ON MINIMAL DECOMPOSITIONS OF LOW RANK SYMMETRIC TENSORS

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This is the final post-print authors' version of the manuscript accepted for publication on Linear Algebra and its Applications
Cite as: Mourrain, B., Oneto, A., On minimal decompositions of low rank symmetric tensors, Linear Algebra and its Applications, Volume 607, 15 December 2020, Pages 347-377
doi.org/10.1016/j.laa.2020.06.029

ABSTRACT. In this paper, we study the set of decompositions of symmetric tensors of low rank of any dimension and of arbitrary order and the Waring loci of points which appear in a decomposition of the tensor. We describe a new and complete stratification of the set of symmetric tensors of rank less than 5 by studying Hilbert function and regularity of ideals of points contained in the apolar ideal. For each stratum, we describe a procedure to compute a minimal apolar set of points, exploiting the algebraic properties of the Waring loci.

1. INTRODUCTION

Tensors are multi-dimensional arrays that can be used to encode large data sets. For applications, it is useful to find convenient ways to represent them and, in the last decades, a lot of research has been focused on *additive decompositions*. For a more extensive survey on the relations between theoretical and applied aspects of tensor decompositions, we refer to the book of J. M. Landsberg [Lan12].

In a space of tensors $V_1 \otimes \ldots \otimes V_d$ (where the V_i's are vector spaces with same ground field), *decomposable* or *rank-one tensors* are the elements of the type $v_1 \otimes \ldots \otimes v_d$, where $v_i \in V_i$. Given a tensor T, a *tensor decomposition* is an expression of T as a linear combination of decomposable tensors. The smallest length of such a decomposition is called *tensor rank* of T. Note that this definition generalizes the classic notion of rank of a matrix. An important family of tensors is the one of *symmetric tensors*, i.e., tensors invariant under the action of the permutation group $\mathfrak{S}_d$ on the space of tensors $V^{\otimes d}$ by permutation of the factors. In this case, we consider additive decompositions as sums of rank-one symmetric tensors. Symmetric tensors can be naturally identified with homogeneous polynomials of degree d in $n+1$ variables and rank-one symmetric tensors are the ones of the type $v^{\otimes d} = v \otimes \ldots \otimes v$, i.e., they correspond to dth powers of linear forms. Hence, in this case, we rephrase the problem on additive decomposition as follows.

Let $S = \mathbb{k}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$ be the standard graded ring of polynomials in $n+1$ variables and with coefficients in a field $\mathbb{k}$. Here, S_d denotes the vector space of degree d homogeneous polynomials, or *forms*.

Definition 1.1. Let $f \in S_d$ be a form of degree d. A **Waring decomposition** of f is an expression as

$$f = \lambda_1 \ell_1^d + \ldots + \lambda_s \ell_s^d, \text{ where the } \ell_i\text{'s are linear forms and } \lambda_i \in \mathbb{k}.$$

The minimal length of such a decomposition is called the **Waring rank**, or **rank**, of f. We denote it $\mathrm{rk}(f)$. We call **minimal Waring decomposition** a Waring decomposition of length equal to the Waring rank.

Question. *Given $f \in S_d$, what is the rank of f? How does a minimal Waring decomposition look like?*

Date: July 16, 2020.

2010 *Mathematics Subject Classification.* Primary 13P05, 14N20 Secondary 13F20, 14N05.

Key words and phrases. Waring rank, symmetric tensors, homogeneous polynomials, ideals of points, Hilbert function, apolarity.

The second author acknowledges a postdoctoral research fellowship at INRIA - Sophia Antipolis Méditerranée (France) in the team AROMATH, during which this project started. The second author also acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the María de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0445) during the final part of the project.

This question has a long history and a very extended literature, especially over complex and real numbers. We refer to [Lan12, CGO14, BCC+18] and their references for a complete exposition of the problem.

Here, we consider the question over the complex numbers. For *general* forms of fixed degree and fixed number of variables, the value of the rank is known due to the result of J. Alexander and A. Hirschowitz [AH95]. In the case of specific polynomials, the question is much more difficult. The case of binary forms (two variables) is very classical and essentially due to J. J. Sylvester [Syl51]. In the case of monomials, E. Carlini, M. V. Catalisano and A. V. Geramita gave an explicit formula just in terms of the exponents of the monomial [CCG12]. In general, several algorithms have been described, but they efficiently work under certain constrains on the given polynomial [BCMT10, BGI11, OO13].

It is well-known that, via apolarity theory, minimal Waring decompositions of a given polynomial correspond to sets of reduced points in projective space *apolar to the polynomial*, i.e., sets of points whose ideal is contained in the so-called *apolar ideal* of the polynomial. This theory is explained in details in the book of A. Iarrobino and V. Kanev [IK06]; see also Section 2.1. Under such a correspondence, the coordinates of the points are (up to a scalar factor) the coefficients of the linear forms that can be used to provide a Waring decomposition of the polynomial. In particular, the minimal cardinality of such a set of points coincides with the Waring rank of the polynomial. Hence, useful information about minimal Waring decompositions of a given form can be deduced by studying algebraic invariants of sets of points apolar to a given polynomial such as their *Hilbert function* and their *regularity*.

Here, we focus on the new concept of *Waring locus* of a form which is defined as the locus of linear forms that may appear in a minimal Waring decomposition (see Section 2.3); this notion has been introduced in [CCO17]. In other words, linear forms in the Waring locus are the ones that can be used to decrease the rank of a given homogeneous polynomial f. For example, if the Waring locus of f is as big as possible, i.e., it is dense in the space of linear forms, then we can pick at *random* a linear form to reduce the rank of f. This is what happen for forms with rank higher than the *generic rank*, i.e., the rank of the general form (see Theorem 2.25). On the other hand, if the Waring locus is (contained in) a proper subvariety of the space of linear forms, we would get conditions on the coefficients of the linear forms we might use.

In Theorem 4.2, we describe a stratification of the set of low rank symmetric tensors with respect to possible configurations of minimal apolar sets of points. In each stratum, our study of Waring loci, allows us to describe a procedure to construct a minimal Waring decomposition: the idea is to iteratively use linear forms picked from the Waring locus to decrease the rank of the given form until we completely find a minimal decomposition. These ideas can be applied to forms of higher rank: however, since the number of configurations to be studied grows very quickly with respect to the rank, we apply them only up to rank 5. In Appendix A, we explain how we implemented this procedure in the algebra software Macaulay2 [GS02].

The problem of finding a minimal Waring decomposition for forms of low rank can be approached with different algorithms (e.g. [BCMT10, OO13]). However, the problem of classification and the analysis of the Waring loci have received much less attention. The related *Varieties of Sums of Powers* which parametrize all minimal apolar sets of points of a homogeneous polynomials has been investigated in [RS00].
Such a classification provides efficient ways to decide which is the rank of a tensor and what are the algebraic properties that are shared by *all* minimal sets of points apolar to the given tensor. This also implies more efficient and optimized techniques to compute decompositions of tensors. We describe a new and complete stratification of the set of tensors of rank less than 5, using algebraic properties of the ideals of points contained in the apolar ideal such as Hilbert function and regularity.

Our investigations lead us to ask the following question.

Question (Question 2.11)**.** *Given a form f, is it true that all minimal apolar sets of points have the same regularity and Hilbert function?*

In all cases we have considered, the question has a positive answer. It would be interesting to know to what extent this is true in general. As far as we know, the only results in this direction regard binary forms (whose minimal apolar sets of points have the same Hilbert function since they are given by principal ideals) and monomials (whose minimal apolar sets of points are complete intersections of the same type [BBT13]).

We want to mention that it is interesting to study also the complement of the Waring locus, called *forbidden locus* in [CCO17]. Linear forms in the forbidden locus might be used to *increase* the rank of a homogeneous polynomial, giving a possible approach to the theoretical question on the existence of forms with rank higher than the generic one. We do not consider this direction here.

Structure of the paper. In Section 2, we introduce the necessary background and the tools we use in our computations. These include apolarity theory (Section 2.1), regularity of ideals of reduced points (e.g., see Theorem 2.14), essential number of variables (Section 2.2) and Waring loci (Section 2.3). In Section 3, we use these tools to study minimal sets of points apolar to polynomial of low rank (e.g., see Proposition 3.5 and Proposition 3.13). In Section 4, we give our main Theorem 4.2 where we describe a iterative procedure to find a minimal set of points apolar to any polynomial of rank at most 5 by using Waring loci. In Appendix A, we describe how we implemented our computations within the algebra software *Macaulay2* [GS02]. The code of the package `ApolarLowRank` can be found as ancillary material accompanying the arXiv and the HAL versions of the paper or on the personal webpage of the second author.

2. Basic definitions and background

We start by recalling some basic definitions and construction.

2.1. Apolarity theory.

One of the most important algebraic tools for studying Waring decompositions of homogeneous polynomials is *apolarity theory*, which relates Waring decompositions of a polynomial f to ideals of reduced points contained in the so-called *apolar ideal* of f. For more details, we refer to [IK06].

Let $T = \mathbb{C}[y_0,\dots,y_n] = \bigoplus_{d\geq 0} T_d$ be a standard graded polynomial ring. We define the **apolar action of T over S** by identifying the polynomials in T with partial differentials over S; namely,

$$\circ : T \times S \longrightarrow S, \quad (G,f) \mapsto G\circ f := G(\partial_0,\dots,\partial_n)\cdot f.$$

Definition 2.1. Let $f\in S_d$. We define the **apolar ideal** of f as $f^\perp := \{G\in T \mid G\circ f = 0\}$. We denote by A_f the quotient ring $T/f^\perp$.

The following lemma is the key of our algebraic approach to Waring decompositions.

Lemma 2.2 (Apolarity lemma, [IK06, Lemma 1.15])**.** *Let $f\in S_d$. Then, the following are equivalent:*

(1) $f = c_1\ell_1^d+\ldots+c_s\ell_s^d$, *for some $c_i\in\mathbb{C}\setminus\{0\}$, $\ell_i\in S_1\setminus\{0\}$;*
(2) $I_{\mathbb{X}}\subset f^\perp$, *where $I_{\mathbb{X}}$ is the defining ideal of s reduced points in $\mathbb{P}^n$.*

In particular, if $\mathbb{X}=\{\xi_1,\dots,\xi_s\}\subset\mathbb{P}^n$, with $\xi_i=(\xi_{i,0}:\ldots:\xi_{i,n})$, then $\ell_i=\ell_{\xi_i}:=\xi_{i,0}x_0+\ldots+\xi_{i,n}x_n\in S_1$.

Definition 2.3. Given $f\in S_d$, a set of points $\mathbb{X}$ such that $I_{\mathbb{X}}\subset f^\perp$ is said to be **apolar** to f. We will call **minimal** a set of points apolar to f of cardinality equal to the Waring rank of f.

Definition 2.4. Given a homogeneous ideal $I\subset S$, the **Hilbert function** in degree i of the quotient ring S/I is the dimension of S_i/I_i as $\mathbb{C}$-vector space, i.e.,

$$h_{S/I}(i) := \dim_{\mathbb{C}}(S/I)_i = \dim_{\mathbb{C}} S_i - \dim_{\mathbb{C}} I_i, \text{ for } i\in\mathbb{N}.$$

We will write the Hilbert function as the sequence of integers $(h_{S/I}(0), h_{S/I}(1), h_{S/I}(2), \cdots)$.

Remark 2.5. An important and useful property of apolar ideals is that, for any $f \in S_d$, the algebra A_f is *artinian Gorenstein* with socle degree d. In particular, the Hilbert function of A_f has finite support, $h_{A_f}(i) = 0$ for $i > d$, and the vector $(h_{A_f}(0), \ldots, h_{A_f}(d))$ is symmetric. Actually, also the viceversa is true, i.e., any artinian Gorenstein algebra is isomorphic to A_f, for some f. This characterization is referred as *Macaulay's duality* [Mac94].

Example 2.6 (Binary forms: *Sylvester algorithm*). We describe here how to compute the Waring rank of a binary form. The idea goes back to J. J. Sylvester [Syl51]. For a modern exposition, we refer to [CS11]. Let $f \in \mathbb{C}[x_0, x_1]$. By Macaulay's duality, we know that $f^\perp$ is artinian Gorenstein and, since we are in codimension 2, it is also a complete intersection, say $f^\perp = (G_1, G_2)$, with $\deg(G_i) = d_i$, $i = 1, 2$, and $d_1 + d_2 = d + 2$. Since ideals of reduced points in $\mathbb{P}^1$ are principal, we look for square-free polynomials in $f^\perp$. In particular, we get the following (we assume $d_1 \leq d_2$):

(1) if G_1 is square-free, then $\mathrm{rk}(f) = d_1$;
(2) otherwise, the general element $H \cdot G_1 + \alpha G_2$, with $H \in T_{d_2 - d_1}$, $\alpha \in \mathbb{C}$, is square-free and $\mathrm{rk}(f) = d_2$.

Remark 2.7. Given a set of reduced points $\mathbb{X}$, we denote the Hilbert function of the quotient ring $S/I_{\mathbb{X}}$ simply by $h_{\mathbb{X}}$. A well-known fact is that this Hilbert function is *strictly increasing* until it reaches the cardinality of the set of points and then it gets constant [IK06, Theorem 1.69].

Given a polynomial $f \in S_d$, the computation of the apolar ideal is a linear algebra exercise. For any $i = 0, \ldots, d$, we construct the **i-th catalecticant matrix** of f as

$$\mathrm{Cat}_i(f) : T_i \longrightarrow S_{d-i}, \quad G \mapsto G \circ f.$$

Then, we have that, $f_i^\perp = \ker \mathrm{Cat}_i(f)$.

Remark 2.8. For any degree d, we consider the standard monomial basis

$$\mathscr{B}_d = \left\{ \mathbf{x}^\alpha := x_0^{\alpha_0} \cdots x_n^{\alpha_n} \mid \alpha \in \mathbb{N}^{n+1}, |\alpha| = \sum_i \alpha_i = d \right\}$$

of S_d, and the dual basis

$$\mathscr{B}_d^\vee = \left\{ \mathbf{y}^{(\alpha)} := \frac{1}{\alpha!}\mathbf{y}^\alpha = \frac{1}{\alpha!} y_0^{\alpha_0} \cdots y_n^{\alpha_n} \mid \alpha \in \mathbb{N}^{n+1}, |\alpha| = \sum_i \alpha_i = d \right\}.$$

Note that $\mathbf{y}^{(\alpha)} \circ \mathbf{x}^\beta = \mathbf{x}^{\beta - \alpha}$. Therefore, with respect to these basis, we have that,

$$\mathrm{Cat}_i(f) = (c_{\alpha+\beta})_{\substack{|\alpha|=i \\ |\beta|=d-i}}, \qquad \text{where } f = \sum_{\alpha \in \mathbb{N}^{n+1},\ |\alpha|=d} c_\alpha \mathbf{x}^\alpha \in S_d.$$

By apolarity lemma, for any set of points $\mathbb{X}$ apolar to f, $h_{\mathbb{X}}(i) \geq h_{A_f}(i) = \operatorname{codim} \ker \mathrm{Cat}_i(f) = \mathrm{rk}\, \mathrm{Cat}_i(f)$. Moreover, for any $i \in \mathbb{N}$, $|\mathbb{X}| \geq h_{\mathbb{X}}(i)$. Therefore, if we denote by $\ell(f) := \max_i\{h_{A_f}(i)\} = \max_i\{\mathrm{rk}\, \mathrm{Cat}_i(f)\}$ the *differential length* of f, we have that $\mathrm{rk}(f) \geq \ell(f)$.

This leads to the following possible algorithm to find the Waring rank of a given polynomial $f \in S_d$:

(1) consider the largest catalecticant $\mathrm{Cat}_m(f)$, for $m = \lfloor \frac{d}{2} \rfloor$ and the ideal I generated by its kernel;
(2) if I does not define a set of reduced points, then we fail;
(3) otherwise, if the zero set of I is a set of reduced points $Z(I) = \{\xi_1, \ldots, \xi_r\}$, then we solve the linear system $f = \sum_{i=1}^r c_i \ell_{\xi_i}^d$ to find suitable c_i's. In this case, this is minimal and unique.

Numerical conditions to ensure that this catalecticant method works have been presented in [IK06, OO13].

In [IK06], A. Iarrobino and V. Kanev analysed the Hilbert function of ideals of sets of reduced points apolar to a given polynomial in order to use apolarity lemma and deduce its rank. We want to continue in this direction and, in the next section, we will classify polynomials with low rank with respect to the algebraic properties of their minimal sets of polynomials.

Definition 2.9 (Regularity). For a family $\mathbb{X} = \{\xi_1, \ldots, \xi_r\}$ of points in $\mathbb{P}^n$, we define the **regularity** of $\mathbb{X}$ as

$$\rho(\mathbb{X}) = \min\{k \in \mathbb{N} \mid \exists U_1, \ldots, U_r \in S_k \text{ s.t. } U_i(\xi_j) = \delta_{i,j}\}, \qquad \text{where } \delta_{i,j} \text{ is the usual Kronecker delta.}$$

Remark 2.10. This regularity is also called the *interpolation degree* of the points $\mathbb{X}$. Let $\mathrm{van}_k(\mathbb{X})$ denotes the Vandermonde matrix of degree k associated to $\mathbb{X}$, i.e., if $\mathbb{X} = \{\xi_1, \ldots, \xi_r\}$, with $\xi_i = (\xi_{i,0} : \ldots : \xi_{i,n}) \in \mathbb{P}^n$,

$$\mathrm{van}_k(\mathbb{X}) = \left(\xi_j^\alpha\right)_{\substack{j=1,\ldots,r\\ |\alpha|=k}},$$

where $\xi_j^\alpha := \xi_{j,0}^{\alpha_0} \cdots \xi_{j,n}^{\alpha_n}$. The regularity $\rho(\mathbb{X})$ is also the minimal k for which, $\mathrm{van}(\mathbb{X})_k$ is of rank $|\mathbb{X}|$.

This regularity coincides with the so-called *regularity index*, i.e., the smallest integer in which the Hilbert function of the ideal of points gets constant. Also, $\rho(\mathbb{X})$ is the *Castelnuovo-Mumford regularity* of $S/I_{\mathbb{X}}$ which is defined as $\min_i\{d_{i,j} - i\}$ where $d_{i,j}$'s are the degrees of generators of the i-th syzygy module in a minimal free resolution of $S/I_{\mathbb{X}}$; see [Eis05, Chapter 4]: $\rho(\mathbb{X}) = \mathrm{reg}(S/I_{\mathbb{X}}) = \mathrm{reg}(I_{\mathbb{X}}) - 1$.

Question 2.11. *Let $\mathbb{X}, \mathbb{X}'$ be minimal set of points apolar to a polynomial $f \in S_d$. Is it true that $\rho(\mathbb{X}) = \rho(\mathbb{X}')$? More generally, is it true that $h_{\mathbb{X}} = h_{\mathbb{X}'}$?*

The latter question has a positive answer for:

(1) binary forms, as described by Sylvester's algorithm;
(2) monomials, since any minimal apolar set of points to a monomial $x_0^{d_0} \cdots x_n^{d_n}$, where the exponents are ordered increasingly, is a complete intersection with n generators of degrees $d_1 + 1, \ldots, d_n + 1$, respectively; see [BBT13],

We now prove that it has an affirmative answer also if the regularity of a minimal set of points is large enough with respect to the degree of the polynomial. In particular, in this case, we have that the catalecticant method works and gives us a minimal apolar set of points.

Lemma 2.12. *Let $f \in S_d$ and let $\mathbb{X}$ be a minimal set of points apolar to f. Assume that $d \geq \rho(\mathbb{X})$. Then,*

$$(I_{\mathbb{X}})_k = f_k^\perp \quad \text{for } 0 \leq k \leq d - \rho(\mathbb{X}).$$

Proof. Let $\mathbb{X} = \{\xi_1, \ldots, \xi_r\}$, where $\xi_i = (\xi_{i,0} : \ldots : \xi_{i,n}) \in \mathbb{P}^n$. We denote by $\ell_{\xi_i} = \xi_0 x_0 + \ldots + \xi_n x_n \in S_1$. By apolarity lemma, we know that $f = \sum_{i=1}^r a_i \ell_{\xi_i}^d$, for some coefficients $a_i \in \mathbb{C}$. Now, for any $\mathbf{y}^{(\alpha)} \in T_k$, we have that

$$\begin{aligned}\mathbf{y}^{(\alpha)} \circ f = \sum_{i=1}^r a_i \mathbf{y}^{(\alpha)} \circ \ell_{\xi_i}^d &= \sum_{i=1}^r \left(a_i \frac{d!}{(d-k)!}\right) \xi_i^\alpha \ell_{\xi_i}^{d-k} = \\ &= \sum_{\substack{\beta \in \mathbb{N}^{n+1}\\ |\beta| = d-k}} \sum_{i=1}^r \left(a_i \frac{d!}{\beta_0! \cdots \beta_n!}\right) \xi_i^{\alpha+\beta} \mathbf{x}^\beta = \sum_{\substack{\beta \in \mathbb{N}^{n+1}\\ |\beta| = d-k}} \sum_{i=1}^r \overline{a}_i \xi_i^{\alpha+\beta} \mathbf{x}^\beta.\end{aligned}$$

Therefore,

$$\mathrm{Cat}_i(f) = \left(\sum_{i=1}^r \overline{a}_i \xi_i^{\alpha+\beta}\right)_{\substack{|\alpha|=k\\ |\beta|=d-k}} = \mathrm{van}_{d-k}(\mathbb{X})^T \cdot D \cdot \mathrm{van}_k(\mathbb{X}),$$

where D is the diagonal matrix $D = \mathrm{diag}(\overline{a}_1, \ldots, \overline{a}_s)$.

Since $d - k \geq \rho$, $\mathrm{van}_{d-k}(\mathbb{X})^T$ is injective. Therefore, we have that the kernel of $\mathrm{Cat}_k(f)$, which is $f_k^\perp$, is equal to the kernel of $\mathrm{van}_k(\mathbb{X})$, which is $(I_\mathbb{X})_k$. □

Remark 2.13. In [IK06, Theorem 5.3(E-ii)], the authors proved a similar statement under a stronger assumption, namely, by assuming that the polynomial admits a *tight* apolar set of points, i.e., a set of points $\mathbb{X}$ apolar to f such that $h_{A_f}(i) = |\mathbb{X}|$, in some degree i.

Theorem 2.14. *Let $f \in S_d$ and let $\mathbb{X}$ be a minimal set of points apolar to f. If $d \geq 2\rho(\mathbb{X}) + 1$, then $I_\mathbb{X} = (f^\perp_{\leq \rho(\mathbb{X})+1})$. Moreover, $\mathbb{X}$ is the unique minimal set of points apolar to f.*

Proof. By Lemma 2.12, for $0 \leq k \leq \rho(\mathbb{X})+1$, we have $(I_\mathbb{X})_k = f_k^\perp$. Since $\rho(\mathbb{X})+1 = \mathrm{reg}(I_\mathbb{X})$ is greater than the degree of a minimal set of generators of $I_\mathbb{X}$, $(f^\perp_{\leq \rho(\mathbb{X})+1}) = I_\mathbb{X}$. □

2.2. **Essential number of variables.** In [Car06], E. Carlini introduced the concept of *essential number of variables* of a polynomial as the smallest number of variables needed to write it.

Definition 2.15. Given a homogeneous polynomial $f \in S$, the **essential number of variables** of f is the smallest number N such that there exists linear forms $\ell_1, \ldots, \ell_N \in S$, such that $f \in \mathbb{C}[\ell_1, \ldots, \ell_N]$. In this case, we call the ℓ_i's the **essential variables** of f. In the literature, a form $f \in \mathbb{C}[x_0, \ldots, x_n]$ with $n+1$ essential variables is also called **concise**.

Lemma 2.16. *Let $f \in S_d$. Then:*

(1) [Car06, Proposition 1] *the number of essential variables is the rank of* $\mathrm{Cat}_1(f)$*, that is* $h_{A_f}(1)$*;*
(2) [Lan12, Exercise 3.2.2.2] *any minimal Waring decomposition of f involves only linear forms in the essential variables.*

For this reason, the first thing we do when we look for a Waring decomposition is to compute the first catalecticant matrix and then we work modulo its kernel.

2.3. **Waring loci and forms of high rank.** In [CCO17], the second author together with E. Carlini and M.V. Catalisano defined the *Waring locus* of a homogeneous polynomial.

Notation 2.17. Given a subset W of elements in a vector space, we denote by $\langle W \rangle$ their linear span. Similarly, if we consider a subset of points in a projective space, it will denote their projective linear span.

Definition 2.18. Let $f \in S_d$. Then, the **Waring locus** of f is the locus of linear forms that can appear in a minimal Waring decomposition of f, i.e.,

$$\mathscr{W}_f := \left\{[\ell] \in \mathbb{P}(S_1) \mid \exists \ell_2, \ldots, \ell_r,\ r = \mathrm{rk}(f),\ \text{s.t. } f \in \left\langle \ell^d, \ell_2^d, \ldots, \ell_r^d \right\rangle \right\};$$

analogously, by apolarity lemma,

$$\mathscr{W}_f := \left\{P \in \mathbb{P}^n \mid \exists P_2, \ldots, P_r,\ r = \mathrm{rk}(f),\ \text{s.t. } I_\mathbb{X} \subset f^\perp,\ \mathbb{X} = \{P, P_2, \ldots, P_r\}\right\}.$$

The complement is called **forbidden locus** of f and denote $\mathscr{F}_f := \mathbb{P}^n \setminus \mathscr{W}_f$.

Remark 2.19. The Waring locus (hence, the forbidden locus) is not necessary open or closed, e.g., in the case of planar cubic cusps it is given by the union of a point and a Zariski open subset of a line; see [CCO17, Theorem 5.1]. We only know that it is constructible since it can be described as a linear projection of (the open part of) the classical *Variety of Sums of Powers* (VSP) defined by K. Ranestad and F.-O. Schreyer [RS00], i.e., $VSP(f, \mathrm{rk}(f)) := \overline{\left\{([\ell_1], \ldots, [\ell_r]) \in Hilb_s(\mathbb{P}(S_1)) \mid f \in \left\langle \ell_1^d, \ldots, \ell_r^d \right\rangle\right\}}$.

One of the motivation to study Waring loci is to look for a *recursive* way to construct Waring decompositions, namely by adding, step-by-step, one power at the time. In [CCO17], Waring loci of quadrics, binary forms, monomials, ternary cubics have been computed.

Example 2.20 (Recursive decomposition of binary forms). In [CCO17, Theorem 3.5], the Waring locus of binary forms has been computed. By Sylvester's algorithm (see Example 2.6), if the rank $r = \mathrm{rk}(f)$ is less than the generic, i.e., $r < \lceil \frac{d+1}{2} \rceil$, or $r = \lceil \frac{d+1}{2} \rceil$ and d is odd, then, we have a unique decomposition and, in particular, the Waring locus is closed and consists of r distinct points. If $r > \lceil \frac{d+1}{2} \rceil$ or $r = \lceil \frac{d+1}{2} \rceil$ and d is even, then, the Waring locus is dense. This means that, in the latter cases, for a general form $\ell \in S_1$, there exists a minimal Waring decomposition of f involving ℓ^d, up to some scalar. Actually, by [CCO17, Proposition 3.8], we know that for a general choice of $\ell_1, \dots, \ell_s \in S_1$, where $s = r - \lceil \frac{d+1}{2} \rceil$, there exist scalars $c_1, \dots, c_s$ such that $f - \sum_{i=1}^{s} c_i \ell_i^d$ has rank $r - s$. At this point, if d is odd, then the decomposition is uniquely determined, and if d is even, then we can chose another linear form at random.

Our first result is a generalization of the fact explained in the latter example in a more general setting.

Definition 2.21. For any projective variety $X \subset \mathbb{P}^N$, we say that X **spans** $\mathbb{P}^N$ if every point of $\mathbb{P}^N$ is in the linear span of points in X. Given a point $P \in \mathbb{P}^N$, the X**-rank** of P is the smallest number of points on X whose linear span contains P. We denote it $\mathrm{rk}_X(P)$. By convention, if P is not in any linear span of points of X, $\mathrm{rk}_X(P) = +\infty$.

Remark 2.22. From this definition, the Waring rank is simply the X-rank inside the space of homogeneous polynomials of $\mathbb{P}(S_d)$ with respect to the *Veronese variety* of d-th powers. Other relevant varieties that have been considered in relation to *tensor decompositions* are *Segre*, *Segre-Veronese varieties* and *Grassmannians*; see [Lan12, CGO14, BCC+18].

Definition 2.23. Given a point $P \in \mathbb{P}^N$, we define the X**-decomposition locus** of P as

$$\mathscr{W}_{X,P} = \{Q \in X \mid \exists Q_2 \in X, \dots, Q_r \in X,\ r = \mathrm{rk}_X(P),\ \ P \in \langle Q, Q_1, \dots, Q_r \rangle\}.$$

The X**-forbidden locus** is $\mathscr{F}_{X,P} = X \setminus \mathscr{W}_{X,P}$.

Remark 2.24. If X is the Veronese variety of d-th powers of linear forms, the X-decomposition locus of a poin $[f] \in \mathbb{P}(S_d)$ corresponds to the image of the Waring locus of f via the d-th Veronese embedding. Analogously for the forbidden locus.

In the following, we prove that the X-decomposition locus of a point with rank higher than the generic is dense in X. The proof follows an idea used in [BHMT18] to study the loci of points with high rank.

Theorem 2.25. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety which spans $\mathbb{P}^N$ and let g be the generic X-rank. Let $P \in \mathbb{P}^N$ with $r = \mathrm{rk}_X(P)$. If $r > g$, then $\mathscr{W}_{X,P}$ is dense in X.*

Proof. We proceed by induction on r. Assume that P has X-rank $r > g+1$. Then it lies on a line $\langle P_1, P_2 \rangle$, where P_1 has X-rank $g+1$ and P_2 has X-rank $r-g-1$. Now, if we assume that the claim holds for P_1, we have that, for a general point $Q \in X$, we have a point $Q' \in \langle P_1, Q \rangle$ of X-rank g. Now, let P' be the point of intersection $\langle P, Q \rangle \cap \langle Q', P_2 \rangle$. Since $P' \in \langle Q', P_2 \rangle$, then $\mathrm{rk}_X(P') \leq r-1$, and, since $P' \in \langle P, Q \rangle$ and $\mathrm{rk}_X(P) = r$, then $\mathrm{rk}_X(P') \geq r-1$. Hence, $P \in \langle P', Q \rangle$ with P' of X-rank $r-1$ and $Q \in X$, so that $Q \in \mathscr{W}_{X,P}$.

Hence, we just need to prove the claim in the case $\mathrm{rk}_X(P) = g+1$. Let σ_g° be the set of points of X-rank equal to g. By definition of the generic rank, we know that σ_g° is a dense subset of $\mathbb{P}^N$. For any $P \in \mathbb{P}^N$ of rank $g+1$, let $C_P = \langle P, X \rangle$ be the union of all lines passing through P and a point on X. As P is of X-rank $g+1$, it is on a line $\langle P', Q \rangle$ with $P' \in \sigma_g^\circ$ of X-rank g and $Q \in X$. Thus $C_P \cap \sigma_g^\circ$ is non-empty. As X and

C_P are irreducible and σ_g° is dense in $\mathbb{P}^N$, the Zariski closure of $C_P \cap \sigma_g^\circ$ is C_P and $C_P \cap \sigma_g^\circ$ is dense in C_P. Therefore, for a generic point $Q \in X$, there is a point $P' \in \sigma_g^\circ$ with X-rank equal to g on the line $\langle P, Q\rangle$. By definition of $\mathscr{W}_{X,P}$, it implies that $Q \in \mathscr{W}_{X,P}$. This concludes the proof. □

Corollary 2.26. *Let g be the generic rank of forms of degree d in $n+1$ variables. Let $f \in S_d$ with $r = \mathrm{rk}(f)$. If $r > g$, then for any general choice of $\ell_1, \ldots, \ell_s \in S_1$, with $s = r - g$, there exists a minimal Waring decomposition involving the ℓ_i's.*

Proof. It directly follows by applying $r - g$ times Theorem 2.25 on Veronese varieties. □

Remark 2.27. As appeared already in other examples, there is a big challenge when we want to use Waring loci to construct minimal Waring decompositions: fixed a linear form ℓ in the Waring locus of f, then, by definition, *there exists* a suitable coefficient c such that $\mathrm{rk}(f + c\ell^d) = \mathrm{rk}(f) - 1$, but computing such scalar c is not trivial. In the case of forms of rank one more than the generic, ℓ can be chosen generically, as well as the scalar c. For higher ranks, Theorem 2.25 tells us that ℓ can be chosen generically, but it is not necessarily true that to impose $\mathrm{rk}(f + c\ell^d) = \mathrm{rk}(f) - 1$ is an open condition on the line $\langle f, \ell^d\rangle$. An example is given by the binary form $f = xy^5$ which has Waring rank equal to 6. In this case the general rank is equal to 4. It is known that binary forms of rank equal to 5 are points of the 3rd secant variety of the rational normal curve of $\mathbb{P}^6$, which is defined by the vanishing of the determinant of the 4×4 3-rd catalecticant matrix; see [CS11]. In particular, this defines a closed condition on the line $\langle f, \ell^6\rangle$, where ℓ is a general linear form. In general, a better knowledge on *high rank loci*, see [BHMT18], is needed in order to solve this challenge.

Remark 2.28. In the proof of Theorem 2.25, the fact that the point has rank *strictly* larger than the generic rank is crucial. Indeed, if we consider forms of smaller rank, anything can happen. For example:

(1) The Waring locus can be *open*. Let f be a general ternary cubic of rank 4, then the apolar ideal is generated by three conics which define a base-point-free linear system $\mathscr{L}$ of conics. Sets of four points apolar to f are complete intersections of two conics in the apolar ideal, i.e., they correspond to lines in $\mathbb{P}(\mathscr{L})$. Conversely a line in $\mathbb{P}(\mathscr{L})$ defines a set of 4 apolar points if the line is a pencil of conics intersecting properly in 4 simple points. Let Δ be the discriminant locus of pencils of conics, which have at least one singular intersection point. It is defined by a single equation called the discriminant of two conics. Imposing the passage through a point P defines a pencil of conics in $\mathbb{P}(\mathscr{L})$. Thus P is in the Waring locus if and only if the corresponding pencil is not a point of the discriminant locus. This is the complement of a closed condition (see for more details [CCO17, Section 3]).

(2) The Waring locus can be *closed* and 0-*dimensional*. If g is the general rank for forms of degree d in $n+1$ variables and $g \cdot (n+1) = \binom{n+d}{n}$, then the general form have finitely many minimal decompositions and the Waring locus is given by the set of points corresponding to these decompositions.

(3) The Waring locus can be *neither closed nor open*. Consider a ternary cuspidal cubic which has rank 4 and, up to a change of variables, it can be written in the form $x_0^3 + x_1^2x_2$. The Waring locus is given by the union of the point $(1:0:0)$ and the pinched line $\mathbb{P}^1_{x_1,x_2} \setminus (0:1)$ (see [CCO17, Section 3]).

In the following lemma, we generalize the latter case to a more general setting.

Lemma 2.29. *Let $f = x_0^d + g(x_1, \ldots, x_n) \in S_d$ of rank $n+2$, with $n \geq 2$, and degree $d \geq 4$. Assume that $g_2^\perp = [(y_0, G_1, \ldots, G_N)]_2$, where the G_i's are quadrics and $N = \binom{n}{2} - 1$. Then, $\mathscr{W}_f = (1:0:\ldots:0) \cup \mathscr{W}_g$. In other words, any minimal Waring decomposition of f is given by x_0^d plus a minimal decomposition of g.*

Proof. By [BBKT15, Lemma 1.12], we know that $f_i^\perp = ((y_0^d)^\perp)_i \cap g_i^\perp$, for $i \leq d-1$. In particular, since $d \geq 4$, we get $f_2^\perp = (y_0y_1, \dots, y_0y_n, G_1, \dots, G_N)$. Hence, we have that

$$h_{A_f}(2) = \binom{n+2}{2} - n - \left(\binom{n}{2} - 1\right) = n+2.$$

Therefore, since any minimal set $\mathbb{X}$ of points apolar to f has rank $n+2$, we obtain $(I_\mathbb{X})_2 = f_2^\perp$. Hence, $\mathbb{X}$ is contained in the variety defined by $(f_2^\perp)$ which is $(1:0:\ldots:0) \cup \mathbb{P}^{n-1}_{x_1,\dots,x_n}$. Hence, any minimal decomposition is of the type $x_0^d + \sum_{i=1}^r \ell_i^d(x_1,\dots,x_n)$, where $r = \mathrm{rk}(g)$. By restricting on $\{x_0 = 0\}$, we get a minimal decomposition of g and the claim follows. □

3. Decompositions of low rank polynomials

In the previous section, we noticed that if the degree of the polynomial is sufficiently large with respect to the regularity of the points of a minimal decomposition, then there is a unique Waring decomposition which can be found directly from the generators of the apolar ideal (Theorem 2.14). Also, we noticed that if the rank is sufficiently large, then we can choose some elements of a minimal Waring decomposition generically and reduce the rank to be equal to the general rank (Theorem 2.25).

In this section, we see how to use these tools to stratify the set of forms of given rank with respect to the possible configurations of minimal apolar sets of points and, for each stratum, how to construct minimal Waring decompositions of polynomials of small rank, for any number of variables and any degree.

Remark 3.1. Any quadric $q(\mathbf{x})$ can be represented by a symmetric matrix Q, i.e., $q(\mathbf{x}) = \mathbf{x}Q\mathbf{x}^T$. Then, it is well known that the Waring rank of q coincides with the rank of Q and a minimal Waring decomposition is obtained by finding a diagonal form of Q. Therefore, we will always assume $d \geq 3$.

We first recall an easy lemma.

Lemma 3.2. *Let $f \in \mathbb{C}[x_0,\dots,x_n]$ be a concise form of degree d. Then, $\mathrm{rk}(f) = n+1$ if and only if $\ell(f) = n+1$ and $f_2^\perp$ defines a set of reduced points. In this case, there is a unique minimal apolar set of points.*

Proof. If f has $n+1$ essential variables and the rank is equal to $n+1$, then, up to a change of coordinate, we can write it as $f = x_0^d + \ldots + x_n^d$. In this case, we know that $h_{A_f} : (1, n+1, n+1, \cdots, n+1, 1)$. Hence, any minimal set of points is such that $(I_\mathbb{X})_2 = (f_2^\perp) = \langle y_iy_j \mid i,j = 0,\dots,n \rangle$. These quadrics define the set of $n+1$ reduced coordinate points; therefore, also uniqueness follows.

Viceversa, if $I_\mathbb{X} = (f_2^\perp)$ is a set of reduced points and $\ell(f) = n+1$, since f is concise, we have that $h_{S/I_\mathbb{X}} : (1, n+1, n+1, n+1, \cdots)$. Therefore, as recalled in Remark 2.7, such an Hilbert function tells us that $|\mathbb{X}| = n+1$ and, by apolarity lemma, we have $\mathrm{rk}(f) = n+1$. □

Now, we can start our analysis of minimal decompositions of low rank polynomials. Since the sets of configurations of one and two points are trivial, we start from the rank 3 case.

3.1. Polynomials of rank 3.

We only have two possible configurations of points:

(3a) three collinear points;
(3b) a non-collinear triple of points.

Proposition 3.3. *Let $f \in S_d$ be a form of rank 3 and $\mathbb{X}$ be a minimal apolar set of points. Then:*

(3a) *f has two essential variables and a minimal decomposition is computed by Sylvester's algorithm; in particular, $f^\perp = (L_1,\dots,L_{n-1}, G_1, G_2)$, where $d_i := \deg(G_i)$ and $d_1 \leq d_2$, and*
   (i) *for $d = 3,4$, $I_\mathbb{X} = (L_1,\dots,L_{n-1}, HG_1 + \alpha G_2)$, for a general choice of $H \in T_{d_2-d_1}$ and $\alpha \in \mathbb{C}$;*

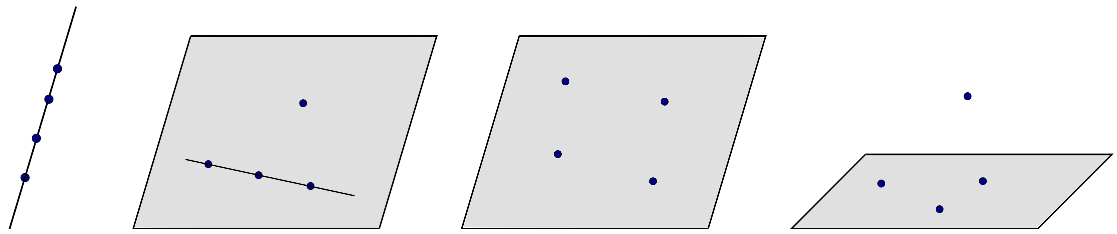
(ii) *if $d \geq 5$, there is a unique minimal apolar set of points given by $I_{\mathbb{X}} = (L_1, \ldots, L_{n-1}, G_1)$.*

(3b) *f has three essential variables and there is a unique minimal apolar set of points given by $I_{\mathbb{X}} = (f_2^{\perp})$.*

Proof. Case (3a) follows from Sylvester algorithm (Example 2.6) and case (3b) from Lemma 3.2. □

Remark 3.4. We observe that for given a polynomial of rank 3, any minimal apolar set of points have the same configuration. In this case, our stratification of the set of rank 3 polynomials is determined by the first catalecticant matrix. It is immediate to see that Question 2.11 has a positive answer for rank 3 polynomials. In particular, if f is of type (3a), then any minimal apolar set of points have Hilbert function $(1, 2, 3, 3, 3, \cdots)$; if f is of type (3b), then any minimal apolar set of points have Hilbert function $(1, 3, 3, 3, 3, \cdots)$.

3.2. **Polynomials of rank 4.** The possible configurations of 4 points in projective spaces are in Figure 1.



(4a) Collinear (4b) Coplanar, with 3 collinear. (4c) General coplanar. (4d) General points.

FIGURE 1. Configurations of 4 points in projective space.

Theorem 3.5. *Let $f \in S_d$ be a form of rank 4 and $\mathbb{X}$ be a minimal apolar set of points. Then:*

(4a) *f has two essential variables and $d \geq 4$. A minimal decomposition is computed by Sylvester's algorithm; in particular, $f^{\perp} = (L_1, \ldots, L_{n-1}, G_1, G_2)$, where $d_i := \deg(G_i)$ and $d_1 \leq d_2$, and*

(i) *for $d = 4, 5, 6$, then $d_2 = 4$, and $I_{\mathbb{X}} = (L_1, \ldots, L_{n-1}, HG_1 + \alpha G_2)$, for general $H \in T_{6-d}, \alpha \in \mathbb{C}$;*

(ii) *for $d \geq 7$, then $d_1 = 4$ and the unique minimal apolar set of points is given by $I_{\mathbb{X}} = (L_1, \ldots, L_{n-1}, G_1)$.*

(4b) *f has three essential variables and $\mathbb{X}$ is of type (4b), then:*

(i) *for $d = 3$, $(f_2^{\perp})$ defines a 0-dimensional scheme $P + D$, where P is a reduced point and D is connected 0-dimensional scheme of length 2 whose linear span is a line L_D; moreover, any minimal apolar set is of the type $P \cup \mathbb{X}'$, with $\mathbb{X}' \subset L_D$;*

(ii) *for $d = 4$, then $h_f(2) = 4$ and $(f_2^{\perp})$ defines a disjoint union $P \cup L$, where P is a reduced point and L is a line not passing through P; moreover, any minimal apolar set is of the type $P \cup \mathbb{X}'$, where $\mathbb{X}' \subset L$.*

(iii) *for $d \geq 5$, then $h_f(2) = 4$ and $(f_2^{\perp})$ defines a disjoint union $P \cup L$, where P is a reduced point and L is a line not passing through P; moreover, the unique minimal apolar set is defined by $(f_3^{\perp})$.*

(4c) *f has three essential variables and $\mathbb{X}$ is of type (4c), then:*

(i) *for $d = 3$, $Z(f_2^{\perp}) = \emptyset$ and $\mathscr{W}_f$ is dense in the plane of essential variables;*

(ii) *if $d \geq 4$, there is a unique minimal apolar set of points given by $I_{\mathbb{X}} = (f_2^{\perp})$.*

(4d) *f has four essential variables and there is a unique minimal apolar set of points given by $I_{\mathbb{X}} = (f_2^{\perp})$.*

Proof. Case (4a). It follows from Sylvester algorithm, Example 2.6.

Case (4b). We may assume that $f = x_0^d + g(x_1, x_2)$, where g is a binary form of rank 3. Then, since $d \geq 3$, by [BBKT15, Lemma 1.12], we have that $f_2^{\perp} = (y_1, y_2)_2 \cap g_2^{\perp}$. If $d = 3$, the claim follows from [CCO17, Section 3]. If $d = 4$, since g is a binary quartic of rank 3, we have $g^{\perp} = (y_0, G_1, G_2)$, where the

G_i's are cubics, and, therefore, $f_2^\perp = (y_0y_1, y_0y_2)$ and $h_f(2) = 4$. Observe that, any set $\mathbb{X}$ of 4 points, non-collinear in $\mathbb{P}^2$, has $\rho(\mathbb{X}) = 2$. Hence, by Lemma 2.12, we have that $(I_\mathbb{X})_2 = f_2^\perp$ and the claim follows. In the case $d \geq 5$, the claim follows from Theorem 2.14.

Case (4c). If $d = 3$, it follows from [CCO17, Section 3]. If $d = 4$, by Lemma 2.12, for any minimal set of points $\mathbb{X}$ apolar to f, we have that $f_2^\perp = (I_\mathbb{X})_2$. Since by assumption there exists a minimal set of points $\mathbb{X}$ which is a complete intersection of two conics, $(f^\perp)_2$ defines the *unique* minimal apolar set of points of f. If $d \geq 5$, then the claim follows from Theorem 2.14.

Case (4d). It follows from Lemma 3.2. □

Remark 3.6. Also in this case, given a homogeneous polynomials of rank 4, any minimal apolar set of points falls in the same configuration. Hence, it is immediate to see that Question 2.11 has a positive answer for polynomials of rank 4. In particular, any minimal apolar set of points of a given rank 4 polynomial have one of the following Hilbert functions:

$$(4a) : (1, 2, 3, 4, 4, 4, \cdots); \quad (4b), (4c) : (1, 3, 4, 4, 4, 4, \cdots); \quad (4d) : (1, 4, 4, 4, 4, 4, \cdots).$$

Note that, as explained in [Eis05, Section 3B.2], in order to distinguish the configurations (4b) and (4c), we should look at finer numerical invariants related to their resolution such as their *graded Betti numbers*. In particular, we have that the Betti tables in these cases have the following forms:

$$(4b) : \begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & 2 & 1 \\ \cdot & 1 & 1 \end{array} \qquad (4c) : \begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 1 \end{array}$$

However, in this case, the stratification is not determined only by the first catalecticant matrix. Indeed, in order to distinguish the cases (4b) and (4c), we need to look at the scheme defined by $f_2^\perp$.

Remark 3.7. As we mentioned, for each stratum of our classification from Theorem 3.5, we want to describe a procedure to determine a minimal Waring decomposition *step-by-step*. The only cases not explained in Theorem 3.5 are (4b-i), (4b-ii) and (4c-i).

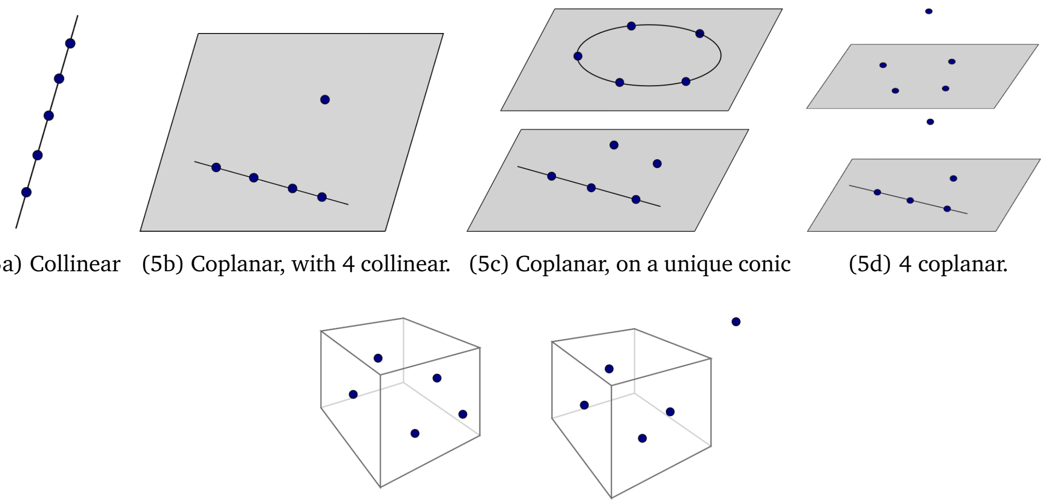
In the cases (4b-i) and (4b-ii), i.e., for $d = 3, 4$, we have seen that the scheme defined by $f_2^\perp$ is the union of a reduced point P and a non-reduced part (for $d = 3$) or a line (for $d = 4$). In both cases, we consider the linear form ℓ_P whose coefficients are the coordinates of P and we find the suitable coefficient c such that $f_c = f - c\ell_P^d$ has two essential variables, i.e., such that the first catalecticant matrix of f_c has rank 2. Then, we find a minimal decomposition of f_c by using Sylvester's algorithm.

In the case (4c-i), for $d = 3$, we can chose a random point P in $\mathbb{P}^2$. Then, we consider the linear form ℓ_P and we find a suitable coefficient c such that $f - c\ell_P^3$ has rank 3. In order to do so, we need to use the equations for the (the closure of) the space of rank 3 polynomials. Assuming that we reduced the problem to the three essential variables, that is a hypersurface given by the so-called *Aronhold invariant* [DK93, p. 250], which has a nice construction as a Pfaffian; see [Ott09]. We refer to [LO13] for an exposition of known equations of secant varieties of Veronese varieties.

3.3. **Polynomials of rank** 5. Possible configurations of 5 points in projective space are in Figure 2.

Since we have several cases to consider, we start with some preliminary lemma. In the first one, we study the case (5b) in more generality, by considering a set of r points $\mathbb{P}^2$ with $r - 1$ collinear points.

Remark 3.8. We want to recall that if $f = x_0^d + g(x_1, \ldots, x_n)$, the fact that $\mathscr{W}_f = (1 : 0 : \ldots : 0) \cup \mathscr{W}_g$, where $\mathscr{W}_g$ is a subspace of in the linear space $\{y_0 = 0\}$, is equivalent to say that any minimal Waring decomposition of f is given by x_0^d plus a minimal Waring decomposition of g. One implication is trivial, the other is proved in [CCO17, Lemma 4.3].



(5a) Collinear (5b) Coplanar, with 4 collinear. (5c) Coplanar, on a unique conic (5d) 4 coplanar.

(5e) 5 general in $\mathbb{P}^3$. (5f) 5 general in $\mathbb{P}^4$.

FIGURE 2. Configurations of 5 points in projective space.

Lemma 3.9. *Let $f = x_0^d + g(x_1,x_2) \in S_d$ of rank ≥ 4. Then, $\mathrm{rk}(f) = \mathrm{rk}(g)+1$ and $\mathscr{W}_f = (1:0:0) \cup \mathscr{W}_g$. In other words, any minimal Waring decomposition of f is given by the sum of x_0^d and a minimal Waring decomposition of g, i.e., it is $x_0^d + \sum_{i=1}^r \ell_i^d(x_1,x_2)$, where $r = \mathrm{rk}(g)$.*

Proof. The fact that $\mathrm{rk}(f) = \mathrm{rk}(g)+1$ is [CCC15, Proposition 3.1].

Since binary forms of degree d have rank at most d, we have that $r = \mathrm{rk}(g) \leq d$. Now, by [BBKT15, Lemma 1.12], we have $f_i^\perp = (y_1,y_2)_i \cap g_i^\perp$, for $i \leq d-1$. If $d = r$, we have that g is a binary form of maximal rank r which, up to a change of variables, can be written in the form $x_1x_2^{r-1}$. Hence, the claim follows from [CCO17, Theorem 5.1]. If $d \geq r+1$, then $g^\perp = (y_0, G_1, G_2)$ where G_1 has degree r and G_2 has degree $d+2-r$. Since $r \geq 3$ and $d \geq r+1$, we have that the G_i's have degree at least 3. In particular, $f_2^\perp = (y_0y_1, y_0y_2)$. Now, observe that any set of $r+1$ points in $\mathbb{P}^2$ with a subset of r collinear points, have $\rho(\mathbb{X}) \leq r-1$. Hence, by Lemma 2.12, for any minimal set of points $\mathbb{X}$ apolar to f, we have $(I_\mathbb{X})_i = f_i^\perp$, for $i \leq d-\rho(\mathbb{X})$. In particular, since $d-\rho(\mathbb{X}) \geq d-r+1 \geq 2$, we have $(I_\mathbb{X})_2 = f_2^\perp = (y_0y_1, y_0y_2)$. □

In the next lemma, we consider the case (5c) in the case of ternary quartics.

Lemma 3.10. *Assume that f is a quartic with three essential variables with $\mathrm{rk}(f) = 5$ and such that the apolar ideal of f has a unique generator g of degree 2 (up to a scalar), i.e., $f_2^\perp = \langle g \rangle$. Let $C = Z(g)$ be the planar conic defined by the vanishing of g. Then:*

1. *if C is irreducible, then $\mathscr{W}_f$ is dense in the conic C;*
2. *if C is reducible, say $C = L_1 \cup L_2$ with $Q = L_1 \cap L_2 = Z(\ell_1) \cap Z(\ell_2)$, then:*
   (a) *if Q is not a forbidden point for f, then $\mathscr{W}_f$ is dense in C;*
   (b) *if Q is forbidden point for f, then, for either $i = 1$ or $i = 2$, $\mathscr{W}_f \cap Z(\ell_i)$ is dense in $Z(\ell_i)$.*

Proof. For any minimal set of points $\mathbb{X}$ apolar to f, since by assumption we have that $h_{A_f}(2) = 5$, we have that $(I_\mathbb{X})_2 = f_2^\perp = \langle g \rangle$. Hence, we have that $\mathscr{W}_f \subset C$.

If $\mathbb{X} = \{\xi_1,\ldots,\xi_5\}$, let u_i be a polynomial vanishing at all the points of $\mathbb{X}$ except for ξ_i. Then, $\{u_1,\ldots,u_5,g\}$ is a basis for T_2 as a vector space, and, with respect to the such basis, we have that

$$\mathrm{Cat}_2(f) = \begin{bmatrix} I_5 & 0 \\ * & 0 \end{bmatrix}.$$

For any point $P \in C$, the catalecticant matrix $\mathrm{Cat}_2(\ell_P^d)$, with respect to the same basis as above, has the last column equal to 0. We write

$$\mathrm{Cat}_2(\ell_P^4) = \begin{bmatrix} M_f & 0 \\ * & 0 \end{bmatrix}.$$

Now, since M_f is a rank 1 symmetric matrix, there exists a (unique) non-zero eigenvalue λ. In particular, for such a λ, we have that $\mathrm{Cat}_2(f + \lambda \ell_P^d)$ has rank 4. Hence, we have a second conic $C' = Z(g')$ such that $(f + \lambda \ell_P^d)_2^\perp = \langle g, g' \rangle$. Moreover, since the coefficients of g' are rational polynomial functions in the coordinates of P, there exists a Zariski open subset $\mathscr{U}$ in C, for which the conics C and C' meet transversally. We need to show when (and where) this open set is non-empty.

(1) *If C is irreducible, then $\mathscr{U}$ is non-empty.* We know that there exists at least one minimal apolar set of points $\mathbb{X}$ for f. This is a set of 5 points lying on the irreducible conic C. In particular, if we assume P to be one of these points, we have that $\mathbb{X} \setminus \{P\}$ is a complete intersection of two conics. In particular, $P \in \mathscr{U}$.

(2-a) *Let $C = L_1 \cup L_2$, $Q = L_1 \cap L_2 = Z(\ell_1) \cap Z(\ell_2)$ and Q is not a forbidden point for f.* By assumption, there exists a minimal apolar set of points $\mathbb{X}$ for f which includes the point Q. By using this set of points, we can write $f = f_1 + f_2 + \ell_Q^4$, where $f_1' = f_1 + \ell_Q^4$ is a rank 3 quartic in the two essential variables of the line $Z(\ell_1)$. By [CCO17, Theorem 3.5], we know that $\mathscr{W}_{f_1'}$ is dense in $Z(\ell_1)$. Since $\mathscr{W}_{f_1'} \subset \mathscr{W}_f$, we conclude that $\mathscr{W}_f$ is dense in $Z(\ell_1)$. By proceeding in the same way with $f_2' = f_2 + \ell_Q^4$, we conclude.

(2-b) *Let $C = L_1 \cup L_2$, $Q = L_1 \cap L_2 = Z(\ell_1) \cap Z(\ell_2)$ and Q is a forbidden point for f.* By assumption, there exists a minimal apolar set of points $\mathbb{X}$ which, since C is reducible, splits as the union of three point on a line, say $Z(\ell_1)$, and two points on the other. Following a similar idea as above, we can write $f = f_1 + f_2$ where f_1 is a quartic in the two essential variables of the line $Z(\ell_1)$ of rank 3. By [CCO17, Theorem 3.5], we know that $\mathscr{W}_{f_1}$ is dense in $Z(\ell_1)$ and this concludes the proof. □

Now, we consider the case of cubics and quartics with four essential variables.

Lemma 3.11. *Let f be a cubic of rank 5 with four essential variables $\{x_0, \dots, x_3\}$ and let $\mathbb{X}$ be a minimal set of points apolar to f. Then:*

1. *if all but one point of $\mathbb{X}$ are coplanar and no three of them collinear, then there exists a unique ternary cubic g and $c \neq 0$ such that $f = c\,x_0^3 + g(x_1, x_2, x_3)$, $\mathrm{rk}(g) = \mathrm{rk}(f) - 1$, and any minimal Waring decomposition of f is obtained from a minimal decomposition of g, i.e., $\mathscr{W}_f = (1:0:0:0) \cup \mathscr{W}_g$;*
2. *if all but two points of $\mathbb{X}$ are collinear then $f = x_0^3 + x_1^3 + g(x_2, x_3)$, where g is a binary cubic of rank 3, and any minimal Waring decomposition of f is obtained from a minimal decomposition of g, i.e., $\mathscr{W}_f = (1:0:0:0) \cup (0:1:0:0) \cup \mathscr{W}_g$;*
3. *otherwise, f has a unique decomposition.*

Proof. If $\mathbb{X}$ has not a subset of four coplanar points, point (3) follows from the classical Sylvester's Pentahedral Theorem. We refer to [Dol12, Theorem 9.4.1] for a modern reference.

If all but one point of $\mathbb{X}$ are coplanar points, then we can write $f = x_0^3 + g(x_1, x_2, x_3)$ where g is a ternary cubic with $4 \leq \mathrm{rk}(g) \leq 5$, being 5 the maximal rank. Then, $f_2^\perp = (y_0y_1, y_0y_2, y_0y_3)_2 + (g_2^\perp \cap \mathbb{C}[y_1, y_2, y_3])_2$. As $\mathrm{rk}(g) \geq 4$, we have $Z(g_2^\perp) = \emptyset$ and $Z(f_2^\perp) = \{P\} \cup H$ where: $P = (1:0:0:0) \in \mathbb{X}$ is one of the apolar point and H is the plane defined by the equation $x_0 = 0$ containing the other apolar points. By Sylvester's Pentahedral Theorem ([Dol12, Theorem 9.4.1]) any minimal apolar set of points contains four coplanar points. Then the non-coplanar point and the plane containing the other points are uniquely determined by $f_2^\perp$. Let ℓ be the linear form corresponding to the non-coplanar point P (or the isolated point of the zero locus of $f_2^\perp$). In a suitable basis of T_1, we can write

$$\mathrm{Cat}_1(f - c\ell^3) = \mathrm{Cat}_1(f) - c\mathrm{Cat}_1(\ell^3) = \begin{bmatrix} \bar{c} - c & 0 \\ 0 & \mathrm{Cat}_1(f') \end{bmatrix}.$$

Therefore, there exists a unique value $c = \bar{c}$ for which $\mathrm{Cat}_1(f - c\ell^3)$ has rank 3 and a unique ternary cubic g such that $f = c\,x_0^3 + g(x_1, x_2, x_3)$. Consequently, $\mathscr{W}_f = (1:0:0:0) \cup \mathscr{W}_{f'}$. This proves (1).

If $\mathbb{X}$ has three collinear points, then we can write $f = x_0^3 + x_1^3 + g(x_2, x_3)$ where g is a binary cubic of rank 3. In this case, we have that the zero locus of $f_2^\perp$ consists of two points $(1:0:0:0)$ and $(0:1:0:0)$. Similarly as above, we conclude that the points and the line where the three collinear points lie is uniquely determined by $f_2^\perp$. This proves (2). □

Lemma 3.12. *Let f be a quartic form with four essential variables $\{x_0, \dots, x_3\}$ and of rank* 5. *Let $\mathbb{X}$ be a minimal set of points apolar to f.*

(1) *If $\mathbb{X}$ contains four coplanar points, then we may assume that $f = x_0^4 + g(x_1, x_2, x_3)$, where g is a ternary quartic of rank* 4 *and we have that $\mathscr{W}_f = (1:0:0:0) \cup \mathscr{W}_g$;*
(2) *otherwise, f has a unique decomposition given by $I_{\mathbb{X}} = (f_2^\perp)$.*

Proof. If $\mathbb{X}$ does not contain 4 coplanar points, then $\mathbb{X}$ has regularity $\rho(\mathbb{X}) = 2$ and is defined by 5 quadrics. Then by Lemma 2.12, $(I_{\mathbb{X}})_2 = f_2^\perp$ and f has a unique decomposition. This proves (2).

If $\mathbb{X}$ contains four coplanar points, we may assume $f = x_0^4 + g(x_1, x_2, x_3)$, where g is a ternary quartic of rank 4. Therefore, we know $f_2^\perp = (y_1, y_2, y_3)_2 \cap g_2^\perp = (y_0y_1, y_0y_2, y_0y_3)_2 + (g_2^\perp \cap \mathbb{C}[y_1, y_2, y_3])$. By Theorem 3.5(4b-ii & 4c-ii), $h_g(2) = 4$, $(g_2^\perp \cap \mathbb{C}[y_1, y_2, y_3])$ is generated by two elements G_1, G_2 defining either the 4 points apolar to g or the union of a point P apolar to g and a line L containing the 3 other points apolar to g. This shows that $(1:0:0:0)$ is a point of any minimal set of points apolar to f and that $\mathscr{W}_f = (1:0:0:0) \cup \mathscr{W}_g$, which proves (1). □

Now, we can give the complete description of rank 5 polynomials.

Theorem 3.13. *Let $f \in S_d$ be a form of rank* 5. *Then:*

(5a) *f has two essential variables and then $d \geq 5$. A minimal decomposition is computed by Sylvester's algorithm; in particular, $f^\perp = (L_1, \dots, L_{n-1}, G_1, G_2)$, where $d_i := \deg(G_i)$ and $d_1 \leq d_2$, and*
   (i) *for $d = 5, \dots, 8$, then $d_2 = 5$, and $I_{\mathbb{X}} = (L_1, \dots, L_{n-1}, HG_1 + \alpha G_2)$, for a general $H \in T_{8-d}$, $\alpha \in \mathbb{C}$;*
   (ii) *for $d \geq 9$, then $d_1 = 5$ and the is a unique minimal apolar set of points given by $I_{\mathbb{X}} = (L_1, \dots, L_{n-1}, G_1)$.*
(5b) *f has three essential variables and $\mathbb{X}$ is of type* (5b), *then, $d \geq 4$ and:*
   (i) *for $d = 4, 5, 6$, any minimal apolar set is of the type $P \cup \mathbb{X}'$, where $\mathbb{X}'$ is a set of four collinear points;*
   (ii) *for $d \geq 7$, the unique minimal apolar set of four points is defined by $(f_4^\perp)$.*
(5c) *f has three essential variables and $\mathbb{X}$ is of type* (5c), *then:*
   (i) *for $d = 3$, then the Waring locus is dense in $\mathbb{P}S_1$;*
   (ii) *for $d = 4$, if $f_2^\perp = \langle C \rangle$, then we have:*
      (a) *if C is irreducible, then $\mathscr{W}_f$ is dense in the conic $Z(C)$;*
      (b) *if C is reducible, say $C = \ell_1\ell_2$, let $Q = Z(\ell_1) \cap Z(\ell_2)$; then:*
         (b1) *if Q is not a forbidden point for f, then $\mathscr{W}_f$ is dense in $Z(C)$;*
         (b2) *otherwise, for either $i = 1$ or $i = 2$, $\mathscr{W}_f \cap Z(\ell_i)$ is dense in $Z(\ell_i)$.*
   (iii) *for $d \geq 5$, then we have a unique minimal apolar set of points given by $(f_3^\perp)$.*

(5d) *f has four essential variables and $\mathbb{X}$ is of type* (5d)*, then it can be written as $f = x_0^d + g(x_1, x_2, x_3)$, where g is a ternary form of rank four; then:*
  (i) *for $d = 3$, then $\mathscr{W}_f = (1 : 0 : 0 : 0) \cup \mathscr{W}_g$;*
  (ii) *for $d \geq 4$, there is unique minimal apolar set given by $I_{\mathbb{X}} = (f_2^\perp)$.*

(5e) *f has four essential variables and $\mathbb{X}$ is of type* (5e)*, then:*
  (i) *if $d = 3$, it is Sylvester Pentahedral Theorem and we have a unique decomposition;*
  (ii) *if $d \geq 4$, then there is a unique decomposition given by $I_{\mathbb{X}} = (f_2^\perp)$.*

(5f) *f has five essential variables, there is a unique minimal apolar set given by $I_{\mathbb{X}} = (f_2^\perp)$.*

Proof. Case (5a). It follows from Sylvester algorithm, Example 2.6.

Case (5b). Up to a change of coordinates, we may assume $f = x_0^d + g(x_1, x_2)$, where g is a binary form of rank 4 then, since binary forms have maximal rank equal to the degree, it has to be $d \geq 4$. The claim follows from Lemma 3.9, in the special case $r = 4$, and by Theorem 3.5(4a).

Case (5c). If $d = 3$, it follows from [CCO17, Section 3.4]. If $d = 4$, it follows from Lemma 3.10. Since 5 general points in $\mathbb{P}^2$ have regularity 2, it follows from Theorem 2.14 in the cases with $d \geq 5$.

Case (5d). It follows from Lemma 3.11 and Lemma 3.12.

Case (5e). If $d = 3$, this is the classical Sylvester Penthaedral Theorem [OO13, Theorem 3.9]. Since the regularity of 5 points in $\mathbb{P}^3$ is equal to 2, by Lemma 2.12, if $d \geq 4$, we have that $(I_{\mathbb{X}})_2 = f_2^\perp$. Since five general points in $\mathbb{P}^3$ are generated by quadrics, the claim follows.

Case (5f). It follows from Lemma 3.2. □

Remark 3.14. Similarly as in the previous cases, this result gives us a stratification of polynomials of rank 5. From Figure 2, the families (5c) and (5d) are subdivided in two cases. In (5c), these two subfamilies are not distinguished from the Betti number since they have a set of 5 general points or a set of 5 points with 3, and only 3, of them collinear, have the same resolution. Also, a ternary cubic of rank 5, i.e., $x_0^2x_1 + x_1^2x_2$, up to change of coordinates, has apolar sets of 5 points of both types.

Instead, in (5d), the two subfamilies should be distinguished by looking at Betti numbers. In this case, the rank 5 form can be written as $f = x_0^d + g(x_1, x_2, x_3)$ and every minimal Waring decomposition of f corresponds to a minimal Waring decomposition of g. From Theorem 3.5, we know that we have two distinguished cases depending whether g is a cuspidal form (and all minimal sets of four points apolar to g have three of them collinear) or is a general form (and all minimal sets of four points apolar to g are a complete intersection). In any case, we get a positive answer to Question 2.11 also for rank 5 polynomials.

4. Low rank symmetric tensor decomposition algorithm

In this section, we summarize our results. We describe our stratification of low rank homogeneous polynomials and we give a procedure, which determines the rank of the tensor when it is ≤ 5 and computes its decomposition. As we have already seen, the stratification depends on the Hilbert function of $A_f = T/f^\perp$ given by the (symmetric) sequence $(h_{A_f}(0), h_{A_f}(1), \ldots, h_{A_f}(d))$ and the scheme defined by $f_i^\perp$, for some i.

Notation 4.1. We write $*$ instead of a finite sequence of values of length at least 1 and by $\underline{k}$ instead of a finite sequence of constant terms equal to k of length at least 1.

We consider here symmetric tensors of degree $d \geq 3$, since the decomposition of quadrics can be done by rank decomposition of symmetric matrices. We implement the procedure described in the following theorem in the algebra software *Macaulay2* [GS02]; see Appendix A.

Theorem 4.2 (and low rank decomposition algorithm). *Let f be a symmetric tensor of degree $d \geq 3$. Then, either one the following points is satisfied or* $\mathrm{rk}(f) > 5$*:*

	Hilbert Sequence	Extra Condition	Algorithm to find a minimal apolar set
1.	$(\underline{1})$		$\mathrm{rk}(f)=1$ *and* $(f_1^{\perp})$ *defines the point apolar to* f
2.	$[1,2,*,2,1]$		f *has two essential variables and Sylvester algorithm is applied:* (i) *if* $f^{\perp}_{l(f)}$ *defines a set of* $l(f)$ *reduced points, then* $\mathrm{rk}(f)=l(f)$; (ii) *otherwise,* $\mathrm{rk}(f)=d+2-l(f)$ *and a minimal apolar set is given by the principal ideal generated by a generic form* $g\in f^{\perp}_{d+2-l(f)}$
3.	$[1,3,3,1]$	$Z(f_2^{\perp})=\emptyset$	*a generic pair of conics* q_1,q_2 *of* $f_2^{\perp}$ *defines* 4 *points and* $\mathrm{rk}(f)=4$
4.	$[1,3,3,1]$	$Z(f_2^{\perp})=P\cup D$, P *is simple point* D *connected, 0-dim* $\deg(D)=2$	$\mathrm{rk}(f)=4$ *and* P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f'=f-c\ell_P^3$ *has two essential variables and we apply Sylvester algorithm to* f' *as in* (2)
5.	$[1,3,3,1]$	$Z(f_2^{\perp})=D$ D *connected, 0-dim* $\deg(D)=3$	$\mathrm{rk}(f)=5$ *and, for a generic* P *and a generic* $c\neq 0$ *such that* $f'=f+c\ell_P^3$ *is a ternary cubic of rank* 4 *and we apply* (4) *to* f'
6.	$[1,3,\underline{3},3,1]$	$Z(f_2^{\perp})=\{P_1,P_2,P_3\}$ P_i*'s are simple points*	$\mathrm{rk}(f)=3$ *and the unique minimal apolar set is* $Z(f_2^{\perp})$
7.	$[1,3,*,3,1]$	$Z(f_2^{\perp})=P\cup L$ P *is simple point* L *is line,* $P\notin L$	P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f'=f-c\ell_P^d$ *has two essential variables and we apply Sylvester algorithm to* f' *as in* (2)
8.	$[1,3,\underline{4},3,1]$	$Z(f_2^{\perp})=\{P_1,\dots,P_4\}$ P_i*'s are simple points*	$\mathrm{rk}(f)=4$ *and the unique minimal apolar set is* $Z(f_2^{\perp})$
9.	$[1,3,5,3,1]$	$Z(f_2^{\perp})=C$ C *is irreducible quadric*	*let* P *be a generic point on* C *and* c *be a scalar such that* $f'=f-c\ell_P^4$ *has* $h_{f'}(2)=4$. (i) *if* $Z((f')_2^{\perp})=\{P_1,\dots,P_4\}$ *is a set of* 4 *reduced points, then,* $\mathrm{rk}(f)=5$, *and a minimal set apolar to* f *is* $\{P,P_1,\dots,P_4\}$; (ii) *otherwise,* $\mathrm{rk}(f)>5$
10.	$[1,3,5,3,1]$	$Z(f_2^{\perp})=L_1\cup L_2$ L_i*'are distinct lines*	*let* P_i *be a generic point on* L_i, *for* $i=1,2$, *respectively, and* c_i *be a scalar such that* $f_i=f-c_i\ell_{P_i}^4$ *has* $h_{f_i}(2)=4$, *for* $i=1,2$. (i) *if* $Z((f_i^{\perp})_2)=\{P_1,\dots,P_4\}$, *for either* $i=1$ *or* $i=2$, *then,* $\mathrm{rk}(f)=5$, *and a minimal apolar set of* f *is* $\{P,P_1,\dots,P_4\}$; (ii) *otherwise,* $\mathrm{rk}(f)>5$
11.	$[1,3,5,\underline{5},3,1]$	$Z(f_3^{\perp})=\{P_1,\dots,P_5\}$ P_i*'s are reduced points*	$\mathrm{rk}(f)=5$ *and the unique minimal apolar set is* $Z(f_3^{\perp})$
12.	$[1,4,4,1]$	$Z(f_2^{\perp})=P\cup H$ P *is a reduced point* H *is a plane,* $P\notin H$	P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f'=f-c\ell_P^3$ *has three essential variables and we apply* (3) *or* (4) *to* f'
13.	$[1,4,\underline{5},4,1]$	$Z(f_2^{\perp})=\{P_1,\dots,P_5\}$	$\mathrm{rk}(f)=5$ *and the unique minimal apolar set is* $Z(f_2^{\perp})$

14. $[1,5,\underline{5},5,1]$ $Z(f_2^\perp)=\{P_1,\dots,P_5\}$ $\mathrm{rk}(f)=5$ *and the unique minimal apolar set is* $Z(f_2^\perp)$

Proof. By the analysis of the previous sections, a symmetric tensor of rank ≤ 5 satisfies one of the listed cases. Let us prove conversely that if one of these cases is satisfied then the rank is determined.

Case 1. f has one essential variable and thus $\mathrm{rk}(f)=1$.

Case 2. f has two essential variables and can be decomposed by Sylvester algorithm; see Example 2.6.

Case 3. 4. and 5. We have $\mathrm{rk}(f)\geq h_f(1)=3$. If $\mathrm{rk}(f)=3$, then by Proposition 3.3, $f_2^\perp$ should define 3 reduced points, which is not the case. Hence, since the maximal rank of ternary cubics is 5, we have $4\leq \mathrm{rk}(f)\leq 5$. Considering the classification of ternary cubics with respect to their rank (see e.g. [LT10]), it is possible to check that we have three possibilities for $Z(f_2^\perp)$.

If $Z(f_2^\perp)=\emptyset$, we know that the Waring locus is dense in the projective plane. If $\mathrm{rk}(f)=4$, for a generic point P, there exists P_1,P_2,P_3 such that $\mathbb{X}=\{P,P_1,P_2,P_3\}$ is a minimal set of points apolar to f. Moreover, they are of type (4c). Then, $(I_{\mathbb{X}})_2\subset f_2^\perp$ is spanned by two quadrics $q_1,q_2\in f_2^\perp$. Thus $(I_{\mathbb{X}})_2$ is the linear space of quadrics in $f_2^\perp$ containing P. Conversely, a generic subset of $f_2^\perp$ of dimension 2 is the space of quadrics in $f_2^\perp$ containing a generic point P. It coincides with $(I_{\mathbb{X}})_2$ for a minimal set of points $\mathbb{X}$ apolar to f. This proves the point (3).

If $Z(f_2^\perp)=P\cup D$, where P is a simple point and D is a degree 2 connected 0-dimensional scheme. As f has three essential variables, we can assume that f is a ternary form in the variables x_0,x_1,x_2. By a change of coordinates, $P=(1:0:0)$ and D lies on the line $y_0=0$, e.g., D is defined by the ideal (y_0,y_1^2). Then, $f=x_0^3+f'(x_1,x_2)$. Since binary cubics have rank at most 3, we have $\mathrm{rk}(f)=4$. By Theorem 3.5(4b-i), P is in any minimal set apolar to f. The other three (collinear) points to get a minimal set of points apolar to f are found by applying (2) to the form $f'=f-c\ell_P^3$, where c is a suitable scalar such that f' has two essential variables, i.e., such that $\mathrm{rk}(\mathrm{Cat}_1(f)-c\mathrm{Cat}_1(\ell_P^3))=2$. This proves the point (4).

If $Z(f_2^\perp)=D$, where D is a degree 3 connected 0-dimensional scheme lying on a plane conic. Since the rank 4 cases have $Z(f_2^\perp)$ which is either empty or the union of a simple point and a degree 2 scheme, we have that $\mathrm{rk}(f)=5$. Then, by Theorem 2.25, for any generic point P and any non-zero $c\in\mathbb{C}$, $f'=f+c\ell_P^3$ is of rank 4 and the previous decomposition applies. This proves the point (5).

Cases 6., 8., 11., 13. and 14. They are the cases where the catalecticant method works (see Section 2.1).

Case 7. As f has 3 essential variables, we can assume that it is a ternary form in the variables x_0,x_1,x_2. By a change of coordinates, we can also assume that $f_2^\perp$ defines $P=(1:0:0)$ and the line L of equation $y_0=0$. Then, f can be written (up to scalar) as $f=x_0^d+f'(x_1,x_2)$ and $f_2^\perp\supset\langle y_0y_1,y_0y_2\rangle$.

If $\mathrm{rk}(f)=3$, then $f_2^\perp$ should define the apolar points, which is not the case. Hence, $\mathrm{rk}(f)\geq 4$ and we deduce the result from Lemma 3.9. In particular, P is in any minimal set of points apolar to f. The other (collinear) points to get a minimal set of points apolar to f are found by applying (2) to the form $f'=f-c\ell_P^3$, where c is a scalar such that f' has two essential variables, i.e., such that $\mathrm{rk}(\mathrm{Cat}_1(f)-c\mathrm{Cat}_1(\ell_P^3))=2$.

Cases 9. and 10. We have $\mathrm{rk}(f)\geq h_f(2)=5$. If $\mathrm{rk}(f)=5$, we deduce the decomposition of f by applying Lemma 3.10. In particular, choosing one of the intersections between a generic line and the irreducible conic Q, in the case (9) or the reducible conic L_1L_2, in the case (10), we can find a scalar c such that $f'=f-c\ell_P^4$ has rank 4, i.e., such that $\mathrm{rk}(\mathrm{Cat}_2(f)-c\mathrm{Cat}_2(\ell_P^4))=4$. Then, we apply (7) to f'.

Case 12. As f has 4 essential variables, we can assume it is a quaternary cubic in the variables x_0,x_1,x_2,x_3. By a change of coordinates, we can assume that the zero locus of $f_2^\perp$ is $P=(1:0:0:0)$ and the plane H defined by $y_0=0$. Then, f can be written (up to a scalar) as $f=x_0^d+f'(x_1,x_2,x_3)$ and

$f_2^{\perp} \supset \langle y_0y_1, y_0y_2, y_0y_3\rangle$. Since ternary cubics have rank at most 5, we have $4 = h_f(1) \leq \mathrm{rk}(f) \leq 6$. If $\mathrm{rk}(f) = 4$, by Theorem 3.5(5d), $f_2^{\perp}$ should define 4 reduced points, which is not the case. Thus $\mathrm{rk}(f) \geq 5$.

If $\mathrm{rk}(f) = 5$, by Lemma 3.11, P is a point of any minimal apolar set of points of X, $\mathrm{rk}(f) = \mathrm{rk}(f') + 1$ and the other points form a minimal set of points apolar to f'.

If $\mathrm{rk}(f) = 6$, then $\mathrm{rk}(f') = 5$, P is one of the apolar points to f and the other are the apolar points to f'.

These points can be computed by finding the scalar c such that $f' = f - c\ell_P^d$ has 3 essential variables, i.e., by imposing $\mathrm{rank}\,(\mathrm{Cat}_1(f) - c\,\mathrm{Cat}_1(\ell_P^d)) = 3$, and by applying (3) and (4) to the cubic f'. □

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APPENDIX A. A *Macaulay2* PACKAGE

The procedure explained in the previous section can be implemented by using computational algebra or computer algebra softwares. We chose to use the algebra software *Macaulay2* [GS02]. The package `ApolarLowRank.m2` here described and a file with the test cases are available on the personal webpage of the second author or in the ancillary files of the arXiv and HAL versions of the article. After downloading it, it can be loaded as

```
i1 : loadPackage "ApolarLowRank"
o1 = ApolarLowRank
o1 : Package
```

For more details, we refer to the documentation

```
i2 : viewHelp ApolarLowRank
```

In the following, we explain some of the main functions and we show how it works in a few examples.

1.1. **Essential variables.** As we have explained in Section 2.2, given a homogeneous polynomial $f \in S$, the essential number of variables of f is the smallest number N such that there exists linear forms $\ell_1, \ldots, \ell_N \in S$ such that $f \in \mathbb{C}[\ell_1, \ldots, \ell_N]$. In our packege, we have implemented the functions:

- `essVar`, which returns the number of variables of f and a list of linear forms generating $f_1^\perp$;

```
i3 : S = QQ[x,y,z,t];
i4 : F = (x+y)^5 + (z-t)^5;
i5 : essVar(F)
o5 = (2, {- x + y, z + t})
o5 : Sequence
```

- `simplifyPoly`, which returns a simplified version of the polynomial in a set of essential variables and a ring map describing the linear change of coordinates needed.

```
i6 : simplifyPoly(symbol Y, F)
           5     5
o6 = (Y   - Y , map(S,QQ[Y , Y ],{x + y, - z + t}))
       0     1            0   1
o6 : Sequence
```

Note that as input in the function `simplifyPoly` it is required also a `Symbol` so that the user can chose a name for the indexed variables for the output.

The output of our main functions will be a set of points apolar to a given homogeneous polynomial f: the ideal will be presented in the essential set of variables. For this reason, as input, the main functions require also a `Symbol` so that the user can chose a name for the indexed variables used in the output. The output is a `ApolarScheme` which is new type of `HashTable` that we have introduced within the package. In particular, an `ApolarScheme` has four attributes:

(1) `hPoly`, which is a homogeneous polynomial;

(2) `idX`, which is the ideal defining a 0-dimensional scheme apolar to the polynomial given by `hPoly`;
(3) `Xdeg`, which is an integer giving the degree of the 0-dimensional scheme;
(4) `Xred`, which is a boolean saying if the 0-dimensional scheme is whether reduced or not.

In the output of our main functions, we also provide the ring map corresponding to the change of coordinates needed to pass from the essential variables to the original set of variables.

1.2. **Two essential variables: Sylvester's algorithm.** In the case of two essential variables, Sylvester's algorithm tells us how to find a minimal set of points apolar to a given form; see Example 2.6.

In our package, we implemented the function `sylvesterApolar` that returns a minimal set of points apolar to a given form with two essential variables.

Hence, the function `sylvesterApolar` works as follows.

```
i7 : sylvesterApolar(symbol Y, F)
                              5     5
o7 = (ApolarScheme{hPoly => Y  - Y   }, map(S,QQ[Y , Y ],{x + y, - z + t}))
                              0     1            0   1
                   idX => ideal(Y Y )
                                 0 1
                   Xdeg => 2
                   Xred => true
o7 : Sequence
```

1.3. **Ternary cubics.** The cases of homogeneous polynomials with three essential variables are dealt with the function `planar5Apolar`. Here, we want to explain how we implemented the cases of ternary cubics, i.e., the cases (3), (4) and (5). As we have seen, the distinction between these cases is given by the vanishing locus of $f_2^\perp$.

```
i8 : S = QQ[x,y,z];
i9 : F = random(3,S);              -- case (3)
i10 : G = random(QQ)*x^3 + y*z^2;   -- case (4)
i11 : H = x*y^2 + y*z^2;            -- case (5)
-- Consider the degree 2 part of the apolar ideal
i12 : Fperp2 = ideal(select(first entries gens perpId(F), i->degree(i)=={2}));
o12 : Ideal of S
i13 : Gperp2 = ideal(select(first entries gens perpId(G), i->degree(i)=={2}));
o13 : Ideal of S
i14 : Hperp2 = ideal(select(first entries gens perpId(H), i->degree(i)=={2}));
o14 : Ideal of S
-- Check the properties of the corresponding vanishing locus
i15 : dim Fperp2
o15 = 0
i16 : primaryDecomposition Gperp2
                     2
o16 = {ideal (x, y ), ideal (y, z)}
o16 : List
i17 : primaryDecomposition Hperp2, radical Hperp2
                              2   2
o17 = ({ideal (x*z, x*y - z , x )}, ideal (z, x))
o17 : Sequence
```

First, we consider the case (3). The general ternary cubic has rank 4 and, as explained in [CCO17, Section 3.4], we know that the Waring locus is dense in the whole plane of ternary linear forms. In other words, given a random ternary cubic F and a random ternary linear form L, there exists a coefficient c such that the cubic F' defined as $F - c * L^3$ has rank 3. Then, F' has a unique decomposition which is easy to compute. In order to compute the suitable value of c, we need to intersect the line spanned by F and the third power of L with the (Zariski closure) of the space of ternary cubics of rank 3 (see Remark 3.7). We use the following function to compute the Aronhold invariant of a given cubic with three essential variables as a Pfaffian as explained in [Ott09].

```
aronhold = method();
aronhold (RingElement) := F -> (
    R := ring F;    V := (entries vars R)_0;
    K := matrix{{0,-V_2,V_1},{V_2,0,-V_0},{-V_1,V_0,0}};
    C := diff(basis(1,R), transpose diff(basis(1,R),F));
    KF := diff(K,C);
    Pf := pfaffians(8,KF);
    if Pf != sub(ideal (),R) then return Pf_0 else return 0_R
 )
```

Now, we can find the suitable coefficient to reduce the rank of the general cubic F in $S = \mathtt{QQ}[x, y, z]$.

```
i19 : L = random(1,S)
      2     7    7
o19 = -x + --y + -z
      7    10    9
o19 : S
i20 : R = QQ[c][x,y,z];
i21 : F' = sub(F,R) - c*(sub(L,R))^3;
i22 : Ic = ideal aronhold F'
              1217672402543    305183
o22 = ideal(- -------------c + ------)
                2083725000       125
o22 : Ideal of R
i23 : F' = sub(sub(F',R/Ic),S);
```

Hence, F' has rank 3 and a unique decomposition, which is given by $Z((f')^{\perp}_{2})$. By adding the point corresponding to the form L, we conclude and find the ideal IX of a minimal set of points apolar to F.

```
i24 : IP = ideal(basis(1,S) * gens kernel transpose (coefficients(L))_1)
o24 = ideal (- 49x + 20y, - 49x + 18z)
o24 : Ideal of S
i25 : IX' = ideal(select(first entries gens perpId(F'), i->degree(i)=={2}));
o25 : Ideal of S
i26 : IX = intersect(IX',IP);
o26 : Ideal of S
-- Check if IX given a minimal set of points apolar to F
i27 : dim IX, degree IX, IX == radical IX, isSubset(IX,perpId(F))
o27 = (1, 4, true, true)
o27 : Sequence
```

Now, we consider the case (4).

```
i28 : G = random(QQ)*x^3+y*z^2
     5 3      2
```

```
o28 = -x  + y*z
     8
o28 : S
i29 : R = QQ[c][x,y,z];
i30 : G' = sub(G,R) - c*x^3;
i31 : Ic = trim minors(3,cat(1,G'))
o31 = ideal(8c - 5)
o31 : Ideal of R
i32 : G' = sub(sub(G',R/Ic),S);
i33 : perpId G'
                   2   3
o33 = ideal (x, y , z )
o33 : Ideal of S
-- Use Sylvester's Algorithm to find a minimal set apolar to G'
i34 : IX' = ideal(x,z^3 - (random(QQ)*y+random(QQ)*z)*y^2);
o34 : Ideal of S
-- Adding the point (1:0:0) corresponding to the linear form 'x', we conclude
i35 : IX = intersect(IX',ideal(y,z));
o35 : Ideal of S
-- Check if IX given a minimal set of points apolar to G
i36 : dim IX, degree IX, IX == radical IX, isSubset(IX,perpId(G))
o36 = (1, 4, true, true)
o36 : Sequence
```

As regards the case (5), if we consider a ternary cubic H of maximal rank 5, as explained in [CCO17, Section 3.4], we can use a random linear form L to reduce the rank. Then, we apply the previous cases. We can check that the rank of H-L^3 drops by looking at the degree 2 part of the apolar ideal, as explained.

```
i37 : H = x*y^2 - y*z^2
           2      2
o37 = x*y  - y*z
i38 : H' = H - L^3;
i39 : Hperp2' = ideal(
                    select(first entries gens perpId(H'), i->degree(i)=={2})
                   )
                   2                    2               2   2
o39 = ideal (24y  + 5x*z - 12y*z - 6z , x*y - x*z + z , x  - 2x*z)
o39 : Ideal of S
i40 : dim Hperp2'
o40 = 0
```

1.4. **Rank 5 ternary quartics.** Here, we want to comment the cases (9) and (10) of ternary quartics f such that $h_{A_f}(2) = 5$, i.e., there is a unique conic in the apolar ideal. If the unique apolar conic C is irreducible (case (9)), then we can reduce the rank of f by taking a generic point on the conic; see Lemma 3.10(a). In our implementation, this is done by considering the intersections of a generic line and the conic C. Since this involves solving a quadratic equation which might not have solution of QQ, in this case, the output of our main function minimalApolar5 depends on a parameter satisfying that quadratic equation and, for this reason, the ideal idX has degree 10 instead of 5.

```
i41 : S = QQ[x,y,z];
i42 : F = sum for i to 4 list (random(1,S))^4;
i43 : first minimalApolar5(symbol Y, F)
```

```
                                        4              3                2 2
o43 = ApolarScheme{hPoly =>14000508577Y +118257495840Y Y +394649884800Y Y ...
                                        0              0 1              0 1
                   idX => ideal (42277476088772685406212514432670091701648...
                   Xdeg => 10
                   Xred => true
 o43 : ApolarScheme
 i44 : ring X#idX
            QQ[a][Y , Y , Y ]
                   0   1   2
 o44 = ----------------------------
        2
       a  - 31293683294493204311665}
o44 : QuotientRing
```

When the unique apolar conic is reducible $C = L_1L_2$ (case (10)), we proceed in a similar way as in the previous case. However, it is not enough to consider a generic point on C because, as we said in Lemma 3.10(2), if the intersection point $Q = L_1 \cap L_2$ is forbidden for the form f, then the Waring locus is dense only in one of the two lines. We see that in an example which also shows how we implemented the procedures explained in Theorem 4.2(9-10) to find a minimal set of points apolar to f.

We consider an example where $f_2^{\perp} = (xz)$.

```
 i56 : G = x^2*y^2;
 i57 : L1 = random(QQ)*y + random(QQ)*z;
 i58 : L2 = random(QQ)*y + random(QQ)*z;
 i59 : F = L1^4 + L2^4 + G;
 i60 : perpId F
                          2            2       3         3            2
 o60 = ideal (x*z, 648y z - 1467y*z  + 560z , 46656y  - 198801y*z  + ...
 o60 : Ideal of S
```

Now, we consider a random point on the line $\{x = 0\}$.

```
 i61 : L = random(QQ)*y + random(QQ)*z
        3
 o61 = -y + 10z
        8
 o61 : S
 i62 : R = QQ[c][x,y,z];
 i63 : F1 = sub(F,R) - c*sub(L^4,R);
 i64 : Ic = radical minors(5,cat(2,F1))
 o64 = ideal(- 10368345145825c + 717382656)
 o64 : Ideal of R
 i65 : F1 = sub(sub(F1,R/Ic),S);
 i66 : F1perp2 = ideal select(first entries gens perpId F1,i->degree(i)=={2})
                                   2                                 2
 o66 = ideal (x*z, 84009951144x  - 26206863345y*z + 26903620240z )
 o66 : Ideal of S
 i67 : dim F1perp2, degree F1perp2
 o67 = (1, 4)
 o67 : Sequence
```

```
i68 : netList primaryDecomposition F1perp2
      +-------------------------------------------+
o68 = |ideal (5241372669y - 5380724048z, x)       |
      +-------------------------------------------+
      |        2                  2               |
      |ideal (z , x*z, 9334439016x  - 2911873705y*z)|
      +-------------------------------------------+
```

Since F1 has the vanishing locus of the homogeneous part of degree 2 which is not among the cases listed in Theorem 4.2, we conclude that it has rank 6. Hence, the random point on the line $\{x = 0\}$ is not in the Waring locus of F. Now, we proceed by considering a random point on the line $\{z = 0\}$.

```
i69 : L = random(QQ)*x + random(QQ)*y
      9    1
o69 = -x + -y
      2    3
o69 : S
i70 : R = QQ[c][x,y,z];
i71 : F2 = sub(F,R) - c*sub(L^4,R);
i72 : Ic = radical minors(5,cat(2,F2))
o72 = ideal(- 81c + 2)
o72 : Ideal of R
i73 : F2 = sub(sub(F2,R/Ic),S);
i74 : F2perp2 = ideal select(first entries gens perpId F2,i->degree(i)=={2})
                        2                 2                   2
o74 = ideal (x*z, 32x  + 432x*y + 5832y  - 13203y*z + 5040z )
o74 : Ideal of S
i74 : dim F2perp2, degree F2perp2, F2perp2 == radical F2perp2
o74 = (1, 4, true)
```

Hence, F2 is a quartic of rank 4 whose apolar ideal in degree 2 defines a minimal apolar set of points. Hence, we we can find a minimal set of 5 points apolar to F.

```
i75 : IP = ideal(basis(1,S) * gens kernel (diff(vars S,L)))
o75 = ideal (- 2x + 27y, z)
o75 : Ideal of S
i76 : IX = intersect(IP,F2perp2)
                         2          2       3      3           3
o76 = ideal (x*z, 648y z - 1467y*z  + 560z , 512x  - 1259712y  + ...
o76 : Ideal of S
i77 : dim IX, degree IX, IX == radical IX, isSubset(IX,perpId(F))
o77 = (1, 5, true, true)
o77 : Sequence
```

1.5. **Main function.** All procedures listed in Theorem 4.2 have been collected in the function `minimalApolar5` that produces a minimal set of points apolar to a given polynomial of rank at most 5 in any number of variables and any degree by using the suitable algorithm, as explained in Theorem 4.2.

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