MINIMAL DEGREE EQUATIONS FOR CURVES AND SURFACES (VARIATIONS ON A THEME OF HALPHEN)

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Abstract. Many classical results in algebraic geometry arise from investigating some extremal behaviors that appear among projective varieties not lying on any hypersurface of fixed degree. We study two numerical invariants attached to such collections of varieties: their minimal degree and their maximal number of linearly independent smallest degree hypersurfaces passing through them. We show results for curves and surfaces, and pose several questions.

1. Introduction

In this note, we study two numerical invariants attached to projective varieties, focusing on the low-dimensional cases of curves and surfaces. To introduce the problem, let X be an m-dimensional integral projective variety in $\mathbb{P}^n$. Let s be an integer with the property that X is not contained in any hypersurface of degree strictly smaller than s, i.e., $h^0(\mathcal{I}_X(s-1)) = 0$. Perhaps, the first basic question one might ask is as follows:

Question 1.1. What is the minimal degree that such an X may have?

Moreover, one might wonder what is the largest number of linearly independent hypersurfaces of degree s passing through such an X, or more formally:

Question 1.2. For such a projective variety X, how big $h^0(\mathcal{I}_X(s))$ may be?

Both Question 1.1 and Question 1.2 concern extremal behaviors, which are often fundamental phenomena in algebraic geometry. When X is a curve, a variant to the questions above is about the maximal genus of a curve given a prescribed linear series on it, or the maximal genus of a curve not lying on a surface of fixed degree: these are the subjects of the famous Castelnuovo's and Halphen's theories respectively, which have been of crucial importance in the theory of curves and much beyond; see, e.g., the works of Chiantini and Ciliberto [12], Di Gennaro and Franco [13], for results on Castelnuovo-Halphen's theory in higher dimensional projective spaces. In the same vein, another line of research investigates the k-normality of a projective variety, i.e., the objective is to find out the least integer k such that the system of degree k hypersurfaces of the ambient projective space cut out a complete linear system on X; see, e.g., the works of Gruson, Lazarsfeld, and Peskine [19], Lazarsfeld [26], Alzati and Russo [1] for developments in this direction.

Question 1.1 may be regarded as the very first step towards classifying non-degenerate minimal degree varieties not lying on any hypersurface of degree $< s$. Therefore the

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classification of the usual non-degenerate minimal degree varieties may be viewed as the case when $s = 2$ [25, 30].

Park [31, Problem A] posed the problem of determining the maximal possible values of $h^0(\mathcal{I}_X(s))$, for every s without further assumptions; thus our Question 1.2 is more restrictive but different, inasmuch as we require the integer s to be the minimal such that $h^0(\mathcal{I}_X(s)) \neq 0$. Park's problem is in fact classical and had previously led, for instance, to significant results such as a characterization of minimal degree varieties by Castelnuovo in terms of the number of linearly independent quadrics passing through them. The answer to this problem in the case of curves is now well-known: an upper bound was first obtained by Harris [20], and later improved with different methods by L'vovsky [27]. Furthermore, Park's extremal cases are related to the study of varieties $X \subset \mathbb{P}^n$ of almost-minimal degree, i.e., $\deg(X) = \mathrm{codim}(X)+2$; see [11, 17, 30, 25].

Contributions and structure of the paper. In §2, we define the collection of projective varieties $\mathcal{A}(n, s, m)$ we are concerned with, and two basic numerical invariants: the minimal degree $d_{n,s,m}$ appearing among all the varieties in $\mathcal{A}(n, s, m)$, and the maximal number $\alpha(n, s, m)$ of linearly independent hypersurfaces of degree s vanishing on a given element of $\mathcal{A}(n, s, m)$. The study of these invariants has classical roots, as pointed out in §1, and yet is new, to the best of our knowledge.

In §3, we focus on the case of curves. Proposition 3.1 establishes the value of the minimal degree $d_{n,s,1}$, whereas Lemma 3.2 gives the possible values of arithmetic genera of curves in $\mathcal{A}(n, s, 1)$ with minimal degree.

In Lemma 3.6, we record the value of $h^0(\mathcal{I}_X(s))$. We conjecture a bound on its maximum value, $\alpha(n, s, 1)$. Question 3.5 asks whether this is the maximum possible. In Remark 3.7, we observe instances where the bound proposed in Question 3.5 does hold. The proof of Proposition 3.1 shows that the minimal degree is reached by some smooth rational curves: Question 3.8 asks whether this is possible for any other degree $d > d_{n,s,1}$. Remark 3.15 points out a numerical range where the latter question has an affirmative answer. In Remark 3.9, we collect our current knowledge around Question 3.8 in the case of $n = 3$. Here we generalize **range A** of curves in $\mathbb{P}^3$ to curves embedded in higher projective spaces and pose several questions about them (Question 3.13). Finally, in Proposition 3.17, using curves in $\mathcal{A}(n, s, 1)$, we produce a non-degenerate irreducible projective variety (that is not a cone) of arbitrarily high degree in $\mathcal{A}(n, m, s)$, for every $m \geq 1$, with $n \geq m + 3$.

In §4, we deal with the case of surfaces. In Theorem 4.1, we show useful upper bounds for the dimension of linear systems on a smooth surface, using one of the building blocks of Mori theory (the *Kawamata Rationality Theorem*). Remark 4.2 and Proposition 4.3 (proven in [4]) point out some circumstances where we can derive some information on $d_{n,s,2}$ and $d_{n,3,2}$, respectively. The findings in Theorem 4.1, along with Proposition 4.3, are summarized in Theorem 4.6. Finally, Remark 4.8 collects some cohomological facts about almost-minimal degree surfaces.

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2. Two numerical invariants

Our varieties are over the complex numbers. We introduce the collections of projective varieties we are concerned with:

Definition 2.1. Let $m \geq 1$, $s \geq 3$ and $n \geq m+2$. Define

$$\mathcal{A}(n,s,m) = \left\{\text{integral } m\text{-dimensional } X \subset \mathbb{P}^n \mid h^0(\mathcal{I}_X(s-1)) = 0\right\}.$$

Since $s > 1$, any $X \in \mathcal{A}(n,s,m)$ is non-degenerate, i.e., it spans the ambient projective space, as one clearly has $h^0(\mathcal{I}_X(1)) = 0$.

Definition 2.2. We introduce two numerical invariants:

$$d_{n,s,m} = \min\left\{\deg(X) \mid X \in \mathcal{A}(n,s,m)\right\}, \text{ and}$$

$$\alpha(n,s,m) = \max\left\{h^0(\mathcal{I}_X(s)) \mid X \in \mathcal{A}(n,s,m)\right\}.$$

Hence $\alpha(n,s,m)$ is the maximal number, over the set $\mathcal{A}(n,s,m)$, of the linearly independent hypersurfaces of degree s vanishing on some $X \in \mathcal{A}(n,s,m)$.

We conclude this preliminary section with an observation about minimal degrees:

Remark 2.3. For $m \geq 2$, we have $d_{n,s,m} \leq d_{n-1,s,m-1}$. To see this, take $X \subset \mathbb{P}^{n-1}$ of dimension $m-1$ with minimal degree $\deg(X) = d_{n-1,s,m-1}$. Let $C(X)$ be the cone over X with a single point as vertex: this sits in $\mathbb{P}^n$, it has dimension m and degree $d_{n-1,s,m-1}$. Moreover, it is clear that $C(X) \in \mathcal{A}(n,s,m)$.

3. Curves

In this section, we focus on the case of curves, i.e., $m = 1$. For a curve X, $p_a(X)$ denotes its arithmetic genus. For the ease of notation, we set

$$\mathcal{A}(n,s) := \mathcal{A}(n,s,1), d_{n,s} := d_{n,s,1}, \text{ and } \alpha(n,s) := \alpha(n,s,1).$$

We introduce another piece of notation:

$$H(d,g;n) = \{\text{smooth, connected, non-deg. } X \subset \mathbb{P}^n, \deg(X) = d, p_a(X) = g\}.$$

In the range of degree $d \geq g+1$, define

$$H(d,g;n)' = \left\{X \in H(d,g;n) \mid h^1(\mathcal{O}_X(1)) = 0\right\}.$$

Note that $H(d,g;n)' = H(d,g;n)$ whenever $d > 2g-2$, as $\mathcal{O}_X(1)$ is non-special.

Proposition 3.1. *The numerical invariant $d_{n,s}$ satisfies*

$$d_{n,s} = \left\lceil \left(\binom{n+s-1}{n} - 1\right)/(s-1)\right\rceil.$$

Proof. Let
$$\delta = \left\lceil \left(\binom{n+s-1}{n} - 1 \right) / (s-1) \right\rceil .$$
We show that $d_{n,s} = \delta$.

For any integer $d \geq n$, the set $H(d,0;n)$ of all smooth and non-degenerate degree d rational curves $X \subset \mathbb{P}^n$ has the structure of a smooth and integral algebraic variety, whose general point X has *maximal rank*: hence $h^0(\mathcal{I}_X(t)) = 0$ for some $t \in \mathbb{N}$ if and only if $dt + 1 \geq h^0(\mathcal{O}_{\mathbb{P}^n}(t))$, i.e., whenever it is possible for the restriction morphism to be injective; see [24, Théorème 0.1] or [6, Theorem, p. 541] for $n = 3$, [5, Theorem 1] for $n = 4$, and [8, Theorem, p. 355] for $n > 4$. Thus a general $X \in H(d,0;n)$ satisfies
$$h^0(\mathcal{I}_X(s-1)) = 0 \quad \text{if and only if} \quad (s-1)d + 1 \geq \binom{n+s-1}{n}.$$
This implies that $d_{n,s} \leq \delta$, as we have a (rational) integral curve of degree δ.

For the converse inequality, consider any integral curve $Y \subset \mathbb{P}^n$ such that $h^0(\mathcal{I}_Y(s-1)) = 0$, and let $k = \deg(Y)$. Since $h^0(\mathcal{I}_Y(s-1)) = 0$, we have $h^0(\mathcal{O}_Y(s-1)) \geq \binom{n+s-1}{n}$. Since Y is an integral curve (of arbitrary arithmetic genus), elementary considerations give
$$h^0(L) \leq \deg(L) + 1,$$
for an arbitrary line bundle L on Y. Thus
$$\deg(\mathcal{O}_Y(s-1)) + 1 = (s-1)k + 1 \geq h^0(\mathcal{O}_Y(s-1)) \geq \binom{n+s-1}{n}.$$
Taking the minimum, the latter inequality implies $d_{n,s} \geq \delta$, which completes the proof. □

Lemma 3.2. *Let $X \subset \mathbb{P}^n$ be an integral curve $X \in \mathcal{A}(n,s)$ such that* $\deg(X) = d_{n,s}$. *Then its arithmetic genus satisfies*
$$p_a(X) \leq (s-1)d_{n,s} + 1 - \binom{n+s-1}{n}.$$

Proof. Since $h^0(\mathcal{I}_X(s-1)) = 0$, Riemann-Roch shows that the statement is true if $h^1(\mathcal{O}_X(s-1)) = 0$. Otherwise, assume $h^1(\mathcal{O}_X(s-1)) > 0$. Since $h^0(\mathcal{O}_X(s-1)) \geq \binom{n+s-1}{n}$, Clifford's theorem gives $(s-1)d_{n,s} \geq 2\binom{n+s-1}{n} - 2$. By Proposition 3.1, we have
$$(s-1)(d_{n,s} - 1) + 2 \leq \binom{n+s-1}{n}.$$
Altogether, they give $\binom{n+s-1}{n} \leq s-1$, contradicting the assumption $s \geq 3$. □

Remark 3.3. Fix an integer g such that $0 \leq g \leq (s-1)d_{n,s} + 1 - \binom{n+s-1}{n}$. Again, as in the proof of Lemma 3.2, Proposition 3.1 yields
$$(s-1)(d_{n,s} - 1) + 2 \leq \binom{n+s-1}{n},$$
and so $g \leq s-2$.

Claim. We have $s-2 \leq d_{n,s}-n$.

Proof of the Claim. By contradiction, assume $d_{n,s} \leq n+s-3$. By Proposition 3.1, in order to obtain a contradiction, it is enough to show

$$(s-1)(n+s-3)+2 \leq \binom{n+s-1}{n}.$$

Let $\varphi(n,s)=\binom{n+s-1}{n}-(s-1)(n+s-3)-2$. This function is non-negative in the desired range. Indeed, consider

$$\varphi(3,s)=(s+2)(s+1)s/6-s(s-1)-2.$$

It is clear this is non-negative for all $s \geq 3$. Moreover:

$$\varphi(n+1,s)-\varphi(n,s)=\binom{n+s-2}{n}-s+1 \geq 0, \quad \text{for all} \quad s \geq 3,$$

which leads to a contradiction.

By the *Claim*, [6, Theorem, p. 541] for $n=3$, [5, Theorem 1] for $n=4$, [8, Theorem, p. 355], a general curve $X \in H(d_{n,s},g;n)'$ has maximal rank. Moreover, for each integer g with $0 \leq g \leq (s-1)d_{n,s}+1-\binom{n+s-1}{n}$, there exists a curve $X \in H(d_{n,s},g;n)$ such that $h^0(\mathcal{I}_X(s-1))=0$. Since $h^1(\mathcal{O}_X(1))=0$, we have $h^0(\omega_X(-1))=0$. The latter condition is equivalent to $h^0(\omega_X(-t))=0$ for all $t \geq 1$. Hence $h^1(\mathcal{O}_X(t))=0$ for $t=s-1,s$. Moreover, we have:

$$h^1(\mathcal{I}_X(s-1))=(s-1)d_{n,s}+1-g-\binom{n+s-1}{n},$$

$$h^0(\mathcal{I}_X(s))=h^1(\mathcal{I}_X(s))+\binom{n+s}{n}-sd_{n,s}-1+g.$$

Now, choose $g_0=(s-1)d_{n,s}+1-\binom{n+s-1}{n}$, so that $h^1(\mathcal{I}_X(s-1))=0$. Furthermore

$$h^2(\mathcal{I}_X(s-2))=h^1(\mathcal{O}_X(s-2))=0,$$

where the right-most equality holds because $s \geq 3$. By the well-known Castelnuovo-Mumford's Lemma, we derive

$$h^1(\mathcal{I}_X(t))=0 \text{ for all } t \geq s.$$

Finally,

$$h^0(\mathcal{I}_X(s))=\binom{n+s}{n}-sd_{n,s}-1+g_0=\binom{n+s-1}{n-1}-d_{n,s}.$$

To summarize, in Remark 3.3, we have shown the following:

Remark 3.4. Setting $g_0=(s-1)d_{n,s}+1-\binom{n+s-1}{n}$, we have:

$$d_{n,s} \geq n+g_0;$$

additionally, a general curve $X \in H(d_{n,s},g_0;n)'$ satisfies

$$h^0(\mathcal{I}_X(s-1))=0 \text{ and } h^0(\mathcal{I}_X(s))=\binom{n+s-1}{n-1}-d_{n,s}.$$

We point out that such curves have an s-linear resolution and were studied in $\mathbb{P}^3$ by Eisenbud and Goto [14, §5]. Whence such curves exist in every $\mathbb{P}^n$ for $n \geq 3$. [14, Corollary 5.1] relates the s-linear resolution of a space curve with the maximal rank property. The latter was established for curves in $\mathbb{P}^3$ in [7, Theorem 1].

Question 3.5. Is it true that

$$\alpha(n,s) = \binom{n+s-1}{n-1} - d_{n,s}?$$

Keep the definition of g_0 from Remark 3.4. Does equality hold if and only if X is a general curve in $H(d_{n,s}, g_0; n)'$?

Lemma 3.6. *Let $n \geq 3$ and $s \geq 3$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve such that $h^0(\mathcal{I}_X(s-1)) = 0$. Fix a general hyperplane $H \subset \mathbb{P}^n$. Then*

$$h^0(\mathcal{I}_X(s)) = \binom{n+s-1}{n-1} - \deg(X) - h^1(\mathcal{I}_X(s-1)) + h^1(\mathcal{I}_X(s)) + \\ + h^1(\mathcal{O}_X(s-1)) - h^1(\mathcal{O}_X(s)). \tag{1}$$

Proof. Fix a general hyperplane $H \subset \mathbb{P}^n$. Note that $h^0(H, \mathcal{I}_{X\cap H}(s)) = \binom{n+s-1}{n-1} - \deg(X) + h^1(H, \mathcal{I}_{X\cap H}(s))$. Since $h^1(\mathcal{O}_X(t)) = h^2(\mathcal{I}_X(t))$ for all $t \in \mathbb{Z}$, in order to conclude it is sufficient to use the long exact sequence in cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_X(s-1) \longrightarrow \mathcal{I}_X(s) \longrightarrow \mathcal{I}_{X\cap H,H}(s) \longrightarrow 0, \tag{2}$$

along with the assumption $h^0(\mathcal{I}_X(s-1)) = 0$. □

Now we discuss the term appearing on the right-hand side of equality (1).

Remark 3.7. In the setting of Lemma 3.6, as usual let $d = \deg(X)$ and $g = p_a(X)$. Since $h^0(\mathcal{I}_X(s-1)) = 0$, by Riemann-Roch, we have

$$h^1(\mathcal{I}_X(s-1)) = (s-1)d + 1 - g - \binom{n+s-1}{n}.$$

Define the numerical function $\psi(t) := h^1(\mathcal{O}_X(t-1)) - h^1(\mathcal{O}_X(t))$. For fixed integers g, $h^1(\mathcal{I}_X(s))$ and $\psi(s)$, the right-hand side of (1) is a strictly decreasing function in d.

Since $\deg(\mathcal{O}_X(t)) = d + \deg(\mathcal{O}_X(t-1))$, Serre duality and Riemann-Roch give

$$0 \leq \psi(t) \leq d.$$

Therefore a class of curves for which the term $h^1(\mathcal{O}_X(s-1)) - h^1(\mathcal{O}_X(s))$ is zero is realized when $h^1(\mathcal{O}_X(s-1)) = 0$.

Moreover, note that the long exact sequence in cohomology of (2) yields

$$h^0(\mathcal{I}_X(s)) \leq \binom{n+s-1}{n-1} - d,$$

whenever $h^1(H, \mathcal{I}_{X\cap H,H}(s)) = 0$. As in [32, Chapter 3], let χ denote the minimal non-negative integer such that $h^1(H, \mathcal{I}_{X\cap H,H}(\chi+1)) = 0$. As classically shown by Castelnuovo, using (2) one obtains $h^1(\mathcal{O}_X(t)) = 0$ for all $t \geq \chi$ [32, Theorem 1, p. 52]. Furthermore, $h^1(\mathcal{I}_X(t)) \geq h^1(\mathcal{I}_X(t+1))$ for all $t \geq \chi$ [32, Lemma 2, p. 53], and $h^1(\mathcal{I}_X(t)) > h^1(\mathcal{I}_X(t+1))$ whenever $t > \chi$ and $h^1(\mathcal{I}_X(t)) \neq 0$.

Note that the proof of Proposition 3.1 shows that a curve in $\mathcal{A}(n,s)$ of minimal degree $d_{n,s}$ can be chosen to be smooth (and rational). More generally, we wonder the following:

Question 3.8. Let $n \geq 3$, $s \geq 3$ and $d > d_{n,s}$.

(i) Is there an integral and non-degenerate curve $X \subset \mathbb{P}^n$ such that $\deg(X) = d$, $h^0(\mathcal{I}_X(s-1)) = 0$ and $h^0(\mathcal{I}_X(s)) \neq 0$?

(ii) Can we choose such an X to be smooth?

In the next two remarks we recall our knowledge around Question 3.8 for $n = 3$.

Remark 3.9. Recall that, by Proposition 3.1,

$$d_{3,s} = \left\lceil \left(\binom{s+2}{3} - 1 \right) / (s-1) \right\rceil .$$

For $d, s \geq 3$, to each pair (d,s), we attach the set of degree d curves in $\mathbb{P}^3$ *not* contained in any surface of degree $< s$. Given (d,s), Halphen's question asks in particular to determine the maximum genus of a curve associated to this pair. It is customary to divide the set of pairs (d,s) into four regions; see [23, §5].

Range $\emptyset$: $d < (s^2+4s+6)/6$. In this range, there is no integral and non-degenerate curve $X \subset \mathbb{P}^3$ such that $\deg(X) = d$ and $h^0(\mathcal{I}_X(s-1)) = 0$ [22, Theorem 3.3].

Range A: $(s^2+4s+6)/6 \leq d < (s^2+4s+6)/3$. In this range, one finds curves with $h^1(\mathcal{O}_X(s-1)) = 0$ and hence $h^1(\mathcal{O}_X(s)) = 0$. Thus if $h^0(\mathcal{I}_X(s-1)) = 0$ and $h^0(\mathcal{I}_X(s)) \neq 0$, by Riemann-Roch one has $(s-1)d+1-g \geq \binom{s+2}{3}$ and $sd+1-g \leq \binom{s+3}{3}$. Moreover, if $\lceil s(s+2)/4 \rceil \leq d < (s^2+4s+6)/3$, it is known such a curve exists for any such choice of (d,s), along with curves reaching the maximal genera allowed; see, e.g., [3, 15, 16]. To fill in all (d,s) such that $(s^2+4s+6)/6 \leq d < \lceil s(s+2)/4 \rceil$ the same result (the existence of curves with (d,s) and $g = (s-1)d+1-\binom{s+2}{3})$ is only known for s large enough, i.e., $s \geq 10.5 \times 10^5$, by the recent [9, Theorem 1].

Range B: $(s^2+4s+6)/3 \leq d < s(s-1)$. Hartshorne and Hirschowitz constructed, for any (d,s) in this range, smooth space curves with very high genera and conjectured that they have the maximal genus for all curves associated to (d,s) [23, Théorème 4.1].

Range C: $d > s(s-1)$. One finds a smooth degree d space curve X such that $\deg(X) = d$ and $h^0(\mathcal{I}_X(s-1)) = 0$; Gruson and Peskine classified all such curves X of maximal genus ([18, Theorem 3.2], [22, Theorem 3.1], [28, Theorem 1.1]): if $d \equiv 0 \pmod{s}$, these curves are complete intersections of a surface of degree s and a surface of degree d/s; in all other cases X is *linked* to a plane curve C of degree $\deg(C) = s\lceil d/s \rceil - d$, by the complete intersection of a surface of degree s and a surface of degree $\lceil d/s \rceil$. For $d \geq s^2$, we have $h^0(\mathcal{I}_X(s)) \leq 2$ with equality if and only if $d = s^2$ and X is the complete intersection of two surfaces of degree s.

Remark 3.10. Assume (d,s) is in **Range A**. Let $X \subset \mathbb{P}^3$ be any integral curve with $\deg(X) = d$, $h^1(\mathcal{O}_X(s-1)) = 0$, $h^0(\mathcal{I}_X(s-1)) = 0$, and $h^0(\mathcal{I}_X(s)) \neq 0$, i.e., by Riemann-Roch one has $(s-1)d+1-g \ge \binom{s+2}{3}$, and $sd+1-g \le \binom{s+3}{3}$, where $g = p_a(X)$ is the arithmetic genus of X. All curves constructed in [3, 9, 10, 15, 16] have $h^1(\mathcal{I}_X(s-1)) = h^2(\mathcal{I}_X(s-1)) = 0$ and hence $h^0(\mathcal{I}_X(s)) = \binom{s+2}{2} - d$. The ones in the main results of [3, 9] have $g = d(s-1)+1-\binom{s+2}{3}$, whereas the ones given in [3, Proposition 4.3] and in [10, Theorem 1.2 and Corollary 1.3] have many different genera for the fixed pair (d, s).

We generalize **Range A** to curves embedded in $\mathbb{P}^n$ for any $n \ge 3$. For all integers $n \ge 3$, $s \ge 2$ and $d \ge d_{n,s}$, let $\mathcal{B}(n,s,d)$ (resp. $\mathcal{B}(n,s,d)'$) denote the set of all smooth and connected (resp. integral) $X \in \mathcal{A}(n,s)$ such that $\deg(X) = d$ and $h^1(\mathcal{O}_X(s-1)) = 0$.

Definition 3.11. Curves in $\mathcal{B}(n,s,d)$ are said to be in the *generalized* **Range A**.

Remark 3.12. Let $X \in \mathcal{B}(n,s,d)'$. Since $h^1(\mathcal{O}_X(s-1)) = 0$, Riemann-Roch gives $h^0(\mathcal{O}_X(s-1)) = d(s-1)+1-p_a(X)$. Since $h^0(\mathcal{I}_X(s-1)) = 0$, one obtains $h^0(\mathcal{O}_X(s-1)) \ge h^0(\mathcal{O}_{\mathbb{P}^n}(s-1)) = \binom{n+s-1}{n}$, and so:

$$p_a(X) \le d(s-1)+1-\binom{n+s-1}{n}.$$

Question 3.13. We pose the following questions:

(i) For which choices of d, s, n, does there exist $X \in \mathcal{B}(n,s,d)$ (or even $X \in \mathcal{B}(n,s,d)'$) such that $p_a(X) = d(s-1)+1-\binom{n+s-1}{n}$?
(ii) What is the maximal arithmetic genus of curves in $\mathcal{B}(n,s,d)$ (or in $\mathcal{B}(n,s,d)'$)?
(iii) For which g_0 and every $0 \le g \le g_0$, does there exist $X \in \mathcal{B}(n,s,d)$ (or $X \in \mathcal{B}(n,s,d)'$) with such arithmetic genus?

Remark 3.14. For any $X \in \mathcal{B}(n,s,d)'$, one has $h^2(\mathcal{I}_X(s-1)) = 0$. Suppose there exists $X \in \mathcal{B}(n,s,d)'$ such that $p_a(X) = d(s-1)+1-\binom{n+s-1}{n}$. Thus $h^0(\mathcal{I}_X(s-1)) = h^2(\mathcal{I}_X(s-1)) = 0$. By Riemann-Roch, $h^0(\mathcal{O}_X(s-1)) = 0$, and hence $h^1(\mathcal{I}_X(s-1)) = 0$. Therefore, these curves are exactly the integral and non-degenerate degree d curves $X \subset \mathbb{P}^n$ such that $h^i(\mathcal{I}_X(s-1)) = 0$ for all $i \ge 0$; this is because for any curve $Y \subset \mathbb{P}^n$, one has $h^i(\mathcal{I}_Y(t)) = h^{i+1}(\mathcal{O}_{\mathbb{P}^n}(t)) = 0$ for all $i \ge 3, i \ne n$ and all $t \in \mathbb{Z}$, and for all $t > -n$ when $i = n$.

Take any hyperplane $H \subset \mathbb{P}^n$. By the exact sequence (2), we have $h^1(\mathcal{I}_X(s)) = h^1(H, \mathcal{I}_{X\cap H,H}(s))$. Assume $h^1(\mathcal{O}_X(s-2)) = 0$, i.e., $h^1(\mathcal{I}_X(s-2)) = 0$. Since $h^i(\mathcal{I}_X(t)) = 0$ for all $i \ge 3$ and $t > -n$, we have $h^i(\mathcal{I}_X(s-1-i)) = 0$ for all $i \ge 0$. In this case, Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge s$: this implies that the homogeneous ideal of X is generated by $\mathrm{H}^0(\mathcal{I}_X(s))$.

Now we drop the assumption $h^1(\mathcal{O}_X(s-2)) = 0$, and instead suppose $h^1(\mathcal{I}_X(s)) = 0$, i.e., $h^1(H, \mathcal{I}_{X\cap H,H}(s)) = 0$. This implies $d \le \binom{n+s-1}{n-1}$. In this case, the Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_X(t)) = 0$ for all $t > s$, which implies that the homogeneous ideal of X is generated by $\mathrm{H}^0(\mathcal{I}_X(s))$ and $\mathrm{H}^0(\mathcal{I}_X(s+1))$.

Remark 3.15. Fix integers n, d, g, s such that $n \ge 3$, $s \ge 3$, with

(i) $0 \le g \le d-n$,
(ii) $(n+1)d \ge ng + n(n+1)$,
(iii) $(s-1)d \ge \binom{n+s-1}{n} + g - 1$, and

(iv) $sd \leq \binom{n+s-1}{n} + g - 2$.

By [6, Theorem, p. 541] for $n = 3$, [5, Theorem 1] for $n = 4$ and [8, Theorem, p. 355] there exists a smooth, connected, and non-degenerate *maximal rank* curve $X \subset \mathbb{P}^n$ with $\deg(X) = d$, arithmetic genus $p_a(X) = g$, and $h^0(\mathcal{I}_X(s)) = \binom{n+s}{n} - sd + g - 1$. Hence, setting the arithmetic genus to be the maximal in this range, i.e., $g = d - n$, (i) and (ii) in Question 3.8 have a positive answer, whenever

$$h^0(\mathcal{I}_X(s)) > 0 \iff (s-1)d < \binom{n+s}{n} - n - 1.$$

Note that inequality (iii) implies $h^0(\mathcal{I}_X(s-1)) = 0$, because X has maximal rank.

Using curves of minimal degree in $\mathcal{A}(n, s)$, for every $m \geq 1$ (with $n \geq m + 3$), we construct an non-degenerate, irreducible variety (that is not a cone) of arbitrarily high degree in $\mathcal{A}(n, s, m)$; this is accomplished in Proposition 3.17, based on the vector bundle construction featured in the next remark.

Remark 3.16. Let $m \geq 1$ and $e \geq 2$. Consider the rank $m + 1$ vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)^{\oplus m}$ on $\mathbb{P}^1$. Define the corresponding projective bundle $T = \mathbb{P}(\mathcal{E})$ and let $\pi : T \longrightarrow \mathbb{P}^1$ be the vector bundle projection on $\mathbb{P}^1$. The map π makes T a $\mathbb{P}^m$-bundle and hence $\mathrm{Pic}(T) \cong \mathbb{Z}^2$, with basis given by a fiber f of π and any line bundle on T whose restriction to any fiber of π has degree one; see, e.g., [21, Ex. II.7.9]. Among these generators we take the only one, say h, such that $|h| = \{h\}$ (this corresponds to the unique surjection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)^{\oplus m} \longrightarrow \mathcal{O}_{\mathbb{P}^1}$, [21, Ch. II]). As an abstract variety, h is the trivial $\mathbb{P}^{m-1}$-bundle over $\mathbb{P}^1$ and hence $h \cong \mathbb{P}^1 \times \mathbb{P}^{m-1}$.

Similarly to the case of surfaces [21, Ch. V], for any $a, b \in \mathbb{Z}$ we have $|ah + bf| \neq \emptyset$ if and only if either $a = 0$ and $b \geq 0$ or $a > 0$ and $b \geq ae$. The line bundle $\mathcal{O}_T(ah + bf)$ is globally generated (resp. ample) if and only if $a \geq 0$ and $b \geq ae$ (resp. $a > 0$ and $b > ae$).

The complete linear system $h + ef$ induces a morphism $u : T \longrightarrow \mathbb{P}^n$, where $n = e + m$, as $h^0(\mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus m}) = e + 1 + m$. Let $W = u(T)$ be the image of T. The morphism u contracts h to an $(m-1)$-dimensional linear space in $E \subset \mathbb{P}^n$, whereas $u|_{T \setminus h}$ is an embedding. Thus u is a morphism birational onto its image, as the support of h is codimension-one.

> *Claim 1.* The variety W has degree e.
> *Proof of Claim 1.* Since u is a morphism birational onto its image, $\deg(W) = (h + ef)^{m+1}$, where the latter integer is the intersection product in the Chow ring of T of $m + 1$ copies of the divisor $h + ef$ defining the morphism. Since any two different fibers of π are disjoint, the class f^2 in the Chow ring is zero. Thus $(h + ef)^{m+1} = h^{m+1} + (m + 1)eh^m f$. Fix a fiber $F \in |f|$ of π. Since $h_{|F}$ is the degree one line bundle on F and $F \cong \mathbb{P}^m$, we have $h^m f = 1$ and hence $(m + 1)eh^m f = (m + 1)e$.
> If $m = 1$, T is a rational ruled surface. The normal bundle of h has degree $h^2 = -e$. Hence the conclusion follows for $m = 1$.
> Assume $m \geq 2$. Since $h \cong \mathbb{P}^1 \times \mathbb{P}^{m-1}$, we have $\mathrm{Pic}(h) \cong \mathbb{Z}^2$. As generators of the lattice $\mathrm{Pic}(h)$, we take the pullbacks of hyperplane classes

of the factors through the projections $\pi_1 : h \longrightarrow \mathbb{P}^1$ and $\pi_2 : h \longrightarrow \mathbb{P}^{m-1}$, i.e., $\mathcal{O}_h(1,0) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathcal{O}_h(0,1) = \pi_2^*(\mathcal{O}_{\mathbb{P}^{m-1}}(1))$.
By definition, the restriction $h_{|h}$ is the normal bundle of h in T and hence $h_{|h} \cong \mathcal{O}_h(-e,1)$, by induction on m and using standard exact sequences on normal bundles. Thus $h^{m+1} = h^m|_h = -me$, concluding the proof.

By *Claim 1*, W is a minimal degree $(m+1)$-dimensional subvariety of $\mathbb{P}^n$, where $n = e+m$. It is a cone whose vertex is the $(m-1)$-dimensional linear space $E = u(h)$, and the base of the cone is a degree e rational normal curve C_e in a linear space $M \subset \mathbb{P}^n$ such that $\dim M = n - m = e$ and $M \cap E = \emptyset$. Thus W is arithmetically Cohen-Macaulay. In particular, for any integer $s > 0$, a Weil divisor $D \subset W$ is contained in a degree s hypersurface of $\mathbb{P}^n$ *not containing* W if and only if its pullback D' to T is a part of the linear system $|sh + sef|$, i.e., there exists a divisor $D'' \geq 0$ such that $D' + D'' \in |sh + sef|$. Thus $h^0(\mathcal{I}_D(s)) = h^0(\mathcal{I}_W(s))$ if $D' \in |h + bf|$ for some integer $b \geq se + 1$. In the Chow ring, we have:

$$\deg(D) = D' \cdot (h+ef)^m = (h+bf)\cdot(h+ef)^m =$$

$$= h^{m+1} + meh^m f + b = b.$$

Moreover, we record the following claims which will be useful in Proposition 3.17:

Claim 2. Every line $L \subset W$ meets the $(m-1)$-dimensional linear space $E = u(h)$. Moreover, either $L \subset E$ (and hence it is contained in every $u(F)$, where F is any fiber of the ruling π) or L is contained in a unique m-dimensional linear space $U = u(F) \subset W$ with F a fiber of π.

Proof of Claim 2. Assume the existence of a line $L \subset W$ such that $L \cap E = \emptyset$. Let $\ell_E : \mathbb{P}^n \setminus E \longrightarrow \mathbb{P}^{n-m}$ denote the linear projection from E. Note that $\ell_E(W \setminus E) = C_e$. The algebraic set $\ell_E(L)$ would be a line, whereas $\ell_E(L) \subseteq \ell_E(W \setminus E) = C_e$ is an irreducible curve of degree $e \geq 2$, a contradiction. Thus $L \cap E \neq \emptyset$.

Assume $L \nsubseteq E$. Thus $L \cap E$ is a single point. Let $\ell_E : \mathbb{P}^n \setminus E \longrightarrow \mathbb{P}^{n-m}$ denote the linear projection from E. Again, $C_e = \ell_E(W \setminus E)$, which is the degree e rational normal curve in $\mathbb{P}^{n-m} = \mathbb{P}^e$. For each $q \in C_e$, the set $\ell_E^{-1}(q) \cap (W \setminus E)$ is of the form $U_q \setminus U_q \cap E$, where U_q is an m-dimensional linear space, which is the image of a unique fiber F_q of π. Since $L \cap E$ is a single point and L is a line, $\ell_E(L \setminus L \cap E)$ is a point $p \in C_e$. Set $U := U_p$ and $F := F_p$.

Claim 3. For a general $D' \in |h + bf|$ with $b \geq m + e + 1$, the Weil divisor $D = u(D')$ of W is not a cone.

Proof of Claim 3. Recall $u(T) = W$ and $u(h) = E$. Up to the identification between h and $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ and between E and $\mathbb{P}^{m-1}$ the morphism $u_{|h} : \mathbb{P}^1 \times \mathbb{P}^{m-1}$ is the projection onto the second factor. Suppose by contradiction that D is a cone and suppose v is a point in its vertex.

First assume $v \in E$. Before taking a general $D' \in |h + bf|$, we fix m different fibers F_i of π, for $1 \leq i \leq m$. Thus $F_i \cap F_j = \emptyset$ for all $i \neq j$. Up to the identification of h with $\mathbb{P}^1 \times \mathbb{P}^{m-1}$, one has $F_i \cap h = \{p_i\} \times \mathbb{P}^{m-1}$ for some $p_i \in \mathbb{P}^1$. Set $\mathcal{F} = \cup_{i=1}^m F_i \subset T$. Since $b \geq m + e + 1$, we have

$h^1(\mathcal{O}_T(h + (b - m)f)) = 0$. Thus the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_T(h + (b - m)f) \longrightarrow \mathcal{O}_T(h + bf) \longrightarrow \mathcal{O}_{\mathcal{F}}(h + bf) \longrightarrow 0$$

shows that for a general $C' \in |\mathcal{O}_{\mathcal{F}}(h + bf)|$ there exists some $D' \in |\mathcal{O}_T(h + bf)|$ with $C' = D' \cap \mathcal{F}$. Note that $\mathcal{O}_{F_i}(h + bf)$ is the degree one line bundle on F_i. For a general $C' \in |\mathcal{O}_{\mathcal{F}}(h + bf)|$, the divisor $C'_{|F_i}$ of $\{p_i\} \times \mathbb{P}^{m-1}$ is of the form $\{p_i\} \times H_i$ with H_i hyperplane of $\mathbb{P}^{m-1}$ and such that $\cap_{i=1}^m H_i = \emptyset$. For a general $D' \in |h + bf|$, we may assume that $D' \cap \mathcal{F}$ is as C'. Recall we are identifying E with $\mathbb{P}^{m-1}$, h with $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ and $u_{|h}$ with the projection onto the second factor. Thus there is an index $i \in \{1, \dots, m\}$ such that $v \notin H_i$, as the intersection of all of them is empty. Then $v \notin u(F_i \cap D') \cap E$ and so $v \notin u(F_i \cap D')$.

Since $D = u(D')$ is not an m-dimensional projective space (it has degree b), and the $(m - 1)$-dimensional projective space $u(F_i \cap D')$ is contained in D and $v \notin u(F_i \cap D')$, it follows that v is not a vertex of D.

Suppose now $v \notin E$. In this case, by *Claim 2* and the first part of this proof, the vertex of D can only consist of the single point v. Therefore, there is a unique fiber G of π such that $v \in u(G)$. Every line through v contained in W is contained in $u(G)$. Since $u(D') \nsubseteq u(G)$, v cannot be the vertex of D.

Finally, observe that in the proof of Proposition 3.17 we utilize the claims above taking a general $D' \in |h + bf|$ with $b \geq \max\{m + e + 1, se + 1\}$, so that *both* the observation after *Claim 1* and *Claim 3* are satisfied.

We are now ready to prove the following

Proposition 3.17. *Let $m \geq 1$, $n \geq m + 3$, $s \geq 3$ and $d \geq \max\{m + d_{n-m,s} + 1, (s-1)d_{n-m,s}+1\}$. There exists an integral and non-degenerate m-dimensional variety $X \subset \mathbb{P}^n$, with $\deg(X) = d$ and $h^0(\mathcal{I}_X(s - 1)) = 0$. Moreover, X is not a cone.*

Proof. Set $\delta := d_{n-m,s}$. Let $H \subset \mathbb{P}^n$ be a linear subspace with $\dim H = n - m$. Then there exists a smooth rational curve $Y \subset H$ such that

$$\deg(Y) = \delta, \quad h^0(H, \mathcal{I}_{Y,H}(s - 1)) = 0, \text{ and } \ h^0(H, \mathcal{I}_{Y,H}(s)) = \binom{n + s - m}{n - m} - \delta;$$

see [24, Théorème 0.1] or [6, Theorem, p. 541] for $n - m = 3$, [5, Theorem 1] for $n - m = 4$, and [8, Theorem, p. 355] for $n - m > 4$.

The curve Y is an isomorphic linear projection of $C_\delta \subset \mathbb{P}^\delta$. We view H as a linear subspace of $\mathbb{P}^\delta$. Let $V \subset \mathbb{P}^\delta$ be a linear subspace such that $\dim V = \delta - n + m - 1$, $V \cap C_\delta = \emptyset$ and $Y = \ell_V(C_\delta)$, where ℓ_V is the projection from V. Therefore, with these conventions, $V \cap H = \emptyset$.

Before we proceed, we introduce another piece of notation: for two varieties X and Y, the variety $J(X, Y)$ denotes their join; when $X = Y$, denote $J(X, X) = \sigma_2(X)$, the secant variety of X.

Regard the linear space $\mathbb{P}^\delta$ as a linear subvariety of $\mathbb{P}^{\delta+m}$. Fix an $(m-1)$-dimensional linear space $E \subset \mathbb{P}^{\delta+m}$ such that $E \cap H = \emptyset$ and $J(H, E) \cap V = \emptyset$; this is possible by dimension count.

Let $W \subset \mathbb{P}^{\delta+m}$ be the cone whose vertex is E and whose base is the rational normal curve C_δ. Such a cone W is a minimal degree $(m+1)$-dimensional cone as the one described in Remark 3.16 (where $\delta = e$). Therefore $\sigma_2(W) \cap V = \emptyset$.

Now, let $\ell_V : \mathbb{P}^{\delta+m} \setminus V \longrightarrow \mathbb{P}^n$ be the linear projection from V, and let $W' = \ell_V(W)$. Since $\sigma_2(W) \cap V = \emptyset$, the linear projection ℓ_V induces an injective map between the two cones W and W' which is an isomorphism outside their vertices. By construction, W' is a cone with the smooth rational curve Y as its base. Note that $h^0(\mathcal{I}_{W'}(s-1)) = h^0(\mathcal{I}_Y(s-1)) = 0$.

Let $D \subset W$ be general as discussed in Remark 3.16: D is such that $\deg(D) = d$, $\dim D = m$ and $h^0(\mathbb{P}^{\delta+m}, \mathcal{I}_D(s-1)) = h^0(\mathbb{P}^{\delta+m}, \mathcal{I}_W(s-1))$. (Here we use $s-1$ instead of s and the inequality $d \geq \max\{m+\delta+1, (s-1)\delta+1\}$ in the statement.)

Set $X = \ell_V(D)$. Since $\ell_{V|W}$ is injective, X is an m-dimensional integral variety and $\deg(X) = d$. Recall $h^0(\mathbb{P}^{\delta+m}, \mathcal{I}_D(s-1)) = h^0(\mathbb{P}^{\delta+m}, \mathcal{I}_W(s-1))$.

Now we show $h^0(\mathbb{P}^n, \mathcal{I}_X(s-1)) = 0$. On the contrary, suppose $h^0(\mathbb{P}^n, \mathcal{I}_X(s-1)) \neq 0$ and take a hypersurface $N \in |\mathcal{I}_X(s-1)|$. Taking the cone N_V over N with vertex V we obtain a hypersurface in $\mathbb{P}^{\delta+m}$ containing D, i.e., $N_V \in |\mathcal{I}_D(s-1)|$. However, N cannot contain W' because $h^0(\mathcal{I}_{W'}(s-1)) = 0$, and so N_V cannot contain W either. This contradicts the equality $h^0(\mathbb{P}^{\delta+m}, \mathcal{I}_D(s-1)) = h^0(\mathbb{P}^{\delta+m}, \mathcal{I}_W(s-1))$.

Recall that the map $\ell_{V|W}$ is a bijection of the cones W and W', which is an isomorphism outside their vertices and induces an isomorphism between the m-dimensional linear subspaces of W and the ones of W'. Thus each line contained in W' is the image of a unique line of W'. If $J \subset W$ is a curve such that $\ell_V(J)$ a line, then J is a line. Since D is not a cone by *Claim 3* of Remark 3.16, X is not a cone. □

4. Surfaces

In this section, we shift gears to surfaces, i.e., $m = 2$. We start by recording an upper bound on the dimension of a linear system on a smooth surface, derived from the Kawamata Rationality Theorem. (For a given projective surface X, denote by $\kappa(X)$ its Kodaira dimension.)

Theorem 4.1. *Let $n \geq 4$, $s \geq 2$ and $d \geq n-1$. Let $X \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate degree d surface. Then:*

(i) *If $\kappa(X) \neq -\infty$, then $h^0(\mathcal{O}_X(s)) \leq 2 + s^2 d/2$.*
(ii) *If $\kappa(X) = -\infty$, $X \ncong \mathbb{P}^2$ and X is $\mathbb{P}^1$-bundle over a smooth curve $\pi : X \to D$ such that the rational fibers have degree one (i.e., X is a scroll), then $h^0(\mathcal{O}_X(s)) \leq 1 + (s^2+2s)d/2$.*
(iii) *We have $h^0(\mathcal{O}_X(s)) \leq 1 + (s^2+3s)d/2$ and equality holds if and only if $d = k^2$ for some k such that $\binom{k+2}{2} \geq n+1$, $X \cong \mathbb{P}^2$ and X is isomorphic to either the degree k Veronese embedding of $\mathbb{P}^2$ or to an isomorphic linear projection of it.*
(iv) *Otherwise, if $\omega_X^{\otimes 2}(3)$ is nef, then $h^0(\mathcal{O}_X(s)) \leq 1 + (s^2 + \frac{3}{2}s)d/2$.*

Proof. Fix a general $C \in |\mathcal{O}_X(s)|$. By Bertini's theorem, C is a smooth and connected curve of degree sd. By the genus formula, C has genus

$$g = 1 + (s^2 d + \omega_X \cdot \mathcal{O}_X(s))/2.$$

Multiplying by the section C yields the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(s) \longrightarrow \mathcal{O}_C(s) \longrightarrow 0. \tag{3}$$

Thus $h^0(\mathcal{O}_X(s)) \leq h^0(\mathcal{O}_C(s)) + 1$, where equality holds if $h^1(\mathcal{O}_X) = 0$.
If $s^2d = \deg(\mathcal{O}_C(s)) \leq 2g-2$, then this divisor on C is special and Clifford's theorem [2, pp. 107–108] gives

$$h^0(\mathcal{O}_C(s)) \leq 1 + s^2d/2.$$

Whence $h^0(\mathcal{O}_X(s)) \leq 2 + s^2d/2$.

From now on, assume $s^2d \geq 2g-1$ (or equivalently, $\mathcal{O}_C(s)$ is non-special). Again, by the genus formula, one has

$$\omega_X \cdot \mathcal{O}_X(1) < 0.$$

Note that Riemann-Roch gives $h^0(\mathcal{O}_C(s)) = s^2d + 1 - g = \frac{1}{2}(s^2d - s\omega_X \cdot \mathcal{O}_X(1))$.

Before we proceed, recall that a line bundle $\mathcal{L}$ on a projective variety X of dimension ≥ 2 is said to be *nef* if $\mathcal{L} \cdot C \geq 0$, for every algebraic curve $C \subset X$.

> *Claim.* If $\omega_X \cdot \mathcal{O}_X(1) < 0$, then $\kappa(X) = -\infty$.
> *Proof of the Claim.* Assume on the contrary that $\kappa(X) \geq 0$. Let X' be the minimal model of X and $\varphi : X \longrightarrow X'$ be the corresponding morphism. Since $\kappa(X') = \kappa(X) \geq 0$, $\omega_{X'}$ is nef. We have $\omega_X \cong \varphi^*\omega_{X'} + E$, where E is an effective divisor and $E = 0$ if and only if X is minimal. Since $\varphi^*\omega_{X'}$ is nef and $\mathcal{O}_X(1)$ is effective, we derive
> $$(\varphi^*\omega_{X'} + E) \cdot \mathcal{O}_X(1) \geq 0,$$
> which is a contradiction.

The *Claim* shows that the Kodaira dimension of X is $\kappa(X) = -\infty$, and statement (i) is proven.

We recall the Kawamata Rationality Theorem in the case of smooth surfaces. Let X be a smooth projective surface such that ω_X is *not* nef. For each ample line bundle $\mathcal{L}$ on X, the *nef-value* τ of $\mathcal{L}$ with respect to ω_X is defined as

$$\tau(\mathcal{L}) = \sup\{t > 0 \mid \mathcal{L} + t\omega_X \text{ is nef}\},$$

where $\mathcal{L} + t\omega_X$ is viewed as a functional on the cone of curves depending on a parameter t. The aforementioned theorem states that the real number τ is rational, $\tau = u/v$, where u is a positive integer and $v \in \{1, 2, 3\}$; see, e.g., [11, Theorem 1.5.2], [29, Corollary 1-2-15]. Moreover, $\mathcal{L} + \tau\omega_X$ is nef [11, Lemma 1.5.5], [29, Theorem 1-2-14]. Thus we have the following cases:

- $u = 1$ and $v = 3$. The cone of curves of X has an *extremal ray* of *length* $3 = \dim X + 1$; see [29] for these notions. This implies $X \cong \mathbb{P}^2$ and $\mathcal{O}_X(1)$ is the generator of $\mathrm{Pic}(X)$. Since $n \geq 4$, X is embedded as described in the statement.

- $u = 1$ and $v = 2$. Since $\mathcal{L} + \frac{1}{2}\omega_X$ is nef, so is the line bundle $2\mathcal{L} + \omega_X$. In this case, there exists a curve $C \subset X$ such that $C \cdot (2\mathcal{L} + \omega_X) = 0$. By [29, Theorem 1-4-8], X is a $\mathbb{P}^1$-bundle over a smooth curve D, $\pi : X \longrightarrow D$, and $\mathcal{L} \cong \mathcal{O}_X(1)$ has degree 1 on each fiber C of π. In such a case, $\omega_X(2)$ is nef and hence $\omega_X \cdot \mathcal{O}_X(1) \geq -2$. This proves statement (ii).

- $u \geq 2$ or $v = 1$. Then $\omega_X^{\otimes 2}(3)$ is nef and therefore $\omega_X \cdot \mathcal{O}_X(1) \geq -3/2$. In this case, we obtain (iv).

In conclusion, the discussion above yields $\omega_X \cdot \mathcal{O}_X(1) \geq -3$ for all X, with equality if and only if $X \cong \mathbb{P}^2$. This proves statement (iii). □

Remark 4.2. Let n, s be integers as in the assumptions of Theorem 4.1, and let k be an integer. Assume $\binom{n+s}{n} \leq 1 + (s^2+3s)k^2/2$. From the ideal sheaf exact sequence and Theorem 4.1 (iii), one has

$$h^1(\mathcal{I}_X(s)) \geq 1 + (s^2+3s)k^2/2 - \binom{n+s}{n},$$

for any isomorphic projection $X \subset \mathbb{P}^n$ of the degree k Veronese embedding of $\mathbb{P}^2$. Assuming the existence of such an X with the additional property that the latter inequality is an equality, one obtains $h^0(\mathcal{I}_X(s)) = 0$. Thus $d_{n,s+1,2} \leq \deg(X) = k^2$.

When $s = 2$, one has the following result that was shown in [4]:

Proposition 4.3 ([4, Theorem 2])**.** *Let $d \geq 2$ and $n \geq 5$. Let $X \subset \mathbb{P}^n$ be a general isomorphic linear projection of a degree k Veronese embedding $Y \subset \mathbb{P}^N$ of $\mathbb{P}^2$, where $N = (k^2+3k)/2$. Then*

$$h^0(\mathcal{I}_X(2)) = \max\left\{0, \binom{n+2}{2} - \binom{2k+2}{2}\right\},$$

$$\textit{and } h^1(\mathcal{I}_X(2)) = \max\left\{0, \binom{2k+2}{2} - \binom{n+2}{2}\right\}.$$

An immediate consequence of Proposition 4.3 and Remark 4.2 is

Corollary 4.4. *Let $n \geq 5$ be an odd integer. Then $d_{n,3,2} \leq (n+1)^2/4$.*

Proof. Let $X \subset \mathbb{P}^n$ be a general linear projection of the degree $(n+1)/2$ Veronese embedding of $\mathbb{P}^2$. By Proposition 4.3, we have

$$h^0(\mathcal{I}_X(2)) = \max\left\{0, \binom{n+2}{2} - \binom{n+3}{2}\right\} = 0.$$

□

Question 4.5. Fix integers $d \geq 2, s \geq 3$, and $n \geq 5$. Let $X \subset \mathbb{P}^n$ be a general isomorphic linear projection of a degree k Veronese embedding $Y \subset \mathbb{P}^N$ of $\mathbb{P}^2$, where $N = (k^2+3k)/2$. Is it true that

$$h^0(\mathcal{I}_X(s)) = \max\left\{0, \binom{n+s}{n} - \binom{sk+2}{2}\right\},$$

$$\text{and } h^1(\mathcal{I}_X(s)) = \max\left\{0, \binom{sk+2}{2} - \binom{n+s}{n}\right\}?$$

Theorem 4.6. *Let $n \geq 6$ be even. Then $h^0(\mathcal{I}_X(2)) \geq \binom{n+2}{2} - 1 - 5d$ for all smooth connected and non-degenerate surfaces of degree d. Moreover, there exists a smooth and connected surface $X \subset \mathbb{P}^n$ such that $\deg(X) = n^2/4$, $h^0(\mathcal{I}_X(2)) = 0$, and $X \cong \mathbb{P}^2$.*

Proof. The first part is statement (iii) of Theorem 4.1. By Proposition 4.3, the general isomorphic linear projection of the degree k Veronese surface has

$$h^0(\mathcal{I}_X(2)) = \max\left\{0, \binom{n+2}{2} - \binom{2k+2}{2}\right\}.$$

Thus the choice $k = n/2$ establishes the second statement. □

Remark 4.7. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate surface of sectional genus g and degree d. Then $h^0(\mathcal{I}_X(2)) \leq \binom{n+1}{2} - 2d - 1 + g$.

Proof. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_X(s-1) \longrightarrow \mathcal{I}_X(s) \longrightarrow \mathcal{I}_{X\cap H,H}(s) \longrightarrow 0. \tag{4}$$

Since H is general, $C = X \cap H$ is a smooth curve of degree d and genus g. Hence, by Riemann-Roch, $h^0(\mathcal{O}_C(2)) \geq 2d + 1 - g$. Thus

$$h^0(H, \mathcal{I}_{X\cap H,H}(2)) \leq \binom{n+1}{2} - 2d - 1 + g.$$

By (4) and $h^0(\mathcal{I}_X(1)) = 0$, the conclusion follows. □

Remark 4.8. A complete classification of surfaces of almost-minimal degree, i.e., all integral and non-degenerate surfaces $X \subset \mathbb{P}^n$ such that $\deg(X) = n$ is known; see, e.g., [25, 30]. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Since $C = X \cap H$ is a degree n integral curve spanning H, there are two possibilities: either C is arithmetically Cohen-Macaulay with $p_a(C) = 1$ or C is a smooth rational curve. In the latter case, X may have only finitely many singular points.

(i) C is arithmetically Cohen-Macaulay with $p_a(C) = 1$. The surface X is linearly normal of degree $\deg(X) \geq 2p_a(C) + 1$. Note that this case may occur for all n: take a cone over a linearly normal elliptic curve $C \subset \mathbb{P}^{n-1}$.

(ii) C is a smooth rational curve. Thus $h^1(H, \mathcal{I}_{X\cap H,H}(1)) = 1$, $h^1(H, \mathcal{I}_{X\cap H,H}(2)) = 0$, and so $h^0(H, \mathcal{I}_{X\cap H,H}(2)) = \binom{n+1}{2} - 2n - 1$. It is clear that

$$h^1(\mathcal{I}_X(1)) \leq h^1(H, \mathcal{I}_{X\cap H,H}(1)) = 1.$$

From (4), we derive $\binom{n+1}{2} - 2n - 2 \leq h^0(\mathcal{I}_X(2)) \leq \binom{n+1}{2} - 2n - 1$. Now, if X is smooth and $X \cap H$ is rational, then the classification of surfaces gives that X is rational. Thus $h^1(\mathcal{O}_X) = 0$. A standard exact sequence gives $h^0(\mathcal{O}_X(1)) = h^0(\mathcal{O}_{X\cap H}(1)) + 1 = n+2$ and $h^0(\mathcal{O}_X(2)) = h^0(\mathcal{O}_{X\cap H}(2)) + h^0(\mathcal{O}_X(1)) = 3n+3$. Thus $h^1(\mathcal{I}_X(1)) = 1$ and so X is not linearly normal. Let $X' \subset \mathbb{P}^{n+1}$ be a smooth surface such that X is an isomorphic linear projection of X' from some $p \in \mathbb{P}^{n+1}$ with p not contained in the secant variety of X'. Thus $\deg(X') = n$ and since $X' \subset \mathbb{P}^{n+1}$, it is either a minimal degree surface scroll or (when $n = 4$) the Veronese surface. In any case, one knows the minimal free resolution of X', and it follows that X' has property K_2, following Alzati and Russo [1, Definition 3.1]. By [1, Theorem 3.2 or Corollary 3.3], we have $h^1(\mathcal{I}_X(2)) = 0$. Hence $h^0(\mathcal{I}_X(2)) = \binom{n+2}{2} - 3n - 3$.

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