# THE IMPACT OF WHITE NOISE ON A SUPERCRITICAL BIFURCATION IN THE SWIFT-HOHENBERG EQUATION

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Abstract. We consider the impact of additive Gaussian white noise on a supercritical pitchfork bifurcation in an unbounded domain. As an example we focus on the stochastic Swift-Hohenberg equation with polynomial nonlinearity. Here we identify the order where small noise first impacts the bifurcation. Using an approximation via modulation equations, we provide a tool to analyse how the noise influences the dynamics close to a change of stability.

## 1. Introduction

In this paper we intend to identify the main impact of an additive Gaussian white noise on the dynamics close to or at a change of stability described by a stochastic partial differential equation with polynomial nonlinearity. For this we will study the reduction of the essential dynamics close to the bifurcation via amplitude or modulation equations. Surprisingly, and in contrast to the strong nonlinear interaction of finitely many Fourier modes, in all our results the additive noise does not add any additional terms to the modulation equation, its nonlinear interaction always disappears via averaging effects and it just shows up as an additive forcing in the amplitude equation.

In order to keep the paper short and to focus on the main results, we do not aim to prove all error estimates in full technical details, but we always state how they can be proven.

As a first problem we consider the following stochastic Swift-Hohenberg equation on $\mathbb{R}^+ \times \mathbb{R}$

$$\partial_t u = -(1+\Delta)^2 u + \nu u^2 - u^3 + \varepsilon^{3/2} \partial_t \widetilde{W}, \tag{1}$$

where $\widetilde{W}$ is a standard cylindrical Wiener process, i.e. $\partial_t \widetilde{W}$ models space-time white noise. This equation was introduced first in the seminal paper by Swift and Hohenberg, [25], where they already discussed the importance of random fluctuations in the context of Rayleigh Bénard convection.

The operator $-(1+\Delta)^2$ is a non-positive self-adjoint operator with spectrum $(-\infty, 0]$. As we do not have an additional linear term in the equation (1), we are exactly at criticality, where the spectrum of the linear operator is non-positive, but it contains 0, which in our case formally corresponds to the complex eigenfunction $e^{ix}$. The parameter $\nu$ in front of the quadratic term in the equation does not change the linearised operator. It will only determine the shape of the bifurcation.

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In [2] we discussed the equation with $\nu = 0$ and an additional linear term in the weakly nonlinear regime close to bifurcation, and we comment on that in more detail below.

In the deterministic case the dynamics of (1) and its importance in pattern formation was studied in numerous publications. See for example [9, 10, 11, 14, 15], where also many examples of a formal derivation of amplitude equations are found.

Rescaling the equation, we will see in our main result that solutions are given by a slow modulation of the dominating solution (or pattern) $e^{ix}$, that is

$$u(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + \text{c.c.}$$

where c.c. denotes the complex conjugate. We denote by $T = \varepsilon^2 t$ the slow time and by $X = \varepsilon x$ the rescaled 'slow' space variable. In the case of (1) we will see that the complex-valued amplitude $A$ function solves

$$\partial_T A = 4\partial_X^2 A - \Big(3 - \frac{38}{9}\nu^2\Big)|A|^2 A + \eta$$

where $\eta$ is a complex-valued space-time white noise. So the presence of the quadratic term in Swift-Hohenberg can change the strength of the cubic in the amplitude equation. It formally can even make the sign of the cubic positive, for $\nu > \sqrt{27/38}$. Although our analysis carries through even in this case, this leads to an unstable cubic in the amplitude equation and would allow for a blow up of solutions in finite time. Our analysis in that case only holds up to times where the solution of the amplitude equation is still of order 1. Similar results on a bounded domain, where the amplitude equation is just a SDE, were derived in [17].

In [2] we studied the classical Swift-Hohenberg equation without a quadratic nonlinearity (i.e. with $\nu = 0$) but with an additional linear term

$$\partial_t u = -(1+\Delta)^2 u + \mu\varepsilon^2 u - u^3 + \varepsilon^{3/2}\partial_t \widetilde{W}.$$

Here the spectrum of the linear operator is $(-\infty, \mu\varepsilon^2]$ and thus changes stability at $\mu = 0$, which means we have a bifurcation here. Further analysis in the deterministic case would reveal, that it is a classical supercritical (i.e., forward) pitchfork bifurcation, where non-trivial stationary states are present only for $\mu > 0$.

Moreover, in [2] we showed that the amplitude in this case solves

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A + \eta.$$

For the effect of a simple scalar valued forcing, which is constant in space, see [18].

In this paper in Section 7, we also briefly consider the Swift-Hohenberg equation with a quintic nonlinearity

$$\partial_t u = -(1+\Delta)^2 u + \nu_2\varepsilon^{1/2} u^2 + \nu_3 \varepsilon u^3 - u^5 + \varepsilon\partial_t \widetilde{W}. \tag{2}$$

As the analysis is quite similar to the cubic case, we will keep the presentation very short here, and only focus on the main differences.

The advantage of adding the quintic is the following. In the setting of (1) without the stable cubic in the case of a subcritical bifurcation, we would have a positive coefficient in front of the highest cubic nonlinear term in the amplitude equation, which thus leads to an equation that might blow up in finite time. In contrast to that the additional quintic leads to a stable quintic in the amplitude equation, which prevents blow up.

Note that due to the quintic nonlinearity, we have a different scaling of the parameters and the quadratic and cubic nonlinearities have to be small in order to not dominate the quintic close to bifurcation. In the scaling

$$u(t,x) = \varepsilon^{1/2} A(\varepsilon^2 t, \varepsilon x) e^{ix} + \text{c.c.}$$

we obtain the following equation for the complex amplitude

$$\partial_T A = 4\partial_X^2 A + \Big(\frac{38}{9}\nu_2^2 + 3\nu_3\Big)|A|^2 A - 10|A|^4 A + \eta\ .$$

If $\nu_2$ is sufficiently large when compared to $\nu_3$ or $\nu_3$ being positive, then the cubic is an unstable subcritical nonlinearity. This means that, if we were to add a linear term $\nu_1\varepsilon^2 u$ to (2) we would obtain also an additional $\nu_1 A$ in the amplitude equation. This equation has for $\nu_1 = 0$ a subcritical backward pitchfork bifurcation if $\nu_2$ is sufficiently large so that the constant in front of the cubic is positive.

Let us also comment that we could also add a quartic nonlinearity $\nu_4\varepsilon^{-1/2}u^4$, to (2) which now leads to an additional quintic nonlinearity with positive coefficients in the amplitude equation. On the expense of overwhelming technical difficulties one could now go to even higher order nonlinearities.

Surprisingly, in all our results the additive noise does not introduce any additional terms to the modulation equation, it just appears as an additive forcing in the amplitude equation. This is in contrast to the strong nonlinear interaction of Fourier modes that, for example, leads to the appearance of cubic terms in the amplitude equation arising from a quadratic nonlinearity in (2). We will however see that in this setting all the nonlinear interactions of noise terms actually vanish due to averaging effects.

The outline of the paper is as follows. In the next Section 2, we briefly discuss the problem of existence and uniqueness of solutions and mainly give references to methods that allow to prove this. In Section 3 we rely on estimates to identify the dominant Fourier modes, which are the ones around the wavenumbers $k \in \{0, \pm1, \pm2\}$ in Fourier space and derive reduced equations for these modes by cutting out all small terms. Using explicit averaging results based on Itô formula in Section 4, we reduce the whole dynamics to the wavenumbers close to $k = \pm1$ in Fourier space and state in Section 5 the final result. Assuming additional regularity of the dominant Fourier modes, we simplify the limiting equation in Section 6. In the final Section 7 we briefly comment on the changes necessary for the result in the quintic case.

## 2. Solutions

Due to a lack of regularity of solutions due to the noise, we consider solutions to our SPDEs in the mild sense. The mild formulation of (1) is given by

$$\begin{aligned} u(t) = e^{tL}u(0) + \int_0^t e^{(t-s)L}[\nu u^2 - u^3](s)ds \\ + \varepsilon^{3/2}\int_0^t e^{(t-s)L} d\widetilde{W}(s), \end{aligned}$$

where $e^{tL}$ is the semigroup generated by the operator $L = -(1+\Delta)^2$. On the unbounded domain we can simply rely on the fact that the linear operator is diagonal in Fourier space and define the semigroup using the standard Fourier transform $\mathcal{F}f = \widehat{f}$. For example,

$$\widehat{Lf}(k) = -(1-|k|^2)^2\widehat{f}(k)$$

and for the semigroup

$$\mathcal{F}[e^{tL}f](k) = \exp\{-(1-|k|^2)^2 t\}\widehat{f}(k).$$

We will now first rescale the equation and then comment on the existence of solutions for the rescaled equation further below.

**Rescaling:** Close to bifurcation we consider small solutions and follow the usual deterministic approach of modulation equations. We rescale small solutions to slow spatial and temporal scales via

$$u(t,x) = \varepsilon v(\varepsilon^2 t, \varepsilon x)$$

to obtain

$$\partial_T v = L_\varepsilon v + \varepsilon^{-1}\nu v^2 - v^3 + \partial_T W, \tag{3}$$

with the rescaled operator $L_\varepsilon = -\varepsilon^{-2}(1+\varepsilon^2\Delta)^2$.

The noise strength is derived using the scaling property of the white noise or, equivalently, the scaling property of the Wiener process $\widetilde{W}$. Here $\partial_T W$ is again space-time white noise and $W$ a standard cylindrical Wiener process. Due to the rescaling $W$ and thus $\partial_T W$ depend path-wise on $\varepsilon$, but as they have the same law as $\widetilde{W}$ and $\partial_t\widetilde{W}$, and we consider error estimates only in law, we ignore this dependence in the following.

The mild formulation of (3) is given by

$$\begin{aligned} v(T) = e^{TL_\varepsilon}v(0) &+ \int_0^T e^{(T-S)L_\varepsilon}[\varepsilon^{-1}\nu v^2 - v^3](S)dS \\ &+ \int_0^T e^{(T-S)L_\varepsilon}dW(S). \end{aligned} \tag{4}$$

We consider solutions in spaces $C^{0,\alpha}_\kappa$, the spaces of $\alpha$-Hölder continuous functions with slow polynomial growth at infinity:

$$C^{0,\alpha}_\kappa = \{u:\mathbb{R}\to\mathbb{R} \;:\; \sup\{L^{-\kappa}\|u\|_{C^{0,\alpha}([-L,L])}\;;\; L>1\}<\infty\}.$$

A more detailed discussion regarding these spaces can be found in [1].

If we consider the stochastic convolution

$$W_{L_\varepsilon}(T) = \int_0^T e^{(T-S)L_\varepsilon}dW(S)$$

we have the following uniform bound in the spaces $C^{0,\alpha}_\kappa$.

**Lemma 1.** *For all $\alpha\in(0,\frac{1}{2})$, $\kappa>0$, the stochastic process is $W_{L_\varepsilon}$ has continuous paths in $C^{0,\alpha}_\kappa$ and for all $T>0$ and $p>1$, we have a constant such that for all $\varepsilon\in(0,1)$*

$$\mathbb{E}\sup_{[0,T]}\|W_{L_\varepsilon}\|^p_{C^{0,\alpha}_\kappa} \le C$$

The proof for this Lemma is quite long but at the same time fairly standard. It can be proven using exactly the same arguments as in the proof of Lemma 3 in [2]. There one considers first bounded spatial domains of length $2L$, and then carefully keeps track of the dependence of various constants on $L$.

For other type of maximal regularity results for the stochastic convolution, for instance in $L^p$ spaces, see [8, 12, 21].

*Remark* 1. Let us remark that $W_{L_\varepsilon}$ is actually more regular than stated in Lemma 1. It is Hölder-continuous with exponent $\alpha$ almost 1. This is due to strong regularization of the fourth order operator in the equation. But in the limit $\varepsilon \to 0$ (see [1]) we lose this property and thus a uniform bound in $\varepsilon$ can only be established for Hölder exponents $\alpha < 1/2$.

In the rest of the paper, we always suppose that we have sufficiently smooth solutions such that the following statement holds.

**Assumption 1.** *The rescaled equation* (3) *has a unique mild solution $u$, which is a stochastic process with continuous paths in $C^{0,\alpha}_\kappa$ for every $\kappa > 0$ and $\alpha \in (0, \frac{1}{2})$.*

*Remark* 2. Before moving on, let us remark that for fixed $\kappa$ and $\alpha$ the standard fixed point argument for the existence and uniqueness of local mild solutions does not work, as the nonlinearity is unbounded in the weight and the semigroup only improves regularity in terms of the Hölder exponent.

We state Assumption 1 as such, and not as a theorem, because its proof would be a paper of its own, so within this paper we just take existence for granted. At the same time, we are quite confident that such result holds: there are some fairly standard approaches we could follow to prove it. Nevertheless this is quite a lot of work, as most results first establish the existence and uniqueness in a weaker topology and then lengthy regularity results are needed.

Among the first results on SPDEs on the whole real line in spatially weighted spaces are the ones of Peszat et. al. [7, 22] using mainly exponential weights but also stating results for polynomial weights.

The complex-valued stochastic Ginzburg-Landau Equation in a weighted $L^2$-space was studied in detail by Blömker and Han [6], but not with regularity in Hölder spaces, which was done in [2], where also Swift-Hohenberg with $\nu = 0$ was discussed.

For recent results on space-time-white noise in weighted Besov spaces see for example Röckner, Zhu, and Zhu [23] or Mourrat and Weber[20].

Let us also mention a recent paper by Moinat and Weber [19] that obtains for the dynamic $\Phi^4_3$ model local regularization on bounded subdomains in case of weaker bounds on the whole domain. Although the model they treat is real-valued the results should hold for the very similar complex Ginzburg Landau model. Moreover, this method should also apply to Swift-Hohenberg.

## 3. A-priori bound

In this section, we show that Fourier modes around $\pm 1$ (or $\pm 1/\varepsilon$ for the rescaled equation) dominate the behaviour.

One can easily argue that the mild solution with initial condition $v(0)$ of order 1 stays of order 1 at least for some time. However, this is not obvious. If we look at the rescaled equation 3, then the quadratic term has an additional $1/\varepsilon$ in front, and due to the rescaled solution and the semigroup being only of order 1, we do not immediately get a bound up to times of order 1, as a direct estimate yields only that the quadratic term is bounded by something of order $1/\varepsilon$. In contrast to that, the cubic term does not cause any difficulties, as everything is order one there. This is also reflected by the fact that we consider small solutions to the original equation (1), so the cubic term should be always smaller than the quadratic one.

With this simple reasoning we can only hope to reach times of order $\varepsilon$, so we need a better estimate.

In order to restrict to regions around $k \approx \pm 1/\varepsilon$ in Fourier space, we consider smooth projectors $P_1$ for a given smooth Fourier kernel $q : \mathbb{R} \to [0,1]$ such that $q = 1$ on the set of $k$ such that $|k \pm 1/\varepsilon| < \delta/\varepsilon$ and $q = 0$ on $|k \pm 1/\varepsilon| > \delta/\varepsilon + 1$, for some $\delta < \delta_0 \leq 1/2$. Hence,

$$P_1 f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} q(k) e^{ik(x-z)} dk f(z) dz \; .$$

Before we move on, let us spend a couple of words on some notation used in the rest of this paper. First, we give the following definition:

**Definition 1.** We say that an $\varepsilon$-dependent event $E_\varepsilon$ has *probability almost 1* or *high probability* if for every $p \geq 1$ there exists a positive constant $C_p$ such that $P(E_\varepsilon) \geq 1 - C_p \varepsilon^p$.

Secondly, let us discuss briefly our use of the $\mathcal{O}$ notation. In the following, we use it in two ways, in both cases considering $\varepsilon \to 0$ or just sufficiently small $\varepsilon$. On one hand it means that the term is bounded for all $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ up to an $\varepsilon$-independent multiplicative constant (as we will see for the semigroups in the next paragraph).

On the other hand, for stochastic processes $w$ (e.g. our solutions) we write $w = \mathcal{O}(\varepsilon^\gamma)$ if for all $c > 0$, $\kappa > 0$, and $\alpha \in (0, 1/2)$ there is a constant $C_{\alpha,\kappa,c}$ such that with probability almost 1 (see Definition 1 above) we have

$$\sup_{T \in [0,T_0]} \|w(T)\|_{C^{0,\alpha}_\kappa} \leq C_{\alpha,\kappa,c} \varepsilon^{\gamma - c} \; , \tag{5}$$

again for all $\varepsilon \in (0, \varepsilon_0)$.

Note that the $c > 0$ allows for small logarithmic corrections to the error bound, which frequently pop up when bounding stochastic convolutions pathwise.

**First estimate:** Note that in Fourier space around $k \approx \pm 1/\varepsilon$ by looking at the eigenvalues we have $L_\varepsilon \leq 0$ and $L_\varepsilon \approx 0$, but for $|k \pm 1/\varepsilon| > \delta/\varepsilon$ we have $L_\varepsilon \leq -C\varepsilon^{-2}$. This also carries over to the semigroups, so if $P_1$ projects to the $\delta/\varepsilon$-neighbourhoods around $k = \pm 1/\varepsilon$ in Fourier space, then we have

$$P_1 e^{TL_\varepsilon} = \mathcal{O}(1) \quad \text{and} \quad (I - P_1) e^{TL_\varepsilon} = \mathcal{O}(e^{-cT/\varepsilon^2}). \tag{6}$$

This result is straightforward to verify, as the operators are all diagonal in Fourier space.

Using the mild formulation, we now aim to show that, for $v_1 := P_1 v$,

$$v = v_1 + \mathcal{O}(\varepsilon)$$

Recall that by Lemma 1 we have $W_{L_\varepsilon} = \mathcal{O}(1)$, but we can improve it with the following:

**Lemma 2.** *For the two projections $P_1$ and $I - P_1$ of the stochastic convolution $W_{L_\varepsilon}$, we have*

$$P_1 W_{L_\varepsilon} = \mathcal{O}(1) \quad \textit{and} \quad (I - P_1) W_{L_\varepsilon} = \mathcal{O}(\varepsilon).$$

*Idea of Proof.* In order to prove this Lemma, one can follow the same ideas as in Lemma 1. The key point is that due to (6) the integrand in one case is still order 1, while it is small in the other. □

Assume that $v(0) = \mathcal{O}(1)$. Then up to times where $v = \mathcal{O}(1)$ we directly obtain from (4)

$$v(T) = \mathcal{O}(1) + \int_0^T \mathcal{O}(\varepsilon^{-1}) dS$$

which is not sufficient for times $T$ of order 1. We need to split $v$ in order to obtain a better estimate. First using the bounds on the semigroup from (6) we can show that

$$(I - P_1) e^{T L_\varepsilon} v(0) = \mathcal{O}(e^{-cT/\varepsilon^2}).$$

For the other terms in the mild formulation, we use a similar estimate, together with the results for the stochastic convolution from the previous Lemma 2 in order to obtain that

$$(I - P_1) v(T) = \mathcal{O}(e^{-cT/\varepsilon^2}) + \mathcal{O}(\varepsilon) + \int_0^T e^{-c(T-S)/\varepsilon^2}) \mathcal{O}(\varepsilon^{-1}) dS.$$

Thus up to times where $v = \mathcal{O}(1)$ we have

$$(I - P_1) v(T) = (I - P_1) e^{T L_\varepsilon} v(0) + \mathcal{O}(\varepsilon).$$

After a short logarithmic time $t_\varepsilon > 0$, we have

$$(I - P_1) v(t_\varepsilon \varepsilon^2) = \mathcal{O}(\varepsilon).$$

Moreover, if we assume that $P_1 v(0) = \mathcal{O}(1)$ and $(I - P_1) v(0) = \mathcal{O}(\varepsilon)$, then $(I - P_1) v = \mathcal{O}(\varepsilon)$ as long as $v = \mathcal{O}(1)$.

Let us now turn to a bound on $v_1 = P_1 v$. Here we rely crucially on the fact that $P_1 (P_1 v)^2 = 0$, if $\delta$ is small, so that

$$P_1 (v_1 + \mathcal{O}(\varepsilon))^2 = \mathcal{O}(\varepsilon).$$

If we now assume that $v = v_1 + \mathcal{O}(\varepsilon)$, then the quadratic term in the nonlinearity is always $\mathcal{O}(1)$ and we obtain from (4) that $v_1 = \mathcal{O}(1)$ up to some times of order 1. To be more precise, the dominant estimate is to the type

$$\|v_1(T)\| \le C \|v_1(0)\| + C \int_0^T (\|v_1(S)\| + \|v_1(S)\|^3) dS + \text{ error terms}$$

Thus we find a time of order one, such that $v_1$ remains of order one if $v_1(0)$ is of order one.

We have thus sketched the proof of the following theorem:

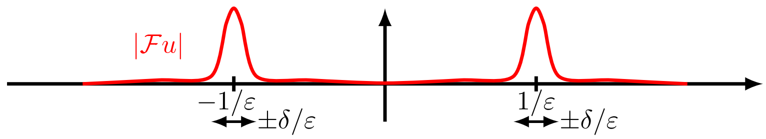


FIGURE 1. Fourier transform of $u(x) = A(x)e^{ix/\varepsilon} + c.c.$ for a not too rough amplitude $A$.

**Theorem 1** (Attractivity)**.** *Consider a solution $v$ of* (3) *with initial conditions of order $\mathcal{O}(1)$, then for a suitable logarithmic time $t_\varepsilon$ the solution is bounded by*

$$v_1(\varepsilon^2 t_\varepsilon) := P_1 v(\varepsilon^2 t_\varepsilon) = \mathcal{O}(1) \text{ and } (1 - P_1)v(\varepsilon^2 t_\varepsilon) = \mathcal{O}(\varepsilon). \tag{7}$$

*Additionally, if we have this bound in (7) for the initial conditions (i.e. $v_1(0) = \mathcal{O}(1)$ and $(1 - P_1)v(0) = \mathcal{O}(\varepsilon)$), then up to some constant time $T_0 > 0$*

$$v_1 = \mathcal{O}(1) \quad \text{and} \quad (1 - P_1)v = \mathcal{O}(\varepsilon). \tag{8}$$

In particular, the Fourier modes around $\pm 1/\varepsilon$ dominate the behaviour close to the bifurcation.

*Remark* 3. Let us remark that with the estimates for the mild solution we cannot rely on any stability of the cubic. From the final result we will see later that $T_0$ might be small if the cubic in the amplitude equation has a positive sign in front of the nonlinearity: in this case the cubic is actually unstable and allows for blow up in finite time (but of order one). On the other hand, if the sign is negative one can show global bounds and thus $T_0$ can be arbitrary.

In the following results, we will assume that a short period of time already has passed, so that the bound (7) is already effective, and we can start a solution in the setting of the second statement of Theorem 1. To me more precise, we will assume in the following estimates that we have a solution $v$ such that (8) holds for some $T_0 > 0$.

*Remark* 4. At the moment each Fourier mode in $v_1$ can have the same order of magnitude, but we can even show that they are given by a modulated wave

$$v_1(T, X) = A(T, X)e^{iX/\epsilon} + \text{c.c.}$$

for an amplitude $A$ having a little bit of regularity. In that case the Fourier transform of $A$ decays for wave-number $|k| \to \infty$, and thus the Fourier modes of $v_1$ are slightly more concentrated in Fourier space around the Fourier modes $\pm 1/\epsilon$. See Figure 1 for a sketch. We will come back to this point in section 6, when we discuss the final approximating equation and identify the terms in it.

**Higher order ansatz:** In order to identify the higher order terms of order $\mathcal{O}(\varepsilon)$, we further split $v$ as follows:

$$v = v_1 + \varepsilon v_0 + \varepsilon v_2 + \varepsilon R, \tag{9}$$

with $v_1 = P_1 v$, as before, concentrated in Fourier space on modes $k$ such that $\left|k \pm \frac{1}{\varepsilon}\right| < \frac{\delta}{\varepsilon}$. For the two new terms we also use smooth Fourier projections $P_0$ and $P_2$ such that $v_0 = \varepsilon^{-1} P_0 v$ is concentrated in Fourier space on modes with $|k| < \frac{2\delta}{\varepsilon}$, and $v_2 = \varepsilon^{-1} P_2 v$ is concentrated on $\left|k \pm \frac{2}{\varepsilon}\right| < \frac{2\delta}{\varepsilon}$. Note that is contrast to $v_1$ we also rescale $v_0$ and $v_2$ by a factor $\varepsilon^{-1}$, so that they are of order 1. Finally, $R$ just collects all the remaining terms.

*Remark* 5. It might seem strange at the first glance that we choose different radii for the regions in Fourier space around $\pm 1$ and for the ones around 0 and $\pm 2$, but the reason we are considering the projections $P_0$ and $P_2$ is to take care of the second order (i.e., the quadratic) terms. But, when we square a term, in Fourier space we also double the size of its support, hence, double the radius.

In other terms, we want $(P_2 + P_0)v_1^2 = v_1^2$, or equivalently $(I - P_2 - P_0)v_1^2 = 0$, so we do not want to cut away some parts of $v_1^2$, which would happen with smaller balls in Fourier space around 0 and 2.

*Remark* 6. Note that the $R$ in the ansatz (9) is simply $R = \varepsilon^{-1}(I - P_1 - P_2 - P_0)v$. Here we cannot show that this term is smaller than $\mathcal{O}(\varepsilon)$, as it contains the term $(I - P_1 - P_2 - P_0)W_{L_\varepsilon}$, which is $\mathcal{O}(\varepsilon)$, from the stochastic convolution and we cannot show that it is smaller.

We will now use also $W_k = P_k W$, for $k = 0, 1, 2$, to shorten the notation a bit.

Let us first check the equation for $v_1$. Simply projecting (4) with $P_1$ we see that $v_1$ is the mild solution of

$$\partial_T v_1 = L_\varepsilon v_1 + \nu\varepsilon^{-1} P_1 v^2 - P_1 v^3 + \partial_T W_1 \ , \tag{10}$$

which we would also obtain by projection (3) directly. Note that we have a bounded linear operator $L_\varepsilon P_1 = P_1 L_\varepsilon = \mathcal{O}(1)$.

Now, by the ansatz (9) we obtain for the cubic

$$P_1 v^3 = P_1(v_1)^3 + \mathcal{O}(\varepsilon) \ ,$$

and for the quadratic term

$$P_1 v^2 = P_1(v_1)^2 + 2\varepsilon P_1(v_1(v_0 + v_2 + R)) + \mathcal{O}(\varepsilon^2) \ .$$

Using the properties of the projectors in Fourier space, we have $P_1(v_1)^2 = 0$ and $P_1(v_1 R) = 0$ so that

$$\varepsilon^{-1} P_1 v^2 = 2P_1(v_1 v_0) + 2P_1(v_1 v_2) + \mathcal{O}(\varepsilon) \ .$$

We can plug this into (10) to finally derive

$$\begin{aligned} \partial_T v_1 = L_\varepsilon v_1 + 2\nu P_1(v_1 v_0) + 2\nu P_1(v_1 v_2) \\ - P_1(v_1)^3 + \mathcal{O}(\varepsilon) + \partial_T W_1 \ . \end{aligned} \tag{11}$$

We would like, however, to have an equation in $v_1$ only, so we need to understand the behaviour of the two mixed products $v_1 v_0$ and $v_1 v_2$. This is the topic for the next section.

## 4. Averaging

Let's go on with the terms $v_0$ and $v_2$ appearing in the ansatz (9) above. The aim of this section is to show that when we consider the two products $v_1 v_k$ for $k = 0, 2$ in (11), their leading order terms are in $v_1$ only. From the rescaled Swift-Hohenberg equation in (3) or (4) we have by projection with $P_0$

$$\partial_T v_0 = L_\varepsilon v_0 + \varepsilon^{-2}\nu P_0 v^2 - \varepsilon^{-1} P_0 v^3 + \varepsilon^{-1}\partial_T W_0$$

with a bounded linear operator $P_0 L_\varepsilon = L_\varepsilon P_0 \approx O(\varepsilon^{-2})$. Recall also that $v_0 = \varepsilon^{-1} P_0 v$, which makes the coefficients different from the equation for $v_1$.

As before we expand the nonlinear terms using (9) together with the properties of Fourier projections to obtain

$$\partial_T v_0 = L_\varepsilon v_0 + \varepsilon^{-2} P_0 \nu v_1^2 + \mathcal{O}(\varepsilon^{-1}) + \varepsilon^{-1} \partial_T W_0$$

Moreover,

$$\partial_T v_2 = L_\varepsilon v_2 + \varepsilon^{-2} P_2 \nu v_1^2 + \mathcal{O}(\varepsilon^{-1}) + \varepsilon^{-1} \partial_T W_2,$$

analogous to the previous one for $v_0$.

Note again that in the two equations above for $v_k$, $k \in \{0, 2\}$, the linear operators are bounded, but also large, as $P_k L_\varepsilon = L_\varepsilon P_k = \mathcal{O}(\epsilon^{-2})$. Nevertheless, for fixed $\varepsilon > 0$ we can consider strong solutions of these equations in order to apply Itô formula.

*Remark* 7. Note that in the mild formulation of the two equations above for both $v_0$ and $v_2$, we have for the stochastic convolution

$$\varepsilon^{-1} \int_0^T e^{(T-S)P_k L_\varepsilon} dW_k(S) = \mathcal{O}(1),$$

so one could conjecture that the noise has an $\mathcal{O}(1)$ contribution to $v_0$ and $v_2$. But it is an Ornstein-Uhlenbeck process on the fast-time scale, so we will see below that its contribution in lowest order is actually negligible due to averaging.

We proceed by an explicit averaging result via Itô formula. The two operators $P_k L_\varepsilon$, $k = 0, 2$, are bounded and invertible. Furthermore, we can use Itô formula and note that we get no correction terms in it, since the noise terms are independent. We thus obtain

$$\begin{aligned} \mathrm{d}[v_1 L_\varepsilon^{-1} v_k] =& L_\varepsilon^{-1} v_k \mathrm{d}v_1 + v_1 L_\varepsilon^{-1} \mathrm{d}v_k \\ =& L_\varepsilon^{-1} v_k (L_\varepsilon v_1 + \mathcal{O}(1))\mathrm{d}t + L_\varepsilon^{-1} v_k \mathrm{d}W_1 \\ &+ v_1 L_\varepsilon^{-1} [L_\varepsilon v_k + \varepsilon^{-2} P_k \nu v_1^2 + \mathcal{O}(\varepsilon^{-1})]\mathrm{d}t \\ &+ \varepsilon^{-1} v_1 L_\varepsilon^{-1} \mathrm{d}W_k. \end{aligned}$$

Since the operator $L_\varepsilon^{-1} P_k = \mathcal{O}(\varepsilon^2)$, we can identify the leading order terms. Only $v_1 v_k \mathrm{d}t$ and $\nu \varepsilon^{-2} L_\varepsilon^{-1} P_k v_1^2 \mathrm{d}t$ are of order 1. All other terms are small in $\varepsilon$.

So we can rewrite the previous equations to obtain

$$\int_0^T v_1 v_k \mathrm{d}t + \nu \int_0^T v_1 \varepsilon^{-2} L_\varepsilon^{-1} P_k v_1^2 \mathrm{d}t = \mathcal{O}(\varepsilon). \tag{12}$$

We have thus identified for both cases $k = 0$ and $k = 2$ the leading order terms in (11).

Let us briefly remark here that in Section 6, when we identify explicitly the terms in the limiting equation, we will see that $\varepsilon^{-2} L_\varepsilon^{-1} P_k$ can be replaced by suitable constants.

We now look at equation (11) for $v_1$:

$$\begin{aligned} \partial_T v_1 = L_\varepsilon v_1 + 2\nu(P_1(v_1 v_0) + P_1(v_1 v_2)) \\ - P_1(v_1)^3 + \mathcal{O}(\varepsilon) + \partial_T W_1 \ , \end{aligned}$$

in integral from in order to plug in the averaging results from (12) to replace the terms including $v_0$ and $v_2$. We obtain

$$(13) \quad v_1(T) = v_1(0) + \mathcal{O}(\varepsilon) + W_1(T) + \int_0^T \Big[ L_\varepsilon v_1 - 2\nu^2 P_1 v_1 \varepsilon^{-2} L_\varepsilon^{-1}(P_0 + P_2) v_1^2 - P_1(v_1)^3 \Big] dS \;.$$

Neglecting the error term gives the final result

$$(14) \quad \partial_T v_1 = L_\varepsilon v_1 - 2\nu^2 P_1 v_1 \varepsilon^{-2} L_\varepsilon^{-1}(P_0 + P_2) v_1^2 - P_1(v_1)^3 + \partial_T W_1 \;.$$

Let us remark that this approximation still depends on $\varepsilon$, but we will see later in Section 6 that in the setting of modulation equations we can further approximate it by an $\varepsilon$-independent Ginzburg-Landau equation. But for our purpose this approximation is sufficient, as it shows that the noise only appears as an additive forcing in the equation for the dominating modes. We will summarise our results in the next section and draw some conclusions.

## 5. Final Result

As we have now the limiting equation (14) for $v_1$, we can prove the following theorem:

**Theorem 2.** *Consider a solution $v$ of the rescaled Swift-Hohenberg equation* (3) *such that the bound* (8) *of Theorem 1 holds up to some $T_0 > 0$, that is, $v_1 = \mathcal{O}(1)$ and $(I - P_1)v = \mathcal{O}(\varepsilon)$.*

*If $w$ is a solution of* (14)*, then*

$$P_1 v - w = \mathcal{O}(\varepsilon)$$

*in the sense given in* (5)*.*

*Idea of proof.* In the previous section we saw in estimate (13) that $P_1 v$ satisfies equation (14) with an additional small residual.

To remove the residual from (14), we rely on the continuous dependence of the solution on an additive forcing. This is a fairly standard argument, but, once again, quite long and technical if all the details are provided. We do not give it here.

Let us only give the key steps in order to motivate the error bound. Consider (13) for $v_1 = P_1 v$ and the time-integrated version of (14) for $w$. With this we build the following equation for the difference

$$e(T) := v_1(T) - w(T) = \mathcal{O}(\varepsilon) + \int_0^T L_\varepsilon e ds - 2\nu^2 \int_0^T \Big[ P_1 v_1 \varepsilon^{-2} L_\varepsilon^{-1}(P_0 + P_2) v_1^2 - w \varepsilon^{-2} L_\varepsilon^{-1}(P_0 + P_2) w^2 \Big] ds - \int_0^T \Big[ P_1(v_1)^3 - P_1 w^3 \Big] dS$$

.

Now we have a deterministic equation, where all cubic terms can be estimated by a global Lipschitz property, which follows from the fact that $v_1 = \mathcal{O}(1)$ by (8) and that we can similarly show $w = \mathcal{O}(1)$. Thus a lengthy computation involving Gronwall's Lemma shows that $e(T) = \mathcal{O}(\varepsilon)$. □

*Remark* 8 (Global estimates). Let us remark, without proof, that when the nonlinearity in (14) is a stable cubic then we can check that the solution of (14) exists for all times $T_0 > 0$ and is order $\mathcal{O}(1)$. The assumption of Theorem 2 remains true for any $T_0 > 0$, and we obtain that even for large times of order one the Fourier modes around $k = \pm 1$ dominate the solution of (3), and their dynamics is given by (14).

*Remark* 9 (No additional impact of noise). Our main result is now a negative one. We consider Swift-Hohenberg in a scaling where small additive noise has an effect on the dynamics. If we take smaller noises, then we would see no contribution at all in the limiting equation.

But even in our scaling, although there is strong nonlinear interaction of Fourier modes, the impact of the noise is actually quite limited, due to the effect of averaging. The noise only appears as an additive forcing on the dominant modes, which is exactly the noise put into the original equation. There is no further effect.

Let us stress that even additive noise still might have a significant impact on the dynamics. See for example [5] where a degenerate noise was able to change the stability of a trivial solution in a Swift-Hohenberg equation. Nevertheless, we do not expect this here with full additive space-time white noise.

*Remark* 10 (Dominant Pattern). Without analysing the dynamics of (14) in detail, we can already draw the implication that all the essential dynamics of our Swift-Hohenberg equation is given by $v_1$, which is concentrated in Fourier space around $\pm 1$, or $\pm 1/\epsilon$ for the rescaled equation. Thus we expect have solutions given by modulated pattern of an underlying $2\pi$-periodic pattern.

## 6. Identifying the limit

The main result, Theorem 2, already shows that the noise in the abstract modulation equation (14) appears only as an additive forcing. Here we want to present some results on how to identify the terms in the equation (14) in the limit $\varepsilon \to 0$.

We will use the ansatz, suggested by the modulation equation approach,

$$v_1(T, X) = A(T, X)e^{iX/\varepsilon} + \text{c.c.}$$

with some smoothness of $A$. Let us remark, that a more detailed analysis as used in Theorem 1 for the attractivity result should justify that after some time this result is typically true for bounded solutions of (3).

Note that the smoothness of $A$ is an assumption here. In space we cannot assume more than weighted Hölder spaces with exponent strictly less than 1/2. See for example [2] or one on the many other results on the (complex or real) Ginzburg-Landau (also called Allen-Cahn or $\Phi^4_3$-model) in 1D, some of which we have mentioned in Section 2.

The crucial term that needs enough smoothness is the linear operator. If we have that $A \in C^4_\kappa$ is order one, then we can evaluate directly as done by Kirrmann, Mielke, and Schneider in [16]

$$L_\varepsilon v_1(T, X) = 4\partial_X^2 A(T, X)e^{iX/\varepsilon} + \text{c.c.} + \mathcal{O}(\varepsilon).$$

In the theory of deterministic modulation equation there are numerous results, which need less regularity than [16]. See for example Part IV of [24] also for many other examples in this direction. But still they need derivatives and moreover $A$ to be uniformly bounded in space.

This is in the stochastic case, however, too much regularity to ask for, so we need to take a different approach. In the setting of weighted Hölder-regularities, using the mild formulation of equation (14) we can replace the semigroups of the Swift-Hohenberg operator $L_\varepsilon$ acting on $v_1$ by the semigroup generated by $4\partial_X^2$ acting on $A$, which is the mild version of the statement we are looking for. This is rigorously proven in the exchange lemmas in [2].

For the noise, we also have to treat the mild formulation of the modulation equation (14). In there we have the stochastic convolution

$$(W_1)_{L_\varepsilon}(T) = P_1 W_{L_\varepsilon}(T) = P_1 \int_0^T e^{(T-S)L_\varepsilon} dW(S).$$

It was proven in [1] that we have

$$P_1 W_{L_\varepsilon}(T,X) \approx \mathcal{W}_{4\partial_X^2}(T,X) e^{iX/\varepsilon}$$

for a complex-valued standard cylindrical Wiener process $\mathcal{W}$ that consists of a rescaling of the Fourier modes of $W$ acting on the dominant modes around $k = 1$, or $k = 1/\varepsilon$ in the rescaled version. Moreover, one can write $\mathcal{W}$ explicitly in terms of $W$. Finally, $\eta = \partial_T \mathcal{W}$ is complex valued space-time white noise.

Let us now turn to the nonlinear terms. For the simple cubic term we obtain, by expanding the cube,

$$-P_1(v_1)^3 \approx -3A|A|^2 e^{iX/\varepsilon} + \text{c.c.}$$

The previous is actually not an identity, but only an approximation, as

$$(1 - P_1)A|A|^2 e^{iX/\varepsilon} \neq 0.$$

This is, on the other hand, a contribution to the non-dominant modes, which are small by Theorem 2.

For the other cubic terms, let us start by considering the one with the projection $P_0$. In the following we are neglecting error terms given by contributions to the non-dominant Fourier modes. For example $(I - P_0)|A|^2$ is non-zero, but small nonetheless, due to the regularity of $A$. We obtain

$$\begin{aligned}
\varepsilon^{-2} L_\varepsilon^{-1} P_0 v_1^2(T,X) &= 2\varepsilon^{-2} L_\varepsilon^{-1} P_0 |A|^2(T,X) \\
&= -2(1 + (\varepsilon^2 \partial_X^2))^{-2} P_0 |A|^2(T,X) \\
&= -2|A|^2(T,X).
\end{aligned}$$

For the step where we replaced $L_\varepsilon^{-1}$ using the eigenvalues of the operator, we can easily see that

$$(1 + (\varepsilon^2 \partial_X^2)^{-1} P_0 = 1 + \mathcal{O}(\delta).$$

Recall that $(1 + (\varepsilon^2 \partial_X^2))^{-2} 1 = 1$. But, using a little bit of regularity of $A$, we can improve this result to an error term that is small in $\varepsilon$. Thus finally,

$$-\nu^2 P_1 v_1 \varepsilon^{-2} L_\varepsilon^{-1} P_0 v_1^2 = 2\nu^2 A|A|^2(T,X) e^{iX/\varepsilon} + \text{c.c.}$$

Similarly, we have for the cubic term involving $P_2$,

$$\begin{aligned}&\varepsilon^{-2}L_\varepsilon^{-1}P_2 v_1^2(T,X)\\&= -(1+(\varepsilon^2\partial_X^2))^{-2}P_2A^2(T,X)e^{i2X/\varepsilon}+\text{c.c.}\\&= -\frac{1}{9}A^2(T,X)e^{i2X/\varepsilon}+\text{c.c.}\end{aligned}$$

The main difference with respect to the previous term is due to the different constant. This can be seen by the fact that $(1+(\varepsilon^2\partial_X^2))^{-2}e^{i2X/\varepsilon} = \frac{1}{9}$. We finally obtain

$$\begin{aligned}&-2\nu^2P_1v_1\varepsilon^{-2}L_\varepsilon^{-1}(P_0+P_2)v_1^2\\&\qquad\qquad = 2(2+1/9)\nu^2A|A|^2(T,X)e^{iX/\varepsilon}+\text{c.c.}\ .\end{aligned}$$

Collecting all cubic terms together with the result on the semigroups and the stochastic convolution, we finally obtain the mild formulation of the Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A - \Big(3-\frac{38}{9}\nu^2\Big)A|A|^2+\eta.$$

So finally this Ginzburg-Landau equation gives the dynamics of the amplitude of the modulated pattern that dominates the behaviour of the Swift-Hohenberg equation. Thus the properties of the change of dynamics when varying $\nu$ should reflect the dynamics of this Ginzburg-Landau equation.

If there is no quadratic nonlinearity in the Swift-Hohenberg equation (i.e. $\nu=0$), then we have a stable cubic $-3A|A|^2$ in Ginzburg-Landau, and due to the presence of noise $A$ should be centred around zero on the order of the noise strength.

Turning on the quadratic nonlinearity weakens this stability and lets $A$ be bigger, until a critical threshold, where the stability breaks down.

It would be interesting to characterize this bifurcation in terms of random attractors of invariant measures. However there seems to be no invariant measure available yet in the setting we are working in.

For random attractors the situation, even on bounded domains, is not that clear, and it might be that for additive noise the random attractor is anyway always a single stable stationary solution. See for example [13] for a quite general result in the case of SDEs, and [3] for a highly degenerate noise in a Swift-Hohenberg equation on an unbounded domain.

In terms of invariant measures, we suppose that, similarly to [4], one should be able to extend our approximation result to invariant measures, provided the Ginzburg-Landau equation is ergodic in the setting we are interested in. Then the qualitative changes in the dynamics, as described heuristically above, could be seen in qualitative changes of the invariant measure.

## 7. Quintic case

Here we comment briefly on the modifications necessary in the quintic case, stated in (2) and rewritten here for ease of reference:

$$\partial_t u = -(1+\Delta)^2u+\nu_2\varepsilon^{1/2}u^2+\nu_3\varepsilon u^3-u^5+\varepsilon\partial_t\widetilde{W}.$$

Let us begin by saying that we do not discuss the existence of solutions. Similarly to the cubic case (1), this can be done using standard methods, and we assume here that an analogue to Assumption 1 holds also for (2).

The scaling

$$u(t,x) = \varepsilon^{1/2} v(\varepsilon^2 t, \varepsilon x)$$

in (2) yields

$$\partial_T v = L_\varepsilon v + \frac{1}{\varepsilon}\nu_2 v^2 + \nu_3 v^3 - v^5 + \partial_T W.$$

**Attractivity:** The attractivity result is now very similar, as apart from the quintic, we have exactly the same terms in the equation. We only have to note that

$$(v_1 + \mathcal{O}(\varepsilon))^5 = (v_1)^5 + \mathcal{O}(\varepsilon)\ .$$

Thus the quintic (as the cubic) does not change any of the estimates and we can assume that $v_1$ is also dominant. In other words,

$$v = v_1 + \mathcal{O}(\varepsilon).$$

**Equation for $v_1$:** Similar to what we had in (10) for the cubic, $v_1$ solves

$$\partial_T v_1 = L_\varepsilon v_1 + \nu_2 \varepsilon^{-1} P_1 v^2 + \nu_3 P_1 v^3 - P_1 v^5 + \partial_T W_1\ .$$

and thus expanding the powers and using as before that $P_1 v_1^2 = 0$ and $P_1(I - P_2 - P_0) = 0$ yields

$$\partial_T v_1 = L_\varepsilon v_1 + 2\nu_2 P_1 v_1 (v_2 + v_0) + \nu_3 P_1 v_1^3 - P_1 v_1^5 + \mathcal{O}(\varepsilon) + \partial_T W_1\ . \tag{15}$$

**Averaging:** In a similar way as the equation for $v_1$ we derive (using $\varepsilon v_k = P_k v$) from (2) that (with $k = 0$ and $k = 2$)

$$\partial_T v_k = L_\varepsilon v_k + \varepsilon^{-2}\nu_2 P_k v^2 + \varepsilon^{-1}\nu_2 P_k v^3 - \varepsilon^{-1} P_k v^5 + \varepsilon^{-1}\partial_T W_k\ .$$

As we did earlier in the cubic case, we expand the nonlinear terms to obtain

$$\partial_T v_k = L_\varepsilon v_k + \varepsilon^{-2}\nu_2 P_k v_1^2 + \mathcal{O}(\varepsilon^{-1}) + \varepsilon^{-1}\partial_T W_k\ .$$

Now the averaging of the quadratic terms in the quintic case (15) is exactly the same as for the cubic case (1) and we obtain

$$\begin{aligned} \partial_T v_1 = L_\varepsilon v_1 &- 2\nu_2^2 P_1 v_1 \varepsilon^{-2} L_\varepsilon^{-1}(P_0 + P_2) v_1^2 \\ &+ \nu_3 P_1 (v_1)^3 - P_1 v_1^5 + \mathcal{O}(\varepsilon) + \partial_T W_1\ . \end{aligned} \tag{16}$$

**Identifying the limit:** Using the ansatz

$$v_1(T,X) = A(T,X) e^{iX/\varepsilon} + \text{c.c.}$$

we see that we can treat almost all terms in (16) in exactly the same way as in (6). Only the term $P_1 v_1^5$ was not present there. Here we obtain similar to the cubic case

$$P_1 (v_1)^5 \approx 10 A|A|^4 e^{iX/\varepsilon} + \text{c.c.}$$

and the final result is thus

$$\partial_T A = 4\partial_X^2 A + \Big(3\nu_3 + \frac{38}{9}\nu_2^2\Big) A|A|^2 - 10A|A|^4 + \eta.$$

Due to the presence of the quintic nonlinearity, even with noise, we expect the solutions to be confined with high probability in a neighbourhood of 0, but the behaviour is quite different depending on whether the coefficient in front of the cubic is positive or negative.

For a negative coefficient we expect solutions to be on the order of noise-strength close to 0. But, if the coefficient is positive, then in the deterministic case there are many stationary solutions $A$, for example the constants defined by $10|A|^2 \approx (3\nu_3 +$

$\frac{38}{9}\nu_2^2$), and in the stochastic case we expect solutions to be with high probability of order noise-strength centred around these stationary solutions.

## 8. Summary and Conclusions

We study as an example the stochastic Swift-Hohenberg equation with additive Gaussian white noise in an unbounded domain, where the deterministic equation exhibits a supercritical pitchfork bifurcation.

As our main result we studied the reduction of dynamics via modulation equation to a stochastic Ginzburg-Landau equation. Here we identify the order where small noise first impacts the dynamics close to a change of stability.

Due to the quadratic and cubic nonlinearities in the Swift-Hohenberg equation there are many nonlinear interactions of Fourier modes, that lead to a cubic nonlinearity in the limiting Ginzburg-Landau equation. Surprisingly, the stochastic effects mostly cancel out due to averaging effects and only an additive forcing survives in the limiting equation.

This result is the first step towards a better understanding of the behaviour of the stochastic bifurcation for SPDEs with additive translation invariant noise on an unbounded domain. Nevertheless, in the setting we are working in here, a full description in terms of random attractors or invariant measures is not yet available, even for the Ginzburg-Landau equation.

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