DEM Discussion Papers

Department of Economics and Management

2015/08

Welfare functions and inequality indices in the binomial decomposition of OWA functions

Silvia Bortot, Ricardo Alberto Marques Pereira,

Thuy H. Nguyen

ISSN 2282-2801 DEM discussion papers [online]

UNIVERSITÀ DEGLI STUDI DI TRENTO

DEM DISCUSSION PAPER

ISSN 2282-2801 DEM Discussion Papers [online]
Università degli Studi di Trento

Department of Economics and Management, University of Trento, Italy.

Guidelines for authors

Papers may be written in Italian or in English. Faculty members of the Department must submit to one of the editors in pdf format. Management papers should be submitted to R. Gabriele. Economics Papers should be submitted to L. Andreozzi. External members should indicate an internal faculty member that acts as a referee of the paper.

Typesetting rules:

1. papers must contain a first page with title, authors, abstract, keywords and codes. Page numbering starts from the following page;
2. a template is available upon request from the managing editors.

Welfare functions and inequality indices in the binomial decomposition of OWA functions

Silvia Bortot*, **Ricardo Alberto Marques Pereira, and Thuy H. Nguyen**

Department of Economics and Management, University of Trento

Abstract

In the context of Choquet integration with respect to symmetric capacities, we consider the binomial decomposition of OWA functions in terms of the binomial Gini welfare functions C_j, $j = 1, ..., n$, and the associated binomial Gini inequality indices G_j, $j = 1, ..., n$, which provide two equivalent descriptions of k-additivity. We illustrate the weights of the binomial Gini welfare functions C_j, $j = 1, ..., n$, and the coefficients of the associated binomial Gini inequality indices G_j, $j = 1, ..., n$, which progressively focus on the poorest part of the population. Moreover, we investigate the numerical behavior of the binomial Gini welfare functions and inequality indices in relation to a family of income distributions described by a parameter related with inequality.

Keywords: Generalized Gini welfare functions and inequality indices, symmetric capacities and Choquet integrals, OWA functions, binomial decomposition and k-additivity

JEL Classification: D31, D63, I31.

1 Introduction

The generalized Gini welfare functions introduced by Weymark [53] and the associated inequality indices in Atkinson-Kolm-Sen's (AKS) framework are related by Blackorby and Donaldson's correspondence formula [5, 6], $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, where $A(\boldsymbol{x})$ denotes a generalized Gini welfare function, $G(\boldsymbol{x})$ is the associated absolute inequality index, and $\bar{x}$ is the plain mean of the income distribution $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{D}^n$ of a population of $n \geq 2$ individuals, with $\mathbb{D} = [0, \infty)$.

The generalized Gini welfare functions [53] have the form $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}$ where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ and, as required by the principle of inequality aversion, $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. These welfare functions correspond to the S-concave class of the ordered weighted averaging (OWA) functions introduced by Yager [56], which in turn correspond [22] to the Choquet integrals associated with symmetric capacities.

The use of non-additivity and Choquet integration [16] in Social Welfare and Decision Theory dates back to the seminal work of Schmeidler [48, 49], Ben Porath and Gilboa [4],

*Corresponding author: silvia.bortot@unitn.it

and Gilboa and Schmeidler [25, 26]. In the discrete case, Choquet integration [46, 14, 17, 27, 28, 40] corresponds to a generalization of both weighted averaging (WA) and ordered weighted averaging (OWA), which remain as special cases. For recent reviews of Choquet integration see Grabisch and Labreuche [33, 34, 35], and Grabisch, Kojadinovich, and Meyer [32].

The complex structure of Choquet capacities can be suitably described in the k-additivity framework introduced by Grabisch [29, 30], see also Calvo and De Baets [11], Cao-Van and De Baets [13], and Miranda, Grabisch, and Gil [45]. The 2-additive case, in particular, has been examined by Miranda, Grabisch, and Gil [45], and Mayag, Grabisch, and Labreuche [42, 43]. Due to its low complexity and versatility the 2-additive case is relevant in a variety of modelling contexts.

The characterization of symmetric Choquet integrals (OWA functions) has been studied by Fodor, Marichal and Roubens [22], Calvo and De Baets [11], Cao-Van and De Baets [13], and Miranda, Grabisch and Gil [45]. It is shown, see Gajdos [24], that in the k-additive case the generating function of the OWA weights is polynomial of degree k, where the weights correspond to differences between consecutive generating function values, as illustrated in (25). In the symmetric 2-additive case, in particular, the generating function is quadratic and thus the weights are equidistant, as in the classical Gini welfare function.

In this paper we review the analysis of symmetric capacities in the Möbius representation framework and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [11], see also [10]. The binomial decomposition can be formulated in terms of two equivalent functional bases, the binomial Gini welfare functions and the Atkinson-Kolm-Sen (AKS) associated binomial Gini inequality indices, according to Blackorby and Donaldson's correspondence formula.

The binomial Gini welfare functions, denoted C_j with $j = 1, \ldots, n$, have null weights associated with the $j-1$ richest individuals in the population and therefore they are progressively focused on the poorest part of the population. Correspondingly, the associated binomial Gini inequality indices, denoted G_j with $j = 1, \ldots, n$, have equal weights associated with the $j-1$ richest individuals in the population and therefore they are progressively insensitive to income transfers within the richest part of the population.

The paper is organized as follows. In Section 2 we review the basic notions of welfare function and inequality index for populations of $n \geq 2$ individuals. In Section 3 we present the basic definitions and results on capacities and Choquet integration, with reference to the Möbius representation framework. In Section 4 we consider the context of symmetric capacities and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [11], see also [10].

In Section 5 we illustrate the weights of the binomial Gini welfare functions C_j, $j = 1, ..., n$, and the coefficients of the associated binomial Gini inequality indices G_j, $j = 1, ..., n$, which progressively focus on the poorest part of the population. In Section 6 we investigate the numerical behavior of the binomial Gini welfare functions and inequality indices in relation to a family of income distributions described by a parameter related with inequality. Finally, Section 7 contains some conclusive remarks.

2 Welfare functions and inequality indices

In this section we consider populations of $n \geq 2$ individuals and we briefly review the notions of welfare function and inequality index in the standard framework of averaging functions on the $\mathbb{D}^n$ domain, with $\mathbb{D} = [0, \infty)$. The income distributions in this framework are represented by points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. In any case, most of our results hold analogously over different domains, for instance the reduced domain $[0, 1]$ or even the extended domain $\mathbb{R}$.

We begin by presenting notation and basic definitions regarding averaging functions on the domain $\mathbb{D}^n$, with $n \geq 2$ throughout the text. Comprehensive reviews of averaging functions can be found in Fodor and Roubens [23], Calvo et al. [12], Beliakov et al. [2], and Grabisch et al. [36].

Notation. Points in $\mathbb{D}^n$ are denoted $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. Accordingly, for every $x \in \mathbb{D}$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i = 1, \ldots, n$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in \mathbb{D}^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\bar{x} = (x_1 + \cdots + x_n)/n$.

Definition 1 *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function.*

1. *A is* monotonic *if $\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is* strictly monotonic *if $\boldsymbol{x} > \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*

2. *A is* idempotent *if $A(x \cdot \mathbf{1}) = x$, for all $x \in \mathbb{D}$. On the other hand, A is* nilpotent *if $A(x \cdot \mathbf{1}) = 0$, for all $x \in \mathbb{D}$.*

3. *A is* symmetric *if $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, for any permutation σ on $\{1, \ldots, n\}$ and all $\boldsymbol{x} \in \mathbb{D}^n$.*

4. *A is* invariant for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is* stable for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*

5. *A is* invariant for dilations *if $A(t \cdot \boldsymbol{x}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\mathbf{x} \in \mathbb{D}^n$. On the other hand, A is* stable for dilations *if $A(t \cdot \boldsymbol{x}) = t\, A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*

We introduce the majorization relation on $\mathbb{D}^n$ and we discuss the concept of income transfer following the approach in Marshall and Olkin [41], focusing on the classical results relating majorization, income transfers, and bistochastic transformations, see Marshall and Olkin [41, Ch. 4, Prop. A.1].

Definition 2 *The* majorization relation $\preceq$ *on $\mathbb{D}^n$ is defined as follows: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that*

$$\boldsymbol{x} \preceq \boldsymbol{y} \quad \text{if} \quad \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \ldots, n \tag{1}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. As usual, we write $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and not $\boldsymbol{y} \preceq \boldsymbol{x}$, and we write $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{y} \preceq \boldsymbol{x}$. We say that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are indifferent if $\boldsymbol{x} \sim \boldsymbol{y}$.

Another traditional reading, which reverses that of majorization, refers to the concept of Lorenz dominance: we say that $\boldsymbol{x}$ is Lorenz superior to $\boldsymbol{y}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ is Lorenz indifferent to $\boldsymbol{y}$ if $\boldsymbol{x} \sim \boldsymbol{y}$.

Given an income distribution $\boldsymbol{x} \in \mathbb{D}^n$, with mean income $\bar{x}$, it holds that $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ since $k\,\bar{x} \geq \sum_{i=1}^{k} x_{(i)}$ for $k = 1, \ldots, n$. The majorization is strict, $\bar{x} \cdot \mathbf{1} \prec \boldsymbol{x}$, when $\boldsymbol{x}$ is not a uniform income distribution. In such case, $\bar{x} \cdot \mathbf{1}$ is Lorenz superior to $\boldsymbol{x}$. Moreover, for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$ with mean income $\bar{x}$ it holds that $\boldsymbol{x} \preceq (0, \ldots, 0, n\bar{x})$, which is strict when $\boldsymbol{x} \neq \mathbf{0}$.

The majorization relation is a partial preorder, a necessary condition for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ to be comparable is that $\bar{x} = \bar{y}$, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ by a permutation. In general, $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if there exists a bistochastic matrix $\mathbf{C}$ (non-negative square matrix of order n where each row and column sums to one) such that $\boldsymbol{x} = \mathbf{C}\boldsymbol{y}$. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if the bistochastic matrix $\mathbf{C}$ is not a permutation matrix.

A particular case of bistochastic transformation is the so-called transfer, also called T-transformation.

Definition 3 *Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that $\boldsymbol{x}$ is derived from $\boldsymbol{y}$ by means of a* transfer *if, for some pair $i, j = 1, \ldots, n$ with $y_i \leq y_j$, we have*

$$x_i = (1 - \varepsilon)\, y_i + \varepsilon y_j \qquad x_j = \varepsilon y_i + (1 - \varepsilon)\, y_j \qquad \varepsilon \in [0, 1] \tag{2}$$

and $x_k = y_k$ for $k \neq i, j$. These formulas express an income transfer, from a richer to a poorer individual, of an income amount $\varepsilon(y_j - y_i)$. The transfer obtains $\boldsymbol{x} = \boldsymbol{y}$ if $\varepsilon = 0$, and exchanges the relative positions of donor and recipient in the income distribution if $\varepsilon = 1$, in which case $\boldsymbol{x} \sim \boldsymbol{y}$. In the intermediate cases $\varepsilon \in (0, 1)$ the transfer produces an income distribution $\boldsymbol{x}$ which is Lorenz superior to the original $\boldsymbol{y}$, that is $\boldsymbol{x} \prec \boldsymbol{y}$.

In general, for the majorization relation $\preceq$ and income distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, it holds that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if $\boldsymbol{x}$ can be derived from $\boldsymbol{y}$ by means of a finite sequence of transfers. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if any of the transfers is not a permutation.

Definition 4 *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function. In relation with the majorization relation $\preceq$, the notions of* Schur-convexity *(S-convexity) and* Schur-concavity *(S-concavity) of the function A are defined as follows:*

1. *A is* S-convex *if $\boldsymbol{x} \preceq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \leq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$*

2. *A is* S-concave *if $\boldsymbol{x} \preceq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*

Moreover, the S-convexity *(resp.* S-concavity*) of a function A is said to be* strict *if $\boldsymbol{x} \prec \boldsymbol{y}$ implies $A(\boldsymbol{x}) < A(\boldsymbol{y})$ (resp. $A(\boldsymbol{x}) > A(\boldsymbol{y})$). Notice that S-convexity (S-concavity) implies symmetry, since $\boldsymbol{x} \sim \boldsymbol{x}_\sigma \;\Rightarrow\; A(\boldsymbol{x}) = A(\boldsymbol{x}_\sigma)$.*

Definition 5 *A function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *is an* n-ary averaging function *if it is monotonic and idempotent. An averaging function is said to be* strict *if it is strictly monotonic. Note that monotonicity and idempotency implies that* $\min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq \max(\boldsymbol{x})$, *for all* $\boldsymbol{x} \in \mathbb{D}^n$.

For simplicity, the n-arity is omitted whenever it is clear from the context. Particular cases of averaging functions are weighted averaging (WA) functions, ordered weighted averaging (OWA) functions, and Choquet integrals, which contain the former as special cases.

Definition 6 *Given a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, *with* $\sum_{i=1}^n w_i = 1$, *the* Weighted Averaging (WA) function *associated with* $\boldsymbol{w}$ *is the averaging function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_i. \tag{3}$$

Definition 7 *Given a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, *with* $\sum_{i=1}^n w_i = 1$, *the* Ordered Weighted Averaging (OWA) function *associated with* $\boldsymbol{w}$ *is the averaging function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}. \tag{4}$$

The traditional form of OWA functions as introduced by Yager [56] is as follows, $A(\boldsymbol{x}) = \sum_{i=1}^n \tilde{w}_i \, x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$. In [57, 58] the theory and applications of OWA functions are discussed in detail.

The following are two classical results particulary relevant in our framework. The first result regards a form of dominance relation between weighting structures and OWA functions, see for instance Bortot and Marques Pereira [10].

Proposition 1 *Consider two OWA functions* $A, B : \mathbb{D}^n \longrightarrow \mathbb{D}$ *associated with weighting vectors* $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n) \in [0,1]^n$, *respectively. It holds that* $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{D}^n$ *if and only if*

$$\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i \qquad \text{for} \quad k = 1, \ldots, n \tag{5}$$

where the case $k = n$ *is an equality due to weight normalization.*

The next result regards the relation between the weighting structure and the S-convexity or S-concavity of the OWA function, see for instance Bortot and Marques Pereira [10].

Proposition 2 *Consider an OWA function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *associated with a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$. *The OWA function* A *is S-convex if and only if the weights are non decreasing,* $w_1 \leq \ldots \leq w_n$, *and* A *is strictly S-convex if and only if the weights are increasing,* $w_1 < \ldots < w_n$. *Analogously, the OWA function* A *is S-concave if and only if the weights are non increasing,* $w_1 \geq \ldots \geq w_n$, *and* A *is strictly S-concave if and only if the weights are decreasing,* $w_1 > \ldots > w_n$.

We will now review the basic concepts and definitions regarding welfare functions and inequality indices. Certain properties which are generally considered to be inherent to the concepts of welfare and inequality are now accepted as basic axioms for welfare and inequality measures, see for instance Kolm [38, 39]. The crucial axiom in this field is the *Pigou-Dalton transfer principle*, which states that welfare (inequality) measures should be non-decreasing (non-increasing) under transfers. This axiom translates directly into the properties of S-concavity and S-convexity in the context of symmetric functions on $\mathbb{D}^n$. In fact, a function is S-concave (S-convex) if and only if it is symmetric and non-decreasing (non-increasing) under transfers, see for instance Marshall and Olkin [41].

Definition 8 *An averaging function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *is a* welfare function *if it is continuous, idempotent, and S-concave. The welfare function is said to be* strict *if it is a strict averaging function which is strictly S-concave.*

Due to monotonicity and idempotency, a welfare function is non decreasing over $\mathbb{D}^n$ but increasing along the diagonal $\boldsymbol{x} = x \cdot \mathbf{1} \in \mathbb{D}^n$, with $x \in \mathbb{D}$. Moreover, notice that S-concavity implies symmetry. Due to S-concavity, a welfare function ranks any Lorenz superior income distribution with the same mean as $\boldsymbol{x}$ as no worse than $\boldsymbol{x}$, whereas a strict welfare function ranks it as better.

Given a welfare function A, the *uniform equivalent income* $\tilde{x}$ associated with an income distribution $\boldsymbol{x}$ is defined as the income level which, if equally distributed among the population, would generate the same welfare value, $A(\tilde{x} \cdot \mathbf{1}) = A(\boldsymbol{x})$. The uniform equivalent concept has been originally proposed by Chisini [15] in the general context of averaging functions, see for instance Bennet et al. [3]. In the welfare context the uniform equivalent income has been considered by Atkinson [1], Kolm [37], and Sen [50] and further elaborated by Blackorby and Donaldson [5, 6, 7] and Blackorby, Donaldson, and Auersperg [8].

Due to the idempotency of A, we obtain $\tilde{x} = A(\boldsymbol{x})$. Since $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$, S-concavity implies $A(\bar{x} \cdot \mathbf{1}) \geq A(\boldsymbol{x})$ and therefore $A(\boldsymbol{x}) \leq \bar{x}$ due to the idempotency of the welfare function. In other words, the mean income $\bar{x}$ and the uniform equivalent income $\tilde{x}$ are related by $0 \leq \tilde{x} \leq \bar{x}$.

We now define the notion of absolute inequality index, introduced by Kolm [38, 39] and developed by Blackorby and Donaldson [6], Blackorby, Donaldson, and Auersperg [8], and Weymark [53]. Following Kolm, inequality measures are described as absolute when they are invariant for additive transformations (translation invariance).

Definition 9 *A function* $G : \mathbb{D}^n \longrightarrow \mathbb{D}$ *is an* absolute inequality index *if it is continuous, nilpotent, S-convex, and invariant for translations. The absolute inequality index is said to be* strict *if it is strictly S-convex.*

In relation with the properties of the majorization relation discussed earlier, it holds that: over all income distributions $\boldsymbol{x} \in \mathbb{D}^n$ with the same mean income $\bar{x}$, a welfare function has minimum value $A(0, \ldots, 0, n\bar{x})$, and an absolute inequality index has maximum value $G(0, \ldots, 0, n\bar{x})$.

In the AKS framework introduced by Atkinson [1], Kolm [37], and Sen [50], a welfare function which is stable for translations induces an associated absolute inequality index by means of the correspondence formula $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, see Blackorby and Donaldson

[6]. The welfare function and the associated inequality index are said to be *ethical*, see also Sen [51], Blackorby, Donaldson, and Auersperg [8], Weymark [53], Blackorby and Donaldson [9], and Ebert [20].

Definition 10 *Given a welfare function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ which is stable for translations, the associated* Atkinson-Kolm-Sen (AKS) absolute inequality index *$G : \mathbb{D}^n \longrightarrow \mathbb{D}$ is defined as*

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) \tag{6}$$

The fact that A is stable for translations ensures the translational invariance of G. The absolute inequality index can be written as $G(\boldsymbol{x}) = \bar{x} - \tilde{x}$ and represents the per capita income that could be saved if society distributed incomes equally without any loss of welfare.

In the AKS framework, a welfare function A which is stable for both translations and dilations is associated with both absolute and relative inequality indices G and G_R, respectively, with $G(\boldsymbol{x}) = \bar{x}\, G_R(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$. In what follows we will omit the term "absolute" when referring to G.

A class of welfare functions which plays a central role in this paper is that of the generalized Gini welfare functions introduced by Weymark [53], see also Mehran [44], Donaldson and Weymark [18, 19], Yaari [54, 55], Ebert [21], Quiggin [47], Ben-Porath and Gilboa [4].

Definition 11 *Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, with $w_1 \geq \cdots \geq w_n \geq 0$ and $\sum_{i=1}^n w_i = 1$, the* generalized Gini welfare function *associated with $\boldsymbol{w}$ is the function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^n w_i x_{(i)} \tag{7}$$

and, in the AKS framework, the associated generalized Gini inequality index *is defined as*

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) = -\sum_{i=1}^n (w_i - \frac{1}{n})\, x_{(i)}\,. \tag{8}$$

The generalized Gini welfare functions, which are strict if and only if $w_1 > \ldots > w_n > 0$, are clearly stable for both translations and dilations. For this reason they have a natural role within the AKS framework and Blackorby and Donaldson's correspondence formula.

An important particular case of the AKS generalized Gini framework is the classical Gini welfare function $A_G^c(\boldsymbol{x})$ and the associated classical Gini inequality index $G^c(\boldsymbol{x}) = \bar{x} - A_G^c(\boldsymbol{x})$,

$$A_G^c(\boldsymbol{x}) = \sum_{i=1}^n \frac{2(n-i)+1}{n^2}\, x_{(i)} \tag{9}$$

where the coefficients of $A^c(\boldsymbol{x})$ have unit sum, $\sum_{i=1}^n (2(n-i)+1) = n^2$, and

$$G^c(\boldsymbol{x}) = -\sum_{i=1}^n \frac{n-2i+1}{n^2}\, x_{(i)} \tag{10}$$

where the coefficients of $G^c(\boldsymbol{x})$ have zero sum, $\sum_{i=1}^n (n-2i+1)=0$. The classical Gini inequality index G^c is traditionally defined as

$$G^c(\boldsymbol{x}) = \frac{1}{2n^2}\sum_{i,j=1}^{n} |x_i - x_j| \,. \tag{11}$$

but in our framework it is convenient to express it as in (10), see [10].

3 Capacities and Choquet integrals

In this section we present a brief review of the basic facts on Choquet integration, focusing on the Möbius representation framework. For recent reviews of Choquet integration see [33, 32, 34, 35] for the general case, and [45, 42, 43] for the 2-additive case in particular.

Consider a finite set of interacting individuals $N = \{1, 2, \dots, n\}$. Any subsets S, $T \subseteq N$ with cardinalities $0 \le s,\, t \le n$ are usually called coalitions. The concepts of capacity and Choquet integral in the definitions below are due to [16, 52, 17, 27, 28].

Definition 12 *A* capacity *on the set N is a set function $\mu : 2^N \longrightarrow [0,1]$ satisfying*

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$ (boundary conditions)

(ii) $S \subseteq T \subseteq N \quad \Rightarrow \quad \mu(S) \le \mu(T)$ (monotonicity conditions).

Definition 13 *Let μ be a capacity on N. The* Choquet integral *$\mathcal{C}_\mu : \mathbb{D}^n \longrightarrow \mathbb{D}$ with respect to μ is defined as*

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\mu(A_{(i)}) - \mu(A_{(i+1)})\right] x_{(i)} \qquad \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{D}^n \tag{12}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \le x_{(2)} \le \dots \le x_{(n)}$. Moreover, $A_{(i)} = \{(i), \dots, (n)\}$ and $A_{(n+1)} = \emptyset$.

Definition 14 *Let μ be a capacity on the set N. The* Möbius transform *$m_\mu : 2^N \longrightarrow \mathbb{R}$ associated with the capacity μ is defined as*

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \qquad\qquad T \subseteq N \tag{13}$$

where s and t denote the cardinality of the coalitions S and T, respectively.

Conversely, given the Möbius transform m_μ, the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad\qquad T \subseteq N \,. \tag{14}$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad\qquad \sum_{T \subseteq N} m_\mu(T) = 1 \tag{15}$$

and the monotonicity conditions can be expressed as follows: for each $i = 1, \ldots, n$ and each coalition $T \subseteq N \setminus \{i\}$, the monotonicity condition is written as

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{16}$$

This form of the monotonicity conditions derives from the original monotonicity conditions in Definition 12, expressed as $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for each $i \in N$ and $T \subseteq N \setminus \{i\}$.

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [29] introduced the concept of k-additive capacities.

Definition 15 *A capacity μ on the set N is said to be* k-additive *if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.*

In the k-additive case, with $k = 1, \ldots, n$, the capacity μ is expressed as follows in terms of the Möbius transform m_μ,

$$\mu(T) = \sum_{S \subseteq T,\, s \leq k} m_\mu(S) \qquad T \subseteq N \tag{17}$$

and the boundary and monotonicity conditions (15) and (16) take the form

$$m_\mu(\emptyset) = 0 \qquad \sum_{T \subseteq N,\, t \leq k} m_\mu(T) = 1 \tag{18}$$

$$\sum_{S \subseteq T,\, s \leq k-1} m_\mu(S \cup \{i\}) \geq 0 \qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{19}$$

Finally, we examine the particular case of symmetric capacities and Choquet integrals, which play a crucial role in this paper.

Definition 16 *A capacity μ is said to be* symmetric *if it depends only on the cardinality of the coalition considered, in which case we use the simplified notation*

$$\mu(T) = \mu(t) \qquad \textit{where} \qquad t = |T|\,. \tag{20}$$

Accordingly, for the Möbius transform m_μ associated with a symmetric capacity μ we use the notation

$$m_\mu(T) = m_\mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{21}$$

In the symmetric case, the expression (14) for the capacity μ in terms of the Möbius transform m_μ reduces to

$$\mu(t) = \sum_{s=1}^{t} \binom{t}{s} m_\mu(s) \qquad t = 1, \ldots, n \tag{22}$$

and the boundary and monotonicity conditions (15) and (16) take the form

$$m_\mu(0) = 0 \qquad \sum_{s=1}^{n} \binom{n}{s} m_\mu(s) = 1 \tag{23}$$

$$\sum_{s=1}^{t} \binom{t-1}{s-1} m_\mu(s) \geq 0 \qquad\qquad t = 1, \ldots, n\,. \tag{24}$$

The monotonicity conditions correspond to $\mu(t) - \mu(t-1) \geq 0$ for $t = 1, \ldots, n$.

The Choquet integral (12) with respect to a symmetric capacity μ reduces to an Ordered Weighted Averaging (OWA) function [22, 56],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(n-i+1) - \mu(n-i)] x_{(i)} = \sum_{i=1}^{n} w_i\, x_{(i)} = A(\boldsymbol{x}) \tag{25}$$

where the weights $w_i = \mu(n-i+1) - \mu(n-i)$ satisfy $w_i \geq 0$ for $i = 1, \ldots, n$ due to the monotonicity of the capacity μ, and $\sum_{i=1}^{n} w_i = 1$ due to the boundary conditions $\mu(0) = 0$ and $\mu(n) = 1$. Comprehensive reviews of OWA functions can be found in [57] and [58].

The weighting structure of the OWA function (25) is of the general form $w_i = f(\frac{n-i+1}{n}) - f(\frac{n-i}{n})$ where f is a continuous and increasing function on the unit interval, with $f(0) = 0$ and $f(1) = 1$. Gajdos [24] shows that the OWA function A is associated with a k-additive capacity μ, with $k = 1, \ldots, n$, if and only if f is polynomial of order k. In fact, in (22), the k-additive case is obtained simply by taking $m_\mu(k+1) = \ldots = m_\mu(n) = 0$, and the binomial coefficient of the Möbius value $m_\mu(k)$ corresponds to $t(t-1)\ldots(t-k+1)/k!$, which is polynomial of order k in the coalition cardinality t.

4 The binomial decomposition

We now consider OWA functions $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [11], with the addition of a uniqueness result, see also [10].

We begin by introducing the convenient notation

$$\alpha_j = \binom{n}{j} m_\mu(j) \qquad j = 1, \ldots, n\,. \tag{26}$$

In this notation the upper boundary condition (23) reduces to

$$\sum_{j=1}^{n} \alpha_j = 1 \tag{27}$$

and the monotonicity conditions (24) take the form

$$\sum_{j=1}^{i} \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \alpha_j \geq 0 \qquad\qquad i = 1, \ldots, n\,. \tag{28}$$

Definition 17 *The* binomial OWA functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$*, with* $j = 1, \ldots, n$*, are defined as*

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}}\, x_{(i)} \qquad j = 1, \ldots, n \tag{29}$$

where the binomial weights w_{ji}*,* $i, j = 1, \ldots, n$ *are null when* $i + j > n + 1$ *according to the usual convention that* $\binom{p}{q} = 0$ *when* $p < q$*, with* $p, q = 0, 1, \ldots$

Except for $C_1(\boldsymbol{x}) = \bar{x}$, the binomial OWA functions C_j, $j = 2, \ldots, n$ have an increasing number of null weights, in correspondence with $x_{(n-j+2)}, \ldots, x_{(n)}$. The weight normalization of the binomial OWA functions, $\sum_{i=1}^n w_{ji} = 1$ for $j = 1, \ldots, n$, is due to the column-sum property of binomial coefficients,

$$\sum_{i=1}^{n} \binom{n-i}{j-1} = \sum_{i=0}^{n-1} \binom{i}{j-1} = \binom{n}{j} \qquad j = 1, \ldots, n\,. \tag{30}$$

Proposition 3 *Any OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \ldots + \alpha_n C_n(\boldsymbol{x}) \tag{31}$$

where the coefficients α_j, $j = 1, \ldots, n$ are subject to conditions (27) and (28). In the binomial decomposition the k-additive case, with $k = 1, \ldots, n$, is obtained simply by taking $\alpha_{k+1} = \ldots = \alpha_n = 0$.

The following interesting result concerning the cumulative properties of binomial weights is due to Calvo and De Baets [11], see also Bortot and Marques Pereira [10].

Proposition 4 *The binomial weights $w_{ji} \in [0, 1]$, with $i, j = 1, \ldots, n$, have the following cumulative property,*

$$\sum_{k=1}^{i} w_{j-1,k} \le \sum_{k=1}^{i} w_{jk} \qquad i = 1, \ldots, n \quad j = 2, \ldots, n\,. \tag{32}$$

Given that binomial weights have the cumulative property (32), Proposition 1 implies that the binomial OWA functions C_j, $j = 1, \ldots, n$ satisfy the relations $\bar{x} = C_1(\boldsymbol{x}) \ge C_2(\boldsymbol{x}) \ge \ldots \ge C_n(\boldsymbol{x}) \ge 0$, for any $\boldsymbol{x} \in \mathbb{D}^n$.

Summarizing, the binomial decomposition (31) holds for any OWA function A in terms of the binomial OWA functions C_j, $j = 1, \ldots, n$ and the corresponding coefficients α_j, $j = 1, \ldots, n$ subject to conditions (27) and (28).

Consider the binomial OWA functions C_j with $j = 1, \ldots, n$. The binomial weights w_{ji}, $i, j = 1, \ldots, n$ as in (29) have regularity properties which have interesting implications at the level of the functions C_j, $j = 1, \ldots, n$, see [10].

Proposition 5 *The binomial weights $w_{ji} \in [0, 1]$, with $i, j = 1, \ldots, n$, have the following properties,*

i. *for* $j = 1$ $\quad 1/n = w_{11} = w_{12} = \ldots = w_{1,n-1} = w_{1n}$

ii. *for* $j = 2$ $\quad 2/n = w_{21} > w_{22} > \ldots > w_{2,n-1} > w_{2n} = 0$

iii. *for* $j = 3, \ldots, n$ $\quad j/n = w_{j1} > w_{j2} > \ldots > w_{j,n-j+2} = \ldots = w_{jn} = 0$

The functions C_j, $j = 1, \ldots, n$ are continuous, idempotent, and stable for translations, where the latter two properties follow immediately from $\sum_{i=1}^n w_{ji} = 1$ for $j = 1, \ldots, n$. Moreover, given that binomial weights are non increasing, $w_{j1} \ge w_{j2} \ge \ldots \ge w_{jn}$ for $j = 1, \ldots, n$, Proposition 2 implies that the functions C_j, $j = 1, \ldots, n$ are S-concave, with strict S-concavity applying only to C_2.

In relation with these properties, we conclude that the functions C_j, $j = 1, \ldots, n$, which we hereafter call *binomial Gini welfare functions*, are generalized Gini welfare functions on the income domain $\boldsymbol{x} \in \mathbb{D}^n$.

Definition 18 *Consider the binomial Gini welfare functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $C_j(\boldsymbol{x}) = \sum_{i=1}^n w_{ji} x_{(i)}$ for $j = 1, \ldots, n$. The* binomial Gini inequality indices *$G_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $j = 1, \ldots, n$, are defined as*

$$G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x}) \qquad j = 1, \ldots, n \tag{33}$$

which means that

$$G_j(\boldsymbol{x}) = -\sum_{i=1}^n v_{ji} x_{(i)} = -\sum_{i=1}^n \left[w_{ji} - \frac{1}{n} \right] x_{(i)} \qquad j = 1, \ldots, n \tag{34}$$

where the coefficients v_{ji}, $i, j = 1, \ldots, n$ are equal to $-1/n$ when $i + j > n + 1$, since in such case the binomial weights w_{ji} are null. The weight normalization of the binomial Gini welfare functions, $\sum_{i=1}^n w_{ji} = 1$ for $j = 1, \ldots, n$, implies that $\sum_{i=1}^n v_{ji} = 0$ for $j = 1, \ldots, n$.

The binomial Gini inequality indices G_j, $j = 1, \ldots, n$ are continuous, nilpotent, and invariant for translations, where the latter two properties follow immediately from $\sum_{i=1}^n v_{ji} = 0$ for $j = 1, \ldots, n$. Moreover, the G_j are S-convex: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we have that $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y}) \Rightarrow G_j(\boldsymbol{x}) \leq G_j(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, due to the S-concavity of the C_j, $j = 1, \ldots, n$.

In fact, the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ in (33) correspond to the Atkinson-Kolm-Sen (AKS) absolute inequality indices associated with the binomial welfare functions C_j, $j = 1, \ldots, n$, in the spirit of Blackorby and Donaldson's correspondence formula. Together, as we discuss below, the binomial Gini welfare functions C_j and the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ can be regarded as two equivalent functional bases for the class of generalized Gini welfare functions and inequality indices.

In analogy with the binomial weights w_{ji}, $i, j = 1, \ldots, n$, their inequality counterparts v_{ji}, $i, j = 1, \ldots, n$ have interesting regularity properties, which follow directly from Proposition 5.

Proposition 6 *The coefficients $v_{ji} \in [-1/n, (n-1)/n]$, with $i, j = 1, \ldots, n$, have the following properties,*

i. *for $j = 1$* $\qquad 0 = v_{11} = v_{12} = \ldots = v_{1,n-1} = v_{1n}$

ii. *for $j = 2$* $\qquad 1/n = v_{21} > v_{22} > \ldots > v_{2,n-1} > v_{2n} = -1/n$

iii. *for $j = 3, \ldots, n$* $\qquad \frac{j-1}{n} = v_{j1} > v_{j2} > \ldots > v_{j,n-j+2} = \ldots = v_{jn} = -1/n$

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $G_1(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, $C_2(\boldsymbol{x})$ has $n-1$ positive linearly decreasing weights and one null last weight, and the associated $G_2(\boldsymbol{x})$ has linearly increasing coefficients and is in fact proportional to the classical Gini index, $G_2(\boldsymbol{x}) = \frac{n}{n-1} G^c(\boldsymbol{x})$. The remaining $C_j(\boldsymbol{x})$, $j = 3, \ldots, n$, have $n-j+1$ positive non-linear decreasing weights and $j-1$ null last weights, and the associated $G_j(\boldsymbol{x})$, $j = 3, \ldots, n$ have $n-j+2$ non-linear increasing weights and $j-1$ equal last weights.

Therefore, the only strict binomial welfare function is $C_1(\boldsymbol{x}) = \bar{x}$ and the only strict binomial inequality index is $G_2(\boldsymbol{x}) = \frac{n}{n-1} G^c(\boldsymbol{x})$. In the remaining $G_j(\boldsymbol{x})$, $j = 3, \ldots, n$ the last $j-1$ coefficients coincide and thus they are non strict absolute inequality indices, in the sense that they are insensitive to income transfers involving only the $j-1$ richest individuals of the population.

5 Numerical and graphical illustration (I)

In this section we compute the weights of the binomial Gini welfare functions and the coefficients of the associated binomial Gini inequality indices in dimensions $n = 2, 3, 4, 5, 6$. We also provide a graphical illustration of weights and coefficients of C_j and G_j for $j = 1, \ldots, 4$ in the case $n = 64$.

In dimensions $n = 2, 3, 4, 5, 6$ the weights $w_{ij} \in [0, 1]$, $i, j = 1, \ldots, n$ of the binomial Gini welfare functions C_j, $j = 1, \ldots, n$ and the coefficients $-v_{ij} \in [-(n-1)/n, 1/n]$, $i, j = 1, \ldots, n$ of the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ are as follows,

$n = 2$

$$\begin{array}{llll} C_1: & (\frac{1}{2}, \frac{1}{2}) & G_1: & (0, 0) \\ C_2: & (1, 0) & G_2: & (-\frac{1}{2}, \frac{1}{2}) \end{array}$$

$n = 3$

$$\begin{array}{llll} C_1: & (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) & G_1: & (0, 0, 0) \\ C_2: & (\frac{2}{3}, \frac{1}{3}, 0) & G_2: & (-\frac{1}{3}, 0, \frac{1}{3}) \\ C_3: & (1, 0, 0) & G_3: & (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \end{array}$$

$n = 4$

$$\begin{array}{llll} C_1: & (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & G_1: & (0, 0, 0, 0) \\ C_2: & (\frac{3}{6}, \frac{2}{6}, \frac{1}{6}, 0) & G_2: & (-\frac{3}{12}, -\frac{1}{12}, \frac{1}{12}, \frac{3}{12}) \\ C_3: & (\frac{3}{4}, \frac{1}{4}, 0, 0) & G_3: & (-\frac{2}{4}, 0, \frac{1}{4}, \frac{1}{4}) \\ C_4: & (1, 0, 0, 0) & G_4: & (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \end{array}$$

$n = 5$

$$\begin{array}{llll} C_1: & (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) & G_1: & (0, 0, 0, 0, 0) \\ C_2: & (\frac{4}{10}, \frac{3}{10}, \frac{2}{10}, \frac{1}{10}, 0) & G_2: & (-\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}) \\ C_3: & (\frac{6}{10}, \frac{3}{10}, \frac{1}{10}, 0, 0) & G_3: & (-\frac{4}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{2}{10}, \frac{2}{10}) \\ C_4: & (\frac{4}{5}, \frac{1}{5}, 0, 0, 0) & G_4: & (-\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \\ C_5: & (1, 0, 0, 0, 0) & G_5: & (-\frac{4}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \end{array}$$

$n = 6$

$$\begin{array}{llll} C_1: & (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) & G_1: & (0, 0, 0, 0, 0, 0) \\ C_2: & (\frac{5}{15}, \frac{4}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}, 0) & G_2: & (-\frac{5}{30}, -\frac{3}{30}, -\frac{1}{30}, \frac{1}{30}, \frac{3}{30}, \frac{5}{30}) \\ C_3: & (\frac{10}{20}, \frac{6}{20}, \frac{3}{20}, \frac{1}{20}, 0, 0) & G_3: & (-\frac{20}{60}, -\frac{8}{60}, \frac{1}{60}, \frac{7}{60}, \frac{10}{60}, \frac{10}{60}) \\ C_4: & (\frac{10}{15}, \frac{4}{15}, \frac{1}{15}, 0, 0, 0) & G_4: & (-\frac{15}{30}, -\frac{3}{30}, \frac{3}{30}, \frac{5}{30}, \frac{5}{30}, \frac{5}{30}) \\ C_5: & (\frac{5}{6}, \frac{1}{6}, 0, 0, 0, 0) & G_5: & (-\frac{4}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \\ C_6: & (1, 0, 0, 0, 0, 0) & G_6: & (-\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \end{array}$$

The binomial Gini welfare functions C_j, $j = 1, \ldots, n$ have null weights associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population. Correspondingly, the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ have equal coefficients associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they are progressively insensitive to income transfers within the richest part of the population.

In the case $n = 64$, the weights of the binomial Gini welfare functions C_j, $j = 1, \ldots, 4$ and the coefficients of the binomial Gini inequality indices G_j, $j = 1, \ldots, 4$ are graphically represented in Fig. 1 and Fig. 2.

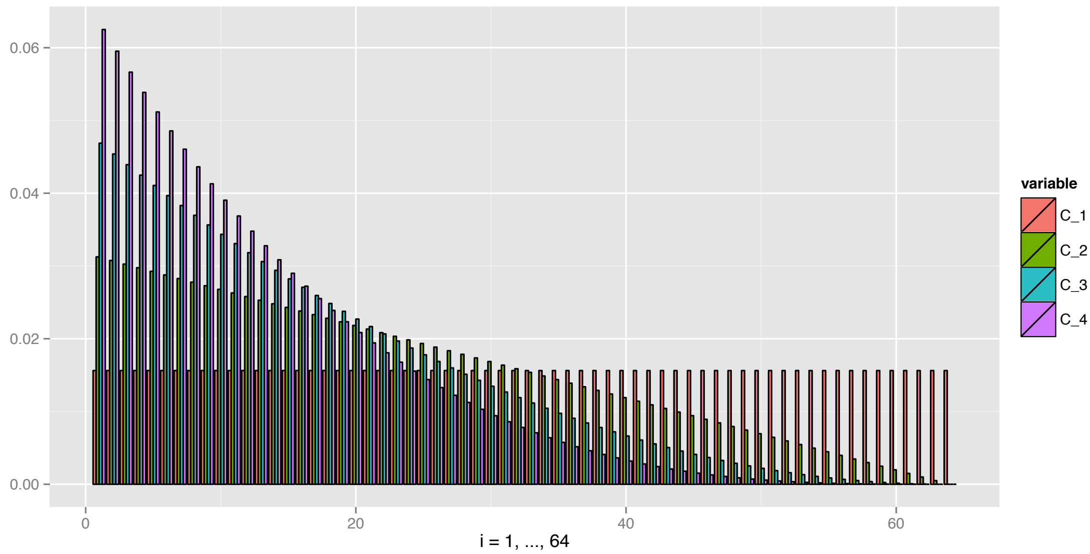

Figure 1: Weights of C_1, C_2, C_3, C_4 with $n = 64$.

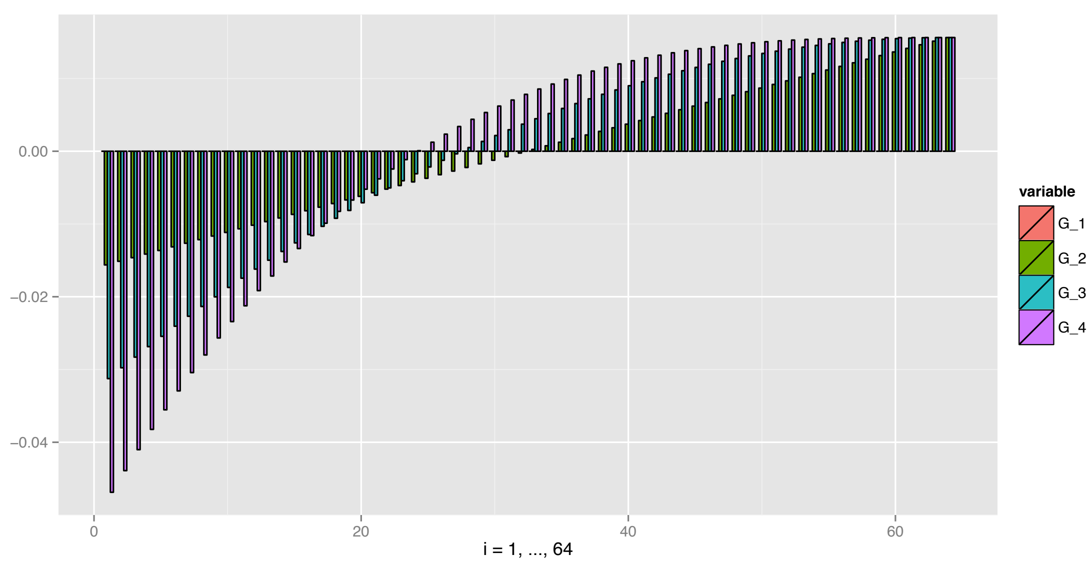

Figure 2: Coefficients of G_1, G_2, G_3, G_4 with $n = 64$.

6 Numerical and graphical illustration (II)

In this section we compute the binomial Gini welfare functions and inequality indices in relation to a family of income distributions with $n = 4$ and $n = 6$. This family of income distributions, each with unit average income, is defined on the basis of the parametric Lorenz curve associated with the generating function

$$f_\alpha(r) = re^{-\alpha(1-r)} \qquad r \in [0,1] \tag{35}$$

where the parameter $\alpha \geq 0$ is related with inequality. Fig. 3 provides a graphical illustration of the parametric Lorenz curve for parameter values $\alpha = 0, 1, \ldots, 8$.

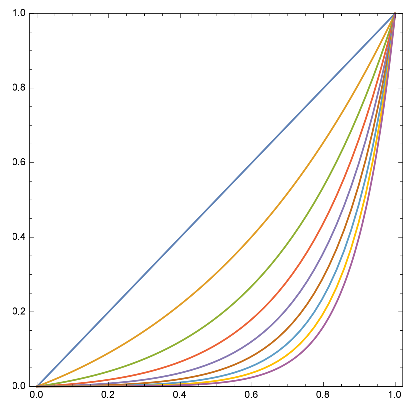

Figure 3: Parametric Lorenz curve for parameter values $\alpha = 0, 1, \ldots, 8$.

Considering a population with n individuals, the family of income distributions with unit average income associated with the parametric Lorenz curve above is given by

$$x_{(i)} = n\Big[f_\alpha\Big(\frac{i}{n}\Big) - f_\alpha\Big(\frac{i-1}{n}\Big)\Big] \qquad i = 1, \ldots n. \tag{36}$$

We now compute the binomial Gini welfare functions and inequality indices in relation to the family of income distributions (36), with $n = 4$ and $n = 6$, for $\alpha = 0, 1, \ldots, 8$.

α	C_1	C_2	C_3	C_4
0	1	1	1	1
1	1	0.670305	0.539449	0.472367
2	1	0.463080	0.295505	0.223130
3	1	0.328127	0.164265	0.105400
4	1	0.237350	0.092561	0.049787
5	1	0.174534	0.052801	0.023518
6	1	0.130012	0.030448	0.011109
7	1	0.097827	0.017722	0.005248
8	1	0.074186	0.010397	0.002479

Table 1: Binomial Gini welfare functions with $n = 4$.

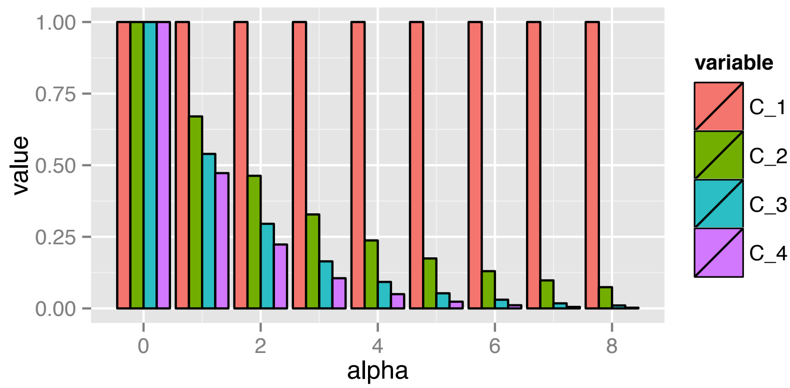

Figure 4: $C_1, \ldots, C_4$ with $\alpha = 0, 1, \ldots 8$.

α	C_1	C_2	C_3	C_4	C_5	C_6
0	1	1	1	1	1	1
1	1	0.691971	0.566210	0.500512	0.460871	0.434598
2	1	0.497069	0.329902	0.254565	0.213783	0.188876
3	1	0.368422	0.197533	0.131594	0.099836	0.082085
4	1	0.280141	0.121300	0.069130	0.046944	0.035674
5	1	0.217440	0.076204	0.036888	0.022227	0.015504
6	1	0.171565	0.048845	0.019979	0.010597	0.006738
7	1	0.137149	0.031860	0.010972	0.005087	0.002928
8	1	0.110786	0.021094	0.006103	0.002458	0.001273

Table 2: Binomial Gini welfare functions with $n = 6$.

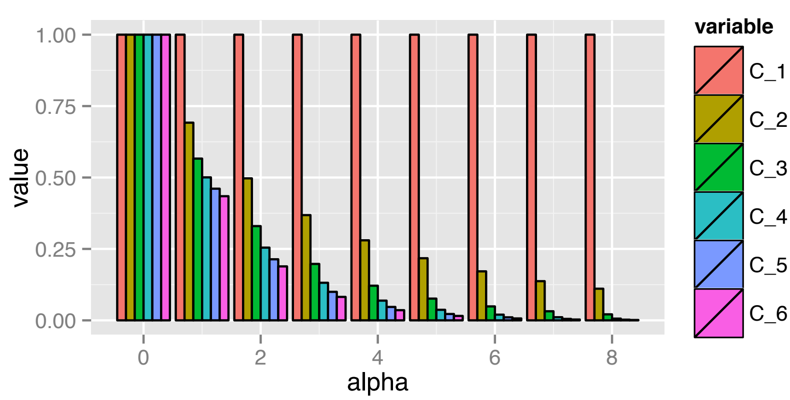

Figure 5: $C_1, \ldots, C_6$ with $\alpha = 0, 1, \ldots 8$.

α	G_1	G_2	G_3	G_4
0	0	0	0	0
1	0	0.329695	0.460551	0.527633
2	0	0.536920	0.704495	0.776870
3	0	0.671873	0.835735	0.894601
4	0	0.762651	0.907439	0.950213
5	0	0.825466	0.947199	0.976482
6	0	0.869988	0.969552	0.988891
7	0	0.902172	0.982278	0.994753
8	0	0.925814	0.989603	0.997521

Table 3: Binomial Gini inequality indices with $n = 4$.

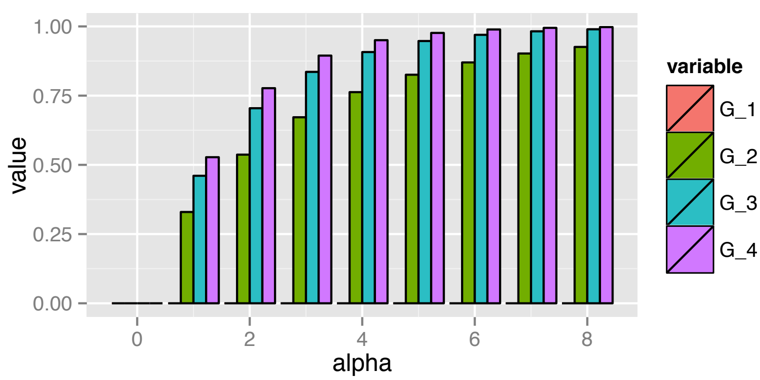

Figure 6: $G_1, \ldots, G_4$ with $\alpha = 0, 1, \ldots 8$.

α	G_1	G_2	G_3	G_4	G_5	G_6
0	0	0	0	0	0	0
1	0	0.308029	0.433790	0.499488	0.539129	0.565402
2	0	0.502931	0.670099	0.745435	0.786217	0.811124
3	0	0.631579	0.802468	0.868406	0.900165	0.917915
4	0	0.719859	0.878700	0.930870	0.953056	0.964326
5	0	0.782560	0.923796	0.963112	0.977773	0.984496
6	0	0.828435	0.951155	0.980021	0.989403	0.993262
7	0	0.862851	0.968140	0.989028	0.994913	0.997072
8	0	0.889214	0.978906	0.993897	0.997542	0.998727

Table 4: Binomial Gini inequality indices with $n = 6$.

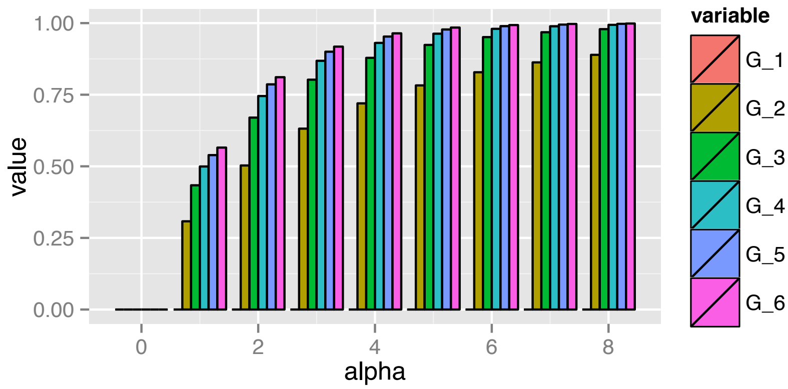

Figure 7: $G_1, \ldots, G_6$ with $\alpha = 0, 1, \ldots 8$.

The pattern of the numerical data in Tables 1-4 and Figures 4-7 reflects the dominance relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \ldots \geq C_n(\boldsymbol{x}) \geq 0$ and $0 = G_1(\boldsymbol{x}) \leq G_2(\boldsymbol{x}) \leq \ldots \leq G_n(\boldsymbol{x}) \leq \bar{x}$ for the income distributions considered here, with $\boldsymbol{x} \in \mathbb{D}^n$ and $\bar{x} = 1$.

Moreover, considering the parametric Lorenz curve depicted in Fig. 3, the values taken by the binomial Gini welfare functions and inequality indices with $n = 4$ and $n = 6$ for $\alpha = 0, 1, \ldots, 8$ illustrate clearly the effect of the parameter $\alpha \geq 0$ in relation with inequality.

7 Conclusions

We consider the binomial decomposition of OWA functions, in terms of the binomial Gini welfare functions $C_j(\boldsymbol{x})$ and the associated binomial Gini inequality indices $G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x})$, with $j = 1, \ldots, n$, for all income distributions $\boldsymbol{x} \in \mathbb{D}^n$. The context is that of Choquet integration with respect to symmetric capacities, in which the binomial Gini welfare functions and associated binomial Gini inequality indices provide two equivalent descriptions of k-additivity.

We illustrate the weights of the binomial Gini welfare functions C_j, $j = 1, ..., n$, and the coefficients of the associated binomial Gini inequality indices G_j, $j = 1, ..., n$. The binomial Gini welfare functions C_j, $j = 1, \ldots, n$ have null weights associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population. Correspondingly, the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ have equal coefficients associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they are progressively insensitive to income transfers within the richest part of the population.

We introduce a family of income distributions described by a parameter $\alpha \geq 0$ related with inequality and we compute the binomial Gini welfare functions and inequality indices with $n = 4$ and $n = 6$ for $\alpha = 0, 1, \ldots, 8$. The data obtained reflects the dominance relations regarding binomial Gini welfare functions and inequality indices and illustrates the effect of the parameter $\alpha \geq 0$ in relation with inequality.

References

[1] A. B. Atkinson, On the measurement of inequality, Journal of Economic Theory 2 (1970) 244–263.

[2] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Studies in Fuzziness and Soft Computing, Vol. 221, Springer, Heidelberg, 2007.

[3] C. D. Bennet, W. C. Holland, G. J. Székely, Integer valued means, Aequationes Mathematicae, (on line since 27 June 2013).

[4] E. Ben Porath, I. Gilboa, Linear measures, the Gini index, and the income-equality trade-off, Journal of Economic Theory 2 (64) (1994) 443–467.

[5] C. Blackorby, D. Donaldson, Measures of relative equality and their meaning in terms of social welfare, Journal of Economic Theory 18 (1978) 59–80.

[6] C. Blackorby, D. Donaldson, A theoretical treatment of indices of absolute inequality, International Economic Review 21 (1) (1980) 107–136.

[7] C. Blackorby, D. Donaldson, Ethical indices for measurement of poverty, Econometrica 48 (4) (1980) 1053–1060.

[8] C. Blackorby, D. Donaldson, M. Auersperg, A new procedure for the measurement of inequality within and among population subgroups, The Canadian Journal of Economics 14 (4) (1981) 665–685.

[9] C. Blackorby, D. Donaldson, Ethical social index numbers and the measurement of effective tax / benefit progressivity, The Canadian Journal of Economics 17 (4) (1984) 683–694.

[10] S. Bortot, R. A. Marques Pereira, The binomial Gini inequality indices and the binomial decomposition of welfare functions, Fuzzy Sets and Systems 255 (2014) 92-114.

[11] T. Calvo, B. De Baets, Aggregation operators defined by k-order additive/maxitive fuzzy measures, International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems 6 (6) (1998) 533–550.

[12] T. Calvo, A. Kolesárova, M. Komorníková, R. Mesiar, Aggregation operators: Properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators: New Trends and Applications, Physica-Verlag, Heidelberg, 2002, pp. 3-104.

[13] K. Cao-Van, B. De Baets, A decomposition of k-additive Choquet and k-maxitive Sugeno integrals, International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems 2 (9) (2001) 127–143.

[14] A. Chateauneuf, Modelling attitudes towards uncertainty and risk through the use of Choquet integral, Annals of Operations Research 52 (1) (1994) 3–20.

[15] O. Chisini, Sul concetto di media, Periodico di Matematiche 4, (1929) 106116.

[16] G. Choquet, Theory of capacities, Annales de l'Institut Fourier 5 (1953) 131-295.

[17] D. Denneberg, Non-additive measure and integral, Kluwer Academic Publishers, Dordrecht, 1994.

[18] D. Donaldson, J. A. Weymark, A single-parameter generalization of the Gini indices of inequality, Journal of Economic Theory 22 (2) (1980) 67–86.

[19] D. Donaldson, J. A. Weymark, Ethically flexible Gini indices for income distributions in the continuum, Journal of Economic Theory 29 (2) (1983) 353–358.

[20] U. Ebert, Size and distribution of incomes as determinants of social welfare, Journal of Economic Theory 41 (1) (1987) 23–33.

[21] U. Ebert, Measurement of inequality: An attempt at unification and generalization, Social Choice Welfare 5 (2) (1988) 147–169.

[22] J. Fodor, J. L. Marichal, M. Roubens, Characterization of the ordered weighted averaging operators, IEEE Trans. on Fuzzy Systems 3 (2) (1995) 236–240.

[23] J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht, 1994.

[24] T. Gajdos, Measuring inequalities without linearity in envy: Choquet integrals for symmetric capacities, Journal of Economic Theory 106 (1) (2002) 190–200.

[25] I. Gilboa, D. Schmeidler, Additive representations of non-additive measures and the Choque integral, Annals of Operations Research 52 (1) (1994) 43–65.

[26] I. Gilboa, D. Schmeidler, Canonical representation of set functions, Mathematics of Operations Research 20 (1) (1995) 197–212.

[27] M. Grabisch, Fuzzy integral in multicriteria decision making, Fuzzy Sets and Systems 69 (3) (1995) 279-298.

[28] M. Grabisch, The application of fuzzy integrals in multicriteria decision making, European Journal of Operational Research 89 (3) (1996) 445-456.

[29] M. Grabisch, k-order additive discrete fuzzy measures and their representation, Fuzzy Sets and Systems 92 (2) (1997) 167-189.

[30] M. Grabisch, Alternative representations of discrete fuzzy measures for decision making, International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems 5 (5) (1997) 587-607.

[31] M. Grabisch, k-Additive measures: recent issues and challenges, in: Proc. 5th International Conference on Soft Computing and Information/Intelligent Systems, Izuka, Japan, 1998, pp.394–397.

[32] M. Grabisch, I. Kojadinovich, P. Meyer, A review of methods for capacity identification in Choquet integral based multi-attribute utility theory: Applications of the Kappalab R package, European Journal of Operational Research 186 (2) (2008) 766-785.

[33] M Grabisch, C. Labreuche, Fuzzy measures and integrals in MCDA, in: J. Figueira, S. Greco, M. Ehrgott (Eds.), Multiple Criteria Decision Analysis, Springer, Heidelberg, 2005, pp. 563–604.

[34] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, 4OR 6 (1) (2008) 1-44.

[35] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, Annals of Operations Research 175 (1) (2010) 247-286.

[36] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Encyclopedia of Mathematics and its Applications, Vol. 127, Cambridge University Press, 2009.

[37] S.C. Kolm, The optimal production of social justice, in: J. Margolis, H. Guitton (Eds.), Public Economics, Macmillan, London, 1969.

[38] S.C. Kolm, Unequal inequalities I, Journal of Economic Theory 12 (1976) 416–442.

[39] S.C. Kolm, Unequal inequalities II, Journal of Economic Theory 13 (1976) 82–111.

[40] J.L. Marichal, Aggregation operators for multicriteria decision aid, Ph.D. Thesis, University of Liège, 1998.

[41] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Mathematics in Science and Engineering, Academic Press, New York, 1979.

[42] B. Mayag, M. Grabisch, C. Labreuche, A representation of preferences by the Choquet integral with respect to a 2-additive capacity, Theory and Decision 71 (3) (2011) 297-324.

[43] B. Mayag, M. Grabisch, C. Labreuche, A characterization of the 2-additive Choquet integral through cardinal information, Fuzzy Sets and Systems 184 (1) (2011) 84-105.

[44] F. Mehran, Linear measures of income inequality, Econometrica 44 (4) (1976) 805–809.

[45] P. Miranda, M. Grabisch, P. Gil, Axiomatic structure of k-additive capacities, Mathematical Social Sciences 49 (2) (2005) 153-178.

[46] T. Murofushi, M. Sugeno, Some quantities represented by the Choquet integral, Fuzzy Sets and Systems 2 (56) (1993) 229–235.

[47] J. Quiggin (Ed.), Generalized expected utility theory: the rank-dependent model, Kluwer Academic Publisher, Dordrecht, 1993.

[48] D. Schmeidler, Integral representation without additivity, Proceedings of the American Mathematical Society 97 (2) (1986) 255–261.

[49] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57 (3) (1989) 571–587.

[50] A. Sen, On Economic Inequality, Clarendon Press, Oxford, 1973.

[51] A. Sen, Ethical measurement of inequality: some difficulties, in: W. Krelle, A. F. Shorrocks (Eds.), Personal Income Distribution, North-Holland, Amsterdam, 1978.

[52] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo Institut of Technology, 1974.

[53] J. A. Weymark, Generalized Gini inequality indices, Mathematical Social Sciences 1 (4) (1981) 409–430.

[54] M. Yaari, The dual theory of choice under risk, Econometrica 55 (1) (1987) 95–115.

[55] M. Yaari, A controversial proposal concerning inequality measurement, Journal of Economic Theory 44 (2) (1988) 381–397.

[56] R. R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. on Systems, Man and Cybernetics 18 (1) (1988) 183–190.

[57] R. R. Yager, J. Kacprzyk (Eds.), The Ordered Weighted Averaging Operators, Theory and Applications, Kluwer Academic Publisher, Dordrecht, 1997.

[58] R. R. Yager, J. Kacprzyk, G. Beliakov (Eds.), Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, Studies in Fuzziness and Soft Computing, Vol. 265, Springer, Heidelberg, 2011.

List of recent papers published in the same series

2015/07 Economic Rebalancing and Growth: the Japanese experience and China's prospects Problem, Andrea Fracasso

2015/06 Ordered Spatial Sampling by Means of the Traveling Salesman Problem, Maria Michela Dickson and Yves Tillé

2015/05 A spatial analysis of health and pharmaceutical firm survival, Giuseppe Arbia, Giuseppe Espa, Diego Giuliani, Rocco Micciolo

2015/04 Innovation, trade and the size of exporting firms, Letizia Montinari, Massimo Riccaboni, Stefano Schiavo

2015/03 La famiglia fa male all'internazionalizzazione dell'impresa?, Mariasole Bannò, Elisa Pozza, Sandro Trento

2015/02 Bilateral netting and contagion dynamics in Financial networks, Edoardo Gaffeo, Lucio Gobbi

2015/01 The Role of Firm R&D Effort and Collaboration as Mediating Drivers of Innovation Policy Effectiveness, Giovanni Cerulli, Roberto Gabriele, Bianca Potì

2014/10 Jointness in Sites: The Case of Migratory Beekeeping, Luciano Pilati, Vasco Boatto

2014/09 Sustainability vs. credibility of fiscal consolidation. A Principal Components test for the Euro Zone, Giuliana Passamani, Roberto Tamborini, Matteo Tomaselli

2014/08 Fitting Spatial Econometric Models through the Unilateral Approximation, Giuseppe Arbia, Marco Bee, Giuseppe Espa, Flavio Santi.

2014/07 Il modello di scoring del Fondo Centrale di Garanzia: un'analisi d'impatto, Davide Panizzolo

2014/06 Going abroad on regional shoulders: The role of spillovers on the composition of regional exports, Mariasole Bannò, Diego Giuliani, Enrico Zaninotto

2014/05 The role of covenants in bond issue and investment policy. The case of Russian companies, Flavio Bazzana, Anna Zadorozhnaya, Roberto Gabriele

2014/04 Efficiency or Bounded Rationality? Drivers of Firm Diversification Strategies in Vietnam, H. Thu Tran, E. Santarelli, E. Zaninotto

2014/03 Where Gibrat meets Zipf: Scale and Scope of French Firms, M. Bee, M. Riccaboni and S. Schiavo

2014/02 Competition in the banking sector and economic growth: panel-based international evidence, E. Gaffeo and R. Mazzocchi

2014/01 Macroprudential Consolidation Policy in Interbank Networks, E. Gaffeo and M. Molinari

2013/18 Bio-Economics of Allocatable Pollination Services: Sequential Choices and Jointness in Sites, L. Pilati and V. Boatto

2013/17 Resilience and specialization in volatile environments: evidence from the Italian Air Force Tornado crews learning practices, M. Laura Frigotto, Marco Zamarian

2013/16 Monetary Policy when the NAIRI is unknown: The Fed and the Great Deviation, Ronny Mazzocchi

2013/15 Intertemporal Coordination Failure and Monetary Policy, Ronny Mazzocchi

2013/14 Investment-Saving Imbalances with Endogenous Capital Stock, Ronny Mazzocchi

2013/13 Scope and Flaws of the New Neoclassical Synthesis, Ronny Mazzocchi

2013/12 Approximate Maximum Likelihood Estimation of the Autologistic Model, Marco Bee, Diego Giuliani, Giuseppe Espa

2013/11 Do middle managers matter?, Elena Feltrinelli, Roberto Gabriele, Sandro Trento

2013/10 Transatlantic contagion 2010-…, Roberto Tamborini

2013/09 Interbank contagion and resolution procedures: inspecting the mechanism, Edoardo Gaffeo e Massimo Molinari

2013/08 Using information markets in grantmaking. An assessment of the issues involved and an application to Italian banking foundations, Edoardo Gaffeo

2013/07 A spatial and sectoral analysis of firm demography in Italy, Giuseppe Espa, Danila Filipponi, Diego Giuliani, Davide Piacentino

2013/06 Reassessing the spatial determinants of the growth of Italian SMEs, Roberto Gabriele, Diego Giuliani, Marco Corsino, Giuseppe Espa

2013/05 At the core of the international financial system, Valentina Feroldi, Edoardo Gaffeo

2013/04 Uses And Motivations For Credit Derivatives: An Empirical Investigation Into Italian Banks, Eleonora Broccardo, Maria Mazzuca and Elmas Yaldiz

2013/03 Origins and prospects of the Euro existential crisis, Luigi Bonatti and Andrea Fracasso

2013/02 Employer moral hazard and wage rigidity. The case of worker-owned and investor-owned firms, Marina Albanese, Cecilia Navarra and Ermanno Tortia

2013/01 A regular multidistance among fuzzy numbers, Franco Molinari.