $1/2$ and radius w_1. Once $I_1, I_2, \ldots, I_n$ are defined, we define I_{n+1} as the union of those open intervals J with the following properties:

i) J has center $k/2^{n+1}$ for some integer k with $0 < k < 2^{n+1}$,

ii) J has radius w_{n+1},

iii) J does not intersect $I_1 \cup I_2 \cup \cdots \cup I_n$.

Define $I = \cup_{n=1}^{\infty} I_n$. The set I_n is a disjoint union of at most 2^{n-1} intervals of length $2w_n$. Hence, since $w_n \leq 1/2^{6n}$,

$$\mathcal{L}^1(I) \leq \sum_{n=1}^{\infty} \mathcal{L}^1(I_n) \leq \sum_{n=1}^{\infty} 2^n w_n \leq 1/31. \tag{2.4.5}$$

Definition of the Horizontal Components

We now define $f, g\colon [0,1] \to \mathbb{R}$ which will be the first two components of the curve. In $[0,1] \setminus I$ we set f and g to be identically 0. Otherwise we proceed as follows. Suppose J is one of the finitely many disjoint open intervals chosen in the definition of I_n for some $n \geq 1$. Divide J into 4 adjacent and equally sized intervals labelled from left to right

$$J_1 = (p_1, p_2), \quad J_2 = [p_2, p_3], \quad J_3 = [p_3, p_4], \quad J_4 = (p_4, p_5).$$

The maps f, g are piecewise linear functions in J defined as follows:

i) In J_1, f is identically 0 and g is linear with $g(p_1) = 0$, $g(p_2) = h_n$.

ii) In J_2, f is linear with $f(p_2) = 0$, $f(p_3) = h_n$ and g is identically h_n.

iii) In J_3, f is identically h_n and g is linear with $g(p_3) = h_n$, $g(p_4) = 0$.

iv) In J_4, f is linear with $f(p_4) = h_n$, $f(p_5) = 0$ and g is identically 0.

Absolute Continuity of the Horizontal Components

Clearly f and g are differentiable at all but finitely many points of I_n for each n, hence at all but countably many points of I. Our first task is to prove that f and g are differentiable at almost every point of $[0, 1] \setminus I$. Before doing so we prove a lemma which roughly states that at almost every point $x \in [0, 1] \setminus I$ the maps f and g do not see 'big jumps' unexpectedly close to x.

For $x \in \mathbb{R}$ and $S \subset \mathbb{R}$ we denote $d(x, S) := \inf\{|x - y| : y \in S\}$. For $n \geq 1$, define

$$A_n = \{x \in [0, 1] \setminus I : d(x, I_1 \cup \cdots \cup I_n) < \lambda_n\}$$

and let

$$A := \limsup A_n \subset [0, 1] \setminus I.$$

By definition of the limit superior, for any $x \in [0, 1] \setminus (I \cup A)$, there exists $N(x) > 0$ such that $n > N(x)$ implies

$$d(x, I_1 \cup \cdots \cup I_n) \geq \lambda_n.$$

Roughly speaking, this states that if $x \in [0, 1] \setminus (I \cup A)$ then on small scales near to x one sees only relatively small intervals. We will use this fact repeatedly later.

Lemma 2.4.2. *The set A has Lebesgue measure zero.*

Proof. The set I_i consists of 2^{i-1} intervals and is contained in I. Hence

$$\mathcal{L}^1(\{x \notin I : d(x, I_i) < \lambda_n\}) \leq 2^{i-1} 2\lambda_n = 2^i \lambda_n.$$

Hence

$$\mathcal{L}^1(A_n) = (2 + 2^2 + \cdots + 2^n)\lambda_n = 2\lambda_n(2^n - 1).$$

Since $\sum_{n=1}^{\infty} 2^n \lambda_n < \infty$ it follows $\sum_{n=1}^{\infty} \mathcal{L}^1(A_n) < \infty$. The Borel Cantelli lemma gives the conclusion. $\square$

Lemma 2.4.3. *For every $x \in (0, 1) \setminus (I \cup A)$, f and g are differentiable at x with $f'(x) = g'(x) = 0$.*

Proof. Fix a point x as in the statement of the lemma and corresponding $N(x) > 0$ such that $n > N(x)$ implies

$$d(x, I_1 \cup \cdots \cup I_n) \geq \lambda_n.$$

For all t sufficiently small there is $n > N(x)$ such that $\lambda_{n+1} \leq |t| < \lambda_n$. Then

$$d(x, I_1 \cup \cdots \cup I_n) \geq \lambda_n > |t|.$$

This implies $x + t \notin I_1 \cup \cdots \cup I_n$. By definition of f we see $0 \leq f(x+t) \leq h_{n+1}$. Since $x \notin I$ we have $f(x) = 0$ and so

$$\left| \frac{f(x+t) - f(x)}{t} \right| \leq \frac{h_{n+1}}{\lambda_{n+1}}.$$

Since $h_n/\lambda_n \to 0$, it follows that f is differentiable at x with $f'(x) = 0$. The argument is the same for g. □

We have now shown that f and g are differentiable almost everywhere on $[0,1]$.

Proposition 2.4.4. *The maps $f, g\colon [0,1] \to \mathbb{R}$ are absolutely continuous.*

Proof. Suppose J is one of the intervals chosen in the construction of I_n for some $n \geq 1$. Then for any $x \in J$ we have

$$|f'(x)| \leq h_n/(w_n/4) = 4h_n/w_n.$$

Since $\mathcal{L}^1(J) \leq 2w_n$ it follows that $\int_J |f'| \leq 8h_n$. Since there are at most 2^{n-1} disjoint intervals in the construction of I_n, we have $\int_{I_n} |f'| = 2^{n+2}h_n$ for every $n \geq 1$. Since $f' = 0$ almost everywhere outside I, we deduce,

$$\int_0^1 |f'| \leq \sum_{n=1}^{\infty} 2^{n+2} h_n < \infty.$$

Hence f' is integrable on $[0,1]$.

We now claim

$$f(b) - f(a) = \int_a^b f' \quad \text{whenever } a < b. \tag{2.4.6}$$

Clearly (2.4.6) is satisfied if a and b belong to a common chosen interval J from the definition of I. Indeed, f is piecewise linear and hence absolutely continuous inside any such interval. Suppose this is not the case. By splitting the integral if necessary, to prove (2.4.6) we may assume $a, b \notin I$. If $J = [c,d]$ is any interval chosen in the construction of I which is contained in (a,b), then

$$\int_J f' = f(d) - f(c) = 0.$$

There are countably many such intervals and $f' = 0$ at almost every point outside I. Hence $\int_a^b f' = 0$. Since $a, b \notin I$ we have $f(b) = f(a) = 0$. Hence (2.4.6) holds. This proves that f is absolutely continuous. The argument for g is the same. □

Vertical Component of the Curve

Since f, g are bounded and f', g' are integrable, the products $f'g$ and $g'f$ are integrable. We define $h\colon [0,1] \to \mathbb{R}$ by

$$h(x) := 2\int_0^x (f'g - g'f) \qquad \text{for } x \in [0,1].$$

Clearly h is absolutely continuous on $[0,1]$. By Lemma 1.1.17, $\Gamma := (f,g,h)$ is an absolutely continuous horizontal curve. It is easy to check that h is piecewise linear since each interval chosen in the construction of I. We also record the following fact for later.

Lemma 2.4.5. *Suppose $J = (a,b)$ is one of the connected components of I_n. Then*

$$h(b) - h(a) = 4h_n^2.$$

Proof. Since $f(a) = f(b) = 0$ we know

$$h(b) - h(a) = 2\int_a^b (f'g - g'f) = 4\int_a^b f'g.$$

From the construction of f and g and the fact $(b-a)/4 = w_n/2$ we obtain

$$h(b) - h(a) = 4(w_n/2)(h_n/(w_n/2))h_n = 4h_n^2.$$

□

2.4.2 Differentiability of the Horizontal Curve

Proposition 2.4.6. *At almost every point $x \in [0,1]$, the maps $f, g, h\colon [0,1] \to \mathbb{R}$ are twice L^p differentiable at x for all $p \geq 1$. For every point $x \in (0,1) \setminus (I \cup A)$, the second order L^p derivatives of f, g, h at x are identically $f(x) = 0$, $g(x) = 0$, and $h(x)$ (possibly non-zero) respectively.*

Proof. Recall that f, g, h are piecewise linear inside each of the countably many intervals whose disjoint union is I. Hence f, g, h are twice L^p differentiable for all $p \geq 1$ at all but countably many points of I. Suppose $x \notin (I \cup A)$. To show f is twice L^p differentiable at x we will show that for every $p \geq 1$

$$\lim_{t \to 0} \frac{1}{t^{2p+1}} \int_{[x-t,x+t]} |f(y)|^p \, \mathrm{d}y = 0.$$

Using the definition of A, we may choose $N(x) > 0$ such that $n > N(x)$ implies

$$d(x, I_1 \cup \cdots \cup I_n) \geq \lambda_n.$$

Recall that $\sum_{n=1}^{\infty} 2^n \lambda_n < \infty$ which implies $\lambda_n \leq 1/2^n$ for all sufficiently large n. Given any $t > 0$ sufficiently small, we may choose $n > N(x)$ with

$$\lambda_{n+1} \leq t < \lambda_n \leq 1/2^n.$$

This implies

$$[x - t, x + t] \cap (I_1 \cup \dots \cup I_n) = \varnothing.$$

The interval $[x - t, x + t]$ has length at most $2\lambda_n \leq 1/2^{n-1}$. Since the intervals in I_k have centers separated by at least distance $1/2^k$, it follows that $[x - t, x + t]$ can intersect at most 2^{k-n+1} intervals from I_k for $k > n$. Recall that $x \notin I$ gives $f(x) = 0$, $|f(y)| \leq h_k$ for y in an interval from I_k, and that $t > \lambda_{n+1}$. We have

$$\begin{aligned} \frac{1}{t^{2p+1}} \int_{[x-t,x+t]} |f(y)|^p \, \mathrm{d}y &= \frac{1}{t^{2p+1}} \int_{[x-t,x+t] \cap I} |f(y)|^p \, \mathrm{d}y \\ &\leq \frac{2}{t^{2p+1}} \sum_{k=n+1}^{\infty} 2^{k-n+1} w_k h_k^p \\ &\leq \frac{2}{\lambda_{n+1}^{2p+1}} \sum_{k=n+1}^{\infty} 2^{k-n+1} w_k h_k^p. \end{aligned}$$

The previous line converges to 0 as $n \to \infty$ for every $p \geq 1$ by definition of the sequences w_k, h_k, λ_k. The argument for g is exactly the same. Finally to show h is twice L^p differentiable at x we will show that for every $p \geq 1$

$$\lim_{t \to 0} \frac{1}{t^{2p+1}} \int_{[x-t,x+t]} |h(y) - h(x)|^p \, \mathrm{d}y = 0.$$

Recall that $[x - t, x + t]$ can intersect at most 2^{k-n+1} intervals from I_k for $k > n$. Hence, using Lemma 2.4.5, for any $y \in [x - t, x + t]$ we have

$$|h(y) - h(x)| \leq \sum_{k=n+1}^{\infty} 2^{k-n+3} h_k^2.$$

Hence

$$\begin{aligned} \frac{1}{t^{2p+1}} \int_{[x-t,x+t]} |h(y) - h(x)|^p \, \mathrm{d}y &\leq \frac{2}{t^{2p}} \left(\sum_{k=n+1}^{\infty} 2^{k-n+3} h_k^2 \right)^p \\ &\leq 2 \left(\frac{8}{\lambda_{n+1}^2} \sum_{k=n+1}^{\infty} 2^{k-n} h_k^2 \right)^p. \end{aligned}$$

We conclude by noticing the last line converges to 0 as $n \to \infty$ for every $p \geq 1$. $\square$

Proposition 2.4.7. *The maps f', g', h' are once approximately differentiable almost everywhere. In particular, f' and g' have approximate derivative* 0 *at every point of* $(0,1)\setminus(I\cup A)$.

Proof. Approximate differentiability of f', g', h' at all but countably many points of I is clear since f, g, h are piecewise linear inside each interval chosen during the construction of I. Recall that $f'(x) = g'(x) = 0$ for every point $x \in (0,1)\setminus(I\cup A)$. Fix such an x. Choose corresponding $N(x) > 0$ such that $n > N(x)$ implies

$$d(x, I_1 \cup \cdots \cup I_n) \geq \lambda_n.$$

As in the proof of Proposition 2.4.6, given any $t > 0$ sufficiently small we may choose $n > N(x)$ such that

$$\lambda_{n+1} \leq t < \lambda_n \leq 1/2^n,$$

which implies

$$[x-t, x+t] \cap (I_1 \cup \cdots \cup I_n) = \varnothing.$$

Again it follows that $[x-t, x+t]$ can intersect at most 2^{k-n+1} intervals from I_k for $k > n$. Recalling that $f'(x) = 0$ at every point of $(0,1)\setminus(I\cup A)$, we have

$$\begin{aligned}\frac{\mathcal{L}^1(\{y \in [x-t,x+t] : f'(y) > 0\})}{2t} &\leq \frac{\mathcal{L}^1([x-t,x+t]\cap I)}{2t}\\ &\leq \frac{1}{2\lambda_{n+1}} \sum_{k=n+1}^{\infty} 2^{k-n+2} w_k.\end{aligned}$$

Since the previous line converges to 0 as $n \to \infty$, it follows f' is approximately differentiable at x with approximate derivative 0. The argument for g is the same. For h we recall that $h' = 2(f'g - g'f)$ almost everywhere. Combining this with the fact $f'(x) = g'(x) = 0$ for every point $x \in (0,1)\setminus(I\cup A)$ gives $h'(x) = 0$ for almost every $x \in (0,1)\setminus(I\cup A)$. For such x the same argument as above applies, giving the desired conclusion. □

2.4.3 No C^2 Horizontal Lusin Approximation

Proposition 2.4.8. *The curve* Γ *does not have the* C^2 *horizontal Lusin approximation property.*

We will prove Proposition 2.4.8 by contradiction. Suppose Γ does have the C^2 horizontal Lusin approximation property. Fix $\theta > 4/5 + 1/31$ and a C^2 horizontal curve $\widetilde{\Gamma} = (F, G, H)\colon [0,1] \to \mathbb{H}$ such that the set

$$E := \{t \in [0,1] : \widetilde{\Gamma}(t) = \Gamma(t)\}$$

satisfies $\mathcal{L}^1(E) > \theta$. Since $\mathcal{L}^1(I) < 1/31$ by (2.4.5), we have $\mathcal{L}^1(E \setminus I) > 4/5$.

Lemma 2.4.9. *Suppose $x \in E \setminus I$ is a Lebesgue density point of $E \setminus I$. Then*

$$F(x) = F'(x) = F''(x) = 0 \quad \textit{and} \quad G(x) = G'(x) = G''(x) = 0.$$

Proof. Let x be as in the statement of the lemma. Then $F(x) = f(x)$ because $x \in E$ and $f(x) = 0$ because $x \notin I$; hence $F(x) = 0$. Since x is a Lebesgue density point of $E \setminus I$ there exist $x_n \in E \setminus I$ with $x_n \to x$. By the same argument as before we have $F(x_n) = 0$ for every n. Hence $x_n \to x$ and $F(x_n) = 0 = F(x)$ for every n. Since F is C^2 this implies $F'(x) = F''(x) = 0$. The argument for G is the same. □

Since $\widetilde{\Gamma}$ is C^2, F'' and G'' are uniformly continuous on $[0, 1]$. Fix $\delta > 0$ such that

$$|F''(x) - F''(y)| < 1 \text{ and } |G''(x) - G''(y)| < 1 \qquad \text{for } |x - y| < \delta. \tag{2.4.7}$$

Lemma 2.4.10. *Suppose $a, b \in E \setminus I$ are Lebesgue density points of $E \setminus I$ and $|b - a| < \delta$. Then*

$$|H(b) - H(a)| \leq 4|b - a|^4.$$

Proof. Since $F(a) = F(b) = 0$ by Lemma 2.4.9 and $\widetilde{\Gamma}$ is horizontal, we have

$$H(b) - H(a) = 2\int_a^b (F'G - G'F) = 4\int_a^b F'G.$$

We have $F(a) = F'(a) = F''(a) = 0$ by Lemma 2.4.9 and $|F''(t) - F''(a)| < 1$ for $t \in [a, b]$ by (2.4.7). Hence $|F'(t)| \leq b - a$ and $|F(t)| \leq (b - a)^2$ for $t \in [a, b]$. The same estimates hold for G. This gives

$$|H(b) - H(a)| \leq 4(b - a)(b - a)(b - a)^2 = 4(b - a)^4.$$

□

Lemma 2.4.11. *For all sufficiently large $n \in \mathbb{N}$, there exists a pair $x, y \in (0, 1)$ with the following properties:*

i) $x, y \in E \setminus I$ and they are Lebesgue density points of $E \setminus I$,

ii) $|x - y| \leq 1/2^n$,

iii) x, y are on opposite sides of an interval chosen in the construction of I_{n+1}.

Proof. We argue by contradiction. Assume there exist arbitrarily large $n \in \mathbb{N}$ for which there is no pair x and y with the desired properties. Fix such an n. We consider intervals of the form $[L/2^n, (L+1)/2^n]$ for different integers $0 \leq L < 2^n$.

Suppose the interval $[L/2^n, (L+1)/2^n]$ has midpoint $(2L+1)/2^{n+1}$ which is the center of an interval J chosen in the construction of I_{n+1}. The interval J separates $[L/2^n, (L+1)/2^n] \setminus J$ into two subintervals J_1 and J_2 each of measure greater than $(1/3)(1/2^n)$. Since there is no pair $x, y \in (0,1)$ with the properties in the statement of the lemma, in particular there is no such pair in the interval $[L/2^n, (L+1)/2^n]$. Hence either J_1 or J_2 does not contain any points of $E \setminus I$ which are Lebesgue density points of $E \setminus I$. Hence we have

$$\mathcal{L}^1((E \setminus I) \cap [L/2^n, (L+1)/2^n]) \leq (2/3)\mathcal{L}^1([L/2^n, (L+1)/2^n]).$$

We now estimate the total measure of those intervals $[L/2^n, (L+1)/2^n]$ whose midpoint $(2L+1)/2^{n+1}$ is not chosen in the construction of I_{n+1}. Fix such an interval $[L/2^n, (L+1)/2^n]$. Then

$$B((2L+1)/2^{n+1}, w_{n+1}) \cap (I_1 \cup \cdots \cup I_n) \neq \varnothing.$$

Different intervals of the form $B(k/2^{n+1}, w_{n+1})$ are separated by a distance

$$1/2^{n+1} - 2w_{n+1} \geq 1/2^{n+1} - 2/2^{6n} \geq 1/2^{n+2}.$$

If an interval of length T intersects K intervals of the form $B(k/2^{n+1}, w_{n+1})$ then we must have $T \geq K/2^{n+2}$, so $K \leq 2^{n+2}T$. The set I_i is a union of 2^{i-1} intervals of length $2w_i$. Hence the number of intervals $B(k/2^{n+1}, w_{n+1})$ which intersect $I_1 \cup \cdots \cup I_n$ can be estimated by

$$\sum_{i=1}^{n} 2^{i-1} 2^{n+2} 2w_i = 2^{n+2} \sum_{i=1}^{n} 2^i w_i.$$

Hence the total measure of all those intervals $[L/2^n, (L+1)/2^n]$ whose midpoint is not chosen in the construction of I_{n+1} can be estimated by

$$(1/2^n)2^{n+2} \sum_{i=1}^{n} 2^i w_i = 4 \sum_{i=1}^{n} 2^i w_i \leq 4 \sum_{i=1}^{\infty} 2^i/2^{6i} = 4/31.$$

Let G be the collection of those integers L such that the midpoint of the interval $[L/2^n, (L+1)/2^n]$ is the center of an interval chosen in the construction of I_{n+1}. Similarly, let B be the collection of those integers L such that the midpoint of

$[L/2^n, (L+1)/2^n]$ is not chosen. We estimate as follows

$$\begin{aligned}\mathcal{L}^1(E \setminus I) &= \sum_{L \in G} \mathcal{L}^1((E \setminus I) \cap [L/2^n, (L+1)/2^n]) \\ &\qquad + \sum_{L \in B} \mathcal{L}^1((E \setminus I) \cap [L/2^n, (L+1)/2^n]) \\ &\leq \sum_{L \in G} (2/3)\mathcal{L}^1([L/2^n, (L+1)/2^n]) + \sum_{L \in B} \mathcal{L}^1([L/2^n, (L+1)/2^n]) \\ &\leq 2/3 + 4/31 \\ &\leq 4/5.\end{aligned}$$

Since $\mathcal{L}^1(E \setminus I) > 4/5$ we obtain a contradiction which proves the lemma. □

We now derive a contradiction which proves Proposition 2.4.8. Recall $\delta > 0$ from (2.4.7) and the fact that $4^n h_n \to \infty$. Using Lemma 2.4.11, we may fix n with $1/2^n < \delta$ and $4^n h_{n+1} \geq 2$ for which there exist points $x, y \in (0,1)$ with $x < y$ such that:

i) $x, y \in E \setminus I$ and are Lebesgue density points of $E \setminus I$,

ii) $|x - y| \leq 1/2^n$,

iii) x, y are on opposite sides of an interval chosen in the construction of I_{n+1}.

Since $|x - y| \leq 1/2^n < \delta$ and $x, y \in E \setminus I$ are Lebesgue density points of $E \setminus I$, we have by Lemma 2.4.10

$$|H(y) - H(x)| \leq 4|y - x|^4 \leq 4/16^n. \tag{2.4.8}$$

Since $x < y$ are on opposite sides of an interval chosen in the construction of I_{n+1}, we have by Lemma 2.4.5

$$h(y) - h(x) \geq 4h_{n+1}^2. \tag{2.4.9}$$

Since $x, y \in E$ we have $H(y) - H(x) = h(y) - h(x)$. Combining this with (2.4.8) and (2.4.9) gives $h_{n+1}^2 \leq 1/16^n$ or equivalently $4^n h_{n+1} \leq 1$. This contradicts the choice of n with $4^n h_{n+1} \geq 2$, proving Proposition 2.4.8 and hence proving Theorem 2.4.1.

Chapter 3

Asymptotic Behaviours in Fractional Orlicz-Sobolev Spaces on Carnot Groups

This chapter corresponds to [21]. In this article we defined a class of fractional Orlicz-Sobolev spaces on Carnot groups and, in the spirit of the celebrated results of Bourgain-Brezis-Mironescu, we studied the asymptotic behavior of the Orlicz functionals when the fractional parameter s goes to 1.

3.1 Motivation

In the seminal paper [12], Bourgain, Brezis and Mironescu proved that for any smooth bounded domain $\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, the well-known fractional Gagliardo seminorm recovers its local counterpart as s goes to 1, in the sense that

$$\lim_{s \uparrow 1}(1-s) \iint_{\Omega \times \Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy = K(n,p) \int_\Omega |\nabla u(x)|^p\, dx, \tag{3.1.1}$$

where the constant $K(n,p)$ is defined as

$$K(n,p) = \frac{1}{p} \int_{\mathbb{S}^{n-1}} |\boldsymbol{\omega} \cdot h|^p\, d\mathcal{H}^{n-1}(h).$$

Here $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ denotes the unit sphere, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $\boldsymbol{\omega}$ is an arbitrary unit vector of $\mathbb{R}^n$. See [13] for a survey on this and related results.

In recent years, there has been an increasing interest towards this kind of results, aiming at extending it in different directions. Despite the literature concerning the generalizations of such kind of results is now pretty vast, let us try to give a brief account of it. The case of BV-functions, originally only partially answered in [12], has been considered independently in [28] and [81]. In [54, 55], the authors covered the case of general open sets $\Omega \subset \mathbb{R}^n$, both for Sobolev and BV-functions. More recently, starting with [67], Nguyen obtained new characterizations of the classical Sobolev space by means of more general nonlocal functionals, providing also Poincaré-type inequalities and several other results, see [68, 69, 70]. The nature of the nonlocal functionals, there considered, differs from the classical Gagliardo seminorm and it has recently found applications in the field of image processing, see [14, 15, 16, 17].

The above mentioned results have also been proved to hold in the case of *magnetic Sobolev spaces*. Roughly speaking, these spaces are the natural functional setting in electromagnetism when dealing with particles interacting with a magnetic field. We refer to [87] for the analogous of (3.1.1) when $p = 2$, and to [80] for the general case and for magnetic BV functions. We finally refer to [71, 72] for similar results for more general nonlocal functionals akin to those considered in [67].

A complementary and natural question is what happens when considering the limit as s goes to 0. The answer is contained in [64], where it is proved that for any $n \geq 1$ and $p \in [1, \infty)$,

$$\lim_{s \downarrow 0} s \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy = C(n,p) \int_{\mathbb{R}^n} |u|^p\, dx, \tag{3.1.2}$$

whenever $u \in \bigcup_{s \in (0,1)} W_0^{s,p}(\mathbb{R}^n)$. Here $C(n,p)$ is a constant that depends only on n and p. We refer to [79] for the magnetic version of (3.1.2).

The asymptotic theory described so far concerns always the classical Euclidean setting. Nevertheless, recent contributions started to attack Bourgain-Brezis-Mironescu-type results even in the case where non–Euclidean geometries appear. We refer, for instance, to [50] for the case of compact Riemannian manifolds.

As for many other problems, one of the non–Euclidean setting where to look for extensions is provided by Carnot groups, that are connected, simply connected and nilpotent Lie groups whose associated Lie algebra is stratified (see the introduction for more details). These spaces are usually the easiest examples of sub–Riemannian manifolds and they are an environment of particular interest for analysis, see e.g. [65]. Fractional Sobolev spaces are now a well established notion even in Carnot groups, see [37, 38] for more details. In this setting, in [6, 61], it was studied the validity of a Bourgain-Brezis-Mironescu-type formula to treat the asymptotic behavior of the corresponding fractional Sobolev seminorm as s goes to 1. We also refer to [26] for the case of more general nonlocal functionals in the spirit of Nguyen.

Following this line of research, one of the most recent contributions, due to Fernández Bonder and Salort, deals with Bourgain-Brezis-Mironescu-type results for the class of Orlicz-Sobolev spaces, see also [10] for similar results in the magnetic setting. The paper [9] is actually one of our motivations in this work.

Finally, we want to recall that the asymptotic behaviour of the perimeter functional has also been addressed, either in Euclidean contexts as well as in Carnot settings, see e.g. [4, 18, 30, 36, 58, 59].

3.2 Preliminaries

The work contained in this chapter aims at extending the validity of Bourgain-Brezis-Mironescu-type formulas in Carnot groups, when behaviours more general than powers are taken into account. In this context, Young functions play a preponderant role and Orlicz-Sobolev spaces become the natural framework to deal with.

The relevants preliminaries about Carnot groups can be found in Chapter 1 and in the references therein.

3.2.1 Introduction to the New Setting

To fix ideas, when speaking of an Orlicz function (or Nice Young function), we will refer to a continuous, non-negative, strictly increasing and convex function φ on $[0,\infty)$ vanishing at the origin. In order to give a well-posed space of definition, φ will be asked to fulfil a structural growth condition, i.e. the existence of constants $p^- \leq p^+$ such that

$$1 < p^- \leq \frac{t\,\varphi'(t)}{\varphi(t)} \leq p^+ < \infty \quad \forall t > 0. \tag{L}$$

We refer the interested reader to the books [29, 49, 74] for a comprehensive introduction to Young functions and Orlicz spaces. Having these definitions in mind, following [38], we define the natural generalization of the fractional Sobolev spaces for Carnot groups in the Orlicz setting.

We recall that the *homogeneous dimension* of a Carnot group is the quantity $Q := \sum_{i=1}^{k} i \dim(V_i)$. It corresponds to the Hausdorff dimension of $\mathbb{G}$ with respect to an appropriate sub–Riemannian distance. This is generally greater than (or equal to) the topological dimension of $\mathbb{G}$ and it coincides with it only when $\mathbb{G}$ is the Euclidean group $(\mathbb{R}^n, +)$, which is the only Abelian Carnot group.

Given a Carnot group $\mathbb{G}$ of homogeneous dimension Q, a Young function φ

and a fractional parameter $0 < s < 1$, we consider the space

$$W^{s,\varphi}(\mathbb{G}) := \{u : \mathbb{G} \to \mathbb{R} \text{ measurable such that } \Phi_\varphi(u) + \Phi_{s,\varphi}(u) < \infty\}$$

where

$$\Phi_\varphi(u) := \int_{\mathbb{G}} \varphi(|u(x)|)\, dx, \qquad \Phi_{s,\varphi}(u) := \iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right) \frac{dx\, dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}.$$

It turns out that $W^{s,\varphi}(\mathbb{G})$ is a reflexive Banach space endowed with the correspondent Luxembourg norm (see Theorem 3.2.17 for more details).

From now on, when needed, a generic $z \in \mathbb{G}$ will be denoted as $z = (z', z'')$ where $z' = (z_1, .., z_m)$ is the horizontal part and $z'' = (z_{m+1}, .., z_n)$ is the vertical one.

We are now ready to state our main result, that is a Bourgain-Brezis-Mironescu-type formula in the case of Orlicz-Sobolev spaces in Carnot groups.

Theorem 3.2.1. *Let φ be an Orlicz function such that the following limit exists*

$$\tilde{\varphi}(t) := \lim_{s\uparrow 1}(1-s)\int_0^1 \left(\int_S \varphi(t\|z'\|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z)\right)\frac{dr}{r},$$

where S is the unit sphere in $\mathbb{G}$, m is the dimension of the horizontal layer and σ is the Radon metric whose existence is granted by Proposition 3.2.6.

Then, for any $u \in L^\varphi(\mathbb{G})$ and $0 < s < 1$

$$\lim_{s\uparrow 1}(1-s)\Phi_{s,\varphi}(u) = \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m}).$$

Let us spend a few words about the proof of Theorem 3.2.1. Roughly speaking, the main technical point concerns a regularization argument in terms of truncations and convolutions (see Lemma 3.2.23 and Lemma 3.2.24), combined with a compactness argument derived from a Rellich-Kondrachov-type result (see Theorem 3.2.20).

Let us remark that, even in the prototype case $\varphi(t) = t^p$, we can prove Theorem 3.2.1 only in the case of homogeneous norms that are invariant under horizontal rotations. See Section 3.2.2 for more details.

3.2.2 Invariant Norms and Radial Functions

For reasons that will be clear later, in this thesis a big role is played by metrics that are induced by homogeneous norms.

Definition 3.2.2. A homogeneous norm $\|\cdot\|_{\mathbb{G}} : \mathbb{G} \to \mathbb{R}_0^+$ is a continuous function with the following properties:

i) $\|x\|_{\mathbb{G}} = 0$ if and only if $x = 0$ for every $x \in \mathbb{G}$;

ii) $\|x^{-1}\|_{\mathbb{G}} = \|x\|_{\mathbb{G}}$ for every $x \in \mathbb{G}$;

iii) $\|\delta_\lambda x\|_{\mathbb{G}} = \lambda\|x\|_{\mathbb{G}}$ for every $\lambda \in \mathbb{R}^+$ and for every $x \in \mathbb{G}$.

We remind that any homogeneous norm induces a left–invariant homogeneous distance by

$$d(x, y) := \|y^{-1} \cdot x\|_{\mathbb{G}} \quad \text{for every } x, y \in \mathbb{G}.$$

A concrete example of such kind of homogeneous distance is given by the Korányi distance, see e.g. [27].

In the proceeding of the chapter we will ask for the following stronger hypotheses on the norm $\| \cdot \|_{\mathbb{G}}$:

iv) invariance under horizontal rotations;

v) the validity of the classical triangular inequality

$$\big|\|y\|_{\mathbb{G}} - \|x\|_{\mathbb{G}}\big| \leq \|y^{-1} \cdot x\|_{\mathbb{G}} \leq \|x\|_{\mathbb{G}} + \|y\|_{\mathbb{G}}.$$

An example of such kind of norm, whose induced distance is equivalent to the well-known Carnot-Carathéodory distance, is given in [39, 40].

Proposition 3.2.3. *Let $\mathbb{G}$ be a Carnot group and $f \in L^1(\mathbb{G})$. Then the Haar measure on $\mathbb{G}$*

i) is invariant under left and right translations:

$$\int_{\mathbb{G}} f(x)\, dx = \int_{\mathbb{G}} f(x \cdot y)\, dx = \int_{\mathbb{G}} f(y \cdot x)\, dx \quad \forall y \in \mathbb{G};$$

ii) scales under group dilations by the homogeneous dimension of $\mathbb{G}$:

$$\int_{\mathbb{G}} f(\delta_\lambda x)\, dx = \lambda^Q \int_{\mathbb{G}} f(x)\, dx \quad \forall \lambda > 0.$$

It trivially follows that $|B(x, r)| = r^Q|B| = r^Q C_b$ for all $x \in \mathbb{G}$ and $r > 0$, where $B = B(0, 1)$ and C_b denotes its Lebesgue measure.

The following three Propositions will be very useful in the sequel.

Proposition 3.2.4. *[38, Proposition 1.13] Let $f \in L^1_{\text{loc}}(\mathbb{G} \setminus \{0\})$ be an homogeneous function of degree $-Q$, i.e., $f(\delta_\lambda x) = \lambda^{-Q} f(x)$. Then, there exists a constant M_f, mean value of f, such that*

$$\int_{\mathbb{G}} f(x) g(\|x\|_{\mathbb{G}})\, dx = M_f \int_0^{+\infty} g(r) \frac{dr}{r}$$

for any $g \in L^1(\mathbb{R}^+, \frac{dr}{r})$.

As a consequence of the previous result, we are able to compute explicitly integrals on balls of functions depending only on the distance from the center of the ball in terms of integrals on the real line.

Proposition 3.2.5. *Let* $f \in L^1(\mathbb{R}^+)$ *and* $R > 0$. *Then*

$$\int_{B(y,R)} f(\|y^{-1}\cdot x\|_{\mathbb{G}})\,dx = \int_{B(0,R)} f(\|x\|_{\mathbb{G}})\,dx = QC_b\int_0^R r^{Q-1}f(r)\,dr$$

and

$$\int_{\mathbb{G}\setminus B(y,R)} f(\|y^{-1}\cdot x\|_{\mathbb{G}})\,dx = \int_{\mathbb{G}\setminus B(0,R)} f(\|x\|_{\mathbb{G}})\,dx = QC_b\int_R^{+\infty} r^{Q-1}f(r)\,dr.$$

Proof. At first, let us compute the constant M_f for the function $f(x) = \|x\|_{\mathbb{G}}^{-Q}$. By Proposition 3.2.4, taking $g(\|x\|_{\mathbb{G}}) = \|x\|_{\mathbb{G}}^{Q}\chi_{[0,1]}(\|x\|_{\mathbb{G}})$, we get

$$C_b = \int_B dx = \int_{\mathbb{G}} \|x\|_{\mathbb{G}}^{-Q}\|x\|_{\mathbb{G}}^{Q}\chi_{[0,1]}(\|x\|_{\mathbb{G}})\,dx = M_{\|x\|_{\mathbb{G}}^{-Q}}\int_0^1 r^{Q-1}\,dr = \frac{M_{\|x\|_{\mathbb{G}}^{-Q}}}{Q},$$

i.e., $M_{\|x\|_{\mathbb{G}}^{-Q}} = QC_b$.

Therefore, still by Proposition 3.2.4, we have

$$\begin{aligned}\int_{B(0,R)} f(\|x\|_{\mathbb{G}})\,dx &= \int_{\mathbb{G}} \|x\|_{\mathbb{G}}^{-Q}\|x\|_{\mathbb{G}}^{Q}\chi_{[0,R]}(\|x\|_{\mathbb{G}})f(\|x\|_{\mathbb{G}})\,dx\\ &= M_{\|x\|_{\mathbb{G}}^{-Q}}\int_0^R r^{Q-1}f(r)\,dr = QC_b\int_0^R r^{Q-1}f(r)\,dr\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{G}\setminus B(0,R)} f(\|x\|_{\mathbb{G}})\,dx &= \int_{\mathbb{G}} \|x\|_{\mathbb{G}}^{-Q}\|x\|_{\mathbb{G}}^{Q}\chi_{[R,+\infty[}(\|x\|_{\mathbb{G}})f(\|x\|_{\mathbb{G}})\,dx\\ &= QC_b\int_R^{+\infty} r^{Q-1}f(r)\,dr.\end{aligned}$$

□

Proposition 3.2.6. *[38, Proposition 1.15] There exists a unique Radon measure* σ *on* S *such that for all* $u \in L^1(\mathbb{G})$

$$\int_{\mathbb{G}} u(x)dx = \int_0^{+\infty}\left(\int_S u(\delta_r z)r^{Q-1}\,d\sigma(z)\right)\,dr$$

where S *is the unit sphere in* $\mathbb{G}$.

We conclude this part recalling the notion of Pansu differentiability, given by Pansu in [73]. First we define the horizontal gradient.

Definition 3.2.7. Let $(X_1, .., X_m)$ be a basis of left-invariant vector fields for the horizontal layer V_1. The vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, \ldots, X_m)$$

is called the *horizontal gradient* or $\mathbb{G}$-*gradient*.

Consequently, for any $u : \mathbb{G} \to \mathbb{R}$ sufficiently smooth, the horizontal gradient of u is defined as

$$\nabla_{\mathbb{G}} u := \sum_{j=1}^{m} (X_j u) X_j = (X_1 u, .., X_m u).$$

Definition 3.2.8. A function $f : \mathbb{G} \to \mathbb{R}$ is said to be *Pansu differentiable* at $x \in \mathbb{G}$ if there exists a $\mathbb{G}-$linear map $L_x^f : \mathbb{G} \to \mathbb{R}$, named Pansu differential, such that

$$\lim_{\|h\|_{\mathbb{G}} \to 0} \frac{f(x \cdot h) - f(x) - L_x^f(h')}{\|h\|_{\mathbb{G}}} = 0.$$

We will say that f is Pansu differentiable in $\mathbb{G}$ if it is Pansu differentiable at any $x \in \mathbb{G}$.

Remark 3.2.9. Let us notice that the Pansu differential L_x^f does not depend on the basis of the Lie algebra $\mathfrak{g}$. In the sequel, we will use the notation

$$L_x^f(h') = \nabla_{\mathbb{G}} f \cdot h'$$

where we recall that $h = (h', h'')$ and h' denotes the horizontal part of h. As noted in [39, Section 5] every function in $C^1(\mathbb{G})$ is also Pansu differentiable and the inclusion is actually strict.

3.2.3 Orlicz Functions

Definition 3.2.10. Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a real valued function such that:

i) $\phi(0) = 0$, and $\phi(t) > 0$ for any $t > 0$;

ii) ϕ is nondecreasing on $\mathbb{R}_0^+$;

iii) ϕ is right-continuous in $\mathbb{R}_0^+$ and $\lim_{t \to +\infty} \phi(t) = +\infty$.

Then, the real valued function defined on $\mathbb{R}_0^+$ by

$$\varphi(t) = \int_0^t \phi(s)\, ds$$

is called Orlicz function (or Nice Young function).

It is easy to show that hypotheses $(i)-(iii)$ imply that φ is continuous, Locally Lipschitz continuous, strictly increasing and convex on $\mathbb{R}_0^+$. Moreover, $\varphi(0) = 0$ and φ is superlinear at zero and at infinity, i.e.,

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \qquad \lim_{t\to +\infty} \frac{\varphi(t)}{t} = +\infty.$$

Up to normalization, we can assume $\varphi(1) = 1$. Hypotheses $(i) - (iii)$ also guarantee the existence of $\varphi^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, which is continuous, concave and strictly increasing, with $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(1) = 1$.

From now on, the following growth condition will be required on φ:

$$p^- \leq \frac{t\,\phi(t)}{\varphi(t)} \leq p^+ \quad \forall t > 0, \tag{L}$$

where $p^- \leq p^+$ are positive constants grater than 1. It holds that

$$s^{\bar{p}}\varphi(t) \leq \varphi(st) \leq s^{\tilde{p}}\varphi(t), \tag{φ_1}$$

$$\varphi(s+t) \leq \frac{2^{p^+}}{2}(\varphi(s) + \varphi(t)). \tag{φ_2}$$

for any $s, t \in \mathbb{R}_0^+$, where $s^{\tilde{p}} := \max\{s^{p^-}, s^{p^+}\}$ and $s^{\bar{p}} := \min\{s^{p^-}, s^{p^+}\}$.

Let us notice that $p^- = p^+$ if and only if $\phi(t) = t^p$, being $\varphi(1) = 1$.
We remind that the conjugate function of φ, defined as its Legendre's transform, is

$$\varphi^*(s) := \sup_{t>0}\{st - \varphi(t)\}.$$

Definition 3.2.11. The smallest $C \in \mathbb{R}^+$ such that the following Δ_2-condition holds

$$\varphi(2t) \leq C\varphi(t) \quad \forall t \in \mathbb{R}_0^+,$$

is called the Δ_2-constant and it is denoted by $\mathbf{C}$. By (φ_2), we have that

$$2 < \mathbf{C} \leq 2^{p^+}. \tag{3.2.1}$$

It is not difficult to show that (L) is equivalent to require the Δ_2-condition both on φ and φ^* (see for instance [74, Chapter 4]).

The following Lemma can be seen as an improvement of (φ_2).

Lemma 3.2.12. *[9, Lemma 2.6] Let φ be an Orlicz function and let $s, t \in \mathbb{R}_0^+$. Then, for any $\delta > 0$, there exists a positive constant C_δ such that*

$$\varphi(s+t) \leq C_\delta \varphi(s) + (1+\delta)^{p^+} \varphi(t).$$

We conclude this section recalling a fundamental definition which is the natural counterpart of [9, Remark 2.15] in the context of Carnot groups.

Definition 3.2.13. For an Orlicz function φ and $t \in \mathbb{R}^+$, we define the bounded function

$$\tilde{\varphi}^+(t) := \limsup_{s\uparrow 1}(1-s) \int_0^1 \left(\int_S \varphi(t\|z'\|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z) \right) \frac{dr}{r}.$$

A similar definition with lim inf instead of lim sup is used to define $\tilde{\varphi}^-$. When they coincide, we will define

$$\tilde{\varphi}(t) := \lim_{s\uparrow 1}(1-s) \int_0^1 \left(\int_S \varphi(t\|z'\|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z) \right) \frac{dr}{r}. \tag{3.2.2}$$

Proposition 3.2.14. *The functions $\tilde{\varphi}^\pm$ are still Orlicz functions, both of them equivalent to φ, i.e., there exist $c_1, c_2 > 0$ such that*

$$c_1\varphi(t) \leq \tilde{\varphi}^\pm(t) \leq c_2 \varphi(t)$$

for any $t \in \mathbb{R}^+$.

Proof. $\tilde{\varphi}^\pm$ are Orlicz functions by similar arguments of [9, Proposition 2.16]. Moreover, by (φ_1), we can notice that

$$\begin{aligned}\int_0^1 \left(\int_S \varphi(t\|z'\|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z) \right) \frac{dr}{r} &\leq \int_S \|z'\|_{\mathbb{R}^m}^{p^-} d\sigma(z) \int_0^1 \varphi(tr^{1-s}) \frac{dr}{r} \\ &\leq QC_b \varphi(t) \int_0^1 r^{(1-s)p^- -1} dr = \frac{QC_b}{(1-s)p^-} \varphi(t)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \left(\int_S \varphi(t\|z'\|_{\mathbb{R}^m} r^{1-s})\, d\sigma(z) \right) \frac{dr}{r} &\geq \int_S \|z'\|_{\mathbb{R}^m}^{p^+} d\sigma(z) \int_0^1 \varphi(tr^{1-s}) \frac{dr}{r} \\ &\geq QC_b \varphi(t) \int_0^1 r^{(1-s)p^+ -1} dr = \frac{QC_b}{(1-s)p^+} \varphi(t).\end{aligned}$$

Thus, taking $c_1 := \frac{QC_b}{p^+}$ and $c_2 := \frac{QC_b}{p^-}$, we get the thesis. □

Remark 3.2.15. Let us notice that $c_1 = c_2 = \frac{QC_b}{p}$ if and only if $\varphi(t) = t^p$. We also remind that explicit examples of $\tilde{\varphi}$, in the Euclidean case, are given in [9, Example 2.17].

3.2.4 The Functional Setting

Definition 3.2.16. Let $\mathbb{G}$ be a Carnot group, let φ be an Orlicz function and let $0 < s \leq 1$. We define, with a little abuse of notation, the Orlicz-Lebesgue space and the Fractional Orlicz-Sobolev spaces, respectively, as

$$\begin{aligned} L^{\varphi}(\mathbb{G}) &:= \{u : \mathbb{G} \to \mathbb{R} \text{ measurables such that } \Phi_{\varphi}(u) < \infty\} \\ W^{s,\varphi}(\mathbb{G}) &:= \{u \in L^{\varphi}(\mathbb{G}) \text{ such that } \Phi_{s,\varphi}(u) < \infty\}, \end{aligned}$$

where

$$\Phi_{\varphi}(u) := \int_{\mathbb{G}} \varphi(|u(x)|)\, dx,$$

$$\Phi_{s,\varphi}(u) := \begin{cases} \displaystyle\iint_{\mathbb{G}\times\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{s}}\right) \frac{dx\, dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{Q}} & \text{if } 0 < s < 1 \\ \Phi_{\varphi}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m}) & \text{if } s = 1 \end{cases}.$$

These spaces are usually endowed with the so-called Luxemburg norms, studied by Luxemburg in [60], and defined as

$$\begin{aligned} \|u\|_{\varphi} &:= \inf\{\lambda > 0 : \Phi_{\varphi}\left(\frac{u}{\lambda}\right) \leq 1\} \\ \|u\|_{s,\varphi} &:= \|u\|_{\varphi} + [u]_{s,\varphi} \end{aligned}$$

where

$$[u]_{s,\varphi} := \inf\{\lambda > 0 : \Phi_{s,\varphi}\left(\frac{u}{\lambda}\right) \leq 1\}$$

is the $(s,\varphi)-$Gagliardo seminorm.

By well-known results given in [29, 42] for the Euclidean case, it is easy to characterize these spaces as follows

Theorem 3.2.17. *Let φ be an Orlicz function, then $L^{\varphi}(\mathbb{G})$ and $W^{1,\varphi}(\mathbb{G})$ are separable Banach spaces. Moreover, if both φ and φ^* satisfy the Δ_2-condition, then the spaces $L^{\varphi}(\mathbb{G})$ and $W^{1,\varphi}(\mathbb{G})$ are also reflexive and the dual space of $L^{\varphi}(\mathbb{G})$ can be identified with $L^{\varphi^*}(\mathbb{G})$. Finally, $C_c^{\infty}(\mathbb{G})$ is dense in both $L^{\varphi}(\mathbb{G})$ and $W^{1,\varphi}(\mathbb{G})$.*

The proof of Theorem 3.2.17 trivially follows from the Euclidean case. The reader can see for instance [29, Theorem 2.3.13, Theorem 2.5.10] and [42, Theorem 5.3, Theorem 5.5, Corollary 3.7, Corollary 3.9], where a more general theory is treated.

Following the same technique of [9, Proposition 2.11], we can also state the following Theorem.

Theorem 3.2.18. *Let us assume the same hypotheses of the previous Theorem. Then, for each* $s \in (0,1)$*, the space* $W^{s,\varphi}(\mathbb{G})$ *is a reflexive and separable Banach space. Moreover,* $C_c^\infty(\mathbb{G})$ *is dense in* $W^{s,\varphi}(\mathbb{G})$*.*

As in the Euclidean case, the immersion of the space $W^{s,\varphi}(\mathbb{G})$ into $L^\varphi(\mathbb{G})$ is compact, as a consequence of the following

Theorem 3.2.19. *[49, Theorem 11.4] Any sequence of functions* $\{v_k\}_k \subset L^\varphi(\mathbb{G})$ *is compact if and only if the following two conditions are satisfied:*

i) $\Phi_\varphi(v_k)$ *is bounded;*

ii) for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\Phi_\varphi(\tau_h v_k - v_k) < \varepsilon$ *for any* $h \in \mathbb{G}$ *such that* $\|h\|_{\mathbb{G}} < \delta$*, where* $\tau_h u(x) := u(x \cdot h)$ *for any* $x \in \mathbb{G}$*.*

Theorem 3.2.20. *Let* $0 < s < 1$ *and* φ *be an Orlicz function. Then, from every bounded sequence* $\{u_n\}_n \subset W^{s,\varphi}(\mathbb{G})$*, there exist* $u \in W^{s,\varphi}(\mathbb{G})$ *and* $\{u_{n_k}\}_k \subset \{u_n\}_n$ *such that* $u_{n_k} \to u$ *in* $L^\varphi(\mathbb{G})$*.*

Proof. Let us fix $u \in W^{s,\varphi}(\mathbb{G})$. In order to apply Theorem 3.2.19, we want to show the existence of a constant $M > 0$ such that

$$\Phi_\varphi(\tau_h u - u) \leq M \|h\|_{\mathbb{G}}^{sp^-} \Phi_{s,\varphi}(u) \tag{3.2.3}$$

for every $h \in \mathbb{G}$ such that $\|h\|_{\mathbb{G}} < \frac{1}{2}$.

For any $y \in B(x, \|h\|_{\mathbb{G}})$, by the monotonicity of φ, the Δ_2-condition and being $|B(x,r)| = r^Q|B| = r^Q C_b$, we have

$$\begin{aligned}
\Phi_\varphi(\tau_h u - u) &= \int_{\mathbb{G}} \varphi(|u(x\cdot h) - u(y) + u(y) - u(x)|)\, dx \\
&\leq \frac{\mathbf{C}}{2}\left[\int_{\mathbb{G}} \varphi(|u(x\cdot h) - u(y)|)\, dx + \int_{\mathbb{G}} \varphi(|u(y) - u(x)|)\, dx\right] \\
&= \frac{\mathbf{C}}{2C_b\|h\|_{\mathbb{G}}^Q} \int_{B(x,\|h\|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi(|u(x\cdot h) - u(y)|)\, dx\right) dy \\
&\quad + \frac{\mathbf{C}}{2C_b\|h\|_{\mathbb{G}}^Q} \int_{B(x,\|h\|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi(|u(y) - u(x)|)\, dx\right) dy \\
&= \frac{\mathbf{C}}{2C_b\|h\|_{\mathbb{G}}^Q}(I_1 + I_2).
\end{aligned} \tag{3.2.4}$$

Let us notice that, by the triangular inequality,

$$\|y^{-1}\cdot x \cdot h\|_{\mathbb{G}} \leq \|y^{-1}\cdot x\|_{\mathbb{G}} + \|h\|_{\mathbb{G}} \leq 2\|h\|_{\mathbb{G}}.$$

Therefore, by (φ_1), the monotonicity of φ and a change of variables, we have

$$\begin{aligned}
I_1 &= \int_{B(x,\|h\|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi \left(\frac{|u(x\cdot h) - u(y)|}{\|y^{-1}\cdot x\cdot h\|_{\mathbb{G}}^s} \|y^{-1}\cdot x\cdot h\|_{\mathbb{G}}^s \right) \frac{\|y^{-1}\cdot x\cdot h\|_{\mathbb{G}}^Q}{\|y^{-1}\cdot x\cdot h\|_{\mathbb{G}}^Q}\, dx \right) dy \\
&\leq 2^Q \|h\|_{\mathbb{G}}^Q \int_{B(x,\|h\|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi \left(\frac{|u(x\cdot h) - u(y)|}{\|y^{-1}\cdot x\cdot h\|_{\mathbb{G}}^s} (2\|h\|_{\mathbb{G}})^s \right) \frac{dx}{\|y^{-1}\cdot x\cdot h\|_{\mathbb{G}}^Q} \right) dy \\
&\leq 2^{sp^-+Q} \|h\|_{\mathbb{G}}^{sp^-+Q} \int_{B(x,\|h\|_{\mathbb{G}})} \left(\int_{\mathbb{G}} \varphi \left(\frac{|u(z) - u(y)|}{\|y^{-1}\cdot z\|_{\mathbb{G}}^s} \right) \frac{dz}{\|y^{-1}\cdot z\|_{\mathbb{G}}^Q} \right) dy \\
&\leq 2^{sp^-+Q} \|h\|_{\mathbb{G}}^{sp^-+Q} \Phi_{s,\varphi}(u).
\end{aligned}$$

Similarly

$$I_2 \leq \|h\|_{\mathbb{G}}^{sp^-+Q} \Phi_{s,\varphi}(u).$$

Thus, by (3.2.4), we finally have

$$\Phi_\varphi(\tau_h u - u) \leq \frac{\mathbf{C}}{2C_b} (2^{sp^-+Q} + 1) \|h\|_{\mathbb{G}}^{sp^-} \Phi_{s,\varphi}(u) := M \|h\|_{\mathbb{G}}^{sp^-} \Phi_{s,\varphi}(u).$$

Let now $\{u_n\}_n \subset W^{s,\varphi}(\mathbb{G})$ be a bounded sequence in $W^{s,\varphi}(\mathbb{G})$. In particular, $\{u_n\}_n$ is bounded in $L^\varphi(\mathbb{G})$. Therefore, by (3.2.3)

$$\sup_{n\in\mathbb{N}} \Phi_\varphi(\tau_h u_n - u_n) \leq \sup_{n\in\mathbb{N}} (\Phi_{s,\varphi}(u_n) + \Phi_\varphi(u_n)) M \|h\|_{\mathbb{G}}^{sp^-}.$$

Thus, by Theorem 3.2.19, there exist $u \in L^\varphi(\mathbb{G})$ and $\{u_{n_k}\}_k \subset \{u_n\}_n$ such that $u_{n_k} \to u$ in $L^\varphi(\mathbb{G})$. In order to conclude the proof, we show that $u \in W^{s,\varphi}(\mathbb{G})$.

By the Fatou's Lemma and the continuity of φ, we have

$$\begin{aligned}
\Phi_{s,\varphi}(u) &= \iint_{\mathbb{G}\times\mathbb{G}} \varphi \left(\frac{|u(x) - u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \right) \frac{dx\, dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} \\
&\leq \liminf_{k\to\infty} \iint_{\mathbb{G}\times\mathbb{G}} \varphi \left(\frac{|u_{n_k}(x) - u_{n_k}(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \right) \frac{dx\, dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} \\
&\leq \sup_{n\in\mathbb{N}} \Phi_{s,\varphi}(u_{n_k}) < \infty.
\end{aligned}$$

□

Taking into account [11, Definition 5.3.6], we state below the notions of convolution and truncation on Carnot groups.

Definition 3.2.21 (Regularization)**.** Let $\rho \in C_c^\infty(\mathbb{G};\mathbb{R}_0^+)$ be a standard mollifier, that is, $\mathrm{supp}(\rho) \subset B(0,1)$ and $\int_{\mathbb{G}} \rho(x)\,dx = 1$. Then, for any $u \in L^\varphi(\mathbb{G})$ and $x \in \mathbb{G}$, considering the sequence of mollifiers

$$\rho_\varepsilon(x) := \varepsilon^{-Q}\rho(\delta_{\varepsilon^{-1}}x) \qquad \varepsilon > 0,$$

we define the regularized functions of u, $\{u_\varepsilon\}_\varepsilon \subset L^\varphi(\mathbb{G}) \cap C^\infty(\mathbb{G})$, as

$$\begin{aligned} u_\varepsilon(x) := (u * \rho_\varepsilon)(x) &:= \int_{\mathbb{G}} u(y)\rho_\varepsilon(x \cdot y^{-1})\,dy = \int_{B(0,\varepsilon)} u(y^{-1}\cdot x)\rho_\varepsilon(y)\,dy \\ &= \int_{B(0,1)} u((\delta_\varepsilon z)^{-1}\cdot x)\rho(z)\,dz \end{aligned}$$

for any $\varepsilon > 0$.

Definition 3.2.22 (Truncation)**.** Given $\eta \in C_c^\infty(\mathbb{G})$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B(0,1)$, $\mathrm{supp}(\eta) \subset B(0,2)$ and $\|\nabla_{\mathbb{G}}\eta\|_{\mathbb{R}^m} \le 2$, we define, for any $k \in \mathbb{N}$, the cut-off functions

$$\eta_k(x) := \eta(\delta_{k^{-1}}x).$$

Let us notice that $0 \le \eta_k \le 1$, $\eta_k = 1$ in $B(0,k)$, $\mathrm{supp}(\eta_k) \subset B(0,2k)$ and $\|\nabla_{\mathbb{G}}\eta_k\|_{\mathbb{R}^m} \le \frac{2}{k}$.

For any $u \in L^\varphi(\mathbb{G})$ we define the truncated functions of u, $\{u_k\}_k$, as

$$u_k := \eta_k u$$

for any $k \in \mathbb{N}$. We remind that $supp(u_k) \subset B(0,2k)$.

The two following Lemmas will be useful in the next section.

Lemma 3.2.23. *Let $u \in L^\varphi(\mathbb{G})$ and let $\{u_\varepsilon\}_\varepsilon$ be a sequence of regularized functions of u, in the sense of Definition 3.2.21. Then*

$$\Phi_{s,\varphi}(u_\varepsilon) \le \Phi_{s,\varphi}(u)$$

for any $\varepsilon > 0$ and $0 < s < 1$.

Proof. Let $x, y \in \mathbb{G}$ and let $h = y^{-1}\cdot x$. Then, by the Jensen's inequality and the monotonicity of φ, we have

$$\begin{aligned} \varphi\left(\frac{|u_\varepsilon(x\cdot h) - u_\varepsilon(x)|}{\|h\|_{\mathbb{G}}^s}\right) &\le \varphi\left(\int_{B(0,\varepsilon)} \frac{|u(y^{-1}\cdot x\cdot h) - u(y^{-1}\cdot x)|}{\|h\|_{\mathbb{G}}^s}\rho_\varepsilon(y)\,dy\right) \\ &\le \int_{B(0,\varepsilon)} \varphi\left(\frac{|u(y^{-1}\cdot x\cdot h) - u(y^{-1}\cdot x)|}{\|h\|_{\mathbb{G}}^s}\right)\rho_\varepsilon(y)\,dy. \end{aligned}$$

Therefore, by the invariance of the norm under translations, we have

$$\begin{aligned}
&\int_{\mathbb{G}} \varphi\left(\frac{|u_\varepsilon(x\cdot h)-u_\varepsilon(x)|}{\|h\|_{\mathbb{G}}^s}\right)\frac{dx}{\|h\|_{\mathbb{G}}^Q}\\
&\le \int_{\mathbb{G}}\left(\int_{B(0,\varepsilon)}\varphi\left(\frac{|u(y^{-1}\cdot x\cdot h)-u(y^{-1}\cdot x)|}{\|h\|_{\mathbb{G}}^s}\right)\rho_\varepsilon(y)\,dy\right)\frac{dx}{\|h\|_{\mathbb{G}}^Q}\\
&=\int_{\mathbb{G}}\left(\int_{\mathbb{G}}\varphi\left(\frac{|u(y^{-1}\cdot x\cdot h)-u(y^{-1}\cdot x)|}{\|h\|_{\mathbb{G}}^s}\right)\frac{dx}{\|h\|_{\mathbb{G}}^Q}\right)\rho_\varepsilon(y)\,dy\\
&=\int_{\mathbb{G}}\varphi\left(\frac{|u(x\cdot h)-u(x)|}{\|h\|_{\mathbb{G}}^s}\right)\frac{dx}{\|h\|_{\mathbb{G}}^Q}.
\end{aligned}$$

Thus, the thesis follows, by integrating in $\mathbb{G}$ with respect to h. □

Lemma 3.2.24. *Let $u \in L^\varphi(\mathbb{G})$ and let $\{u_k\}_k$ be the sequence of truncated functions of u, in the sense of Definition 3.2.22. Then*

$$\Phi_{s,\varphi}(u_k) \le \frac{\mathbf{C}}{2}\left(\Phi_{s,\varphi}(u) + \left(\frac{2}{k}\right)^{p^-}\frac{QC_b}{(1-s)p^+}\Phi_\varphi(u) + 2^{p^+}\frac{QC_b}{sp^-}\Phi_\varphi(u)\right)$$

for any $k \in \mathbb{N}$ and $0 < s < 1$.

Proof. Let us fix $x, y \in \mathbb{G}$. Then, by the Δ_2-condition and the monotonicity of φ, we have

$$\begin{aligned}
&\varphi\left(\frac{|u_k(x)-u_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\\
&=\varphi\left(\frac{|\eta_k(x)u(x)-\eta_k(y)u(x)+\eta_k(y)u(x)-\eta_k(y)u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\\
&\le \frac{\mathbf{C}}{2}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)+\frac{\mathbf{C}}{2}\varphi\left(\frac{|\eta_k(y)||u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right).
\end{aligned}$$

Hence, being $\eta_k \le 1$ for any $k \in \mathbb{N}$, we get

$$\begin{aligned}
\Phi_{s,\varphi}(u_k) &= \iint_{\mathbb{G}\times\mathbb{G}}\varphi\left(\frac{|u_k(x)-u_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\\
&\le \frac{\mathbf{C}}{2}\Phi_{s,\varphi}(u)+\frac{\mathbf{C}}{2}\iint_{\mathbb{G}\times\mathbb{G}}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\\
&=\frac{\mathbf{C}}{2}\Phi_{s,\varphi}(u)+\frac{\mathbf{C}}{2}\left[\int_{\mathbb{G}}\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}<1\}}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\right.\\
&\quad\left.+\int_{\mathbb{G}}\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\ge 1\}}\varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\right].
\end{aligned}$$

Since $\|\nabla\eta_k\|_{\mathbb{R}^m} \le \frac{2}{k}$, then, by (φ_1), assuming without loss of generality $k > 2$, and by Proposition 3.2.5, we have

$$\begin{aligned}
&\int_{\mathbb{G}}\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\\
&\qquad\le \int_{\mathbb{G}}\left(\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\le 1\}}\varphi\left(\frac{2}{k}\frac{|u(x)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{s-1}}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\right)dx\\
&\qquad\le \left(\frac{2}{k}\right)^{p^-}\int_{\mathbb{G}}\left(\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\le 1\}}\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{(s-1)p^++Q}}\right)\varphi(|u(x)|)\,dx\\
&\qquad= \Phi_\varphi(u)\left(\frac{2}{k}\right)^{p^-} QC_b\int_0^1 r^{(1-s)p^+-1}\,dr\\
&\qquad= \left(\frac{2}{k}\right)^{p^-}\frac{QC_b}{(1-s)p^+}\Phi_\varphi(u).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_{\mathbb{G}}\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\ge 1\}} \varphi\left(\frac{|u(x)||\eta_k(x)-\eta_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\\
&\qquad\le \int_{\mathbb{G}}\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\ge 1\}}\varphi\left(\frac{2|u(x)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{s}}\right)\frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\\
&\qquad\le 2^{p^+}\int_{\mathbb{G}}\left(\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\ge 1\}}\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{sp^-+Q}}\right)\varphi(|u(x)|)\,dx\\
&\qquad= \Phi_\varphi(u)2^{p^+}QC_b\int_1^\infty r^{-sp^--1}\,dr\\
&\qquad= 2^{p^+}\frac{QC_b}{sp^-}\Phi_\varphi(u).
\end{aligned}$$

□

3.3 A Bougain-Brezis-Mironescu-type Theorem

In order to prove Theorem 3.2.1, we need the two following fundamental Lemmas.

Lemma 3.3.1. *Let $u \in W^{1,\varphi}(\mathbb{G})$. Then, for any $0 < s < 1$,*

$$\Phi_{s,\varphi}(u) \le \frac{QC_b}{p^-}\left(\frac{1}{1-s}\Phi_\varphi(\|\nabla_{\mathbb{G}}u\|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s}\Phi_\varphi(u)\right),$$

where $\mathbf{C}$ is the Δ_2-constant given in (3.2.1) *and p^- is given in* (L).

Proof. Let $u \in C_c^2(\mathbb{G})$ and denote for simplicity $h = y^{-1} \cdot x$. We can compute $\Phi_{s,\varphi}(u)$ as follows

$$\Phi_{s,\varphi}(u) = \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x\cdot h) - u(x)|}{\|h\|_{\mathbb{G}}^s} \right) \frac{dh}{\|h\|_{\mathbb{G}}^Q} \right) dx$$
$$+ \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}}\geq 1\}} \varphi\left(\frac{|u(x\cdot h) - u(x)|}{\|h\|_{\mathbb{G}}^s} \right) \frac{dh}{\|h\|_{\mathbb{G}}^Q} \right) dx = I_1 + I_2.$$

Let us start by expanding I_1. First observe that $u \in C_c^2(\mathbb{G})$ implies that u is Pansu differentiable (see also [6, Section 2]). We now define the auxiliary function $\xi : [0,1] \to \mathbb{R}$ as

$$\xi(t) := u(x \cdot \delta_t h).$$

Then we can write

$$u(x\cdot h) - u(x) = \xi(1) - \xi(0) = \int_0^1 \frac{d}{dt}\xi(t)\,dt = \int_0^1 \nabla_{\mathbb{G}} u(x\cdot \delta_t h)\cdot h'\,dt.$$

Therefore, by the monotonicity and the convexity of φ, we get

$$\begin{aligned} \varphi\left(\frac{|u(x\cdot h) - u(x)|}{\|h\|_{\mathbb{G}}^s} \right) &\leq \varphi\left(\int_0^1 \frac{|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)\cdot h'|}{\|h\|_{\mathbb{G}}^s}\,dt \right) \\ &\leq \int_0^1 \varphi\left(\frac{|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)\cdot h'|}{\|h\|_{\mathbb{G}}^s} \right) dt \qquad (3.3.1) \\ &\leq \int_0^1 \varphi(\|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)\|_{\mathbb{R}^m}\|h\|_{\mathbb{G}}^{1-s})\,dt. \end{aligned}$$

Thus, by (φ_1) and Proposition 3.2.5

$$\begin{aligned} I_1 &\leq \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(\|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)\|_{\mathbb{R}^m}\|h\|_{\mathbb{G}}^{1-s})\,dt \right) \frac{dh}{\|h\|_{\mathbb{G}}^Q} \right) dx \\ &\leq \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(\|\nabla_{\mathbb{G}} u(x\cdot\delta_t h)\|_{\mathbb{R}^m})\,dt \right) \frac{\|h\|_{\mathbb{G}}^{(1-s)p^-}}{\|h\|_{\mathbb{G}}^Q}\,dh \right) dx \\ &= \int_{\{\|h\|_{\mathbb{G}}<1\}} \|h\|_{\mathbb{G}}^{(1-s)p^- - Q}\,dh \int_{\mathbb{G}} \varphi(\|\nabla_{\mathbb{G}} u(x)\|_{\mathbb{R}^m})\,dx \\ &= QC_b \int_0^1 r^{(1-s)p^- - 1}\,dr\, \Phi_\varphi(\|\nabla u\|_{\mathbb{R}^m}) \\ &= \frac{QC_b}{(1-s)p^-}\Phi_\varphi(\|\nabla u\|_{\mathbb{R}^m}). \end{aligned}$$

Moreover, by (φ_1), (φ_2), Proposition 3.2.5, the monotonicity of φ and by a change of variables, we have

$$
\begin{aligned}
I_2 &\le \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}} \ge 1\}} \varphi\left(|u(x\cdot h)| + |u(x)|\right) \frac{dh}{\|h\|_{\mathbb{G}}^{sp^-+Q}} \right) dx \\
&\le \frac{\mathbf{C}}{2} \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}} \ge 1\}} \varphi\left(|u(x\cdot h)|\right) \frac{dh}{\|h\|_{\mathbb{G}}^{sp^-+Q}} \right) dx \\
&\quad + \frac{\mathbf{C}}{2} \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}} \ge 1\}} \varphi\left(|u(x)|\right) \frac{dh}{\|h\|_{\mathbb{G}}^{sp^-+Q}} \right) dx \\
&= \mathbf{C} \int_{\mathbb{G}} \left(\int_{\{\|h\|_{\mathbb{G}} \ge 1\}} \varphi\left(|u(x)|\right) \frac{dh}{\|h\|_{\mathbb{G}}^{sp^-+Q}} \right) dx \\
&= \mathbf{C} \int_{\{\|h\|_{\mathbb{G}} \ge 1\}} \frac{dh}{\|h\|_{\mathbb{G}}^{sp^-+Q}} \int_{\mathbb{G}} \varphi\left(|u(x)|\right) dx \\
&= \mathbf{C} Q C_b \int_1^{+\infty} r^{-sp^- -1}\, dr\, \Phi_\varphi(u) = \mathbf{C} \frac{Q C_b}{sp^-} \Phi_\varphi(u).
\end{aligned}
$$

Finally, for any $u \in W^{1,\varphi}(\mathbb{G})$, let $\{u_k\}_k \subset C_c^2(\mathbb{G})$ be convergent to u in $W^{1,\varphi}(\mathbb{G})$. Thus, by the Fatou's Lemma and the continuity of φ, we get

$$
\begin{aligned}
\Phi_{s,\varphi}(u) &\le \liminf_{k\to\infty} \Phi_{s,\varphi}(u_k) \le \lim_{k\to\infty} \left[\frac{QC_b}{p^-} \left(\frac{1}{1-s} \Phi_\varphi(\|\nabla_{\mathbb{G}} u_k\|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s} \Phi_\varphi(u_k) \right) \right] \\
&= \frac{QC_b}{p^-} \left(\frac{1}{1-s} \Phi_\varphi(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s} \Phi_\varphi(u) \right)
\end{aligned}
$$

as desired. □

Lemma 3.3.2. *Let φ be an Orlicz function such that $\tilde{\varphi}$ exists and let $u \in C_c^2(\mathbb{G})$. Then, for every fixed $x \in \mathbb{G}$, we have that*

$$
\lim_{s\uparrow 1} (1-s) \int_{\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \right) \frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} = \tilde{\varphi}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m}).
$$

Proof. For each fixed $x \in \mathbb{G}$ we have

$$
\begin{aligned}
\int_{\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \right) \frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} &= \int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}} < 1\}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \right) \frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} \\
&\quad + \int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}} \ge 1\}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \right) \frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} \\
&= I_1 + I_2.
\end{aligned}
$$

Let us first notice that

$$\lim_{s\uparrow 1}(1-s)I_2 = 0.$$

In fact, by (φ_1) and Proposition 3.2.5, we have

$$\begin{aligned}\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\geq 1\}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} &\leq \varphi(2\|u\|_\infty)\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\geq 1\}}\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{sp^-+Q}}\\ &= \varphi(2\|u\|_\infty)QC_b\int_1^{+\infty} r^{-sp^--1}\,dr\\ &= \frac{QC_b}{sp^-}\varphi(2\|u\|_\infty).\end{aligned}$$

Now, by the local Lipschitzianity of φ, for any $x,y\in\mathbb{G}$ such that $x\neq y$, we have

$$\begin{aligned}\left|\varphi\left(\frac{|u(x)-u(y)|}{\|h\|_{\mathbb{G}}^s}\right)-\varphi\left(\frac{|\nabla_{\mathbb{G}}u(x)\cdot h'|}{\|h\|_{\mathbb{G}}^s}\right)\right| &\leq L\frac{|u(x)-u(y)-\nabla_{\mathbb{G}}u(x)\cdot h'|}{\|h\|_{\mathbb{G}}^s}\\ &\leq C\|h\|_{\mathbb{G}}^{2-s}\end{aligned}$$

where L is the Lipschitz constant of φ in the interval $[0,\|\nabla_{\mathbb{G}}u\|_\infty]$, C is a constant depending on the C^2-norm of u and $h=y^{-1}\cdot x$. (The last inequality follows from standard results about the Taylor polynomial that can be found, for instance, in [11, Chapter 20]).

Moreover, by Proposition 3.2.5, it follows that

$$\int_{\{\|h\|_{\mathbb{G}}<1\}}\|h\|_{\mathbb{G}}^{2-s}\frac{dy}{\|h\|_{\mathbb{G}}^Q} = QC_b\int_0^1 r^{1-s}\,dr = \frac{QC_b}{2-s}.$$

Thus

$$\begin{aligned}&\lim_{s\uparrow 1}(1-s)\int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}<1\}}\varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}\\ &=\lim_{s\uparrow 1}(1-s)\int_{\{\|h\|_{\mathbb{G}}<1\}}\varphi\left(\frac{|\nabla_{\mathbb{G}}u(x)\cdot h'|}{\|h\|_{\mathbb{G}}^s}\right)\frac{dy}{\|h\|_{\mathbb{G}}^Q}.\end{aligned}$$

Finally, by Proposition 3.2.6 and the invariance of $\|\cdot\|_{\mathbb{G}}$ under horizontal rotations, we have

$$\begin{aligned}\int_{\{\|h\|_{\mathbb{G}}<1\}}\varphi\left(\frac{|\nabla_{\mathbb{G}}u(x)\cdot h'|}{\|h\|_{\mathbb{G}}^s}\right)\frac{dy}{\|h\|_{\mathbb{G}}^Q} &= \int_0^1\left(\int_S\varphi\left(\frac{|\nabla_{\mathbb{G}}u(x)\cdot\delta_r z'|}{\|\delta_r z\|_{\mathbb{G}}^s}\right)\frac{d\sigma(z)}{\|\delta_r z\|_{\mathbb{G}}^Q}\right)r^{Q-1}\,dr\\ &= \int_0^1\left(\int_S\varphi\left(\frac{|\nabla_{\mathbb{G}}u(x)\cdot z'|}{r^s\|z\|_{\mathbb{G}}^s}r\right)\frac{d\sigma(z)}{\|z\|_{\mathbb{G}}^Q}\right)\frac{r^{Q-1}}{r^Q}\,dr\\ &= \int_0^1\left(\int_S\varphi(\|\nabla_{\mathbb{G}}u(x)\|_{\mathbb{R}^m}\|z'\|_{\mathbb{R}^m}r^{1-s})d\sigma(z)\right)\frac{dr}{r},\end{aligned}$$

i.e.,

$$\lim_{s\uparrow 1}(1-s)I_1 = \tilde{\varphi}(\|\nabla_{\mathbb{G}}u(x)\|_{\mathbb{R}^m}).$$

□

Finally, we are ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. We divide the proof of the Theorem in three steps.

Step 1: Let us prove the Theorem for any function $u \in C_c^2(\mathbb{G})$ whose support is contained in $B(0,R)$. Let

$$F_s(x) := \int_{\mathbb{G}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}.$$

In virtue of Lemma 3.3.2, in order to apply the Dominated Convergence Theorem, it is enough to show the existence of an integrable function in $\mathbb{G}$ that dominates the sequence $\{(1-s)F_s\}_{s\in(0,1)}$.

Let us fix $R>1$. For any $x\in\mathbb{G}$ such that $\|x\|_{\mathbb{G}}<2R$, we have

$$\begin{aligned} F_s(x) &= \int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}<1\}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} \\ &\quad + \int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\geq 1\}} \varphi\left(\frac{|u(x)-u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} = I_1 + I_2. \end{aligned}$$

By (3.3.1), (φ_1), Proposition 3.2.5 and the monotonicity of φ, called $h = y^{-1}\cdot x$, we have

$$\begin{aligned} I_1 &\leq \int_{\{\|h\|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(\|\nabla_{\mathbb{G}}u(x\cdot\delta_t h)\|_{\mathbb{R}^m}\|h\|_{\mathbb{G}}^{1-s})\,dt\right)\frac{dh}{\|h\|_{\mathbb{G}}^Q} \\ &\leq \int_{\{\|h\|_{\mathbb{G}}<1\}} \left(\int_0^1 \varphi(\|\nabla_{\mathbb{G}}u(x\cdot\delta_t h)\|_{\mathbb{R}^m})\,dt\right)\frac{\|h\|_{\mathbb{G}}^{(1-s)p^-}}{\|h\|_{\mathbb{G}}^Q}\,dh \\ &\leq \varphi(||\nabla_{\mathbb{G}}u||_\infty)QC_b\int_0^1 r^{(1-s)p^- -1}\,dr = \frac{QC_b}{(1-s)p^-}\varphi(||\nabla_{\mathbb{G}}u||_\infty). \end{aligned}$$

Moreover, by (φ_1) and Proposition 3.2.5

$$\begin{aligned} I_2 &\leq \int_{\{\|y^{-1}\cdot x\|_{\mathbb{G}}\geq 1\}} \varphi\left(|u(x)|+|u(y)|\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^{sp^-+Q}} \\ &\leq \varphi(2||u||_\infty)QC_b\int_1^\infty r^{-sp^- -1}\,dr = \frac{QC_b}{sp^-}\varphi(2||u||_\infty). \end{aligned}$$

Thus

$$F_s(x) \leq \frac{QC_b}{(1-s)p^-}\varphi(||\nabla_{\mathbb{G}} u||_\infty) + \frac{QC_b}{sp^-}\varphi(2||u||_\infty) \tag{3.3.2}$$

for every $\|x\|_{\mathbb{G}} < 2R$.

Let $\|x\|_{\mathbb{G}} \geq 2R$. Since the support of u is contained in $B(0,R)$, then $u(z) = 0$ for any $\|z\|_{\mathbb{G}} > R$. Thus

$$F_s(x) = \int_{\{\|y\|_{\mathbb{G}} \leq R\}} \varphi\left(\frac{|u(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\frac{dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q}.$$

Now, since $\|y^{-1}\cdot x\|_{\mathbb{G}} \geq \|x\|_{\mathbb{G}} - \|y\|_{\mathbb{G}} \geq \|x\|_{\mathbb{G}} - R \geq \frac{1}{2}\|x\|_{\mathbb{G}}$, then by the monotonicity of φ, the Δ_2-condition and (φ_1), we get

$$\begin{aligned}
F_s(x) &\leq \int_{\{\|y\|_{\mathbb{G}} \leq R\}} \varphi\left(\frac{|u(y)|}{(\frac{1}{2}\|x\|_{\mathbb{G}})^s}\right)\frac{dy}{(\frac{1}{2}\|x\|_{\mathbb{G}})^Q} \\
&\leq 2^Q \int_{\{\|y\|_{\mathbb{G}} \leq R\}} \varphi\left(2^s|u(y)|\right)\frac{dy}{\|x\|_{\mathbb{G}}^{sp^-+Q}} \\
&\leq \mathbf{C}\frac{2^Q}{\|x\|_{\mathbb{G}}^{sp^-+Q}} \int_{\{\|y\|_{\mathbb{G}} \leq R\}} \varphi\left(|u(y)|\right) dy \\
&\leq \mathbf{C}\frac{2^Q}{\|x\|_{\mathbb{G}}^{\frac{1}{2}p^-+Q}} \int_{\{\|y\|_{\mathbb{G}} \leq R\}} \varphi\left(|u(y)|\right) dy,
\end{aligned}$$

that is

$$F_s(x) \leq \frac{K}{\|x\|_{\mathbb{G}}^{\frac{1}{2}p^-+Q}} \tag{3.3.3}$$

for every $\|x\|_{\mathbb{G}} \geq 2R$, where we assumed $s \geq 1/2$. Here K is a constant independent of s.

Therefore, by (3.3.2) and (3.3.3), we have

$$\begin{aligned}
F_s(x) \leq &\left(\frac{QC_b}{(1-s)p^-}\varphi(||\nabla_{\mathbb{G}} u||_\infty) + \frac{QC_b}{sp^-}\varphi(2||u||_\infty)\right)\chi_{B(0,2R)}(x) \\
&+ \frac{K}{\|x\|_{\mathbb{G}}^{\frac{1}{2}p^-+Q}}\chi_{\mathbb{G}\setminus B(0,2R)}(x) =: H(x),
\end{aligned}$$

i.e.,

$$(1-s)F_s(x) \leq (1-s)H(x) \in L^1(\mathbb{G}),$$

as desired.

Step 2: Let $u \in W^{1,\varphi}(\mathbb{G})$. Then, thanks to Theorem 3.2.17, there exists $\{u_k\}_{k\in\mathbb{N}} \subset C_c^2(\mathbb{G})$ such that $u_k \to u$ in $W^{1,\varphi}(\mathbb{G})$. Let us show that

$$\lim_{s\uparrow 1}(1-s)\Phi_{s,\varphi}(u) = \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m}).$$

Being

$$\begin{aligned}|(1-s)\Phi_{s,\varphi}(u) - \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m})| &\le |(1-s)\Phi_{s,\varphi}(u) - (1-s)\Phi_{s,\varphi}(u_k)| \\ &\quad + |(1-s)\Phi_{s,\varphi}(u_k) - \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u_k\|_{\mathbb{R}^m})| \\ &\quad + |\Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u_k\|_{\mathbb{R}^m}) - \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m})|\end{aligned}$$

then, by Lemma 3.3.2, it only remains to show that for any $s \in (0,1)$ and for any $\varepsilon > 0$ there exists $\overline{k} \in \mathbb{N}$ such that

$$(1-s)|\Phi_{s,\varphi}(u) - \Phi_{s,\varphi}(u_k)| + |\Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u_k\|_{\mathbb{R}^m}) - \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m})| < \varepsilon$$

$\forall k \ge \overline{k}$.

Fixed $\varepsilon > 0$, by Theorem 3.2.17, there exists $k_0 \in \mathbb{N}$ such that

$$|\Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u_k\|_{\mathbb{R}^m}) - \Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u\|_{\mathbb{R}^m})| < \frac{\varepsilon}{2}$$

for any $k \ge k_0$. Moreover, by Lemma 3.2.12, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\varphi(s+t) \le C_\delta \varphi(s) + (1+\delta)^{p^+}\varphi(t) \text{ for any } s,t \ge 0.$$

Moreover, there exists $\overline{\delta} > 0$ such that $(1+\delta)^{p^+} \le 1 + \overline{\delta}$. Therefore

$$\begin{aligned}|\Phi_{s,\varphi}(u) - \Phi_{s,\varphi}(u_k)| &\le \iint_{\mathbb{G}\times\mathbb{G}} \Bigg|\varphi\Bigg(\frac{|(u-u_k)(x) - (u-u_k)(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s} \\ &\quad + \frac{|u_k(x)-u_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\Bigg) - \varphi\left(\frac{|u_k(x)-u_k(y)|}{\|y^{-1}\cdot x\|_{\mathbb{G}}^s}\right)\Bigg| \frac{dx\,dy}{\|y^{-1}\cdot x\|_{\mathbb{G}}^Q} \\ &\le C_\delta \Phi_{s,\varphi}(u-u_k) + \overline{\delta}\Phi_{s,\varphi}(u_k).\end{aligned}$$

Taking into account Lemma 3.3.1, and being $u_k \to u$ in $W^{1,\varphi}(\mathbb{G})$, then there exists $k_1 \in \mathbb{N}$ such that

$$\Phi_{s,\varphi}(u-u_k) \le \frac{QC_b}{p^-}\left(\frac{1}{1-s}\Phi_\varphi(\|\nabla_{\mathbb{G}}(u-u_k)\|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s}\Phi_\varphi(u-u_k)\right)$$

for any $k \ge k_1$, that is,

$$(1-s)\Phi_{s,\varphi}(u-u_k) \le \frac{\varepsilon}{4C_\delta}.$$

Moreover, still by Lemma 3.3.1, there exists a positive constant M such that

$$\Phi_{s,\varphi}(u_k) \leq \frac{QC_b}{p^-}\left(\frac{1}{1-s}\Phi_\varphi(\|\nabla_{\mathbb{G}}(u_k)\|_{\mathbb{R}^m}) + \frac{\mathbf{C}}{s}\Phi_\varphi(u_k)\right) \leq M.$$

Then, taking $\overline{\delta} \leq \dfrac{\varepsilon}{4M(1-s)}$, we get the thesis by choosing $k \geq \max\{k_0, k_1\}$.

Step 3: In order to conclude the proof of the Theorem, let us prove the result for any $u \in L^\varphi(\mathbb{G})$.

Let us fix $u \in L^\varphi(\mathbb{G})$, $k \in \mathbb{N}$ and $\varepsilon > 0$ and let us define

$$u_{k,\varepsilon} := (\eta_k u) * \rho_\epsilon \in C_c^\infty(\mathbb{G})$$

where $\{\rho_\varepsilon\}_\varepsilon$ is a sequence of mollifiers and $\{\eta_k\}_k$ is a truncated sequence, in the sense of Definition 3.2.21 and Definition 3.2.22.

Then, by Lemma 3.2.23 and Lemma 3.2.24, there exists a positive constant N, independent of k and ε, such that

$$\liminf_{s\uparrow 1}(1-s)\Phi_{s,\varphi}(u_{k,\varepsilon}) < N.$$

Therefore, by *Step 1*

$$\Phi_{\tilde{\varphi}}(\|\nabla_{\mathbb{G}} u_{k,\varepsilon}\|_{\mathbb{R}^m})| < \infty,$$

i.e., the sequence $\{u_{k,\varepsilon}\}_{k,\varepsilon}$ is bounded in $W^{1,\tilde{\varphi}}(\mathbb{G})$ and then, in virtue of Proposition 3.2.14, $\{u_{k,\varepsilon}\}_{k,\varepsilon}$ is bounded in $W^{1,\varphi}(\mathbb{G})$.

Thus, by the reflexivity of the space $W^{1,\varphi}(\mathbb{G})$, there exists $\tilde{u} \in W^{1,\varphi}(\mathbb{G})$ such that, up to subsequence,

$$u_{k,\varepsilon} \rightharpoonup \tilde{u} \text{ weakly in } W^{1,\varphi}(\mathbb{G})$$

as $k \uparrow \infty$ and $\varepsilon \downarrow 0$. Thus, being $u_{k,\varepsilon} \to u$ in $L^\varphi(\mathbb{G})$, it follows that $\tilde{u} = u$ in $W^{1,\varphi}(\mathbb{G})$. Finally, the thesis holds by *Step 2.* □

Chapter 4

Maximal Directional Derivatives in Laakso Space

This chapter corresponds to an ongoing work in collaboration with Andrea Pinamonti (University of Trento) and Gareth Speight (University of Cincinnati, US). Our goal is to investigate the connection between maximal directional derivatives and differentiability for Lipschitz functions defined on Laakso spaces. We show that maximality implies differentiability only at points in a σ-porous set. We also obtain results about the differentiability of the distance function that are different from what one would expect from the settings of Euclidean spaces or Carnot groups. The content of this chapter will appear in [23].

4.1 Motivation

Rademacher's theorem states that a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere with respect to the Lebesgue measure.

The study of a converse of Rademacher's theorem dates back to [90], in which the simplest case $n = m = 1$ is solved. However the complete resolution was not achieved until [82, 83], where the case $n > m$ is covered, and [2, 25], where the remaining case $n \leq m$ is proved. Further developments are [31, 32, 34], in which the case $n > m = 1$ is strengthened, showing that $\mathbb{R}^n$ contains a compact set with Hausdorff (and even Minkowski) dimension one containing a point of differentiability for all real-valued Lipschitz functions. Sets with this property are called universal differentiability sets (UDS).

In more recent years the existence of UDS has been studied in more general settings like Banach spaces [33] or Carnot groups [75, 77].

The techniques used in the above mentioned papers to construct measure zero UDS originate from a work by Preiss [82]. The main tool introduced in that paper is a result that ensures the Frechet differentiability of a Lipschitz functions f on Banach spaces E in a dense set of points, provided that the norm of the space is Frechet differentiable away from the origin. Another important tool introduced in [82] is the relation between the concepts of maximality and differentiability. Preiss proved that if the directional derivative at a point $x \in E$ along a direction e is maximal in the sense that it coincides with the Lipschitz constant $\mathrm{Lip}(f)$, then f is differentiable at x. However, since Lipschitz functions does not admit, in general, such maximal directional derivative, Preiss introduced the concept of *almost maximal* directional derivative. He then showed that these exist for each Lipschitz function and they suffice to prove differentiability.

Following this line of research, in this chapter we investigate whether Preiss' techniques can be implemented in more general metric measure spaces. In particular we found Laakso spaces to be a perfect candidate for our speculations. They are one of the most important example of metric space with a differentiable structure other than Euclidean spaces and Carnot groups.

We recall that Laakso spaces were first introduced by Laakso in [51] as an example of a PI space that cannot be bi-Lipschitz embedded in $\mathbb{R}^n$ for any n. Despite being metric measure spaces with a doubling measure and supporting a Poincaré inequality, they do not posses many of the properties that Euclidean spaces or Carnot groups have. For example, Laakso spaces do not have an underlying group structure. Due to this fact it is not possible to define translations and dilations in a Laakso space. For this reason we had to define from the start basic concepts like directional derivatives and differentiability of Lipschitz functions. For the former we based on the classical concept of directional derivative that comes from $\mathbb{R}^n$, for the latter we adopted Cheeger's definition of differentiability. In particular, Cheeger's definition relies on the existence of a linear function that plays the role of the gradient. We will show how this role is played by the directional derivative in accordance with the intuition that the two concepts should be related.

There are two main results in this chapter. The first one is Theorem 4.3.2, which states a connection between existence of directional derivatives (Definition 4.2.1) and differentiability of Lipschitz functions.

The second is Theorem 4.4.9, that gives a criterion for the distance function to be differentiable. Laakso spaces does not have a privileged point that plays the role of the origin and for this reason the definition of the distance function relies on the starting point. This is particularly relevant because the set of points in which the distance function is not differentiable depends on the choice of the starting point.

These result are different from what one would expect from the Euclidean or the Carnot settings and enlighten how the geometry of the underlying space plays a fundamental role for questions related to differentiability.

4.2 Preliminaries

For the remainder of this chapter, instead of a generic Cantor-like set K we will use the classical middle third Cantor set. It corresponds to the choice of $Q = 1+\log_3 2$, and hence $t = \frac{1}{3}$, in the construction presented in 1.1.3. We refer to that section for the preliminary definitions about wormhole and wormholes levels.

4.2.1 Differentiability of Functions on a Laakso Space

Before proceeding with the definition of differentiability we need to introduce two quantities that will play an important role through the chapter. Roughly speaking they are used to measure the nearest wormhole level of a given depth to a fixed height.

From now on we will denote with $J_n \subset I$ the set of all wormhole levels of depth n. For any $t \in I \setminus \{0, 1\}$, the following quantities are well defined and non-zero for all sufficiently large n:

$$D_n^+(t) := \inf\{s > 0 : t + s \in J_n\} \tag{4.2.1}$$

and

$$D_n^-(t) := \inf\{s > 0 : t - s \in J_n\}. \tag{4.2.2}$$

We want to clarify the different meaning that the concept of derivative can have in this setting. The first one is reminiscent of the definition of derivative for functions from $\mathbb{R}$ to $\mathbb{R}$, however in Laakso spaces we must pay attention to wormholes.

Definition 4.2.1. Let $f : F \to \mathbb{R}$ and $x = [x_1, x_2] \in F$.

If x is not a wormhole, we define

$$f_I(x) := \lim_{t \to 0} \frac{f([x_1 + t, x_2]) - f(x)}{t}$$

whenever the limit exists, where the limit is one-sided if $x_1 = 0$ or $x_1 = 1$.

Suppose x is a wormhole of order n and $(x_1, x_2) \in I \times K$ is the representative of x with the smaller value of x_2. Then we define

$$f_L(x) := \lim_{t \to 0} \frac{f([x_1 + t, x_2]) - f([x_1, x_2])}{t}$$

and

$$f_R(x) := \lim_{t \to 0} \frac{f([x_1 + t, x_2 + \frac{2}{3^n}]) - f([x_1, x_2 + \frac{2}{3^n}])}{t}$$

whenever the relevant limit exists, where the limit is one-sided if $x_1 = 0$ or $x_1 = 1$. If $f_L(x)$ and $f_R(x)$ exist and are equal, we say that $f_I(x)$ exists and define it to be the common value.

Definition 4.2.2. Let $f\colon F \to \mathbb{R}$ and $x \in F$. We say that f is differentiable at x if there exists $Df(x) \in \mathbb{R}$ such that

$$\lim_{\substack{y \to x \\ y \in F}} \frac{f(y) - f(x) - Df(x)(h(y) - h(x))}{d(y, x)} = 0.$$

Lemma 4.2.3. *If $f\colon F \to \mathbb{R}$ is differentiable at a point $x \in F$ with derivative $Df(x)$, then $f_I(x)$ exists and equals $Df(x)$.*

For any $x \in F$, there exists a Lipschitz function $f\colon F \to \mathbb{R}$ such that $f_I(x)$ exists but f is not differentiable at x.

Proof. For the first part, first suppose that f is differentiable at $x = [x_1, x_2] \in F$ with derivative $Df(x)$ and assume that x is not a wormhole. Then we have

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{f([x_1 + t, x_2]) - f(x) - Df(x)(h([x_1 + t, x_2]) - h([x_1, x_2]))}{d([x_1 + t, x_2], [x_1, x_2])} \\ &= \lim_{t \to 0} \frac{f([x_1 + t, x_2]) - f(x) - t \cdot Df(x)}{t} \\ &= \lim_{t \to 0} \frac{f([x_1, x_2 + t]) - f(x)}{t} - Df(x). \end{aligned}$$

The case in which x is a wormhole is done by computing f_L and f_R separately.

For the second part, first fix $x \in F$ and assume $x_1 \notin \{0, 1\}$. Let V denote the line through x in the I direction. Fix $N \in \mathbb{N}$ sufficiently large that

$$D_n^+(x_1), D_n^-(x_1) \neq 0 \quad \text{and} \quad (x_1 - D_n^-(x_1), x_1 + D_n^+(x_1)) \subset (0, 1) \quad \text{for all } n \geq N.$$

If x is a wormhole, so that $x_1 \in J_M$ for some M, then we additionally choose N such that $N > M$. Next, for each $n \geq N$ we define the following points:

- u_n is the point vertically above x at a vertical distance $D_n^+(x_1)$,
- d_n is the point vertically below x at a vertical distance $D_n^-(x_1)$,
- y_n is the point obtained by starting at x, travelling up to u_n, using the wormhole to jump to the identified point, then travelling back down to the point with the same height as x.

Let $A = V \cup \{y_n : n \geq N\}$ and define $f\colon A \to \mathbb{R}$ by

$$f|_V = 0 \qquad \text{and} \qquad f(y_n) = \min(D_n^+(x_1), D_n^-(x_1)) \text{ for } n \geq N.$$

Clearly the directional derivative $f_I(x)$ exists and equals 0.

Notice $d(x, y_n) = \min(2D_n^+(x_1), 2D_n^-(x_1))$ for all $n \geq N$. This gives

$$\frac{f(y_n) - f(x)}{d(y_n, x)} = \frac{\min(D_n^+(x_1), D_n^-(x_1))}{\min(2D_n^+(x_1), 2D_n^-(x_1))} = \frac{1}{2} \not\to 0.$$

Hence f is not differentiable at x.

To see that f is Lipschitz it will suffice to estimate the values of $|f(y_n) - f(u_n)|$, $|f(y_n) - f(d_n)|$, and $|f(y_n) - f(y_m)|$ for $n, m \geq N$. First notice that for every $n \geq N$.

$$|f(y_n) - f(u_n)| = \min(D_n^+(x_1), D_n^-(x_1)) \leq D_n^+(x_1) = d(y_n, u_n)$$

and

$$|f(y_n) - f(d_n)| = \min(D_n^+(x_1), D_n^-(x_1)) \leq D_n^-(x_1) = d(y_n, d_n).$$

Now suppose $n, m \geq N$. Notice that we can choose a geodesic from y_n to y_m which passes either through u_n or through d_n. Suppose one passes through u_n. Then

$$\begin{aligned} |f(y_n) - f(y_m)| &\leq |f(y_n) - f(u_n)| + |f(u_n) - f(y_m)| \\ &\leq d(y_n, u_n) + d(u_n, y_m). \\ &= d(y_n, y_m). \end{aligned}$$

This shows that $f \colon A \to \mathbb{R}$ is 1-Lipschitz. Extending f to a Lipschitz function on F proves the second part of the lemma in the case $x_1 \notin \{0, 1\}$. The proof is similar if $x_1 = 0$ or $x_1 = 1$, with appropriate adjustments to make the construction one-sided. □

It is well known that the Laakso space is a PI space, so admits a differentiable structure giving charts with respect to which Lipschitz functions are almost everywhere differentiable. It also seems to be understood by those working in the field that one can choose the chart to be the whole Laakso space together with the height map, which yields the notion of differentiability described above.

Theorem 4.2.4. *Every Lipschitz function $f \colon F \to \mathbb{R}$ is differentiable almost everywhere.*

Recall that, for a function $f : F \to \mathbb{R}$, $\mathrm{Lip}(f)$ is defined as

$$\mathrm{Lip}(f) = \sup_{x,y \in F} \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \right\}.$$

Proposition 4.2.5. *Let $f \colon F \to \mathbb{R}$ be Lipschitz. Then*

$$\mathrm{Lip}(f) = \sup \{ f_I(x) : x \in F \text{ and } f_I(x) \text{ exists} \}. \tag{4.2.3}$$

Proof. We start by proving $\geq$. Temporarily denote by $\mathrm{Lip}_D(f)$ the right side of (4.2.3). Fix $x = [x_1, x_2] \in F$ such that $f_I(x)$ exists. If x is not a wormhole then we have

$$\begin{aligned} f_I(x) &= \lim_{t\to 0} \frac{f([x_1+t, x_2]) - f(x)}{t} \\ &\leq \limsup_{t\to 0} \frac{\mathrm{Lip}(f)\cdot d(x,[x_1+t,x_2])}{t} \\ &= \mathrm{Lip}(f). \end{aligned}$$

The same holds when $[x_1, x_2]$ is a wormhole by recalling the fact that, in this case, $f_I = f_R = f_L$. This proves $\mathrm{Lip}(f) \geq \mathrm{Lip}_D(f)$.

Next fix $x, y \in F$ and let $L = d(x, y)$. Choose a geodesic $\gamma\colon [0, L] \to F$ from x to y for which there is a decomposition $[0, L] = \cup_{i=1}^{\infty} I_i$ with the following properties:

i) I_i for $i \geq 1$ are closed intervals overlapping only pairwise on their endpoints.

ii) For every $i \geq 1$ there exists $a_i \in [0, 1]$ and $x_i \in K$ such that

$$\gamma|_{I_i}(t) = [a_i \pm t, x_i] \qquad \text{for some } a_i \in I, x_i \in K.$$

In other words, γ is the concatenation of countably many lines in the I-direction. Next we use the fact that $f \circ \gamma$ is absolutely continuous to estimate as follows:

$$\begin{aligned} f(y) - f(x) &= f(\gamma(L)) - f(\gamma(0)) \\ &= \int_0^L (f\circ\gamma)'(s)\,\mathrm{d}s \\ &\leq L \sup\{|(f\circ\gamma)'(s)| : s \in (0, L) \text{ and } (f\circ\gamma)'(s) \text{ exists}\} \\ &\leq L\mathrm{Lip}_D(f). \end{aligned}$$

Since $L = d(x, y)$, this yields $\mathrm{Lip}(f) \leq \mathrm{Lip}_D(f)$ so completes the proof.

□

Definition 4.2.6. Let $f\colon F \to \mathbb{R}$ be Lipschitz and $x \in F$. Suppose that the directional derivative $f_I(x)$ exists and $|f_I(x)| = \mathrm{Lip}(f)$. Then we say that f has a *maximal directional derivative* at x.

We define M to be the set of $x \in F$ with the following property: every Lipschitz map $f\colon F \to \mathbb{R}$ with a maximal directional derivative at x is necessarily differentiable at x.

4.2.2 Porous Sets

A concept that will play an important role in the main results is the concept of porous set. Roughly speaking a set is porous if its point are "sparse" enough.

Recall that a subset of a metric space is said to be of first category or meager if it is a countable union of nowhere dense sets. A property of points in a metric space holds for typical points if the set where it does not hold is of first category.

Definition 4.2.7. A set P in a metric space (X, d) is called porous if there exists $0 < \rho < 1$ such that for all $x \in P$ and $\delta > 0$, there exists $y \in X$ with $d(y, x) < \delta$ such that

$$B(y, \rho d(y, x)) \cap P = \varnothing.$$

A set is called σ-porous if it is a countable union of porous sets.

Clearly every porous set is nowhere dense and every σ-porous set is of first category. In the case of a metric measure space (X, d, μ) equipped with a doubling measure μ, porous sets have measure zero. This is well known. For an explicit proof one could follow the steps in [76], which do not rely on the Carnot group structure in that paper.

4.3 Maximal Derivatives and Differentiability

Recall the definitions of D_n^+ and D_n^- from (4.2.1) and (4.2.2). Let S be the set of $t \in (0, 1)$ for which there is $C(t) \geq 1$ and $N(t) \in \mathbb{N}$ so that

$$C(t)^{-1} \leq \frac{D_n^+(t)}{D_n^-(t)} \leq C(t) \qquad \text{for } n \geq N(t). \tag{4.3.1}$$

Notice that $S \neq \varnothing$; for instance $J_n \subset S$ for all $n \geq 1$.

Lemma 4.3.1. *The following statements hold:*

i) For any $t \in (0, 1)$, we have for all sufficiently large n

$$D_n^+(t) \leq 2/3^n \qquad \text{and} \qquad D_n^-(t) \leq 2/3^n.$$

ii) For any $t \in S$, we have for all sufficiently large n

$$D_n^+(t) \geq c(t)/3^n \qquad \text{and} \qquad D_n^-(t) \geq c(t)/3^n,$$

where $c(t) = 1/(1 + C(t))$.

iii) For any $t \in (0,1) \setminus S$, we have

$$\limsup_{n\to\infty} \frac{D_n^+(t)}{D_n^-(t)} = \infty \qquad \textit{or} \qquad \limsup_{n\to\infty} \frac{D_n^-(t)}{D_n^+(t)} = \infty.$$

Proof. Statement i) holds because adjacent elements of J_n are separated by at most a distance $2/3^n$; the factor 2 is necessary because of the requirement $m_n > 0$ in the definition of J_n.

Statement ii) follows from the estimate

$$1/3^n \leq D_n^+(t) + D_n^-(t) \leq (1 + C(t))D_n^+(t),$$

which yields $D_n^+(t) \geq c(t)/3^n$ with $c(t) = 1/(1 + C(t))$. A similar argument yields the estimate for $D_n^-(t)$.

Statement iii) follows from negating the definition of S. □

Recall that, at the end of Section 4.2.1, we defined the set M to be the set of $x \in F$ with the following property: every Lipschitz map $f\colon F \to \mathbb{R}$ with a maximal directional derivative at x is necessarily differentiable at x.

Theorem 4.3.2. *Suppose $x = [x_1, x_2] \in F$ with $x_1 \in (0,1) \cap S$. Then $x \in M$. In other words, whenever a Lipschitz map $f\colon F \to \mathbb{R}$ has a maximal directional derivative at x, necessarily f must be differentiable at x.*

Proof. Let $f\colon F \to \mathbb{R}$ be Lipschitz with $f_I(x) = \mathrm{Lip}(f)$. Let $L := \mathrm{Lip}(f)$. We will show that

$$\lim_{y\to x} \frac{f(y) - f(x) - L(h(y) - h(x))}{d(y,x)} = 0.$$

Fix $N(x_1) \in \mathbb{N}$, $C(x_1) \geq 1$, and $0 < c(x_1) < 1$ such that (4.3.1), Lemma 4.3.1(1), and Lemma 4.3.1(2) hold with $t = x_1$ for all $n \geq N(x_1)$.

Case 1: Suppose x is not a wormhole level. Fix $\varepsilon > 0$. Let $y \in F$ be sufficiently close to x in a sense to be made precise below. Assume $y_1 \geq x_1$; the case $y_1 < x_1$ is similar. Let N be the minimal $n \in \mathbb{N}$ for which every path connecting x and y must pass through a point of F whose height belongs to J_n. By making y sufficiently close to x, we may assume that $N \geq N(x_1)$. Since every path connecting x and y must pass through a height in J_N, it follows that

$$d(x,y) \geq \min\{D_N^+(x_1), D_N^-(x_1)\} \geq c(x_1)/3^N, \tag{4.3.2}$$

where the fact $N \geq N(x_1)$ and Lemma 4.3.1(2) was used for the second inequality.

Let $z_N := [x_1 - 2/3^N, x_2]$. Using $f_I(x) = L$ and assuming y is sufficiently close to x (which makes N sufficiently large), we can ensure

$$f(z_N) - f(x) \leq L(h(z_N) - h(x)) + \frac{\varepsilon}{3^N}.$$

Notice that
$$h(y) - h(z_N) \geq h(x) - h(z_N) = 2/3^N$$
and in any interval of length $2/3^N$ we can find elements of J_n for all $n \geq N$. Hence $d(y, z_N) = h(y) - h(z_N)$. Using (4.3.2) for the final line, we now estimate as follows

$$\begin{aligned} f(y) - f(x) &= (f(y) - f(z_N)) + (f(z_N) - f(x)) \\ &\leq Ld(y, z_N) + L(h(z_N) - h(x)) + \frac{\varepsilon}{3^N} \\ &= L(h(y) - h(z_N)) + L(h(z_N) - h(x)) + \frac{\varepsilon}{3^N} \\ &= L(h(y) - h(x)) + \frac{\varepsilon}{c(x_1)} d(x, y). \end{aligned}$$

For the opposite inequality, let $w_N := [y_1 + 2/3^N, x_2]$. Provided y is sufficiently close to x (which ensures y_1 is close to x_1 and N is sufficiently large), we may use $f_I(x) = L$ to obtain

$$f(w_N) - f(x) \geq L(h(w_N) - h(x)) - \varepsilon(h(w_N) - h(x)).$$

Since any interval of length $2/3^N$ contains elements of J_n for all $n \geq N$, we have $d(y, w_N) = h(w_N) - h(y)$. Using (4.3.2) for the final line, we estimate as follows

$$\begin{aligned} f(y) - f(x) &= (f(y) - f(w_N)) + (f(w_N) - f(x)) \\ &\geq -Ld(y, w_N) + L(h(w_N) - h(x)) - \varepsilon(h(w_N) - h(x)) \\ &= L(h(y) - h(w_N) + h(w_N) - h(x)) - \varepsilon\left(h(y) - h(x) + \frac{2}{3^N}\right) \\ &\geq L(h(y) - h(x)) - \varepsilon d(x, y)\left(1 + \frac{2}{c(x_1)}\right). \end{aligned}$$

Case 2: Suppose x is a wormhole level of order $K \geq 1$, so $x_1 \in J_K$. The argument is similar to that of Case 1, but we give the details for completeness. Let x_2 be the smaller of the two elements of K satisfying $x = [x_1, x_2]$. Fix $\varepsilon > 0$. Let $y \in F$ be sufficiently close to x in a sense to be made precise below. Assume $y_1 \geq x_1$. Let N be the minimal $n \in \mathbb{N}$ for which every path connecting x and y must pass through a height in $J_n \setminus J_K$. By making y sufficiently close to x, we may assume that $N \geq N(x_1)$. Since every path connecting x and y must pass through a height in J_N, it follows as before that

$$d(x, y) \geq \min\{D_N^+(x_1), D_N^-(x_1)\} \geq c(x_1)/3^N.$$

Let $z_N = [x_1 - 2/3^N, x_2]$. Using $f_I(x) = L$ and assuming y is sufficiently close to x, we have
$$f(z_N) - f(x) \leq L(h(z_N) - h(x)) + \frac{\varepsilon}{3^N}.$$

Notice that

$$h(y) - h(z_N) \geq h(x) - h(z_N) = 2/3^N$$

and in any interval of length $2/3^N$ we can find elements of J_n for all $n \geq N$. Hence $d(y, z_N) = h(y) - h(z_N)$, so we can estimate as follows

$$\begin{aligned} f(y) - f(x) &= (f(y) - f(z_N)) + (f(z_N) - f(x)) \\ &\leq Ld(y, z_N) + L(h(z_N) - h(x)) + \frac{\varepsilon}{3^N} \\ &= L(h(y) - h(z_N)) + L(h(z_N) - h(x)) + \frac{\varepsilon}{3^N} \\ &= L(h(y) - h(x)) + \frac{\varepsilon}{c(x_1)} d(x, y). \end{aligned}$$

For the opposite inequality, fix $p \subset I \times K$ such that $s(p)$ is a path joining x with y and $\mathcal{H}^1(p) = d(x, y)$. If $(x_1, x_2) \in p$ we let $w_N := [y_1 + 2/3^N, x_2]$. Otherwise p contains $(x_1, x_2 + 2/3^K)$ and we let $w_N := [y_1 + 2/3^N, x_2 + 2/3^K]$. With this choice of w_N, the points y and w_N are separated only by jumps of level $n \geq N$. Assuming y is sufficiently close to x, we may use $f_I(x) = L$ to obtain

$$f(w_N) - f(x) \geq L(h(w_N) - h(x)) - \varepsilon(h(w_N) - h(x)).$$

Since any interval of length $2/3^N$ contains elements of J_n for all $n \geq N$, we have $d(y, w_N) = h(w_N) - h(y)$. Using (4.3.2) for the final line, we estimate as follows

$$\begin{aligned} f(y) - f(x) &= (f(y) - f(w_N)) + (f(w_N) - f(x)) \\ &\geq -Ld(y, w_N) + L(h(w_N) - h(x)) - \varepsilon(h(w_N) - h(x)) \\ &= L(h(y) - h(w_N) + h(w_N) - h(x)) - \varepsilon \left(h(y) - h(x) + \frac{2}{3^N} \right) \\ &\geq L(h(y) - h(x)) - \varepsilon d(x, y) \left(1 + \frac{2}{c(x_1)} \right). \end{aligned}$$

A similar argument holds if $y_1 < x_1$. Hence f is differentiable at x. □

Theorem 4.3.3. *Suppose $x = [x_1, x_2] \in F$ with $x_1 \in (0,1) \setminus S$. Then $x \notin M$. In other words, there exists a Lipschitz map $f\colon F \to \mathbb{R}$ with a maximal directional derivative at x which is not differentiable at x.*

Proof. For simplicity let $D_n^+ := D_n^+(x_1)$ and $D_n^- := D_n^-(x_1)$, which exist and are non-zero for all sufficiently large n. Since $x_1 \notin S$, it follows that either

$$\limsup_{n\to\infty} D_n^-/D_n^+ = \infty \qquad \text{or} \qquad \limsup_{n\to\infty} D_n^+/D_n^- = \infty.$$

We assume that $\limsup_{n\to\infty} D_n^-/D_n^+ = \infty$; the argument in the other case is similar with the construction inverted in the I direction.

Choose a strictly increasing sequence n_k such that $D_{n_k}^-/D_{n_k}^+ \to \infty$. Since $D_n^+ \to 0$ and $D_n^- \to 0$ as $n \to \infty$, by taking a subsequence if necessary we may assume $D_{n_k}^-$ and $D_{n_k}^+$ are each strictly decreasing and for every $k \geq 1$,

$$D_{n_{k+1}}^-/D_{n_k}^- < 1/n_k \qquad \text{and} \qquad 2D_{n_k}^+/D_{n_k}^- < 1/n_k.$$

Fix a sequence $0 < \theta_{n_k} < 1$ which satisfies $\theta_{n_k} \to 1$ as $k \to \infty$ and

$$\theta_{n_k} \leq \left(1 - \frac{D_{n_k}^+ + D_{n_{k+1}}^-}{D_{n_k}^-}\right) / \left(1 - \frac{D_{n_{k+1}}^-}{D_{n_k}^-}\right) \text{ for all } k \geq 1. \tag{4.3.3}$$

Note that, since the right hand side converges to 1 as $k \to \infty$, these conditions on θ_{n_k} can be realized. Let $I := [x_1 - D_{n_1}^-, x_1 + D_{n_1}^+] \subset \mathbb{R}$. Define $\varphi\colon I \to [0,1]$ by

$$\varphi(t) = 1 \qquad \text{if } t \geq x_1$$

and

$$\varphi(t) = \theta_{n_k} \qquad \text{if } x_1 - D_{n_k}^- \leq t < x_1 - D_{n_{k+1}}^- \text{ for some } k \geq 1.$$

Since $x_1 \notin S$, we know that x is not a wormhole level. For all $k \geq 1$, let $y_{n_k} \in F$ be the endpoint of the path which starts at x, travels vertically up along the line segment to height $x_1 + D_{n_k}^+$, jumps using the level $J_{n_k}^+$, then travels vertically down along the line segment to height x_1. Thus y_{n_k} is either $[x_1, x_2 + 2/3^{n_k}]$ or $[x_1, x_2 - 2/3^{n_k}]$, where the choice of sign may depend on k. Since $D_{n_k}^+ < D_{n_k}^-$, we have $d(y_{n_k}, x) = 2D_{n_k}^+$. Now let

$$A := \{[t, x_2] \in F \colon t \in I\} \cup \{y_{n_k} : k \in \mathbb{N}\}.$$

Since x is not a wormhole level, the sets $\{[t, x_2] \in F \colon t \in I\}$ and $\{y_{n_k} : k \in \mathbb{N}\}$ are disjoint. Hence we may define $f\colon A \to \mathbb{R}$ by

$$f[t, x_2] = \int_{x_1}^{t} \varphi(s)\, \mathrm{d}s \qquad \text{for } t \in I$$

and

$$f(y_{n_k}) = D_{n_k}^+ \qquad \text{for } k \in \mathbb{N}.$$

Notice that $f(x) = 0$.

The function f will be used to construct an example of function with maximal directional derivative at x that is not differentiable in that point. This will suffice to prove the theorem.

Claim 4.3.4. *f is 1-Lipschitz with respect to the restriction to A.*

Proof of Claim. Suppose $a = [t, x_2]$ and $b = [s, x_2]$ for some $t, s \in I$. Then $|\varphi| \leq 1$ implies

$$|f(a) - f(b)| \leq |t - s| = d(a, b).$$

Hence f is 1-Lipschitz restricted to the set $\{[t, x_2] \in F : t \in I\}$.

Next let u_{n_k} be the point reached by starting at x and travelling vertically up along the line segment to height $x_1 + D^+_{n_k}$. Similarly let d_{n_k} be the point reached by starting at x and travelling vertically down along the line segment to height $x_1 - D^-_{n_k}$. It follows from the definition of f that

$$f(u_{n_k}) = D^+_{n_k} = f(y_{n_k}).$$

On the other hand we have, using the definitions of f and φ,

$$\begin{aligned} f(d_{n_k}) - f(y_{n_k}) &= f(d_{n_k}) - f(u_{n_k}) \\ &= \int_{x_1 - D^-_{n_k}}^{x_1 + D^+_{n_k}} \varphi(s)\ \mathrm{d}s \\ &= \theta_{n_k}(D^-_{n_k} - D^-_{n_{k+1}}) + \int_{x_1 - D^-_{n_{k+1}}}^{x_1} \varphi(s)\ \mathrm{d}s + D^+_{n_k}. \end{aligned}$$

Using $|\varphi| \leq 1$, $d(d_{n_k}, y_{n_k}) = D^-_{n_k}$, and the estimate of θ_{n_k} in (4.3.3), we obtain

$$\frac{|f(d_{n_k}) - f(y_{n_k})|}{d(d_{n_k}, y_{n_k})} \leq \theta_{n_k}\left(1 - \frac{D^-_{n_{k+1}}}{D^-_{n_k}}\right) + \frac{D^-_{n_{k+1}}}{D^-_{n_k}} + \frac{D^+_{n_k}}{D^-_{n_k}} \leq 1.$$

Suppose $a = [t, x_2]$ for some $t \in I$ and $k \geq 1$. Every geodesic from y_{n_k} to a must pass through either u_{n_k} or d_{n_k}. Denote such a point by z_{n_k}; the argument will be the same in either case. Then we have

$$d(y_{n_k}, a) = d(y_{n_k}, z_{n_k}) + d(z_{n_k}, a).$$

Using also what was proved above, we have

$$\begin{aligned} |f(y_{n_k}) - f(a)| &\leq |f(y_{n_k}) - f(z_{n_k})| + |f(z_{n_k}) - f(a)| \\ &\leq d(y_{n_k}, z_{n_k}) + d(z_{n_k}, a) \\ &= d(y_{n_k}, a). \end{aligned}$$

It remains to estimate $|f(y_{n_k}) - f(y_{n_l})|$ for $k > l \geq 1$. Define the points u_{n_l}, u_{n_k} and d_{n_l}, d_{n_k} as before in the proof of Claim 4.3.4.

A geodesic from y_{n_l} to y_{n_k} can be constructed by the following procedure:

i) Travel vertically, starting from y_{n_l}, upwards to the wormhole u_{n_l},

ii) Jump using wormhole u_{n_l} and travel downwards to the wormhole u_{n_k},

iii) Jump using wormhole u_{n_k} and travel downwards to the point y_{n_k}.

Thus $d(y_{n_l}, y_{n_k}) = 2D^+_{n_l}$ for $k > l$. Hence, since $D^+_{n_l} > D^+_{n_k}$, we can estimate

$$\begin{aligned} |f(y_{n_l}) - f(y_{n_k})| &= D^+_{n_l} - D^+_{n_k} \\ &\leq d(y_{n_l}, y_{n_k}). \end{aligned}$$

This concludes the proof of the claim. □

Now extend f arbitrarily to a 1-Lipschitz function $f\colon F \to \mathbb{R}$.

Claim 4.3.5. *The directional derivative $f_I(x)$ exists and equals 1.*

Proof of Claim. If $t \in [-D^-_{n_1}, D^+_{n_1}]$ then

$$\frac{f[x_1 + t, x_2] - f[x_1, x_2]}{t} = 1 + \frac{1}{t}\int_{x_1}^{x_1+t} (\varphi(s) - 1)\, \mathrm{d}s.$$

Since $\varphi(s) = 1$ for $s \geq x_1$ we obtain

$$\frac{f[x_1 + t, x_2] - f[x_1, x_2]}{t} = 1 \qquad \text{for all } t \geq 0.$$

Fix $\varepsilon > 0$ and fix K large enough so that $|\theta_{n_k} - 1| < \varepsilon$ for all $k \geq K$. Then

$$\left|\frac{1}{t}\int_{x_1}^{x_1+t} (\varphi(s) - 1)\, \mathrm{d}s\right| \leq \varepsilon \qquad \text{for } -D^-_{n_K} < t < 0.$$

This proves the claim. □

Claim 4.3.6. *f is not differentiable at x.*

Proof of Claim. Recall that $f(x) = 0$, $f(y_{n_k}) = D^+_{n_k}$ and $d(y_{n_k}, x) = 2D^+_{n_k}$. Hence for any $\lambda \in \mathbb{R}$ we have,

$$\frac{f(y_{n_k}) - f(x) - \lambda(h(y_{n_k}) - h(x))}{d(y_{n_k}, x)} = \frac{f(y_{n_k}) - f(x)}{d(y_{n_k}, x)} = 1/2.$$

Since $y_{n_k} \to x$ as $k \to \infty$, this shows that f is not differentiable at x. □

This third claim concludes the proof of the theorem. □

Theorem 4.3.7. *Suppose $x = [x_1, x_2] \in F$ with $x_1 \in \{0, 1\}$. Then $x \notin M$. In other words, there exists a Lipschitz map $f: F \to \mathbb{R}$ with a maximal directional derivative at x which is not differentiable at x.*

The proof of the previous theorem is largely the same as that of Theorem 4.3.3, except the construction is made only on one side of x; vertically above if $x_1 = 0$ and vertically below if $x_1 = 1$.

Combining Theorem 4.3.2, Theorem 4.3.3 and Theorem 4.3.7 yields the following.

Corollary 4.3.8. *Maximality implies differentiability at $x = [x_1, x_2]$ if and only if $x_1 \in S$. In other words,*

$$M = \{[x_1, x_2] : x_1 \in S\}.$$

In the following two results we use the Euclidean distance on I and the usual metric d on F.

Lemma 4.3.9. *If $A \subset I$ is porous, then $h^{-1}(A) \subset F$ is also porous.*

Proof. Since A is porous in I, there exists $C > 0$ such that for every $t \in A$ there is a sequence $t_n \in I$ with $t_n \to t$ such that

$$B(t_n, C|t_n - t|) \cap A = \varnothing \qquad \text{for every } n \in \mathbb{N}. \tag{4.3.4}$$

Fix $[t, x] \in h^{-1}(A)$. Then $t \in A$. Hence there exists a sequence $t_n \in I$ with $t_n \to t$ such that (4.3.4) holds. Consider the sequence $[t_n, x] \in F$. Clearly $[t_n, x] \to [t, x]$ with respect to the natural metric on F. Let

$$B_n = B([t_n, x], Cd([t_n, x], [t, x])).$$

We claim that

$$B_n \cap h^{-1}(A) = \varnothing. \qquad \text{for every } n \in \mathbb{N} \tag{4.3.5}$$

To this end, fix $n \in \mathbb{N}$ and suppose $[s, y] \in B_n$. Then

$$\begin{aligned} |s - t_n| &\le d([s, y], [t_n, x]) \\ &\le Cd([t_n, x], [t, x]) \\ &= C|t_n - t|. \end{aligned}$$

Hence

$$s \in B(t_n, C|t_n - t|).$$

By (4.3.4), this implies $s \notin A$. Hence $[s, y] \notin h^{-1}(A)$. This verified (4.3.5), so $h^{-1}(A)$ is porous in F as required. □

Proposition 4.3.10. *The set $S \subset I$ is σ-porous in I, hence first category and of Lebesgue measure zero.*

The set $M \subset F$ is σ-porous in F, hence first category and of $\mathcal{H}^Q$ measure zero.

Proof. We can write

$$S = \bigcup_{\substack{C \in \mathbb{Q} \\ C > 1}} \bigcup_{N \in \mathbb{N}} S_{C,N}$$

where

$$S_{C,N} = \left\{ t \in (0,1) \colon D_n^-(t) \neq 0 \text{ and } C^{-1} \leq \frac{D_n^+(t)}{D_n^-(t)} \leq C \text{ for all } n \geq N \right\}.$$

Fix $C \in \mathbb{Q}$ with $C > 1$ and $N \in \mathbb{N}$. We show that the set $S_{C,N} \subset (0,1)$ is porous. Fix $0 < \lambda < 1/2$ such that $(1-\lambda)/\lambda > C$. Let $t \in J_n$ for some $n \geq N$. We will show that:

$$\left(t - \frac{\lambda}{3^n}, t + \frac{\lambda}{3^n} \right) \cap S_{C,N} = \varnothing. \tag{4.3.6}$$

First fix $s \in [t, t + \lambda/3^n]$. Then $D_n^-(s) \leq \lambda/3^n$ and $D_n^+(s) \geq 1/3^n - \lambda/3^n$. Combining these inequalities gives

$$\frac{D_n^+(s)}{D_n^-(s)} \geq \frac{1-\lambda}{\lambda} > C.$$

Hence $s \notin S_{C,N}$, so $[t, t+\lambda/3^n] \cap S_{C,N} = \varnothing$. Similarly $[t - \lambda/3^n, t] \cap S_{C,N} = \varnothing$, which establishes (4.3.6).

Next fix $t_0 \in S_{C,N}$ and $\delta > 0$. Choose $n > N$ with $2/3^n < \delta$ and $t \in J_n$ with $|t - t_0| < 2/3^n$. Then $(t - \lambda/3^n, t + \lambda/3^n) \cap S_{C,N} = \varnothing$. This shows that $S_{C,N}$ is porous and hence S is σ-porous. The implication follows because every σ-porous set is first category and has Lebesgue measure zero.

The second part of the proposition follows by Lemma 4.3.9 and the fact that $M = h^{-1}(S)$. Note that $\mathcal{H}^Q(S) = 0$ because $\mathcal{H}^Q$ is a doubling measure on F and doubling measures assign measure zero to porous sets. □

4.4 Differentiability of the Distance Function

For the remainder of this chapter we will ignore what happens on the top edge $x_1 = 1$ and the bottom edge $x_1 = 0$. Definition 4.2.1 was given also in the case of a one-sided limit, so all the results below apply in these cases too with slight modifications.

For $p \in F$ we denote by d_p the map from F to $\mathbb{R}$ given by $y \mapsto d_p(y) := d(y,p)$.

Theorem 4.4.1. *Fix $p \in F$. Then the map $d_p \colon F \to \mathbb{R}$ is differentiable at a point $x \in F$ if and only if the directional derivative $(d_p)_I(x)$ exists.*

Proof. Fix $p \in F$. Clearly if d_p is differentiable at a point $x \in F$, then the directional derivative $(d_p)_I(x)$ exists. Suppose $x \in F$ and $D := (d_p)_I(x)$ exists. We must show that d_p is differentiable at x, namely

$$\lim_{y \to x} \frac{d_p(y) - d_p(x) - D(h(y) - h(x))}{d(y, x)} = 0.$$

Case 1: Suppose x is not a wormhole.

Claim 4.4.2. *There is $\Delta > 0$ (depending on p and x) such that if $t \in (-\Delta, \Delta)$ and $y_2 \in K$ with $|y_2 - x_2| < \Delta$, then $d_p([x_1 + t, x_2]) = d_p([x_1 + t, y_2])$.*

Proof of Claim. Define

$$N := \min\{n \in \mathbb{N} : \text{every geodesic from } p \text{ to } x \text{ passes through a height in } J_n\}.$$

Choose $\Delta > 0$ sufficiently small that the following properties hold for every point $y_2 \in K$ with $|y_2 - x_2| < \Delta$:

i) Every jump level required to connect x_2 to y_2 can be found both in the interval $(p_1, p_1 + D_N^+(p_1))$ and in the interval $(p_1 - D_N^-(p_1), p_1)$.

ii) A jump level in J_N is required to connect p_2 to y_2.

Now suppose $t \in (-\Delta, \Delta)$ and $y_2 \in K$ with $|y_2 - x_2| < \Delta$.

Let γ be a geodesic from p to $[x_1 + t, x_2]$. Using the definition of N, γ must pass through all heights in either the interval $(p_1, p_1 + D_N^+(p_1))$ or in the interval $(p_1 - D_N^-(p_1), p_1)$. Using i), we may modify γ without changing its length to obtain a curve $\widetilde{\gamma}$ connecting p to $[x_1 + t, y_2]$. This gives the inequality

$$d_p([x_1 + t, y_2]) \le d_p([x_1 + t, x_2]).$$

Conversely, let η be a geodesic from p to $[x_1 + t, y_2]$. Using ii), we know that η must pass through all heights in either the interval $(p_1, p_1 + D_N^+(p_1))$ or in the interval $(p_1 - D_N^-(p_1), p_1)$. Using i), we can modify η without changing its length to obtain a curve $\widetilde{\eta}$ connecting p to $[x_1 + t, x_2]$. This gives the inequality

$$d_p([x_1 + t, y_2]) \ge d_p([x_1 + t, x_2]).$$

Combining the two inequalities concludes the proof. □

Fix $\Delta > 0$ as in the claim. Let $\varepsilon > 0$. Using the definition of the directional derivative, we can find $\delta > 0$ such that whenever $t \in I$ with $0 < |t| < \delta$ we have

$$|d_p([x_1 + t, x_2]) - d_p([x_1, x_2]) - tD| < \varepsilon|t|. \tag{4.4.1}$$

If $0 < |t| < \min(\delta, \Delta)$ and $y_2 \in K$ with $|y_2 - x_2| < \Delta$ then

$$|d_p([x_1 + t, y_2]) - d_p([x_1, x_2]) - tD| < \varepsilon|t|.$$

Writing $y_1 = x_1 + t$, $y = [y_1, y_2]$, and using $t \leq d(y, x)$ then gives

$$|d_p(y) - d_p(x) - D(h(y) - h(x))| < \varepsilon|t|. \tag{4.4.2}$$

If $y \in F$ is sufficiently close to x then (4.4.2) holds. Hence d_p is differentiable at x.

Case 2: Suppose x is a wormhole. Write $x = [x_1, x_2] = [x_1, x_2']$ where $x_1 \in I$ and $x_2, x_2' \in K$ with $x_2 < x_2'$. The proof of the following claim is similar to the one we just finished.

Claim 4.4.3. *There is $\Delta > 0$ (depending on p and x) such that the following implications hold for every $t \in (-\Delta, \Delta)$.*

i) If $y_2 \in K$ with $|y_2 - x_2| < \Delta$ then $d_p([x_1 + t, x_2]) = d_p([x_1 + t, y_2])$.

ii) If $y_2 \in K$ with $|y_2 - x_2'| < \Delta$ then $d_p([x_1 + t, x_2']) = d_p([x_1 + t, y_2])$.

The argument in Case 2 is now similar to that of Case 1, using the fact that existence of $(d_p)_I(x)$ gives (4.4.1) also with x_2 replaced by x_2' in the case that x is a wormhole. □

For the function $d_p(\cdot)$ we can compute the value of $(d_p)_I(x)$, when it exists. Indeed if $p \in F$ is fixed then, with the help of [51, Propositions 1.1 and 1.2] we can prove the following proposition.

Proposition 4.4.4. *For every $x = [x_1, x_2] \in F$ there exists $\delta > 0$ such that:*

i) Either $d_p([x_1+t, x_2]) = d_p(x)+t$ for all $t \in (0, \delta)$ or $d_p([x_1+t, x_2]) = d_p(x)-t$ for each $t \in (0, \delta)$

ii) Either $d_p([x_1+t, x_2]) = d_p(x)+t$ for all $t \in (-\delta, 0)$ or $d_p([x_1+t, x_2]) = d_p(x)-t$ for each $t \in (-\delta, 0)$.

In other words if we move close to x along the vertical direction then the change in the distance is exactly the change in the height, up to a sign.

Proof. The idea is to repeatedly apply Propositions 1.1.21 and 1.1.22. We prove only i), i.e. the case of $t > 0$, as the proof of ii) is very similar with the relevant signs changed. We distinguish three possible cases: $h(x) > h(p)$, $h(x) = h(p)$ and $h(x) < h(p)$.

Case 1: $h(x) > h(p)$. Let $[a, b] \subset I$ be a minimal height interval for x and p. We further distinguish two sub-cases: $h(x) = b$ and $h(x) < b$.

- If $h(x) = b$ then in order to compute $d_p([x_1 + t, x_2])$ we have to specify a new minimal height interval $[a', b']$.

  If $a = h(p)$ then we define $\delta = 1 - b$ and for any $t \in (0, \delta)$ we can choose $a' = a$ and $b' = h(x)+t$. This is because moving in the vertical direction does not change the wormhole needed. Since they where already in the interval $[a, b]$, the only change we need to cover with the new interval is the change in the height of the final point. Hence from Proposition 1.1.22 we get

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h([x_1 + t, x_2]) + h(p) \\ &= 2h(x) + 2t - 2h(p) - h(x) - t + h(p) \\ &= h(x) + t - h(p). \end{aligned}$$

  But we also have $d_p(x) = 2b - 2a - h(x) + h(p) = h(x) - h(p)$, from which we conclude that $d_p([x_1 + t, x_2]) = d_p(x) + t$.

  On the other hand, still assuming $h(x) = b$, if $a < h(p)$ then we have to make sure that by going up on the vertical of x we do not encounter any wormhole level that normally we would encounter only between heights $h(p)$ and a. Such a wormhole level must exists, otherwise we would be again in the case $a = h(p)$, and moreover a is one of these wormhole level. Indeed if it were not the case then we could replace the minimal height interval with $[\tilde{a}, b] \subset [a, b]$, but this would contradict the minimality of $[a, b]$. So let us assume that a is a wormhole level of level N and define $\delta = D_N^+(h(x))$. Then for $t \in (0, \delta)$ a minimal height interval $[a', b']$ for the points p and $[x_1 + t, x_2]$ is given by $a' = a$ and $b' = h(x) + t$, hence

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h([x_1 + t, x_2]) + h(p) \\ &= 2h(x) + 2t - 2a - h(x) - t + h(p) \\ &= h(x) + t - 2a + h(p). \end{aligned}$$

  But we also have $d_p(x) = 2b - 2a - h(x) + h(p) = h(x) - 2a + h(p)$, from which we conclude that $d_p([x_1 + t, x_2]) = d_p(x) + t$. This concludes the proof of i) for the first sub-case.

- If $h(x) < b$ then we define $\delta = b - h(x)$. For $t \in (0, \delta)$ we apply Proposition 1.1.21 to find a new minimal height interval $[a', b']$. Moving in the vertical direction does not change the wormhole levels needed to go from p to $[x_1+t, x_2]$, because they only depends on the second coordinate x_2. This in particular implies that, if we choose $a' = a$ and $b' = b$, the resulting interval contains heights of $h(p), h([x_1 + t, x_2])$ and all the wormhole levels needed to connect the two point with a path. We claim that with this choice $[a', b']$ is a minimal height interval. Indeed if we were able to find another interval $[a'', b'']$ with the same properties of $[a', b']$ for which $b'' - a'' < b' - a' = b - a$ holds, then the same interval would be a minimal height interval for the points p and x,. But this is in contradiction with the definition of $[a, b]$, hence $[a', b']$ is a minimal height interval. From Proposition 1.1.22 we have

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h([x_1 + t, x_2]) + h(p) \\ &= 2b - 2a - h(x) - t + h(p). \end{aligned}$$

  Since $d_p(x) = 2b - 2a - h(x) + h(p)$, then $d_p([x_1 + t, x_2]) = d_p(x) - t$ and this concludes the proof in the first case.

Case 2: $h(x) = h(p)$. Let $[a, b] \subset I$ be the minimal height interval. We further distinguish three sub-cases:

$$a = h(p) = h(x) < b; \;\; a < h(x) = h(x) = b \text{ and } a < h(p) = h(x) < b.$$

Notice that the fourth case, $a = h(p) = h(x) = b$ boils down to $x = p$, in which case the statement of the proposition is trivially true.

- If $a = h(p) = h(x) < b$ we define $\delta = b - h(x)$. For $t \in (0, \delta)$ when looking for a minimal height interval $[a', b'] \subset I$ we can choose $a' = a$ and $b' = b$. Hence

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h([x_1 + t, x_2]) + h(p) \\ &= 2b - 2a - h(x) - t + h(p) \\ &= 2b - 2a - t. \end{aligned}$$

  But we also know that $d_p(x) = 2b - 2a$, from which we conclude that $d_p([x_1 + t, x_2]) = d_p(x) - t$.

- If $a < h(p) = h(x) = b$ then we observe that a must be a wormhole of level N for a certain $N > 0$ and moreover no others wormhole of level N can be found between heights $h(p)$ and a. Indeed if this were not the case then we could find $a < a' < h(p)$ such that $[a', b] \subset [a, b]$ is a minimal height interval, but this would contradict the minimality of $[a, b]$.

If $D_N^+(h(p)) = D_N^-(h(p))$ (i.e. if $h(p)$ is exactly in between two wormholes of level N, one of them being a) then $h(p) - a = D_N^-(h(p)) = D_N^+(h(p))$ and another minimal height interval is given by $[a', b']$ with $a' = h(p) = h(x)$ and $b' = h(x) + D_N^+(h(p))$. Hence $a' = h(p) = h(x) < b'$ and we can repeat the argument of the first sub-case.

On the other hand if $D_N^+(h(p)) \neq D_N^-(h(p))$ we define δ to be their difference: $\delta = |D_N^+(h(x)) - D_N^-(h(p))|$. For $t \in (0, \delta)$ we have that the new minimal height interval $[a', b']$ is given by $a' = a$ and $b' = b + t$. Hence

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h([x_1 + t, x_2]) + h(p) \\ &= 2b + 2t - 2a - h(x) - t + h(p) \\ &= 2b - 2a + t \end{aligned}$$

and since $d_p(x) = 2b - 2a - h(x) + h(p) = 2b - 2a$ we finally get that $d_p([x_1 + t, x_2]) = d_p(x) + t$.

- If $a < h(p) = h(x) < b$ then clearly $d_p(x) = 2(b-a)$. We repeat the argument used in the second sub-case of case 1. By choosing $\delta = b - h(x)$ we get that for $t \in (0, \delta)$ the a', b' obtained from the application of [51, Proposition 1.1] must verify $b' - a' = b - a$. Hence

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h([x_1 + t, x_2]) + h(p) \\ &= 2(b' - a') - h(x) - t + h(p) \\ &= 2(b - a) - t = d_p(x) - t \end{aligned}$$

and this concludes the proof for case 2.

Case 3: $h(x) < h(p)$. Let $[a, b] \subset I$ be the minimal height interval. We have that $a \leq h(x) < h(p) \leq b$, hence we distinguish two sub-cases: $h(x) = a$ and $h(x) > a$.

- If $h(x) = a$ then using Proposition 1.1.21 and Proposition 1.1.22 we can compute $d_p(x) = 2b - 2a - h(p) + h(x) = 2b - h(x) - h(p)$. If all the wormhole levels needed to connect p and x are between heights $h(p)$ and b then we can choose $\delta = h(p) - h(x)$ and for $t \in (0, \delta)$ a new minimal height interval $[a', b'] \subset I$ is given by $a' = h(x) + t$ and $b' = b$. Hence

$$\begin{aligned} d_p([x_1 + t, x_2]) &= 2b' - 2a' - h(p) + h([x_1 + t, x_2]) \\ &= 2b - 2(h(x) + t) - h(p) + h(x) + t \\ &= 2b - h(x) - h(p) - t \\ &= d_p(x) - t. \end{aligned}$$

On the other hand if not all the wormhole levels needed to connect p and x can be found between heights $h(p)$ and b then some of them must be between heights $h(x)$ and $h(p)$. We define

$$\begin{aligned}\eta = \sup\{s \geq 0 &: \text{all the wormhole levels needed to connect} \\ &p \text{ to } x \text{ are between heights } h(x)+s \text{ and } b\}.\end{aligned}$$

If $\eta = 0$ then $h(x) = a$ is itself a wormhole level needed to go from p to x. Hence we define $\delta = h(p) - h(x)$ and for $t \in (0, \delta)$ a new minimal height interval is given by $a' = a$ and $b' = b$ and we get

$$\begin{aligned}d_p([x_1+t, x_2]) &= 2b' - 2a' - h(p) + h([x_1+t, x_2]) \\ &= 2b - 2a - h(p) + h(x) + t \\ &= 2b - h(x) - h(p) + t \\ &= d_p(x) + t.\end{aligned}$$

If $\eta > 0$ then we choose $\delta = \eta$ and for $t \in (0, \delta)$ we a new minimal height interval is given by $a' = h(x) + t$ and $b' = b$ and we get

$$\begin{aligned}d_p([x_1+t, x_2]) &= 2b' - 2a' - h(p) + h([x_1+t, x_2]) \\ &= 2b - 2(h(x)+t) - h(p) + h(x) + t \\ &= 2b - h(x) - h(p) - t \\ &= d_p(x) - t.\end{aligned}$$

- If $h(x) > a$ then $d_p(x) = 2b - 2a - h(p) + h(x)$. For $\delta = h(p) - h(x)$ and $t \in (0, \delta)$ we have to find a new minimal height segment $[a', b'] \subset I$ to connect p to $[x_1 + t, x_2]$. It is readily verified that we can choose $a' = a$ and $b' = b$. Hence

$$\begin{aligned}d_p([x_1+t, x_2]) &= 2b' - 2a' - h(p) + h([x_1+t, x_2]) \\ &= 2b - 2a - h(p) + h(x) + t \\ &= d_p(x) + t\end{aligned}$$

and this concludes the proof for the final case.

□

Thanks to Proposition 4.4.4 we can narrow down the possible values that $(d_p)_I(x)$ can assume.

Corollary 4.4.5. *Fix $p \in F$. Then for every $x = [x_1, x_2] \in F$ we have that both*

$$\lim_{t\uparrow 0} \frac{d_p([x_1 + t, x_2]) - d_p(x)}{t} \quad \textit{and} \quad \lim_{t\downarrow 0} \frac{d_p([x_1 + t, x_2]) - d_p(x)}{t}$$

exist and they are either 1 *or* -1.

Notice that this does not give a precise value for $(d_p)_I(x)$, because we are not assuming that the directional derivative of $d_p(\cdot)$ exists at x. What we can say is that, when it exists, it is either 1 or -1.

Definition 4.4.6. An upward going segment in F is a curve γ of the form

$$\gamma(t) = [\lambda + t, \mu]$$

with $\lambda \in [0, 1]$ and $\mu \in K$. Similarly a downward going segment is a curve γ of the form

$$\gamma(t) = [\lambda - t, \mu].$$

In both cases t varies in some interval and λ, μ are constants fixed for the segment.

We say that a curve $\gamma\colon [a, b] \to F$ ends in an upward (downward) going segment if there exists $\delta > 0$ such that the restriction of γ to the interval $[b - \delta, b]$ is an upward (downward) going segment.

Note that for every pair of distinct points $x, y \in F$, we can always find either an upward ending or a downward ending geodesic from x to y. For some pairs of points there may be both.

The position of the geodesic gives information about the value of $(d_p)_I(x)$.

Proposition 4.4.7. *Fix $p \in F$. Then for every $x = [x_1, x_2] \in F$ the following holds*

i) there exists an upward ending geodesic from p to x if and only if

$$\lim_{t\uparrow 0} \frac{d_p([x_1 + t, x_2]) - d_p([x_1, x_2])}{t} = 1$$

ii) there exists a downward ending geodesic from p to x if and only if

$$\lim_{t\downarrow 0} \frac{d_p([x_1 + t, x_2]) - d_p([x_1, x_2])}{t} = -1.$$

Proof. We prove i), the proof of ii) is the same with the relevant signs changed.

$(\Rightarrow)$: Let $\gamma : [a, b] \to F$ be an upward ending geodesic from p to x. Then by the definition of upward there exists $\delta > 0$ such that the restriction of γ to $[b - \delta, b]$

is an upward going segment. For $-\delta < t < 0$ we have that the point $[x_1 + t, x_2]$ belongs to that segment. This in particular implies that we can define a geodesic $\hat{\gamma} : [a, b - (\delta + t)] \to F$ connecting p to $[x_1 + t, x_2]$ as $\hat{\gamma}(s) = \gamma(s)$. Hence, for $t \uparrow 0$, $d_p([x_1 + t, x_2]) = d_p(x) + t$ and this proves the implication.

($\Leftarrow$) : We can find $\delta > 0$ small enough such that

$$\frac{d_p([x_1 + t, x_2]) - d_p([x_1, x_2])}{t} > 0 \quad \text{for} \quad t \in (-\delta, 0).$$

Since t is negative then $d_p([x_1 + t, x_2]) < d_p([x_1, x_2])$. From Proposition 4.4.4, possibly making δ smaller if necessary, for $t \in (-\delta, 0)$, $d_p([x_1 + t, x_2]) = d_p(x) - |t|$. Now fix $t \in (-\delta, 0)$ and take a geodesic $\gamma : [a, b] \to F$ connecting p to $[x_1 + t, x_2]$ (it does not matter if this geodesic ends upward or downward). Define the path $\overline{\gamma} : [a, b + |t|] \to F$ as

$$\overline{\gamma}(s) = \begin{cases} \gamma(s) & s \in [a, b] \\ [x_1 + t - b + s, x_2] & s \in [b, b + |t|]. \end{cases} \tag{4.4.3}$$

Clearly $\overline{\gamma}$ is an upward ending path connecting p to x. To prove that it is a geodesic we have to show that its length equals the distance from p to x. But the length of $\overline{\gamma}$ is exactly the length of γ plus the length of the vertical segment added at the end of γ. In other words, by denoting with $L(\cdot)$ the length of a path in F, we have that

$$\begin{aligned} L(\overline{\gamma}) &= L(\gamma) + L(\{[x_1 + t - b + s, x_2] \,:\, s \in [b, b + |t|]\}) \\ &= d_p([x_1 + t, x_2]) - |t| + L(\{[x_1 + t - b + s, x_2] \,:\, s \in [b, b + |t|]\}) \\ &= d_p([x_1 + t, x_2]) - |t| + |t| \\ &= d_p(x) \end{aligned}$$

and this concludes the proof. □

We recall that the goal of this section is to study the differentiability of the distance function $d_p(\cdot)$ for any $p \in F$. We already established the relation between the existence of $(d_p)_I(x)$ and the differentiability at x in Theorem 4.4.1. Now we want to be able to tell at which point of F the map $d_p(\cdot)$ is differentiable. As one might expect, it will turn out that these points will depend on the choice of the base point$p \in F$.

In the following we describe the set B_p, a candidate to be the set of non-differentiability points of $d_p(\cdot)$ for a given $p \in F$. As one should expect, it will depend on the base point p. However in the final result of this section we will prove the general fact that B_p is always a σ-porous set.

Definition 4.4.8. Fix $p = [p_1, p_2] \in F$. Define $B_p \subset F$ to be the set of points $x \in F$ satisfying at least one of the following conditions:

i) $x = p$,

ii) x is a wormhole of level N for some $N \geq 1$ and every geodesic from p to x does not pass through a wormhole of level N until it reaches x,

iii) there exists both a geodesic from p to x which ends in an upward going segment and a geodesic from p to x which ends in a downward going segment (does not matter whether x is a wormhole or not).

Some comments on the structure of B_p are in order. First we notice that $p \in B_p$. This fact is reminiscent of the classical result that, in Euclidean spaces and Carnot groups, the distance (considered from the origin) is not differentiable at the origin.

Point iii) is also similar to other know cases. Take for example a point p on a sphere and its antipodal point $-p$. If we compute the distance function with base point p we get that p and $-p$ are points of non differentiability. Indeed there are multiple geodesics connecting p to $-p$.

Point ii) on the other hand is not similar to any other example that we were able to think of. It is our opinion that this particular behaviour is caused by the presence of wormholes and by the very peculiar structure of Laakso spaces.

Theorem 4.4.9. *Fix $p = [p_1, p_2] \in F$. Then the map $y \mapsto d_p(y)$ is differentiable at a point $x \in F$ if and only if $x \notin B_p$.*

We divide the proof of the theorem into two claims.

Claim 4.4.10. *Suppose that $x \in B_p$. Then $d_p(y)$ is not differentiable at x.*

Proof of Claim. If $x \in B_p$ there are three possibilities as in the definition of B_p; we address each one separately.

Case 1: Suppose $[x_1, x_2] = x = p$. In this case it is clear that $y \mapsto d_p(y)$ is not differentiable at x. Indeed for $\delta > 0$ and $t \in (-\delta, \delta)$, $d_p([x_1 + t, x_2]) = |t|$ and this suffices for the non existence of $(d_p)_I(x)$. Theorem 4.4.1 then implies the non differentiability.

Case 2: Suppose $x = [x_1, x_2] = [x_1, x_2 + \frac{2}{3^N}]$ is a wormhole of level N for some $N \geq 1$ and every geodesic from p to x does not pass through a wormhole of level N until it reaches x. In this case geodesics from p to x reach x through only one of the vertical lines through x determined by the identification (namely the one determined by p). Let us assume that is the vertical line above x_2. After x if we move in either direction along the opposite vertical line the distance will increase and in particular for $t > 0$ we have that

$$d_p\left(\left[x_1 + t, x_2 + \frac{2}{3^N}\right]\right) = d_p\left(\left[x_1 - t, x_2 + \frac{2}{3^N}\right]\right) = d_p([x_1, x_2]) + t$$

This yields non differentiability as in case 1.

Case 3: Suppose there exists both a geodesic from p to x which ends in an upward going segment and a geodesic from p to x which ends in a downward going segment. Then thanks to Proposition 4.4.7 we infer that the limit in Definition 4.2.1 does not exists. Hence $(d_p)_I(x)$ does not exists and this suffices for non differentiability. □

Claim 4.4.11. *Suppose $x \notin B_p$. Then $d_p(y)$ is differentiable at x.*

Proof of Claim. Assume that i) and iii) in the definition of B_p fail. We further distinguish two cases.

Case 1: x is not a wormhole. Let us also assume the we have an upward ending geodesic that connects p and x, the other case is similar. We claim that $(d_p)_I(x) = 1$. This will suffice to prove differentiability.

Since x can be reached by an upward going geodesic, Proposition 4.4.7 ensures that

$$\lim_{t \uparrow 0} \frac{d_p([x_1 + t, x_2]) - d_p(x)}{t} = 1.$$

We claim that also

$$\lim_{t \downarrow 0} \frac{d_p([x_1 + t, x_2]) - d_p(x)}{t} = 1.$$

Indeed if this were not the case then from Corollary 4.4.5 we would have that

$$\lim_{t \downarrow 0} \frac{d_p([x_1 + t, x_2]) - d_p(x)}{t} = -1.$$

However this is not possible because Proposition 4.4.7 would then imply that the existence of a downward ending geodesic from p to x, which is not possible because we assumed $x \notin B_p$ and in particular that (3) fails. From this we can conclude that $(d_p)_I(x) = 1$ and this implies differentiability thanks to Theorem 4.4.1.

Case 2: $x = [x_1, x_2] = [x_1, x_2 + \frac{2}{3^N}]$ is a wormhole of level N. Let $\gamma : [a, b] \to F$ be a geodesic that connects p to x and assume that is upward ending. Hence there exists a $\delta > 0$ such that the restriction of γ to $[b - \delta, b]$ is a segment of the form $[\lambda + t, \mu]$ where λ is a constant depending on γ and μ is either x_2 or $x_2 + \frac{2}{3^N}$. Let us assume that $\mu = x_2$, the proof in the other case is the same.

Since $\mu = x_2$ we can repeat the argument used in the previous case to prove that

$$(d_p)_L(x) = \lim_{t \to 0} \frac{d_p([x_1 + t, x_2]) - d_p(x)}{t} = 1$$

and we want to prove that $(d_p)_R(x)$ exists and has the same value. Notice that if we assumed $\mu = x_2 + \frac{2}{3^N}$ then we would have had $(d_p)_R(x) = 1$ and we should prove that $(d_p)_L(x)$ exists and has the same value.

Since $x \notin B_p$, by negating ii) in Definition 4.4.8 we get that γ must pass through at least one other wormhole of level N, different from x, before reaching x. Call y one of those wormholes. We define a new curve $\tilde{\gamma} : [a, b] \to F$ as follow:

i) $\tilde{\gamma}$ coincides with γ from p to y,

ii) in y, $\tilde{\gamma}$ has the opposite behaviour of γ: if γ jumps using the wormhole y, then $\tilde{\gamma}$ does not jump in y and vice versa,

iii) in every other wormhole $[z_1, z_2]$ that $\tilde{\gamma}$ meets after y, it does the same thing (jump or not jump) that γ does in the corresponding wormhole $[z_1, z_2 \pm \frac{2}{3^N}]$. Here the sign depends on whether $\tilde{\gamma}$ jumped or not at y: .

The curve $\tilde{\gamma}$ connects p to x. Moreover, since jumping through wormholes does not change the length of a curve, $L(\gamma) = L(\tilde{\gamma})$. From this we conclude that $\tilde{\gamma}$ is an upward ending geodesic that connects p to x. Hence there exists $\delta' > 0$, possibly smaller than δ, such that the restriction of $\tilde{\gamma}$ to $[b - \delta', b]$ is a segment of the form $[\lambda' + t, \mu']$ with λ' a constant depending on $\tilde{\gamma}$ and $\mu' = x_2 + \frac{2}{3^N}$. From this we can compute

$$(d_p)_R(x) = \lim_{t \to 0} \frac{d_p\left(\left[x_1 + t, x_2 + \frac{2}{3^N}\right]\right) - d_p(x)}{t} = 1.$$

Hence $(d_p)_(x) = (d_p)_R(x)$ and this implies differentiability thanks to Theorem 4.4.1. □

Before proceeding with the final result about the porosity of the set B_p we need to give a definition.

Definition 4.4.12. Given $p = [p_1, p_2] \in F$ not a wormhole and N a positive integer, we define

$$V_N^p = \left\{ \left[t, p_2 \pm \frac{2}{3^N} \right] \,:\, t \in [0, 1] \right\}$$

where the choice of $+$ or $-$ is uniquely determined by p_2. It is the vertical line of all the points in F that can be reached from p by jumping only once and only through a wormhole of level N.

Similarly, if $p = [p_1, p_2] = [p_1, p_2']$ is a wormhole of level M and N is a positive integer different from M then V_N^p will be composed of two vertical lines: one above p_2 and one above p_2'.

We can also define lines reached with multiple jumps. Let $N_1 < N_2 < \dots$ be a sequence of positive integers such that, if p is a wormhole of level M, $N_i \neq M$ for each i. We define the quantity

$$\Delta = \pm\frac{2}{3^{N_1}} \pm \frac{2}{3^{N_2}} \pm \dots$$

where all the signs are uniquely determined by p_2. Then

$$V^p_\Delta = \{[t, p_2 + \Delta] \,:\, t \in [0,1]\}$$

is the vertical line (or the two vertical lines, if p is a wormhole) of all the points in F that can be reached from p by jumping only on wormholes of level $N_1, N_2, \dots$, using each wormhole level exactly once.

Proposition 4.4.13. *Fix $p = [p_1, p_2] \in F$. Then there exists a countable set $\mathcal{N} \subset [0,1]$ such that for each $x \in B_p$, $h(x) \in \mathcal{N}$. In particular B_p has measure zero and is σ-porous.*

Proof. To prove the first part we have to count the number of points that are in $h(B_p)$ for a fixed p. The idea for doing that is to count how many points are added to the total from each of the three cases in Definition 4.4.8.

i) From this point we have to add only p itself.

ii) Points that satisfies ii) in the definition of B_p are wormholes. Since the set of all wormhole levels is countable then also the contribution from heights of points that satisfy ii) is countable.

iii) Points that satisfies iii) are the most difficult to count. The process of counting them will be divided in three steps.

  Step 1: Count the points that are on lines of the form V^p_N. For a positive integer N (different from M, if p is a wormhole of level M) we look for $D^+_N(p_1)$ and $D^-_N(p_1)$. If one of the two does not exists then we will not find any point in V^p_N to add to the count of B_p. Indeed if only $D^+_N(p_1)$ (or $D^-_N(p_1)$) exists then it is easy to verify that the only point in V^p_N in which $(d_p)_I$ does not exists is $[p_1 + D^+_N(p_1), p_2 + \frac{2}{3^N}]$ (or $[p_1 - D^-_N(p_1), p_2 + \frac{2}{3^N}]$). This point satisfies (2) in the definition of B_p, hence we already counted it.

  On the other hand if both $D^+_N(p_1)$ and $D^-_N(p_1)$ exist then we define

$$c_N = D^+_N(p_1) - D^-_N(p_1) \tag{4.4.4}$$

  and we claim that $x = [p_1 + c_N, p_2 + \frac{2}{3^N}]$ is the only point in V^p_N that satisfies (3) in the definition of B_p.

To prove that, we construct a geodesic from p to x by starting at p, going up to height $p_1 + D_N^+(p_1)$, jumping in the wormhole of level N found there and then going down to height $p_1 + c_N$. This is a geodesic from p to x that ends downward and if we use this path to compute $d_p(x)$ we get $d_p(x) = D_N^+(p_1) + D_N^-(p_1)$. However we could also start from p, go downward to height $p_1 - D_N^-(p_1)$, jump in the wormhole found there and then go up to height $p_1 + c_N$. Also with this path we get $d_p(x) = D_N^+(p_1) + D_N^-(p_1)$, hence it is a geodesic from p to x and it ends upward. We are left to show that x is the only point in V_N^p that satisfies iii). To this end let us suppose that there is another $y \in V_N^p$ that satisfies iii). Clearly we must have that $p_1 - D_N^-(p_1) < h(y) < p_1 + D_N^+(p_1)$ because if $y \in V_N^p$ is such that $h(y) > p_1 + D_N^+(p_1)$ (or $h(y) < p_1 - D_N^-(p_1)$) then $(d_p)_I(y)$ exists and Theorem 4.4.9 implies that $y \notin B_p$. Let us also assume that $h(y) > h(x)$, the proof in the other case is the same. Since we are assuming that $y \in B_p$ then there must be a downward ending geodesic γ_y^d from p to y. From the construction of V_N^p and x we get that this geodesic must coincide, up to y, with the one that connects p to x, let us call it γ_x. Hence $L(\gamma_y^d) < L(\gamma_x)$. However, for the upward ending geodesic γ_y^u, we have the opposite relation $L(\gamma_y^u) > L(\gamma_x)$. But this is in contrast with the fact that, being y a point that satisfies iii), we must impose $L(\gamma_y^d) = L(\gamma_y^u)$, proving the uniqueness of x.

We can now conclude the count. Indeed for each N we found at most one point in V_N^p that satisfies iii), which means that in this step we are adding only countably many contributions to the total of $h(B_p)$.

Step 2: Count the points that are on lines of the form V_Δ^p where Δ is as in Definition 4.4.12 but comes from a sequence of only two values $N < M$. There are three possible cases, based on the position of p_1 with respect to the nearest wormholes of level N and M above and below it.

(a) $p_1 - D_N^-(p_1) < p_1 - D_M^-(p_1) < p_1 < p_1 + D_M^+(p_1) < p_1 + D_N^+(p_1)$

(b) $p_1 - D_N^-(p_1) < p_1 - D_M^-(p_1) < p_1 < p_1 + D_N^+(p_1) < p_1 + D_M^+(p_1)$

(c) $p_1 - D_M^-(p_1) < p_1 - D_N^-(p_1) < p_1 < p_1 + D_M^+(p_1) < p_1 + D_N^+(p_1)$.

Let us start by case (a). In this case we claim that, for each $x \in V_\Delta^p$, there exists a $\overline{x} \in V_N^p$ such that $d_p(x) = d_p(\overline{x})$ and $h(\overline{x}) = h(x)$. This comes from the fact that any geodesic that connects p to a point in V_Δ^p must jump at a wormhole of level N. But in the configuration (a) any such path must pass through a wormhole of level M before reaching one of level N. This in particular implies that $x \in B_p$ if and only if $\overline{x} \in B_p$. Hence, since $h(\overline{x}) = h(x)$, from this case we are not adding any new point to the count of $h(B_p)$.

Cases (b) and (c) are very similar. We count only the points that came from case (b) for brevity reasons. Let us define c_N and c_M as in (4.4.4). We claim that the only points in V^p_Δ that satisfy (3) in the definition of B_p are

$$x = [p_1 + c_N, p_2 + \Delta]; \quad y = [p_1 + c_M, p_2 + \Delta] \text{ and } z = [p_1, p_2 + \Delta].$$

To prove this we exhibit, for each point, an upward ending geodesic and a downward ending one from p to that point. Let us start from x. To get the downward ending geodesic γ^d_x we proceed as follow:

(i) we start from p and we go up to height $p_1 + D^+_M(p_1)$;

(ii) we jump with the wormhole of lever M found there;

(iii) we go down to height $p_1 + D^+_N(p_1)$;

(iv) we jump with the wormhole of level N we found there;

(v) we go down to height $h(x) = p_1 + c_N$.

To get the upward ending geodesic γ^u_x we proceed as follow:

(i) we start from p and we go up to height $p_1 + D^+_N(p_1)$;

(ii) we jump with the wormhole of lever N found there;

(iii) we go down to height $p_1 + D^-_M(p_1)$;

(iv) we jump with the wormhole of level M we found there;

(v) we go up to height $h(x) = p_1 + c_N$.

An easy computation shows that $L(\gamma^d_x) = L(\gamma^u_x) = d_p(x) = D^+_N(p_1) + D^-_N(p_1)$. The construction of the geodesics for the other points is analogous, but we avoid the details because they would take too much space.

The only thing we are left to prove is that x, y and z are the only three points in V^p_Δ that satisfies iii) in the definition of B_p, but this can be done as in the previous step by showing that there can not be intermediate points with the same property.

We have proved that for each couple of positive integers N, M with $N < M$ there are at most three points in V^p_Δ that satisfy iii) in the definition of B_p. Hence, since there are countably many of such couples, we are adding only countably many contributions to the total of $h(B_p)$.

Step 3: Count the points that are on lines of the form V^p_Δ where Δ comes from any sequence $N_1 < N_2 < \ldots$. The strategy in this step is similar to the strategy of case (a) in step 2. We start by defining Δ_2 to be the Δ of

Definition 4.4.12 for the sequence with only two values: N_1 and N_2. We claim that, for each $x \in V^p_\Delta$, there exists a $\overline{x} \in V^p_{\Delta_2}$ such that $d_p(x) = d_p(\overline{x})$ and $h(\overline{x}) = h(x)$. This comes from the fact that, between any two wormhole levels $N_1 < N_2$, we can find wormhole levels of order M for each $M > N_2$. This in particular implies that $x \in B_p$ if and only if $\overline{x} \in B_p$. Hence, since $h(\overline{x}) = h(x)$, in this step we are not adding any new point to the count of $h(B_p)$.

This concludes the counting and proves that B_p consists of points with at most countably many different heights, i.e. $h(B_p) = \mathcal{N} \subset I$ with $\mathcal{N}$ countable. Hence

$$B_p \subseteq h^{-1}(h(B_p)) = h^{-1}\left(\bigcup_{n\in\mathcal{N}} \{n\}\right) = \bigcup_{n\in\mathcal{N}} h^{-1}(\{n\})$$

and this, using Lemma 4.3.9 and thanks to the fact that singletons are porous, concludes the proof. $\square$

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