Γ-CONVERGENCE FOR FUNCTIONALS DEPENDING ON VECTOR FIELDS. I. INTEGRAL REPRESENTATION AND COMPACTNESS.

A. MAIONE, A. PINAMONTI, AND F. SERRA CASSANO*

Abstract. Given a family of locally Lipschitz vector fields $X(x) = (X_1(x), \dots, X_m(x))$ on $\mathbb{R}^n$, $m \le n$, we study functionals depending on X. We prove an integral representation for local functionals with respect to X and a result of Γ-compactness for a class of integral functionals depending on X.

Abstract. [Résumé] Dans le présent article, nous considérons une famille de champs vectoriels localement lipschitziens $X(x) = (X_1(x), \dots, X_m(x))$ sur $\mathbb{R}^n$, $m \le n$, et nous étudions les fonctionnelles dépendantes de X. Nous démontrons un théorème de représentation intégrale des fonctionnelles locales par rapport à X et un résultat de Γ-compacité pour une classe de fonctionnelles intégrales dépendantes de X.

1. Introduction

In this paper we will deal with the Γ-convergence, with respect to $L^p(\Omega)$-topology, for integral functionals $F, F_1 : L^p(\Omega) \to [0, \infty]$, $1 < p < \infty$, defined by

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in \mathbf{C}^1(\Omega) \\ \infty & \text{otherwise} \end{cases} \tag{1}$$

and

$$F_1(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,1}_{\text{loc}}(\Omega) \\ \infty & \text{otherwise} \end{cases}, \tag{2}$$

Date: June 7, 2020.

2010 *Mathematics Subject Classification.* 49J45, 35C15, 53C17 .

Key words and phrases. Functionals,Vector Fields, Γ-convergence, Sub-Riemannian Geometry.

A.M is supported by MIUR, Italy, GNAMPA of INDAM and University of Trento, Italy.

A.P. is supported by MIUR, Italy, GNAMPA of INDAM and University of Trento, Italy.

F.S.C. is supported by MIUR, Italy, GNAMPA of INDAM and University of Trento, Italy.

*Corresponding author.

where $X(x) := (X_1(x), \dots, X_m(x))$ is a given family of first order differential operators, with Lipschitz coefficients on a bounded open set $\Omega \subset \mathbb{R}^n$, that is,

$$X_j(x) = \sum_{i=1}^{n} c_{ji}(x)\partial_i \quad j = 1, \dots, m$$

with $c_{ji}(x) \in Lip(\Omega)$ for $j = 1, \dots, m$, $i = 1, \dots, n$ and where $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ is a Borel function. In the following, we will refer to X and f as *X-gradient* and *integrand function*, respectively. As usual, we will identify each X_j with the vector field $(c_{j1}(x), \dots, c_{jn}(x)) \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$. Moreover, we set

$$C(x) = [c_{ji}(x)]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}, \tag{3}$$

and we will call $C(x)$ the *coefficient matrix of the X-gradient.*

Throughtout the paper the class of integrand functions will tipically satisfy the following structural conditions:

(I_1) for a.e. $x \in \Omega$, the function $f(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is convex;

(I_2) there exist constants $c_1 > c_0 \geq 0$ such that

$$c_0 \, |\eta|^p \leq f(x, \eta) \leq c_1 \, (|\eta|^p + 1) \, ,$$

for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$.

We will denote by $I_{m,p}(\Omega, c_0, c_1)$ the class of such integrand functions. Notice that both functionals (1) and (2) always admit an integral representation with respect to the Euclidean gradient. Indeed, for instance, functional (1) can be represented as follows

$$F(u) = \int_\Omega f_e(x, Du) \, dx \text{ for each } u \in \mathbf{C}^1(\Omega)$$

where $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$ now denotes the *Euclidean integrand* defined as

$$f_e(x, \xi) := f(x, C(x)\xi) \quad \text{for a.e. } x \in \Omega \, , \text{for each } \xi \in \mathbb{R}^n. \tag{4}$$

Notice also that, in general, we cannot reverse this representation (see Counterexample 3.14). Moreover the representation with respect to the Euclidean gradient could yield a loss of coercivity. Indeed, for instance, let us consider as X-gradient the *Grushin* and *Heisenberg* vector fields in Example 2.2 (ii) and (iii), respectively, and let $f(x, \eta) = |\eta|^2$. Then, it is easy to see that there are no positive constants $c > 0$ such that the associated Euclidean integrand $f_e(x, \xi) = f(x, C(x)\xi) = |C(x)\xi|^2$ satisfies

$$f_e(x, \xi) \geq c \, |\xi|^2 \text{ for a.e. } x \in \Omega, \, \forall \, \xi \in \mathbb{R}^n \, ,$$

if the open set $\Omega \subset \mathbb{R}^2$ contains some segment of the line $\{x_1 = 0\}$, for the Grushin vector field, and for each open set $\Omega \subset \mathbb{R}^3$, for the Heisenberg vector fields. Nonetheless, we will show that, by replacing the Euclidean gradient with the X-gradient, we can get rid of this drawback.

Functional (1) was studied in [FSSC1] as far as its relaxation and in connection with the so-called Meyers-Serrin theorem for Sobolev spaces associated with the X-gradient, denoted $W^{1,p}_X(\Omega)$ (see Definition 2.3 and [FS]). As a consequence, the following characterization of relaxed functionals $\bar{F}$ and $\bar{F}_1$ can be given (see (22) and Theorem 3.1): if $f \in I_{m,p}(\Omega, c_0, c_1)$ with $c_1 \geq c_0 > 0$ and $F^* : L^p(\Omega) \to [0, \infty]$ denotes the functional

$$F^*(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}, \tag{5}$$

then

$$\bar{F}(u) = \bar{F}_1(u) = F^*(u) \quad \forall\, u \in L^p(\Omega)\,. \tag{6}$$

By (6) and a well-known property of Γ-convergence (see [DM, Propostion 6.11]), the characterization of Γ-limits for functionals of type (1) or (2), associated to integrand functions in $I_{m,p}(\Omega, c_0, c_1)$, can be reduced to the one for functionals of type (5) still associated to integrand functions in $I_{m,p}(\Omega, c_0, c_1)$. For getting such a characterization, the following structure assumption on the X-gradient turns out to be a key point.

1.1. **Definition.** We say that the family of vector fields $X(x) = (X_1(x), \dots, X_m(x))$ on an open set $\Omega \subset \mathbb{R}^n$ satisfies the *linear independence condition* (LIC) if there exists a set $\mathcal{N}_X \subset \Omega$, closed in the topology of Ω, such that $|\mathcal{N}_X| = 0$ and, for each $x \in \Omega_X := \Omega \setminus \mathcal{N}_X$, $X_1(x), \dots, X_m(x)$ are linearly independent as vectors of $\mathbb{R}^n$.

Let us point out that (LIC) condition embraces many relevant families of vector fields studied in literature (see Example 2.2). In particular neither the *Hörmander condition* for X, that is, vector fields X_j's are smooth and the rank of the Lie algebra generated by $X_1, \dots, X_m$ equals n at any point of Ω, nor the (weaker) assumption that the X-gradient induces a *Carnot-Carathéodory metric* in Ω is requested. An exhaustive account of these topics can be found in [BLU].

The main results of this paper are the following (see Theorems 3.12 and 4.10).

- Assume that the X-gradient satisfies (LIC) on Ω and let us denote by $\mathcal{A}$ the class of open sets contained in Ω. Then an integral representation result, with respect to the X-gradient, is provided for a local functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ satisfying suitable assumptions.
- Assume that the X-gradient satisfies (LIC) on Ω, and let $F^*_h : L^p(\Omega) \to [0, \infty]$ $(h = 1, 2, \dots)$ be a sequence of integral functionals of the form (5) with $f \equiv f_h$, where $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$ for given constants $0 < c_0 \leq c_1$. Then, up to a subsequence, $(F^*_h)_h$ Γ-converges, in $L^p(\Omega)$-topology, to a functional $F^* : L^p(\Omega) \to [0, \infty]$, and F^* can be still represented as in (5), for a suitable integrand function $f \in I_{m,p}(\Omega, c_0, c_1)$.

We will also single out two significant integrand function subclasses $J_i \subset I_{m,p}(\Omega, c_0, c_1)$ $(i = 1, 2)$ for which the associated functionals in (5) are still compact with respect to Γ- convergence with respect to $L^p(\Omega)$-topology (see Theorem 4.19).

The techniques for showing the integral representation Theorem 3.12 rely on the analogous classical integral representation result for the Euclidean gradient (see [DM, Theorem 20.1]), together with a characterization of integral functionals depending on the Euclidean gradient which can be also represented with respect to a given X-gradient (see Theorem 3.5). Let us stress that we cannot here exploit, as in the case of the Euclidean gradient, the approximation by piecewise-affine functions in classical Sobolev space $W^{1,p}(\Omega)$, since it could not work in Sobolev space $W^{1,p}_X(\Omega)$ (see section 2.3). The strategy for showing the Γ-compactness Theorem 4.10 will consist of two steps.

1st step. By applying classical results contained in [DM], we will prove the following result (see Theorem 4.17): let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$, let $(F_h)_h$ be a sequence of integral functionals on $L^p(\Omega) \times \mathcal{A}$, $1 < p < \infty$, of the form

$$F_h(u, A) := \begin{cases} \int_A f_{h,e}(x, Du(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,1}_{\text{loc}}(A) \\ \infty & \text{otherwise} \end{cases}, \tag{7}$$

where

$$f_{h,e}(x, \xi) := f_h(x, C(x)\xi) \quad x \in \Omega,\ \xi \in \mathbb{R}^n\,. \tag{8}$$

Then, up to a subsequence, there exists $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that

$$F(\cdot, A) = \Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}\,, \tag{9}$$

and F can be represented by an integral form on $W^{1,p}(A)$ by means of an **Euclidean integrand function**, that is,

$$F(u, A) := \int_A f_e(x, Du(x))\, dx \tag{10}$$

for every $A \in \mathcal{A}$, for every $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}(A)$ for a suitable Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$.

2nd step. We will show that the class $I_{m,p}(\Omega, c_0, c_1)$ satisfies the following closure property with respect to $\Gamma(L^p(\Omega))$-convergence (see Theorem 4.18): assume that $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1)$ and (9) and (10) hold, then F satisfies the assumptions of the integral representation Theorem 3.12. Thus F can be also represented in the integral form (5), by means of an integrand function $f \in I_{m,p}(\Omega, c_0, c_1)$.

Eventually let us point out that the Γ-convergence for functionals such as in (1) have been studied in the framework of Dirichlet forms [MR, Fu], but for special integrand functions f and X-gradient satisfying the Hörmander condition (see, for instance, [Dr, Mo, MV, BT] and references there in). Other variational convergences, such as homogenization and H-convergence

([Mu]) for subelliptic PDEs have been also widely studied, always assuming the X-gradient satisfying the Hörmander condition (see, for instance, [BMT, BPT1, BPT2, FT, FGVN, FTT, BFT, BFTT] and the references there in). In the subsequent paper [MPSC] we will be concerned with relationships between Γ-convergence of functionals depending on vector fields and convergence of their minimizers. Thus, we will refer to [MPSC] for a comparison among our results with those already present in literature.

2. Vector fields and Sobolev spaces depending on vector fields

2.1. **Notation and definitions.** Through this paper $\Omega \subset \mathbb{R}^n$ is a fixed open set and $\overline{\mathbb{R}} = [-\infty, \infty]$. If $v, w \in \mathbb{R}^n$, we denote by $|v|$ and $\langle v, w\rangle$ the Euclidean norm and the scalar product, respectively. If Ω and Ω' are subsets of $\mathbb{R}^n$ then $\Omega' \Subset \Omega$ means that Ω' is compactly contained in Ω. Moreover, $B(x, r)$ is the open Euclidean ball of radius r centered at x. Sometimes we will denote by $B^k(x, r)$ the open Euclidean ball of radius r centered at $x \in \mathbb{R}^k$ in $\mathbb{R}^k$. If $A \subset \mathbb{R}^n$ then χ_A is the characteristic function of A, $\mathcal{L}^n(A) = |A|$ is its n-dimensional Lebesgue measure $\mathcal{L}^n$ and by notation *a.e.* $x \in A$, we will simply mean $\mathcal{L}^n$-a.e. $x \in A$.

In the sequel we denote by $\mathbf{C}^k(\Omega)$ the space of $\mathbb{R}$-valued functions k times continuously differentiable and by $\mathbf{C}^k_c(\Omega)$ the subspace of $\mathbf{C}^k(\Omega)$ whose functions have support compactly contained in Ω.

We will use spherically symmetric mollifiers ρ_ϵ defined by $\rho_\epsilon(x) := \epsilon^{-n}\rho(\epsilon^{-1}|x|)$, where $\rho \in \mathbf{C}^\infty_c((-1, 1))$, $\rho \geq 0$ and $\int_{B(0,1)} \rho(|x|)dt = 1$.

For any $u \in L^1(\Omega)$ define Xu as an element of $\mathcal{D}'(\Omega; \mathbb{R}^m)$ as follows

$$\begin{aligned} Xu(\psi) :&= (X_1u(\psi_1), \dots, X_mu(\psi_m)) \\ &= -\int_\Omega u \left(\sum_{i=1}^n \partial_{x_i}(c_{1,i}\,\psi_1), \dots, \sum_{i=1}^n \partial_{x_i}(c_{m,i}\,\psi_m) \right) dx \end{aligned}$$

$\forall \psi = (\psi_1, \dots, \psi_m) \in \mathbf{C}^\infty_c(\Omega; \mathbb{R}^m)$. If we set $X^T\psi := (X_1^T\psi_1, \dots, X_m^T\psi_m)$ with

$$X_j^T\varphi := \int_\Omega \sum_{i=1}^n \partial_{x_i}(c_{j,i}\,\varphi)\, dx \quad \forall\, \varphi \in \mathbf{C}^\infty_c(\Omega),\ \forall\, j = 1, \dots, m\,,$$

the aspect of the definition is even more familiar

$$Xu(\psi) := -\int_\Omega uX^T\psi\, dx \qquad \forall \psi \in \mathbf{C}^\infty_c(\Omega; \mathbb{R}^m).$$

2.1. *Remark.* If $X = (X_1, \dots, X_m)$ satisfies (LIC) on an open set $\Omega \subset \mathbb{R}^n$, then $m \leq n$. Moreover, by the well-known extension result for Lipschitz functions, without loss of generality, we can assume that vector fields' coefficients $c_{ji} \in Lip_{\mathrm{loc}}(\mathbb{R}^n)$ for each $j = 1, \dots, m$, $i = 1, \dots, n$.

2.2. **Example** (Relevant vector fields)**.**

(i) (Euclidean gradient) Let $X = (X_1, \dots, X_n) = D := (\partial_1, \dots, \partial_n)$. In this case the coefficients matrix $C(x)$ of X is a $n \times n$ matrix and

$$C(x) = \mathrm{I_n} \quad \forall\, x \in \mathbb{R}^n\,, \tag{11}$$

denoting I_n the identity matrix of order n.

(ii) (Grushin vector fields) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^2$ defined as

$$X_1(x) := \partial_1\,, \quad X_2(x) := x_1\,\partial_2 \text{ if } x = (x_1, x_2) \in \mathbb{R}^2\,.$$

In this case the coefficients matrix $C(x)$ of X is a 2×2 matrix and

$$C(x) := \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix} \tag{12}$$

(iii) (Heisenberg vector fields) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^3$ defined as

$$X_1(x) := \partial_1 - \frac{x_2}{2}\partial_3,\ X_2(x) := \partial_2 + \frac{x_1}{2}\partial_3 \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3\,.$$

In this case the coefficients matrix $C(x)$ of X is a 2×3 matrix and

$$C(x) := \begin{bmatrix} 1 & 0 & -x_2/2 \\ 0 & 1 & x_1/2 \end{bmatrix} \tag{13}$$

Notice that all three families of vector fields satisfy (LIC) respectively in $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^2$ and $\Omega = \mathbb{R}^3$. Indeed, it suffices to take $\Omega_X = \Omega$ in (i) and (ii) and $\Omega_X = \Omega \setminus \mathcal{N}_X$ with $\mathcal{N}_X := \{(0, x_2) : x_2 \in \mathbb{R}\}$ in (iii). Moreover they are locally Lipschitz continuous in Ω.

2.3. **Definition.** For $1 \le p \le \infty$ we set

$$W^{1,p}_X(\Omega) := \{u \in L^p(\Omega) : X_j u \in L^p(\Omega) \text{ for } j = 1, \dots, m\}$$
$$W^{1,p}_{X;loc}(\Omega) := \Big\{u : u|_{\Omega'} \in W^{1,p}_X(\Omega') \text{ for every open set } \Omega' \Subset \Omega\Big\}$$

2.4. *Remark.* Since vector fields X_j have locally Lipschitz continuous coefficients, $\partial_i c_{j,i} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ for each $j = 1, \dots, m$, $i = 1, \dots, n$, thus, by definition, it is immediate that, for each open bounded set $\Omega \subset \mathbb{R}^n$,

$$W^{1,p}(\Omega) \subset W^{1,p}_X(\Omega) \quad \forall\, p \in [1, \infty]\,, \tag{14}$$

and for any $u \in W^{1,p}(\Omega)$

$$Xu(x) = C(x)\, Du(x) \quad \text{ for a.e. } x \in \Omega\,, \tag{15}$$

where $W^{1,p}(\Omega)$ denotes the classical Sobolev space, or, equivalently, the space $W^{1,p}_X(\Omega)$ associated to $X = D := (\partial_{x_1}, \dots, \partial_{x_n})$ (see Example 2.2 (i)). Moreover it is easy to see that inclusion (14) can be strict and turns out to be continuous. As well, there is the inclusion

$$W^{1,p}_{\mathrm{loc}}(\Omega) \subset W^{1,p}_{X;\mathrm{loc}}(\Omega) \quad \forall\, p \in [1, \infty]\,. \tag{16}$$

The following Proposition is proved in [FS]

2.5. **Proposition.** *$W^{1,p}_X(\Omega)$ endowed with the norm*

$$\|u\|_{W^{1,p}_X(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^{m} \|X_j u\|_{L^p(\Omega)}$$

is a Banach space, reflexive if $1 < p < \infty$.

2.6. *Remark.* The following properties hold for functions in $W^{1,p}_{X;\mathrm{loc}}(\Omega)$:

(i) let $u \in L^p(\Omega)$ and assume there exists an open set $A \subset \Omega$ such that $u|_A \in W^{1,p}_{X;\mathrm{loc}}(A)$. Then, for every open set $A' \Subset A$, there exists

$$w \in W^{1,p}_X(\Omega) \text{ such that } u|_{A'} = w|_{A'}\,. \tag{17}$$

Indeed, there exists a cut-off function $\varphi \in \mathbf{C}^1_c(A)$ such that $\varphi \equiv 1$ in A'. If

$$w(x) := u(x)\,\varphi(x) \text{ if } x \in \Omega\,,$$

then it is easy to see that w satisfies (17).

(ii) Let $A \subset \Omega$ be an open subset and let $u \in L^p(A)$ be such that there exists $M > 0$, $\|u\|_{W^{1,p}_X(A')} \le M$ for any $A' \Subset A$, then $u \in W^{1,p}_X(A)$. It is easy to see that u admits the weak gradient Xu. Consider a sequence of open subsets of A, $\{A_i\}_{i\in\mathbb{N}}$ with $A_i \Subset A_{i+1}$ and $A \subseteq \bigcup_{i=1}^{\infty} A_i$

$$\int_A |Xu|^p\,dx \le \int_{\bigcup_{i=1}^{\infty} A_i} |Xu|^p\,dx = \lim_{i\to\infty} \int_{A_i} |Xu|^p\,dx \le M$$

and the conclusion follows.

(iii) Let $\{A_1, \dots, A_N\}$ be a finite family of open subsets of Ω and let $u \in L^p(\Omega)$. If $u_{|A_i} \in W^{1,p}_X(A_i)$ for all $i = 1, \dots, N$ then $u \in W^{1,p}_X\left(\bigcup_{i=1}^{N} A_i\right)$. Indeed, for every $A' \Subset \bigcup_{i=1}^{N} A_i$ there exists a partition of unity subordinate to the covering $\{A_1, \dots, A_N\}$, i.e., non-negative functions $\{\eta_1, \dots, \eta_N\} \subset C^\infty_c\left(\bigcup_{i=1}^{N} A_i\right)$ such that each η_j has support in some A_i and $\sum_{j=1}^{N} \eta_j(x) = 1$ for all $x \in A'$. Set $u_j := u\eta_j$. Since the support of η_j is contained in some A_i, it is clear that $u_j \in W^{1,p}_X\left(\bigcup_{i=1}^{N} A_i\right)$ and $u \in W^{1,p}_X(A')$. Let us estimate

$$\|u\|_{W^{1,p}_X(A')} \le \sum_{j=1}^{N} \|u\eta_j\|_{W^{1,p}_X(A')} \le C\|u\|_{W^{1,p}_X(A_i)}$$

where $C > 0$ is independent of A'. The conclusion then follows using (ii).

(iv) Let $A \subset \Omega$ be an open subset and $u \in W^{1,p}_X(A)$, then $u_{|B} \in W^{1,p}_X(B)$ for any open set $B \subseteq A$. The thesis follows easily observing that $C^\infty_c(B) \subseteq C^\infty_c(A)$.

2.2. **Approximation by regular functions.** Let us recall in this section some results of approximation by regular functions in these anisotropic Sobolev spaces. In particular the analogous of the celebrated Meyers-Serrin theorem, proved, independently, in [FSSC1] and [GN]. Analogous results (under some additional assumptions) in the weighted cases are proved in [FSSC2], see also [APS] for a generalization to metric measure spaces.

Here and in the sequel, if $u : \Omega \to \bar{\mathbb{R}}$, we will denote by $\bar{u} : \mathbb{R}^n \to \bar{\mathbb{R}}$ its extension to the whole $\mathbb{R}^n$ being 0 outside of Ω.

2.7. **Proposition.** *Assume $u \in W^{1,p}_X(\Omega)$ for $1 \le p < \infty$. Then if $\Omega' \Subset \Omega$*

$$\lim_{\epsilon \to 0} \|\bar{u} * \rho_\epsilon - u\|_{W^{1,p}_X(\Omega')} = 0,$$

where $\rho_\epsilon(x) = \epsilon^{-n}\rho(\epsilon^{-1}|x|)$ is a mollifier supported in $B(0,\epsilon)$.

2.8. **Definition.** For $1 \le p \le \infty$ we set

$$H^{1,p}_X(\Omega) := \text{closure of } \mathbf{C}^1(\Omega) \cap W^{1,p}_X(\Omega) \text{ in } W^{1,p}_X(\Omega)$$

As for the usual Sobolev spaces $H^{1,p}_X(\Omega) \subset W^{1,p}_X(\Omega)$. The classical result '$H = W$'of Meyers and Serrin ([MS]) still holds for these anisotropic Sobolev spaces.

2.9. **Theorem.** *Let Ω be an open subset of $\mathbb{R}^n$ and $1 \le p < \infty$. Then*

$$H^{1,p}_X(\Omega) = W^{1,p}_X(\Omega).$$

The proofs of Proposition 2.7 and Theorem 2.9 can be found in [FSSC1] and [GN].

Let us collect below some well-known properties about approximation by convolution and convex functions.

2.10. **Proposition.** (i) *Let $(u_h)_h$ and u be in $L^p_{loc}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that*

$$u_h \to u \text{ in } L^1_{loc}(\Omega) \text{ as } h \to \infty\,.$$

Then, for each open set $\Omega' \Subset \Omega$, for given $0 < \epsilon < \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,

$$\rho_\epsilon * u_h \to \rho_\epsilon * u \text{ uniformly on } \Omega', \text{ as } h \to \infty\,. \tag{18}$$

(ii) *Let $f : \mathbb{R}^m \to [0,\infty)$ be a convex function and let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. Then, for each bounded open sets Ω' and Ω with $\Omega' \Subset \Omega$, for each $0 < \epsilon < \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$,*

$$\int_{\Omega'} f(\rho_\epsilon * w)\,dx \le \int_\Omega f(w)\,dx.$$

Proof. (i) See, for instance, [DM, Proof of Theorem 23.1]
(ii) See, for instance, [DM, (23.5)]. □

2.3. **Approximation by piecewise affine functions.** It is well known (see, for instance, [ET, Chap. X, Proposition 2.9]) that the class of piecewise affine functions is dense in the classical Sobolev space $W^{1,p}(\Omega)$, provided that Ω is a bounded open set with Lipschitz boundary. This result is crucial in the proof of the classical integral representation theorem with respect to the Euclidean gradient (see, for instance, [DM, Theorem 20.1]). The aim of this section is to prove that no results of this kind are available for a general family $X = (X_1, \dots, X_m)$ in $\mathbb{R}^n$, by extending, in a natural way, the notion to be affine with respect to the X-gradient. We say that $u \in \mathbf{C}^\infty(\mathbb{R}^n)$ is $X-$*affine* if there exists $c \in \mathbb{R}^n$ such that $Xu(x) = c$ for all $x \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be open. We say that $u : \Omega \to \mathbb{R}$ is $X-$affine if it is the restriction to Ω of a $X-$affine function over $\mathbb{R}^n$. Moreover, we say that $u : \mathbb{R}^n \to \mathbb{R}$ is $X-$piecewise affine if it is continuous and there is a partition of $\mathbb{R}^n$ into a negligible set and a finite number of open sets on which u is $X-$affine. We prove that for Grushin and Heisenberg vector fields the approximation of functions in $W^{1,p}_X(\Omega)$ using $X-$piecewise affine functions does not hold. It is easy to see that, if $X = (X_1, X_2)$ is the Heisenberg vector field on $\mathbb{R}^3$ (see Example 2.2 (iii)), then a function $u \in \mathbf{C}^\infty(\mathbb{R}^3)$ is $X-$affine if and only if

$$u(x) = c_1x_1 + c_2x_2 + c_3 \text{ for each } x = (x_1, x_2, x_3) \in \mathbb{R}^3\,, \tag{19}$$

for suitable constants $c_i \in \mathbb{R}$ $i = 1, 2, 3$. Indeed, it is trivial that a function u in (19) is X- affine. Conversely, if $X_1u = c_1$ and $X_2u = c_2$ on $\mathbb{R}^3$, for some $u \in \mathbf{C}^\infty(\mathbb{R}^3)$, then the commutator $[X_1, X_2]u := (X_1X_2 - X_2X_1)u = \partial_3 u = 0$ on $\mathbb{R}^3$, which gives $u(x) = c_1x_1 + c_2x_2 + c_3$ for each $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, for some $c_3 \in \mathbb{R}$.

Let $u(x) = x_3$, then $u \in W^{1,p}_X(\Omega)$ whenever $|\Omega| < \infty$. Since any $X-$piecewise affine function does not depend on x_3, there cannot be any sequence of $X-$piecewise affine functions $(u_h)_h$ such that $u_h(x_1, x_2, x_3) \to u(x_1, x_2, x_3)$ for a.e. $(x_1, x_2, x_3) \in \Omega$.

Let $X = (X_1, X_2)$ be the Grushin vector fields on $\mathbb{R}^2$ (see Example 2.2 (ii)). Let $u \in \mathbf{C}^\infty(\mathbb{R}^2)$ be such that $X_1u = c_1$ and $X_2u = c_2$ on $\mathbb{R}^2$. Then it is easy to prove, arguing as before, that $u(x) = c_1x_1 + c_3$ for each $x = (x_1, x_2) \in \mathbb{R}^2$, for some $c_3 \in \mathbb{R}$. The conclusion follows as in the previous case taking $u(x_1, x_2) = x_2$, which belongs to $W^{1,p}_X(\Omega)$ for any $p \geq 1$ and any bounded open set $\Omega \subset \mathbb{R}^2$.

3. Relaxation and characterization of integral functionals depending on vector fields

In the study of the $\Gamma-$convergence it will be helpful to consider F and F_1 as *local functionals*. Namely, according to [DM, Chap. 15], we will consider the functionals F, $F_1 : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$

$$F(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A} \text{ and } u \in \mathbf{C}^1(A) \cap L^p(A) \\ \infty & \text{otherwise} \end{cases} \tag{20}$$

(21)

$$F_1(u,A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A} \text{ and } u \in W^{1,1}_{\text{loc}}(A) \cap L^p(A) \\ \infty & \text{otherwise} \end{cases}$$

For future use, we denote by $\mathcal{A}_0$ the class of all open sets compactly contained in Ω.

3.1. **Characterization of the relaxed functional and its finiteness domain.** We are going to characterize the *relaxed functionals of* F in (1) and F_1 in (2) with respect to the topology of $L^p(\Omega)$. Let us recall that the relaxed functional of a given functional $G : L^p(\Omega) \to [0,\infty]$ is defined as follows (see, for instance, [B]):

$$\bar{G}(u) := \inf \left\{ \liminf_{h\to\infty} G(u_h) : (u_h)_h \subset L^p(\Omega), u_h \to u \text{ in } L^p(\Omega) \right\} . \tag{22}$$

Then it is well known (see, for instance, [B]) that $\bar{G}$ is the greatest $L^p(\Omega)$-lower semicontinuous functional smaller or equal to G.

The relaxed functionals $\bar{F}$ and $\bar{F}_1$ can be characterized as follows:

3.1. **Theorem.** *Let $p > 1$ and let Ω be an open subset of $\mathbb{R}^n$; let $f : \Omega\times\mathbb{R}^m \to [0,\infty)$ be an integrand function in $I_{m,p}(\Omega, c_1, c_0)$ with $c_1 \geq c_0 > 0$. Then*
(i) $\operatorname{dom}\bar{F} := \left\{u \in L^p(\Omega) : \bar{F}(u) < \infty\right\} = W^{1,p}_X(\Omega)$ *;*
(ii) $\bar{F}(u) = \int_\Omega f(x, Xu(x))\, dx$ *for every* $u \in W^{1,p}_X(\Omega)$*;*
(iii) $\bar{F}(u) = \bar{F}_1(u)$ *for each* $u \in L^p(\Omega)$*.*

Proof. Claims (i) and (ii) are proved in [FSSC1, Theorem 3.1.1]. Let us prove (iii). Let $u \in L^p(\Omega)$ and $(u_h)_h \subset \mathbf{C}^1(\Omega) \cap L^p(\Omega)$ with $u_h \to u$ in $L^p(\Omega)$. Since, in particular, $(u_h)_h \subset W^{1,1}_{loc}(\Omega) \cap L^p(\Omega)$ we get

$$\bar{F}_1(u) \leq \liminf_{h\to\infty} \bar{F}_1(u_h) = \liminf_{h\to\infty} \int_\Omega f(x, Xu_h)\, dx = \liminf_{h\to\infty} F(u_h) \tag{23}$$

which implies

$$\bar{F}_1(u) \leq \bar{F}(u). \tag{24}$$

Let $F^* : L^p(\Omega) \to [0,\infty]$ denote the functional in (5). By [B, Theorem 2.3.1], F^* is $L^p(\Omega)$-lower semicontinuous. Let $u \in \operatorname{dom}(F_1) := \{v \in W^{1,1}_{loc}(\Omega) \cap L^p(\Omega) \mid F_1(u) < \infty\}$, then, by (I_3), we have

$$c_0 \int_\Omega |Xu|^p\, dx \leq F_1(u) < \infty,$$

thus $u \in W^{1,p}_X(\Omega)$ and $\operatorname{dom} F_1 \subset W^{1,p}_X(\Omega)$. This implies $F^* \leq F_1$ on $L^p(\Omega)$ and consequently $F^* \leq \bar{F}_1$ on $L^p(\Omega)$. Using (24) and (ii) we conclude

$$F^* \leq \bar{F}_1 \leq \bar{F} = F^* \quad \text{on} \quad L^p(\Omega)$$

which completes the proof.

□

When $p = 1$ the domain of relaxed functional $\bar{F}$ gives rise to the space of functions of bounded variation function associated to X, $BV_X(\Omega)$ (see [FSSC1, Theorem 3.2.3]).

3.2. **A characterization of functionals depending on vector fields.** We are going to study when a local functional $F : \mathbf{C}^1(\Omega) \times \mathcal{A} \to [0, \infty]$ can be equivalently represented both with respect to a family of vector fields X and the Euclidean gradient D.

We already stressed that the functional F in (1) can be always represented with respect to the Euclidean gradient on $\mathbf{C}^1(\Omega)$ by means of the Euclidean integrand (4).

Then, it is clear that, for each $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$,

$$\begin{aligned} F(u, A) &= \int_A f(x, Xu)\, dx = \int_A f(x, C(x)Du)\, dx \\ &= \int_A f_e(x, Du)\, dx\,. \end{aligned} \tag{25}$$

Viceversa, we are going to study when, given an X-gradient and a functional $F : \mathbf{C}^1(\Omega) \times \mathcal{A} \to [0, \infty]$

$$F(u, A) = \int_A f_e(x, Du)\, dx \quad u \in \mathbf{C}^1(A)\,, \tag{26}$$

there exist a function $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ such that

$$F(u, A) = \int_A f(x, Xu)\, dx \quad \text{for every } u \in W^{1,p}_X(A).$$

Let us begin with some preliminaries of linear algebra.

In the following, we identify the space of real matrices of order $m \times n$ with $\mathbb{R}^{mn}$ or $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ denotes the class of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ endowed with its operator norm. Given a matrix $A = [a_{ij}]$ of order $m \times n$ its *operator norm* is defined as

$$\|A\| := \sup_{|z|=1} |Az|$$

and its *Hilbert-Schmidt norm* as

$$\|A\|_{\mathbb{R}^{mn}} := \sqrt{\sum_{i,j} a_{ij}^2}$$

(see [La, Chap. 7]). Since the norms are equivalent, we can also identify the spaces

$$\mathbf{C}^0(\Omega_X, \mathbb{R}^{mn}) \equiv \mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))\,, \tag{27}$$

where we recall that $\Omega_X = \Omega \setminus \mathcal{N}_X$ (see Definition 1.1). For each $x \in \Omega$, let $L_x : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map

$$L_x(v) := C(x)v \text{ if } v \in \mathbb{R}^n \tag{28}$$

where $C(x)$ denotes the matrix in (3). Let N_x and V_x respectively denote the subspaces of $\mathbb{R}^n$ defined as

$$N_x := \ker(L_x), \quad V_x := \{C(x)^T z \ : \ z \in \mathbb{R}^m\}. \tag{29}$$

It is well-known that N_x and V_x are orthogonal complements in $\mathbb{R}^n$, that is

$$\mathbb{R}^n = N_x \oplus V_x \,. \tag{30}$$

Moreover, for each $x \in \Omega$ and $\xi \in \mathbb{R}^n$, let us define $\xi_{N_x} \in N_x$ and $\xi_{V_x} \in V_x$ as the unique vectors of $\mathbb{R}^n$ such that

$$\xi = \xi_{N_x} + \xi_{V_x} \tag{31}$$

and let $\Pi_x : \mathbb{R}^n \to V_x \subset \mathbb{R}^n$ be the projection

$$\Pi_x(\xi) := \xi_{V_x}. \tag{32}$$

3.2. **Proposition.** *Assume that the family X of vector fields satisfies (LIC) on Ω. Let $C(x)$ be the matrix in* (3) *and L_x be the map in* (28)*. Then $L_x : V_x \to \mathbb{R}^m$ is invertible and the map $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ defined as*

$$L^{-1}(x) := L_x^{-1} \ \text{ if } x \in \Omega_X \tag{33}$$

belongs to $\mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.

Before giving the proof of Proposition 3.2, let us prove a preliminary technical lemma.

3.3. **Lemma.** *Under the same assumptions of Proposition 3.2,*

(i) $\dim V_x = m$ *for each $x \in \Omega_X$ and $L_x(V_x) = \mathrm{range}(L_x) = \mathbb{R}^m$ where $\mathrm{range}(L_x)$ denotes the range of L_x, that is, $\mathrm{range}(L_x) := \{L_x(v) : v \in \mathbb{R}^n\}$. In particular $L_x : V_x \to \mathbb{R}^m$ is an isomorphism.*

(ii) *Let*

$$B(x) := C(x)\, C^T(x) \quad x \in \Omega \,. \tag{34}$$

Then, for each $x \in \Omega_X$, $B(x)$ is a symmetric invertible matrix of order m. Moreover the map $B^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, defined as

$$B^{-1}(x)(z) := B(x)^{-1} z \quad \text{if } z \in \mathbb{R}^m \,, \tag{35}$$

is continuous.

(iii) *For each $x \in \Omega_X$, the projection Π_x in* (32) *can be represented as*

$$\Pi_x(\xi) = \xi_{V_x} = C(x)^T B(x)^{-1} C(x)\, \xi, \quad \forall\, \xi \in \mathbb{R}^n \,.$$

If $m = n$, then, $\Pi_x = \mathrm{Id}_n : \mathbb{R}^n \to \mathbb{R}^n$, the identity map in $\mathbb{R}^n$.

3.4. *Remark.* Using the definition of V_x it is easy to see that

$$V_x = \mathrm{span}_{\mathbb{R}} \{X_1(x), \ldots, X_m(x)\} \ ,$$

i.e., the so-called *horizontal bundle*, denoted also by H_x.

Proof. (i) The claim is a well-known result of basic linear algebra.

(ii) It is straightforward that $B(x)$ a symmetric matrix of order m for each $x \in \Omega$. We have only to show that it is invertible for each $x \in \Omega_X$ or, equivalently, that

$$\text{if } B(x)z = 0 \text{ for some } z \in \mathbb{R}^m, \text{ then } z = 0. \tag{36}$$

Let z^T denotes the transpose of a column vector $z \in \mathbb{R}^m$. If $B(x)z = 0$, then

$$\begin{aligned} 0 = z^T B(x) z &= z^T C(x) C^T(x) z \\ &= \left| C^T(x) z \right|^2_{\mathbb{R}^m} \quad \Longleftrightarrow \quad C^T(x) z = 0\,. \end{aligned} \tag{37}$$

By (LIC), since

$$\operatorname{rank} C(x) = \operatorname{rank} C^T(x) = m \quad \forall x \in \Omega_X\,,$$

from (37) we get that $z = 0$ and (36) follows. Let us now prove that the map (35) is continuous. Let us recall that, given a matrix $A \in \mathbf{C}^0(\Omega_X, \mathbb{R}^{m^2})$, by the definition of determinant (see, for instance, [La, Chap.3, Theorem 6]), the determinant map

$$\det A : \Omega_X \to \mathbb{R}, \quad (\det A)(x) := \det(A(x))$$

is continuous. Moreover

$$A(x) \text{ is invertible} \iff \det A(x) \neq 0\,.$$

By Cramer's rule (see, for instance, [La, Chap.3,Theorem 7]), if $B(x)^{-1} = [b^*_{ij}(x)]$, then

$$b^*_{ij}(x) = (-1)^{i+j} \frac{\det B_{ij}(x)}{\det B(x)} \quad x \in \Omega_X,\ i, j = 1, \dots, m\,,$$

where B_{ij} is the $(m-1) \times (m-1)$ matrix obtained by striking out the ith row and jth column of B, i.e., the (ij)th minor of B. This implies that $B^{-1} \in \mathbf{C}^0(\Omega_X, \mathbb{R}^{m^2}) \equiv \mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m))$.

(iii) We have

$$\Pi_x(\xi) = \xi_{V_x} = C(x)^T w \tag{38}$$

for a suitable (unique) $w = w(x, \xi) \in \mathbb{R}^m$ depending on x and ξ. On the other hand, by (38),

$$\begin{aligned} C(x)\xi = L_x(\xi) &= L_x(\xi_{N_x}) + L_x(\xi_{V_x}) \\ &= C(x)\xi_{V_x} = C(x)C(x)^T w = B(x) w\,. \end{aligned} \tag{39}$$

Since $B(x)$ is invertible, by (39), we get the desired conclusion. □

Proof of Proposition 3.2. The fact that the map $L_x : V_x \to \mathbb{R}^m$ is invertible follows from Lemma 3.3 (i). Let us now prove that

$$L_x^{-1}(z) = C^T(x) B(x)^{-1} z \quad \forall\, z \in \mathbb{R}^m\,, \tag{40}$$

where $B(x)$ is the matrix in (34). Let us fix $z \in \mathbb{R}^m$ and let $v = L_x^{-1}(z) \in V_x$. By Lemma 3.3 (iii), there exists $w \in \mathbb{R}^m$ such that $v = C^T(x)w$. Thus

$$z = L_x(v) = C(x)C^T(x)w = B(x)w\,.$$

By Lemma 3.3 (ii), it holds $w = B(x)^{-1}z$. Therefore we get

$$L_x^{-1}(z) = v = C^T(x)B(x)^{-1}z \tag{41}$$

and (40) follows. Let us define

$$A(x) := C^T(x)B(x)^{-1} \quad \text{if } x \in \Omega_X\,.$$

Then, from Lemma 3.3 (ii), $A \in \mathbf{C}^0(\Omega_X, \mathbb{R}^{mn}) \equiv \mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$. Thus, by (41), we get the desired conclusion. □

3.5. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and assume that X satisfies (LIC) on Ω. Let $F : \mathbf{C}^1(\Omega) \times \mathcal{A} \to [0,\infty]$ be the functional in* (26) *with $f_e : \Omega \times \mathbb{R}^n \to [0,\infty]$ a Borel measurable function satisfying*

$$\text{for each } \xi \in \mathbb{R}^n,\ f_e(\cdot,\xi) \in L^1_{\mathrm{loc}}(\Omega) \tag{42}$$

and

$$f_e(x,\cdot) : \mathbb{R}^n \to [0,\infty) \text{ convex for a.e. } x \in \Omega\,. \tag{43}$$

Define $f : \Omega \times \mathbb{R}^m \to [0,\infty)$ as

$$f(x,\eta) := \begin{cases} f_e(x, L^{-1}(x)(\eta)) & \text{if } (x,\eta) \in \Omega_X \times \mathbb{R}^m \\ 0 & \text{otherwise} \end{cases}, \tag{44}$$

where $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is the map in (33)*. Then, f is a Borel measurable function satisfying*

$$f(x,\cdot) : \mathbb{R}^m \to [0,\infty) \text{ convex for a.e. } x \in \Omega\,. \tag{45}$$

Moreover,

$$\begin{aligned} F(u,A) &= \int_A f_e(x, Du)\,dx \\ &= \int_A f(x, Xu)\,dx \quad \forall A \in \mathcal{A},\ u \in \mathbf{C}^1(A) \end{aligned} \tag{46}$$

if and only if

$$f_e(x,\xi) = f_e(x, \Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n\,, \tag{47}$$

where $\{V_x : x \in \Omega_X\}$ is the distribution of m-planes in $\mathbb{R}^n$ defined in Proposition 3.2 and $\Pi_x : \mathbb{R}^n \to V_x$ denotes the projection of $\mathbb{R}^n$ on V_x in (32)*. In addition, the function f for which* (46) *holds is unique, that is, if there exists another Borel measurable function $f^* : \Omega \times \mathbb{R}^m \to [0,\infty)$ satisfying $f^*(x,\cdot) : \mathbb{R}^m \to [0,\infty)$ convex a.e. $x \in \Omega$ and* (46) *holds, then $f(x,\eta) = f^*(x,\eta)$ for a.e. $x \in \Omega$ and $\eta \in \mathbb{R}^m$.*

3.6. *Remark.* If the X-gradient does not satisfy (LIC) condition, the uniqueness of representation (46) may trivially fail. For instance, let $X = (X_1, X_2) := (\partial_1, 0)$ be the family of vector fields on $\Omega = \mathbb{R}^2$ and let $f(\eta) := \eta_1^2 + g(\eta_2)$ and $f^*(\eta) := \eta_1^2 + g^*(\eta_2)$ for each $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, where $g, g^* : \mathbb{R} \to [0, \infty)$ are convex functions satisfying $g(0) = g^*(0) = 0$, but $g \neq g^*$. Then it is clear that f and f^* are integrand functions of the same functional F defined in (46) , even though $f \neq f^*$.

3.7. *Remark.* Notice that, in the case $m = n$ and X satisfies (LIC) on Ω, condition (47) always holds, since, by Lemma 3.3 (iii), $\Pi_x \equiv \mathrm{Id}_n$.

Proof. **1st step.** Let us prove that f is Borel measurable. Let $\Psi : \Omega_X \times \mathbb{R}^m \to \Omega_X \times \mathbb{R}^n$ denote the map

$$\Psi(x, \eta) := (x, L^{-1}(x)(\eta)) \quad \text{if } (x, \eta) \in \Omega_X \times \mathbb{R}^m.$$

By Proposition 3.2, Ψ is continuous, then it is also Borel measurable. Since $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$ is Borel measurable, the composition $f = f_e \circ \Psi : \Omega_X \times \mathbb{R}^m \to [0, \infty]$ is still Borel measurable.

To prove (45) it is sufficient to notice that

$$f(x, \cdot) = f_e(x, \cdot) \circ L^{-1}(x) \quad \forall\, x \in \Omega_X$$

indeed $f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is convex for a.e. $x \in \Omega$ and $L^{-1}(x) : \mathbb{R}^m \to \mathbb{R}^n$ is linear for $x \in \Omega_X$.

2nd step. Let us prove the uniqueness of representation in (46). Assume that

$$\begin{aligned} \int_A f(x, Xu)\, dx &= \int_A f^*(x, Xu)\, dx \\ &= \int_A f_e(x, Du)\, dx \quad \forall\, u \in \mathbf{C}^1(A),\ A \in \mathcal{A} \end{aligned} \tag{48}$$

for given Borel measurable functions $f,\ f^* : \Omega \times \mathbb{R}^m \to [0, \infty)$, with $f(x, \cdot), f^*(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ convex a.e. $x \in \Omega$. Let us choose as

$$u(x) = u_\xi(x) := \langle \xi, x \rangle \quad x \in \mathbb{R}^n\,, \tag{49}$$

for fixed $\xi \in \mathbb{Q}^n$, in the previous equality. By (48) and (42), it follows that the functions

$$\Omega \ni y \mapsto f(y, C(y)\xi) \text{ and } \Omega \ni y \mapsto f^*(y, C(y)\xi) \text{ are in } L^1_{\mathrm{loc}}(\Omega)\,.$$

Choosing $A = B(x, r)$ in (48), by Lebesgue's differentiation theorem, we get that there exists a negligible set $\mathcal{N}_\xi \subset \Omega$ such that $\forall\, x \in \Omega \setminus \mathcal{N}_\xi$

$$\begin{aligned} f(x, L_x(\xi)) &= f(x, C(x)\xi) = f^*(x, C(x)\xi) \\ &= f^*(x, L_x(\xi))\,. \end{aligned} \tag{50}$$

If $\mathcal{N} := \cup_{\xi \in \mathbb{Q}^n} \mathcal{N}_\xi$, then (50) holds for each $x \in \Omega \setminus \mathcal{N}$ and $\xi \in \mathbb{Q}^n$. Since, for each $x \in \Omega \setminus \mathcal{N}$, $f(x, \cdot), f^*(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ are continuous, it follows that (50) holds for each $x \in \Omega \setminus \mathcal{N}$ and $\xi \in \mathbb{R}^n$. Being the map $L_x : \mathbb{R}^n \to \mathbb{R}^m$

onto, we get the desired conclusion.

3nd step. Let us assume (47). To prove (46) it is sufficient to prove that, for each $A \in \mathcal{A}$, $u \in \mathbf{C}^1(A)$

$$f(x, Xu(x)) = f_e(x, Du(x)) \quad \text{a.e. } x \in \Omega\,. \tag{51}$$

Given $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$, let us recall that

$$Xu(x) = C(x)Du(x) \quad \forall\, x \in A\,.$$

Thus, by (47), Lemma 3.3 (iii) and the definition of V_x , a.e. $x \in \Omega$, if $v_x := Du(x)$

$$\begin{aligned} f(x, Xu(x)) &= f(x, C(x)v_x) = f(x, L_x(\Pi_x(v_x))) \\ &= f_e(x, L_x^{-1}(L_x(\Pi_x(v_x))) = f_e(x, \Pi_x(v_x)) \\ &= f_e(x, v_x) = f_e(x, Du(x)) \end{aligned} \tag{52}$$

and (51) follows. On the other hand, let us assume that for every $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$

$$\int_A f_e(x, Du)\, dx = \int_A f(x, Xu)\, dx$$

where f is the function in (44). By (52), for every $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$,

$$f(x, Xu(x)) = f_e(x, \Pi_x(Du(x))) \quad \forall\, x \in A\,,$$

which implies

$$\int_A f(x, Xu(x))\, dx = \int_A f_e(x, \Pi_x(Du(x)))\, dx\,.$$

Thus, for every $A \in \mathcal{A}$ and $u \in \mathbf{C}^1(A)$,

$$\int_A f_e(x, \Pi_x(Du(x)))\, dx = \int_A f_e(x, Du)\, dx$$

and the conclusion now follows by proceeding as in the second step of the proof. □

3.8. *Remark.* Observe that (51) actually holds for each $u \in W^{1,p}(A)$. As a consequence, (46) holds for each $A \in \mathcal{A}$ and $u \in W^{1,p}(A)$.

3.3. **Integral representation for local functionals with respect to vector fields.** Let us recall, for reader's convenience, some notation about set functions on $\mathcal{A}$ and local functionals on $L^p(\Omega) \times \mathcal{A}$, according to [DM]. Let $\Omega \subset \mathbb{R}^n$ be an open set.

3.9. **Definition.** Let $\alpha : \mathcal{A} \to [0, \infty]$ be a set function. We say that:

(i) α is *increasing* if $\alpha(A) \leq \alpha(B)$, for each A, $B \in \mathcal{A}$ with $A \subseteq B$;

(ii) α is *inner regular* if

$$\alpha(A) = \sup\{\alpha(B) : B \in \mathcal{A},\, B \Subset A\} \text{ for each } A \in \mathcal{A};$$

(iii) α is *subadditive* if $\alpha(A) \leq \alpha(A_1) + \alpha(A_2)$ for every A, A_1, $A_2 \in \mathcal{A}$ with $A \subset A_1 \cup A_2$;

(iv) α is *superadditive* if $\alpha(A) \geq \alpha(A_1) + \alpha(A_2)$ for every A, A_1, $A_2 \in \mathcal{A}$ with $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$;

(v) α is a *measure* if there exists a Borel measure $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ such that $\alpha(A) = \mu(A)$ for every $A \in \mathcal{A}$.

3.10. *Remark.* Let us recall that, if $\alpha : \mathcal{A} \to [0, \infty]$ is an increasing set function, then it is a measure if and only if it is subadditive, superadditive and inner regular (see [DM, Theorem 14.23]).

3.11. **Definition.** Let

$$F : L^p(\Omega) \times \mathcal{A} \to [0, \infty] .$$

We say that:

(i) F is *increasing* if, for every $u \in L^p(\Omega)$, $F(u, \cdot) : \mathcal{A} \to [0, \infty]$ is increasing as set function;

(i) F is *inner regular* (on $\mathcal{A}$) if it is increasing and, for each $u \in L^p(\Omega)$, $F(u, \cdot) : \mathcal{A} \to [0, \infty]$ is inner regular as set function;

(iii) F is a *measure*, if for every $u \in L^p(\Omega)$, $F(u, \cdot) : \mathcal{A} \to [0, \infty]$ is a measure as set function;

(iv) F is *local* if

$$F(u, A) = F(v, A)$$

for each $A \in \mathcal{A}$, $u, v \in L^p(\Omega)$ such that $u = v$ a.e. on A;

(v) F is *lower semicontinuous* (lsc), if for every $A \in \mathcal{A}$, $F(\cdot, A) : L^p(\Omega) \to [0, \infty]$ is lower semicontinuous.

3.12. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that X satisfies (LIC) on Ω. Let $p > 1$ and*

$$F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$$

be an increasing functional satisfying the following properties:

(a) *F is local;*

(b) *F is a measure;*

(c) *F is lsc;*

(d) *$F(u + c, A) = F(u, A)$ for each $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $c \in \mathbb{R}$;*

(e) *there exist a non negative function $a \in L^1_{\mathrm{loc}}(\Omega)$ and a positive constant b such that*

$$0 \leq F(u, A) \leq \int_A (a(x) + b\,|Xu(x)|^p)\, dx$$

for each $u \in \mathbf{C}^1(A)$, $A \in \mathcal{A}$.

Then, there exists a Borel function $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ such that:

(i) *for each $u \in L^p(\Omega)$, for each $A \in \mathcal{A}$ with $u|_A \in W^{1,p}_{X;\mathrm{loc}}(A)$, we have*

$$F(u, A) = \int_A f(x, Xu(x))\, dx\,;$$

(ii) *for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is convex;*

(iii) *for a.e.* $x \in \Omega$,

$$0 \leq f(x,\eta) \leq a(x) + b\,|\eta|^p \quad \forall\, \eta \in \mathbb{R}^m\,.$$

In order to prove Theorem 3.12, we need an auxiliary lemma that will turn out to be a key result through the paper and provides when a Euclidean integrand can be represented as an integrand respect to X-gradient.

3.13. **Lemma.** *Let* $f_e : \Omega \times \mathbb{R}^n \to [0,\infty]$ *be a Borel measurable function. Suppose that*

(i) *for a.e.* $x \in \Omega$, $f_e(x,\cdot) : \mathbb{R}^n \to [0,\infty)$ *is convex;*

(ii) *there exist a non negative function* $a \in L^1_{\mathrm{loc}}(\Omega)$ *and a positive constant* b *such that for a.e.* $x \in \Omega$

$$f_e(x,\xi) \leq a(x) + b|C(x)\xi|^p \quad \forall\, \xi \in \mathbb{R}^n\,,$$

where $C(x)$ *denotes the coefficient matrix of* X*-gradient in* (3).

Then, f_e *satisfies* (47).

Proof. Let us prove that, for a.e. $x \in \Omega$,

$$f_e(x,\xi_{N_x} + \zeta) = f_e(x,\zeta) \quad \forall\, \xi,\, \zeta \in \mathbb{R}^n\,, \tag{53}$$

according to notation in Section 3.2. Notice that (53) is equivalent to (47), that is, for a.e. $x \in \Omega$

$$f_e(x,\xi) = f_e(x,\xi_{V_x}) \quad \forall\, \xi \in \mathbb{R}^n\,.$$

Since $t\xi_{N_x} \in N_x$ for every $t \in \mathbb{R}$, by our assumptions for a.e. $x \in \Omega$ we have

$$f(x, t\xi_{N_x} + \zeta) \leq a(x) + b|C(x)\zeta|^p \quad \text{for every } t \in \mathbb{R}.$$

This implies that for a.e. $x \in \Omega$ the convex function $t \to f(x, t\xi_{N_x} + \zeta)$ is bounded from above, and hence it is constant. Considering its values at $t = 1$ and $t = 0$ we obtain (53). □

Proof Theorem 3.12. Let us first observe that inequality in assumption (e) can be extended to each $u \in W^{1,p}_X(A)$, $A \in \mathcal{A}$. Indeed, let $A \in \mathcal{A}$, let $u \in W^{1,p}_X(A)$ and let $\bar{u}$ be the extension of u to $\mathbb{R}^n$ which vanishes outside Ω, let $(\rho_\varepsilon)_\varepsilon$ be a family of mollifiers, and let

$$u_\varepsilon = \bar{u} * \rho_\varepsilon(x) \quad x \in \mathbb{R}^n.$$

By Proposition 2.7 for each $A' \in \mathcal{A}$ with $A' \Subset A$, we have

$$u_\varepsilon \to u \text{ in } L^p(\Omega)\,; \tag{54}$$

$$u_\varepsilon|_{A'} \in W^{1,p}_X(A') \text{ and } u_\varepsilon \to u \text{ in } W^{1,p}_X(A')\,. \tag{55}$$

Let $u \in L^p(\Omega)$ be such that $u|_A \in W^{1,p}(A)$ for some $A \in \mathcal{A}$. For each $A' \Subset A$, by assumption (c), (54) and (55), it follows that

$$\begin{aligned} F(u,A') &\leq \liminf_{\varepsilon\to 0} F(u_\varepsilon, A') \leq \lim_{\varepsilon\to 0}\left(\int_{A'} (a(x) + b\,|Xu_\varepsilon(x)|^p)\; dx\right) \\ &= \int_{A'} (a(x) + b\,|Xu(x)|^p)\; dx. \end{aligned}$$

Since $F(u,\cdot)$ is a measure, it is also inner regular (see Remark 3.10). Thus, taking the supremum on all $A' \in \mathcal{A}$ with $A' \Subset A$, we get the desired conclusion. We will now divide the proof into three steps.

1st step. Let us first prove that there exists an integral representation of F with respect to a Euclidean integrand, that is, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty]$ and a positive constant b_2 such that

$$F(u,A) = \int_A f_e(x, Du)\, dx\,, \tag{56}$$

for each $u \in L^p(\Omega)$, $A \in \mathcal{A}$ with $u|_A \in W^{1,p}_{\mathrm{loc}}(A)$;

$$\text{for a.e. } x \in \Omega,\ f_e(x,\cdot) : \mathbb{R}^n \to [0,\infty) \text{ is convex;} \tag{57}$$

$$\text{for a.e. } x \in \Omega,\ 0 \le f_e(x,\xi) \le a(x) + b_2\,|\xi|^p \quad \forall\, \xi \in \mathbb{R}^n; \tag{58}$$

(59) (47) holds, that is, for a.e. $x \in \Omega$ $f_e(x,\xi) = f_e(x, \Pi_x(\xi)) \quad \forall\, \xi \in \mathbb{R}^n$.

By (15), if $u \in W^{1.p}(\Omega)$, then, for a.e. $x \in \Omega$, we have that

$$|Xu(x)|^p \le \sup_{x\in\Omega} \|C(x)\|^p\, |Du(x)|^p = \bar{b}\, |Du(x)|^p\,, \text{with } \bar{b} < \infty \tag{60}$$

since the coefficients of X-gradient are Lipschitz on Ω. By (60) and assumption (e), it follows that

$$0 \le F(u,A) \le \int_A (a(x) + b_2 |Du(x)|^p)\ dx\,, \tag{61}$$

for every $u \in W^{1.p}(\Omega)$ and for every $A \in \mathcal{A}$, with $b_2 := b\bar{b}$. Therefore by (a), (b), (c), (d) and (61), by applying [DM, Theorem 20.1], there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty]$ satisfying (56), (57) and (58). Observe now that, by (56) and assumption (e), if $u = u_\xi$, if follows that, for each $x \in \mathbb{R}^n$,

$$\int_A f_e(x,\xi)\, dx \le \int_A (a(x) + b|C(x)\xi|^p)\, dx \quad \forall\, A \in \mathcal{A}.$$

From this integral inequality, arguing as in section 3.2, we can infer the pointwise inequality, that is, there exists a negligible set $\mathcal{N} \subset \Omega$, such that, for each $x \in \Omega \setminus \mathcal{N}$,

$$f_e(x,\xi) \le a(x) + b|C(x)\xi|^p \quad \forall\, \xi \in \mathbb{R}^n\,, \tag{62}$$

From (57), (62) and Lemma 3.13, (59) holds.

2nd step. Let us prove that there exists a Borel function $f : \Omega \times \mathbb{R}^m \to [0,\infty]$ such that

$$F(u,A) = \int_A f(x, Xu)\, dx\,, \tag{63}$$

for each $A \in \mathcal{A}$, $u \in \mathbf{C}^1(A)$ satisfying claims (ii) and (iii). By (57), (58) and (59), we can apply Theorem 3.5 and (63) follows at once with $f : \Omega \times \mathbb{R}^m \to [0,\infty]$ defined as in (44), which also satisfies claim (ii).

From assumption (e) and (63) with $u = u_\xi$, it follows that

$$0 \le \int_A f(y, C(y)\xi)\, dy \le \int_A (a(y) + b\,|C(y)\xi|^p)\, dy \quad \text{for each } A \in \mathcal{A},\ \xi \in \mathbb{R}^n\,.$$

Taking $A = B(x,r)$, applying Lebesgue's differentiation theorem and arguing as before, from the previous inequality, we can get the following pointwise estimate: for a.e. $x \in \Omega$ it holds that

$$0 \le\ f(x, C(x)\xi) \le\ a(x) +\ b\,|C(x)\xi|^p \quad \forall\, \xi \in \mathbb{R}^n\,.$$

Observe now that, by (LIC), for a.e. $x \in \Omega$, the map $L_x : \mathbb{R}^n \to \mathbb{R}^m$, $L_x(\xi) := C(x)\xi$, is surjective. Then claim (iii) also follows.

3rd step. Let us prove that the integral representation in (63) can be extended to functions $u \in W^{1,p}_{X,\mathrm{loc}}(A)$. Therefore claim (i) will follow.

Let us begin to observe that, given $A \in \mathcal{A}_0$, the functional

$$W^{1,p}_X(A) \ni u \mapsto \int_A f(x, Xu)\, dx \text{ is (strongly) continuous.} \tag{64}$$

Indeed, since for a.e. $x \in \Omega$, $f(x,\cdot) : \mathbb{R}^m \to [0,\infty)$ is continuous and claim (iii) holds, we can apply the Carathéodory continuity theorem (see, for instance, [DM, Example 1.22]).

Let $u \in W^{1,p}_X(\Omega)$ and let $A,\ A' \in \mathcal{A}$ with $A' \Subset A$. Since $F(\cdot, A') :\ L^p(\Omega) \to [0,\infty]$, by (54), it follows that

$$F(u, A') \le \liminf_{\varepsilon\to 0^+} F(u_\varepsilon, A') = \lim_{\varepsilon\to 0^+} \int_{A'} f(x, Xu_\varepsilon)\, dx = \int_{A'} f(x, Xu)\, dx\,.$$

As F is a measure, taking the limit as $A' \uparrow A$, we get

$$F(u, A) \le \int_A f(x, Xu)\, dx\,, \tag{65}$$

for every $u \in W^{1,p}_X(\Omega)$, for each $A \in \mathcal{A}$.

Let us fix $w \in W^{1,p}_X(\Omega)$ and let us consider the functional $G :\ L^p(\Omega)\times\mathcal{A} \to [0,\infty]$

$$G(u, A) :=\ F(u + w, A)\,.$$

It is easy to show that G still satisfies assumptions (a)-(e). Thus, by the second step, there exists a Borel function $g :\ \Omega \times \mathbb{R}^m \to$ satisfying claims (ii) and (iii) with $f \equiv g$, for suitable $a \in L^1_{\mathrm{loc}}(\Omega)$ and $b >\ 0$ such that

$$G(u, A) = \int_A g(x, Xu)\, dx\,, \tag{66}$$

for each $A \in \mathcal{A}$, $u \in \mathbf{C}^1(A)$ and

$$G(u, A) \le \int_A g(x, Xu)\, dx\,, \tag{67}$$

for every $u \in W^{1,p}_X(\Omega)$, for each $A \in \mathcal{A}$. Moreover, arguing as in (64), one can prove that, for each $A \in \mathcal{A}_0$, the functional

$$W^{1,p}_X(A) \ni u \mapsto \int_A g(x, Xu)\, dx \text{ is (strongly) continuous.} \tag{68}$$

Let

$$w_\varepsilon := \bar{w} * \rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}$$

and fix $A \in \mathcal{A}$. Then, for every $A' \in \mathcal{A}$ with $A' \Subset A$, as $\varepsilon \to 0^+$,

$$w_\varepsilon \to w \text{ in } L^p(\Omega) \text{ and } w_\varepsilon \to w \text{ in } W^{1,p}_X(A')\,.$$

Thus, by (64), (65), (66), (67), (68) we obtain

$$\begin{aligned}
\int_{A'} g(x,0)\, dx = G(0, A') = F(w, A') &\le \int_{A'} f(x, Xw)\, dx \\
&= \lim_{\varepsilon \to 0^+} \int_{A'} f(x, Xw_\varepsilon)\, dx = \lim_{\varepsilon \to 0^+} F(w_\varepsilon, A') \\
&= \lim_{\varepsilon \to 0^+} G(w_\varepsilon - w, A') = \lim_{\varepsilon \to 0^+} \int_{A'} g(x, Xw_\varepsilon - Xw)\, dx \\
&= \int_{A'} g(x, 0)\,.
\end{aligned}$$

This implies that

$$F(w, A') = \int_{A'} f(x, Xw)\, dx \text{ for each } A' \in \mathcal{A} \text{ with } A' \Subset A\,.$$

Taking the limit as $A' \uparrow A$ in the previous identity, we get that

$$F(w, A) = \int_A f(x, Xw)\, dx \text{ for each } w \in W^{1,p}_X(\Omega) \text{ and } A \in \mathcal{A}\,. \tag{69}$$

If $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $u|_A \in W^{1,p}_{X;\mathrm{loc}}(A)$ then, for every $A' \in \mathcal{A}$ with $A' \Subset A$, by Remark 2.6, there exists $w \in W^{1,p}_X(\Omega)$ such that

$$u|_{A'} = w|_{A'}\,.$$

Since F is local, by (69), we obtain that

$$F(u, A') = F(w, A') = \int_{A'} f(x, Xw)\, dx = \int_{A'} f(x, Xu)\, dx\,.$$

Taking the limit as $A' \uparrow A$ we get

$$F(u, A) = \int_A f(x, Xu)\, dx\,,$$

which concludes the proof. □

3.14. **Counterexample.** If X agrees with the Euclidean gradient (Example 2.2 (i)), there are well-known examples that, dropping one of the assumptions among (a)-(e) in Theorem 3.12, then the conclusion may fail (see, for instance, [B]). Let X be the Heisenberg vector fields in $\mathbb{R}^3$ (Example 2.2

(iii)), let $\Omega \subset \mathbb{R}^3$ be a bounded open set containing the origin and $p = 2$. Then we give an instance that, dropping assumption (e), the conclusion of Theorem 3.12 may fail. Let $F : L^2(\Omega) \times \mathcal{A} \to [0, \infty]$ be the local functional defined as

$$F(u, A) := \begin{cases} \int_A |Du|^2 \, dx & \text{if } u \in W^{1,2}(A) \\ \infty & \text{otherwise} \end{cases} .$$

Then, it is clear that F satisfies (a)-(d). Let us prove that functional F cannot satisfy claim (i). Indeed, by contradiction, if there is some integrand $f : \Omega \times \mathbb{R}^2 \to [0, \infty]$ for which (i) holds, then, by Theorem 3.5, the compatibility condition (47) must be satisfied, that is,

$$|\xi|^2 = f_e(x, \xi) = f(x, C(x)\xi) = f_e(x, \Pi_x(\xi)) = |\Pi_x(\xi)|^2$$

for a.e. $x \in \Omega, \forall \, \xi \in \mathbb{R}^3$. Since, by Lemma 3.3 (iii), function $\Omega \ni x \mapsto \Pi_x(\xi)$ is continuous, the previous identity must hold for each $x \in \Omega$ and $\xi \in \mathbb{R}^3$. Let $x = 0$, then a simple calculation yields that $\Pi_0(\xi) = (\xi_1, \xi_2, 0)$ for each $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Thus, if we choose $\xi = (0, 0, 1)$, the previous identity is not satisfied and then we have a contradiction. This example also shows that the correspondence which maps integrand $f(x, \eta)$ to Euclidean integrand $f_e(x, \xi) := f(x, C(x)\xi)$ cannot be reversed.

4. Γ- for integral functionals depending on vector fields

In this section we are going to show some results concerning Γ-convergence of integral functionals depending on vector fields, in the strong and weak topology of $W^{1,p}_X(\Omega)$ and in the strong one of $L^p(\Omega)$. In particular, we will prove a Γ-compactness result for a class of integral functionals depending on vector fields with respect to $L^p(\Omega)$-topology (see Theorem 4.10).

Let us first recall some notions and results concerning Γ-convergence theory, which are contained in the fundamental monograph [DM] and to which we will refer through this section. We also recommend monograph [Bra] as exastuive account on this topic, containing also interesting applications of Γ-convergence.

Let (X, τ) be a topological space and let $(F_h)_h$ be a sequence of functionals from the space (X, τ) to $\bar{\mathbb{R}}$. Let $\mathcal{U}(x)$ be the family of open neighborhoods of $x \in X$. Then we pose for every $x \in X$

$$(\Gamma(\tau)\text{-}\liminf_{h\to\infty} F_h)(x) = \sup_{U \in \mathcal{U}(x)} \liminf_{h\to\infty} \inf_U F_h \, .$$

$$(\Gamma(\tau)\text{-}\limsup_{h\to\infty} F_h)(x) = \sup_{U \in \mathcal{U}(x)} \limsup_{h\to\infty} \inf_U F_h \, .$$

They are called, respectively, the Γ-*lower limit* and Γ-*upper limit* of the sequence $(F_h)_h$ in the topology τ.

Then, we give the following definition.

4.1. **Definition.** Let $(F_h)_h$ and F be functionals from space (X,τ) to $\bar{\mathbb{R}}$. We say that $(F_h)_h$ $\Gamma(\tau)$*-converges to* F, or also that $(F_h)_h$ Γ*-converges to* F *in the topology* τ*, at* $x \in X$*,* if

$$(\Gamma(\tau)\text{-}\liminf_{h\to\infty} F_h)(x) = (\Gamma(\tau)\text{-}\limsup_{h\to\infty} F_h)(x) = F(x)$$

and we write

$$F(x) = (\Gamma(\tau)\text{-}\lim_{h\to\infty} F_h)(x)\,.$$

Let us recall below some relevant properties concerning Γ-convergence that we will need later.

4.2. **Theorem.** *Let* F_h *and* F *be functionals from space* (X,τ) *to* $\bar{\mathbb{R}}$*.*

(i) ([DM, Proposition 6.1]) *If* $(F_h)_h$ $\Gamma(\tau)$*-converges to* F*, then each of its subsequence* $(F_{h_k})_k$ *still* $\Gamma(\tau)$*-converges to* F*.*

(ii) ([DM, Proposition 6.3]) *Let* τ_i*,* $i = 1, 2$*, be two topologies on* X *and suppose that* τ_1 *is weaker than* τ_2*. If* $(F_h)_h$ $\Gamma(\tau_1)$*-converges to* F_1 *and* $\Gamma(\tau_2)$*-converges to* F_2*, then* $F_1 \le F_2$*.*

(iii) ([DM, Theorem 7.8]) (*Fundamental Theorem of* Γ*-convergence*) *Assume that the sequence* $(F_h)_h$ *is equicoercive* (*on* X)*, that is, for each* $t \in \mathbb{R}$ *there exists a closed countably compact* $K_t \subset X$ *such that*

$$\{x \in X : F_h(x) \le t\} \subset K_t \quad \text{for each } h\,.$$

Let us also assume that $(F_h)_h$ $\Gamma(\tau)$*-converges to* F*. Then* F *is coercive and*

$$\min_{x\in X} F(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x)\,.$$

(iv) ([DM, Proposition 8.1]) *Assume that* (X,τ) *satisfies the first countability axiom. Then* $(F_h)_h$ $\Gamma(\tau)$*-converges to* F *if and only if the following two conditions hold:*

(1) ($\Gamma - \liminf$ inequality) *for any* $x \in X$ *and for any sequence* $(x_h)_h$ *converging to* x *in* X *one has*

$$F(x) \le \liminf_{h\to\infty} F_h(x_h)\,;$$

(2) ($\Gamma - \lim$ equality) *for any* $x \in X$*, there exists a sequence* $(x_h)_h$ *converging to* x *in* X *such that*

$$F(x) = \lim_{h\to\infty} F_h(x_h)\,.$$

(v) ([DM, Theorem 8.5]) *Assume that* (X,τ) *satisfies the second countability axiom, that is, there is a countable base for the topology* τ*. Then every sequence* $(F_h)_h$ *of functionals from* X *to* $\bar{\mathbb{R}}$ *has a* $\Gamma(\tau)$*-convergent subsequence.*

4.3. *Remark.* It is well-known that inequality in Theorem 4.2 (ii) can be strict, even in the case of a (infinite dimensional) Banach space X, $\tau_1 \equiv$ weak topology of X and $\tau_2 \equiv$ strong topology of X (see, for instance, [DM, Example 6.6]). An instance of such a phenomenon can occur in the case of non-coercive quadratic integral functionals [ACM].

4.4. **Definition** ($\bar\Gamma$-convergence for local functional on $L^p(\Omega)\times\mathcal{A}$)**.** Let $F_h : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ $(h=1,2,\dots)$ be a sequence of increasing functionals. We say that the sequence $(F_h)_h$ $\bar\Gamma$-converges to a functional $F:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$, and we will write $F=\ \bar\Gamma-\lim_{h\to\infty}F_h$, if F is increasing, inner regular and lsc and the following conditions are satisfied:

[$\bar\Gamma-\liminf$ inequality] for each $u\in L^p(\Omega)$, for every $A\in\mathcal{A}$ and $(u_h)_h\subset L^p(\Omega)$ converging to u in $L^p(\Omega)$, it holds

$$F(u,A)\le\liminf_{h\to\infty}F_h(u_h,A)\,;$$

[$\bar\Gamma-\limsup$ inequality] for each $u\in L^p(\Omega)$, for each A, $B\in\mathcal{A}$ with $A\Subset B$, there exists $(u_h)_h\subset L^p(\Omega)$ converging to u in $L^p(\Omega)$ with

$$F(u,B)\ge\limsup_{h\to\infty}F_h(u_h,A)\,.$$

4.5. *Remark.* Let us consider a sequence of increasing functionals $F_h : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ $(h=1,2,\dots)$. Assume that there exists a measure functional $F:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ such that $(F_h(\cdot,A))_h$ Γ-converges to $F(\cdot,A)$ for each $A\in\mathcal{A}$. Then $(F_h)_h$ $\bar\Gamma$-converges to F. Indeed, being F a Γ-limit, it is lsc (see [DM, Propostion 6.8]) and it is increasing and inner regular, because it is a measure. Moreover the $\bar\Gamma$-$\liminf$ and $\bar\Gamma$-$\limsup$ inequalities immediately follows by the characterization of Γ-limit in Theorem 4.2 (iv).

4.6. **Definition.** Let $F:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ be a non-negative functional. We say that F satisfies the *fundamental estimate* if, for every $\varepsilon>0$ and for every A', A'', $B\in\mathcal{A}$, with $A'\Subset A''$, there exists a constant $M>0$ with the following property: for every u, $v\in L^p(\Omega)$, there exists a function $\varphi\in\mathbf{C}^\infty_c(A'')$, with $0\le\varphi\le1$ on A'', $\varphi=1$ in a neighborhood of A' , such that

$$F\Big(\varphi u+(1-\varphi)v,A'\cup B\Big)\le\ (1+\varepsilon)\Big(F(u,A'')+F(v,B)\Big)+$$
$$+\,\varepsilon\Big(\|u\|^p_{L^p(S)}+\|v\|^p_{L^p(S)}+1\Big)+M\|u-v\|_{L^p(S)}\,,$$

where $S=(A''\setminus A')\cap B$. Moreover, if $\mathcal{F}$ is a class of non-negative functional on $L^p(\Omega)\times\mathcal{A}$, we say that the *fundamental estimate holds uniformly in* $\mathcal{F}$ if each element F of $\mathcal{F}$ satisfies the fundamental estimate with M depending only on ε, A', A'', B while φ may depend also on F, u, v.

4.7. *Remark.* Let us recall that, if $F=\bar\Gamma\text{-}\lim_{h\to\infty}F_h$ and $F_h:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ are *measures*, then F need not be a measure (see [DM, Examples 16.13 and 16.14]). If the sequence $(F_h)_h$ satisfies the fundamental estimates uniformly with respect to h, then F is a measure (see [DM, Theorem 18.5]).

Let us now state a result which assures the coincidence between the $\bar\Gamma\text{-}\lim F_h$ and $\Gamma\text{-}\lim F_h$ for a sequence of local functional $F_h:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$, provided that the fundamental estimate holds uniformly for the sequence $(F_h)_h$ [DM, Theorem 18.7].

4.8. **Theorem.** *Let $(F_h)_h$ be a sequence of non-negative increasing functionals on $L^p(\Omega)\times\mathcal{A}$ which $\bar{\Gamma}$ -converges to a functional F. Assume that there exist two constants $c_1\geq 1$ and $c_2\geq 0$, a non-negative increasing functional $G:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$, and a non-negative Radon measure $\mu:\mathcal{B}(\Omega)\to[0,\infty]$ such that*

$$G(u,A)\leq\ F_h(u,A)\leq\ c_1\,G(u,A)+\ c_2\|u\|^p_{L^p(A)}+\ \mu(A)$$

for every $u\in L^p(\Omega)$, $A\in\mathcal{A}$ and $h\in\mathbb{N}$. Assume, in addition, that G is a lower semicontinuous measure and that the fundamental estimate holds uniformly for the sequence $(F_h)_h$. Then, $(F_h(\cdot,A))_h$ Γ-converges in $L^p(\Omega)$ to $F(\cdot,A)$ for every $A\in\mathcal{A}$ such that $\mu(A)<\infty$.

Moreover, let us remind that, if we will deal with integral functionals $F:W^{1,p}_X(\Omega)\to\mathbb{R}$, with Ω bounded open subset of $\mathbb{R}^n$ and $p>1$, of the form

$$F(u):=\int_\Omega f(x,Xu)\,dx \tag{70}$$

where the integrand $f:\ \Omega\times\mathbb{R}^m\to\mathbb{R}$ belongs to class $I_{m,p}(\Omega,c_0,c_1)$, then it is easy to show, taking [DM, Proposition 5.12] into account, that the following Γ-convergence results still hold.

4.9. **Proposition.** *Let $(f_h)_h$ and f be functions in $I_{m,p}(\Omega,0,c_1)$. Let F_h, $F:W^{1,p}_X(\Omega)\to\mathbb{R}$ be the corresponding integral functionals in (70). Assume that*

$$F_h\to F\ (\textit{pointwise})\ \textit{in}\ W^{1.p}_X(\Omega)\,. \tag{71}$$

Then $(F_h)_h$ Γ-converges to F in $W^{1.p}_X(\Omega)$, i.e.,

$$F(u)=\ (\Gamma(W^{1,p}_X(\Omega))\text{-}\lim_{h\to\infty}F_h)(u)\quad\forall\,u\in W^{1,p}_X(\Omega)\,. \tag{72}$$

4.1. **Γ-compactness results for integral functional depending on vector fields.** The main result of this section is the following.

4.10. **Theorem.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and let $X=(X_1,\dots,X_m)$ satisfy (LIC) on Ω. Let $(f_h)_h\subset I_{m,p}(\Omega,c_0,c_1)$ and, for each h, let $F^*_h:L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ be the local functional defined as*

$$F^*_h(u,A):=\begin{cases}\int_A f_h(x,Xu(x))dx & \textit{if}\ A\in\mathcal{A},\ u\in W^{1,p}_X(A)\\ \infty & \textit{otherwise}\end{cases}\,. \tag{73}$$

Then, up to a subsequence, there exist a local functional $F:\ L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ and $f\in I_{m,p}(\Omega,c_0,c_1)$ such that

(i) $F(\cdot,A)=\Gamma(L^p(\Omega))-\lim_{h\to\infty}F_h(\cdot,A)$ *for each $A\in\mathcal{A}$;*
(ii) *F admits the following representation*

$$F(u,A):=\begin{cases}\int_A f(x,Xu(x))dx & \textit{if}\ A\in\mathcal{A},u\in W^{1,p}_X(A)\\ \infty & \textit{otherwise}\end{cases}\,. \tag{74}$$

Let us begin to recall a fundamental result about the representantion of the $\bar{\Gamma}$-limit with respect to a Euclidean integrand [DM, Theorem 20.3], which applies to a large class of integral functionals. Let c_1, c_2, c_3 be real numbers with $c_i \geq 0$ $i = 1, 2, 3$. Let us denote by $\mathcal{H} = \mathcal{H}(p, c_1, c_2, c_3)$ the class of all local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ for which there exist two Borel functions f_e, $g : \Omega \times \mathbb{R}^n \to [0, \infty)$ (depending on F) such that

(a) $F(u, A) := \begin{cases} \int_A f_e(x, Du)dx & \text{if } A \in \mathcal{A}, u \in W^{1,1}_{\text{loc}}(A) \\ \infty & \text{otherwise} \end{cases}$;

(b) $g(x, \xi) \leq f_e(x, \xi) \leq c_1 (g(x, \xi) + 1)$;

(c) $0 \leq g(x, \xi) \leq c_2 (|\xi|^p + 1)$;

(d) $g(x, \cdot)$ is convex on $\mathbb{R}^n$;

(e) $g(x, 2\xi) \leq c_3 (g(x, \xi) + 1)$,

for every $u \in L^p(\Omega)$, $x \in \Omega$, $\xi \in \mathbb{R}^n$.

4.11. **Theorem.** *For every sequence $(F_h)_h$ of functionals of the class $\mathcal{H}$ there exist a subsequence $(F_{h_k})_k$ and an increasing functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $(F_{h_k})_k$ $\bar{\Gamma}$-converges to F. The functional F can be represented in integral form by a Euclidean integrand, that is, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty]$ verifying*

(i) *$f_e(x, \cdot)$ is convex on $\mathbb{R}^n$;*

(ii) *$0 \leq f_e(x, \xi) \leq c_1(c_2 + 1) + c_1 c_2 |\xi|^p$ for a.e. $x \in \Omega$, for each $\xi \in \mathbb{R}^n$,*

such that (10) *holds.*

Let us also recall an useful criterion for proving that a class of local functionals on $L^p(\Omega) \times \mathcal{A}$ satisfies the fundamental estimate uniformly [DM, Theorem 19.4] and a $\bar{\Gamma}$-compacness result in this class [DM, Theorem 19.5].

4.12. **Theorem.** *Let c_i $(i = 1, 2, 3, 4)$ be non negative real numbers and let $\sigma : \mathcal{A} \to [0, \infty]$ be a superadditive increasing set function such that $\sigma(A) < \infty$ for each $A \Subset \Omega$. Let $\mathcal{F}' = \mathcal{F}'(p, c_1, c_2, c_3, c_4)$ be the class of all non-negative increasing local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ with the following properties: F is a measure and there exists a non-negative increasing local functional $G : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ (depending on F) such that G is a measure and*

$$G(u, A) \leq F(u, A) \leq c_1 G(u, A) + c_2 \|u\|^p_{L^p(A)} + \sigma(A) ; \tag{75}$$

$$\begin{aligned} G(\varphi u + (1 - \varphi)v, A) \leq\ & c_4 (G(u, A) + G(v, A)) + \\ & + c_3 c_4 \max_{\Omega} |D\varphi|^p \|u - v\|^p_{L^p(A)} + 2c_4 \sigma(A) , \end{aligned} \tag{76}$$

for every u, $v \in L^p(\Omega)$, $A \in \mathcal{A}$, $\varphi \in \mathbf{C}^\infty_c(\Omega)$ with $0 \leq \varphi \leq 1$. Then, the fundamental estimate holds uniformly on $\mathcal{F}'$.

4.13. **Theorem.** *Let $\mathcal{F}' = \mathcal{F}'(p, c_1, c_2, c_3, c_4)$ be the class of local functionals defined in Theorem 4.12. For every sequence $(F_h)_h \subset \mathcal{F}'$, there exists a subsequence $(F_{h_k})_k$ which $\bar{\Gamma}$-converges to a lower semicontinuous functional $F \in \mathcal{F}'$.*

Let us now introduce some results concerning functionals depending on vector fields. Let us first prove a Γ-compactness result (see Theorem 4.15) for a class of local functional on $L^p(\Omega)\times\mathcal{A}$ satisfying suitable growth conditions with respect to the local functional $\Psi_p : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ defined as

$$\Psi_p(u,A) := \begin{cases} \int_A |Xu|^p\,dx & \text{if } A\in\mathcal{A}, u\in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases}. \tag{77}$$

As a consequence, we will get a Γ-compactness result for a class of integral functionals represented with respect to Euclidean integrands, but still with growth condition with respect to Ψ_p (see Theorem 4.16). The former is an extension of [DM, Theorem 19.6], the latter of [DM, Theorem 20.4].

4.14. **Lemma.** *Let $p>1$. Then $\Psi_p : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ is a measure and lsc.*

Proof. Let us start by proving that for any $A\in\mathcal{A}$ the function $u\to\Psi_p(u,A)$ is L^p-lsc, i.e., for any $A\in\mathcal{A}$ and $(u_h)_h\subset L^p(\Omega)$, $u_h\to u$ in $L^p(\Omega)$, it satisfies

$$\Psi_p(u,A)\le\liminf_{h\to\infty}\Psi_p(u_h,A). \tag{78}$$

We can assume $\liminf_{h\to\infty}\Psi_p(u_h,A)<\infty$. Therefore, up to a subsequence, we can also assume that $\lim_{h\to\infty}\Psi_p(u_h,A)$ exists in $\mathbb{R}$. Hence $(u_h)_h$ is bounded in $W^{1,p}_X(A)$ and, since $W^{1,p}_X(A)$ is reflexive (recall Proposition 2.5 and that $p>1$), we get a subsequence $u_h\rightharpoonup u$ in $W^{1,p}_X(A)$ and, in particular, $Xu_h\rightharpoonup Xu$ in $L^p(A)$, which implies the conclusion, recalling the lower semicontinuity of the L^p-norm with respect to the weak convergence.

We now prove that for any $u\in L^p(\Omega)$ the function $\Psi_p(u,\cdot):\mathcal{A}\to[0,\infty]$ is a measure, i.e., there exists a Borel measure $\mu_u:\mathcal{B}(\Omega)\to[0,\infty]$ such that $\Psi_p(u,A)=\mu_u(A)$ for every $A\in\mathcal{A}$. Since, by Remark 3.10, $\Psi_p(u,\cdot)$ is nonnegative, increasing and such that $\Psi_p(u,\emptyset)=0$, it suffices to prove that $\Psi_p(u,\cdot)$ is subadditive, superadditive and inner regular.

$\Psi_p(u,\cdot)$ is subadditive, namely for every $A,A_1,A_2\in\mathcal{A}$ with $A\subseteq A_1\cup A_2$

$$\Psi_p(u,A)\le\Psi_p(u,A_1)+\Psi_p(u,A_2). \tag{79}$$

We can assume $u\in W^{1,p}_X(A_1)\cap W^{1,p}_X(A_2)$ and $A_1,A_2\in\mathcal{A}$, otherwise the conclusion is trivial. Remark 2.6 (iii) gives $u\in W^{1,p}_X(A_1\cup A_2)$, therefore $\Psi_p(u,A_1\cup A_2)=\int_{A_1\cup A_2}|Xu|^p\,dx$ and (79) follows.

$\Psi_p(u,\cdot)$ is superadditive, namely for every $A,A_1,A_2\in\mathcal{A}$ with $A_1\cup A_2\subseteq A$ and $A_1\cap A_2=\emptyset$

$$\Psi_p(u,A)\ge\Psi_p(u,A_1)+\Psi_p(u,A_2). \tag{80}$$

We can assume $u\in W^{1,p}_X(A)$ and $A\in\mathcal{A}$, otherwise the conclusion is trivial. Remark 2.6 (iv) gives $u\in W^{1,p}_X(B)$ for any open set $B\subseteq A$. Let $A,A_1,A_2\in$

$\mathcal{A}$, $A_1 \cup A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$. Then

$$\Psi_p(u, A_1) + \Psi_p(u, A_2) = \int_{A_1 \cup A_2} |Xu|^p \, dx \leq \int_A |Xu|^p \, dx$$

and (80) follows.

$\Psi_p(u, \cdot)$ is inner regular, namely for every $A \in \mathcal{A}$

$$\Psi_p(u, A) = \sup \{\Psi_p(u, B) \mid B \in \mathcal{A},\ B \Subset A\}. \tag{81}$$

Let $M := \sup \{\Psi_p(u, B) \mid B \in \mathcal{A},\ B \Subset A\} \in [0, +\infty]$. If $M = +\infty$, there exists $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$, $B_i \Subset A$ such that $\Psi_p(u, B_i) \to \infty$ as $i \to +\infty$ and the conclusion follows by observing that for all $i \in \mathbb{N}$, $\Psi_p(u, B_i) \leq \Psi_p(u, A)$. If $M \in [0, \infty)$, then $\|u\|_{W^{1,p}_X(B)} \leq M$ for any $B \in \mathcal{A}$, $B \Subset A$. Then, Remark 2.6 (ii) gives $u \in W^{1,p}_X(A)$ and, by definition, $\Psi_p(u, A) = \int_A |Xu|^p \, dx$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |Xu|^p \, dx \leq \varepsilon$ for any $E \in \mathcal{A}$ with $|E| \leq \delta$. Let $B \Subset A$ such that $|A \setminus \overline{B}| \leq \delta$, then

$$\int_A |Xu|^p \, dx = \int_B |Xu| \, dx + \int_{A \setminus \overline{B}} |Xu|^p \, dx \leq \int_B |Xu|^p \, dx + \varepsilon$$

and the thesis follows. □

4.15. **Theorem.** *Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $c_1 \geq c_0 > 0$. Denote by $\mathcal{M} = \mathcal{M}(p, c_0, c_1)$ the class of local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that F is a measure and*

$$c_0 \, \Psi_p(u, A) \leq F(u, A) \leq c_1 \left(\Psi_p(u, A) + \|u\|^p_{L^p(A)} + |A| \right) \tag{82}$$

for every $u \in L^p(\Omega)$ and for every $A \in \mathcal{A}$. Then, the fundamental estimate holds uniformly in $\mathcal{M}$ and every sequence $(F_h)_h \subset \mathcal{M}$ has a subsequence $(F_{h_k})_k$ which $\bar{\Gamma}$-converges to a functional F of the class $\mathcal{M}$. Moreover, $(F_{h_k}(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ and

$$\mathrm{dom} F(\cdot, A) := \{u \in L^p(\Omega) : F(u, A) < \infty\} = W^{1,p}_X(A) \tag{83}$$

for every $A \in \mathcal{A}$.

Proof. Let us begin to prove that the fundamental estimate holds uniformly for the sequence $(\Psi_h)_h$. Let

$$g(x, \xi) := c_0 \, |C(x)\xi|^p \quad \text{if } x \in \Omega,\ \xi \in \mathbb{R}^n. \tag{84}$$

Notice that, since the entries of matrix $C(x)$ are Lipschitz continuous functions,

$$g(x, \xi) \leq c_0 \sup_\Omega \|C(x)\|^p \, |\xi|^p = c_2 |\xi|^p \quad \text{if } x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{85}$$

$$g(x, 2\xi) = 2^{p-1} \, 2g(x, \xi) = c_3 \, 2g(x, \xi) \quad \text{if } x \in \Omega,\ \xi \in \mathbb{R}^n \tag{86}$$

and

$$g(x, \cdot) \text{ is convex on } \mathbb{R}^n. \tag{87}$$

Thus, from (85), (86) and (87), arguing as in [DM, (19.6)], it follows that

$$g(x, t\xi + (1-t)\eta + \zeta) \leq c_3 \left[g(x,\xi) + g(x,\eta) + c_2 |\zeta|^p\right] \tag{88}$$

for every $x \in \Omega$, $t \in [0,1]$, ξ, $\eta \in \mathbb{R}^n$. We are going to apply Theorem 4.12. Observe that, choosing $G = c_0 \Psi_p$, from (82), (75) immediately holds with

$$c_1 \equiv \frac{c_1}{c_0},\ c_2 \equiv c_1,\ \sigma(A) = c_1 |A|\,.$$

Let us show (76). By (88), it follows that

$$\begin{aligned} G(\varphi u + (1-\varphi)v, A) &= \int_A g\left(x, \varphi Du + (1-\varphi)Dv + (u-v)D\varphi\right) dx \\ &\leq \int_A c_3 \left[g(x, Du) + g(x, Dv) + c_2 |D\varphi|^p |u-v|^p\right] dx \\ &\leq c_3 \left(G(u,A) + G(v,A)\right) + c_3 c_2 \left(\max_\Omega |D\varphi|^p\right) \|u-v\|^p_{L^p(A)} \end{aligned}$$

for each u, $v \in L^p(\Omega)$, $A \in \mathcal{A}$, $\varphi \in \mathbf{C}^\infty_c(\Omega)$ with $0 \leq \varphi \leq 1$. Thus (76) holds with

$$c_4 \equiv c_3 \text{ and } c_3 c_4 \equiv c_3 c_2.$$

Thus we get the desired conclusion. From Theorem 4.13, every sequence $(F_h)_h \subset \mathcal{M}$ has a subsequence $(F_{h_k})_k$ $\bar{\Gamma}$-converging to a functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ which is a measure. As each functional F_h satisfies (82), the functional F satisfies (82), since Ψ_p is lsc and inner regular by Lemma 4.14 and Remark 3.10. By applying Theorem 4.8, we get that $(F_{h_k}(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for each $A \in \mathcal{A}$, since Ω is bounded. Finally, by (82), (83) follows. □

Let $p > 1$ and let $c_1 \geq c_0$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let us denote by $\mathcal{I} = \mathcal{I}(p, c_0.c_1)$ the class of local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ for which there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty)$ such that

(i) claim (a) of properties defining $\mathcal{H}$ holds;
(ii) $c_0 |C(x)\xi|^p \leq f_e(x,\xi) \leq c_1 \left(|C(x)\xi|^p + 1\right)$ a.e. $x \in \Omega$, for each $\xi \in \mathbb{R}^n$.

4.16. **Theorem.** *For every sequence $(F_h)_h \subset \mathcal{I}$ there exist a subsequence $(F_{h_k})_k$ and a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ such that $(F_{h_k}(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ and* (83) *holds for every $A \in \mathcal{A}$. Moreover there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty)$, convex in the second variable and satisfying (ii), for which* (10) *holds.*

Proof. By Theorem 4.15, for each $(F_h)_h \subset \mathcal{I}$ there exist a subsequence $(F_{h_k})_k$ and an inner regular functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ such that $(F_{h_k}(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for every $A \in \mathcal{A}$. Moreover, since Ψ_p is lsc and inner regular, for each $u \in L^p(\Omega)$, $A \in \mathcal{A}$,

$$c_0 \Psi_p(u, A) \leq F(u, A) \leq c_1 \left(\Psi_p(u, A) + |A|\right) \tag{89}$$

where Ψ_p is the local functional in (77). If $g(x,\xi)$ is as in (84), $\mathcal{I}(p, c_0.c_1) \subset \mathcal{H}(p, c_1', c_2', c_3')$, for suitable c_i' $(i = 1, 2, 3)$. From Theorem 4.11 , there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, also convex in the second variable, for which (10) holds.

Let us now prove that (ii) of properties defining $\mathcal{I}$ holds. Let u_ξ be the function in (49). From (89), it follows that

$$c_0 \int_A |C(x)\xi|^p \, dx \leq \int_A f_e(x,\xi) \, dx \leq c_1 \left(|A| + \int_A |C(x)\xi|^p \, dx \right)$$

for each $\xi \in \mathbb{R}^n$ and $A \in \mathcal{A}$. By means of the usual procedure, we can infer that there exists a negligeble set $\mathcal{N} \subset \Omega$ such that, for each $x \in \Omega \setminus \mathcal{N}$,

$$c_0 \, |C(x)\xi|^p \leq f_e(x,\xi) \leq c_1 \, (|C(x)\xi|^p + 1) \quad \forall \, \xi \in \mathbb{Q}^n \, .$$

Then, since $f_e(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is continuous a.e. $x \in \Omega$, we can extend the previous inequality to all $\xi \in \mathbb{R}^n$. □

4.17. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $(f_h)_h \subset I_{m.p}(\Omega, c_0, c_1)$ and, for each h, let $F_h^* : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be the local functional defined in* (73)*. Then, there exist a subsequence $(F_{h_k}^*)_k$ and a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $(F_{h_k}^*(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ and* (83) *holds for every $A \in \mathcal{A}$. Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which* (10) *holds.*

Proof. Let $(f_{h,e})_h$ denote the sequence of Euclidean integrands in (8) and let $(F_h)_h$ be the sequence of local functionals in (7). Since $(f_{h,e})_h \subset \mathcal{I}$, by applying Theorem 4.16, there exist a subsequence $(F_{h_k})_k$ and a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $(F_{h_k}(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for every $A \in \mathcal{A}$. Moreover, there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which (10) holds.

By Theorem 3.1 (iii), it follows that, for each $h \in \mathbb{N}$, $A \in \mathcal{A}$,

$$F_h^*(\cdot, A) = \bar{F}_h(\cdot, A) \tag{90}$$

where $\bar{F}_h(\cdot, A) : L^p(\Omega) \to [0, \infty]$ denotes the relaxed functional of $F_h(\cdot, A) : L^p(\Omega) \to [0, \infty]$ with respect to the $L^p(\Omega)$ topology (see (22)). By (90) and a well-known property of Γ-convergence (see [DM, Propostion 6.11]), we also get that $(F_{h_k}^*(\cdot, A))_k$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for every $A \in \mathcal{A}$. □

4.18. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $X = (X_1, \ldots, X_m)$ satisfy (LIC) on Ω. Let $(f_h)_h \subset I_{m.p}(\Omega, c_0, c_1)$ and, for each h, let $F_h^* : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be the local functional defined in* (73)*. Assume that:*

(i) *there exists a measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that $(F_h^*(\cdot, A))_h$ Γ-converges to $F(\cdot, A)$ in $L^p(\Omega)$ for each $A \in \mathcal{A}$;*

(ii) *there exists a Borel function $f_e : \Omega \times \mathbb{R}^n \to [0, \infty)$, convex in the second variable, satisfying (ii) of properties defining $\mathcal{I}$, for which F admits the integral representation in* (10)*.*

(iii) (83) *holds for every* $A \in \mathcal{A}$.

Then, there exists $f \in I_{m,p}(\Omega, c_0, c_1)$ *for which* F *admits the integral representation* (74).

Proof. Let us first notice that f_e satisfies the assumptions of Lemma 3.13. Thus we can assume that it satisfies (47).

Let $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ be the function in (44). Let us prove that $f \in I_{m,p}(\Omega, c_0, c_1)$. Property (I_1) follows from Therem 3.5. Since f_e satisfies (ii) of properties defining class $\mathcal{I}$, from (40), we can infer (I_2).

From Theorem 3.5 and Remark 3.8, F admits the integral representation (74), but only for functions $u \in W^{1,p}(A)$. We are going to extend this representation to all functions $u \in W^{1,p}_X(A)$, by means of Theorem 3.12 about the integral representation of local functionals with respect to X-gradient. Being F a Γ-limit, it is lsc (see [DM, Proposition 6.8]) and, by [DM, Proposition 16.15], it is also local and, by assumptions, a measure. Thus assumptions (a), (b) and (c) of Theorem 3.12 are satisfied. Let us prove assumtion (d). For every $h \in \mathbb{N}$, we have $F^*_h(u + c, A) = F^*_h(u, A)$ whenever $u \in L^p(\Omega)$, $c \in \mathbb{R}$. Then it is easy to see that this property also holds for the Γ-limit F. Let us now prove assumption (e). By the integral representation (10) and Remark 3.8, it follows that, for each $A \in \mathcal{A}$, $u \in W^{1,p}(A)$

$$
\begin{aligned}
F(u, A) &= \int_A f_e(x, Du)\, dx = \int_A f(x, Xu)\, dx \\
&\leq c_1 \left(\int_A |Xu|^p + |A| \right)
\end{aligned}
\tag{91}
$$

which implies property (e). Thus there exists a Borel function $f^* : \Omega \times \mathbb{R}^m \to [0, \infty]$ satisfying property (i) and (ii) of Theorem 3.12. In particular, for each $A \in \mathcal{A}$, $u \in W^{1,p}_X(A)$

$$
F(u, A) = \int_A f^*(x, Xu)\, dx\,.
$$

By (91) and Theorem 3.5, we get that $f(x, \eta) = f^*(x, \eta)$ for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$. This concludes the proof. □

Proof of Theorem 4.10. The proof immediately follows from Theorems 4.17 and 4.18. □

We now introduce two integrand function subclasses $J_i \subset I_{m,p}(\Omega, c_0, c_1)$ $(i = 1, 2)$ for which the associated functionals in (5) are still compact with respect to Γ- convergence in $L^p(\Omega)$-topology. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let us fix $0 < c_0 \leq c_1$.

- $J_1 \equiv J_1(\Omega, c_0, c_1)$ is the subclass of $I_{m,2}(\Omega, c_0, c_1)$ composed of integrand functions $f \in I_{m,2}(\Omega, c_0, c_1)$ which are quadratic forms with

respect to η, that is,

$$f(x,\eta) = \langle a(x)\eta,\eta\rangle = \sum_{i,j=1}^{m} a_{ij}(x)\eta_i\eta_j \quad \text{a.e. } x\in\Omega, \forall\,\eta\in\mathbb{R}^m\,,$$

with $a(x) = [a_{ij}(x)]$ $m\times m$ symmetric matrix .

- The subclass $J_2 \equiv J_2(\Omega, c_0, c_1)$ is composed by integrand functions $f \in I_{m,p}(\Omega, c_0, c_1)$ such that $f = f(\eta)$, that is, f is independent of x.

4.19. **Theorem.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and let $X = (X_1,\dots,X_m)$ satisfy (LIC) on Ω. Let $(f_h)_h \subset J_i(\Omega, c_0, c_1)$ $(i = 1, 2)$ and, for each h, let $F_h^* : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ be the local functional defined in* (73)*. Then, up to a subsequence, there exist a local functional $F : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ and $f\in J_i(\Omega, c_0, c_1)$ such that*

(i) (9) *holds;*
(ii) *F admits representation* (74)*.*

Proof. **1st case.** Let us first show the conclusion for the subclass J_1.

Let $(f_h)_h \subset J_1$. By definition, we can assume that

$$f_h(x,\eta) := \langle a_h(x)\eta,\eta\rangle \quad x\in\Omega,\ \eta\in\mathbb{R}^m\,,$$

where $a_h(x) = [a_{h,ij}(x)]$ is a $m\times m$ symmetric matrix satisfying

$$c_0\,|\eta|^2 \le \langle a_h(x)\eta,\eta\rangle \le c_1\left(|\eta|^2+1\right) \text{ a.e. } x\in\Omega,\ \forall\eta\in\mathbb{R}^m \tag{92}$$

$$a_{h,ij}\in L^\infty(\Omega) \text{ for each } i,j = 1,\dots,m,\ h\in\mathbb{N}\,. \tag{93}$$

Applying Theorem 4.10, up to a subsequence, there exist a local functional $F : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ and $f\in I_{m,2}(\Omega, c_0, c_1)$ such that (9) holds and F admits representation (74). We have only to prove that

$$f\in J_1\,. \tag{94}$$

Notice that we can also assume that F admits representation (10) with

$$f_e(x,\xi) := f(x, C(x)\xi) \text{ for a.e. } x\in\Omega, \text{ for each } \xi\in\mathbb{R}^n\,.$$

Moreover, by Theorem 3.5 (see (44) and (40)), it also holds the opposite representation, that is, for each $x\in\Omega_X$,

$$f(x,\eta) = f_e(x, L_x^{-1}(\eta)) \quad \forall\,\eta\in\mathbb{R}^m\,, \tag{95}$$

with

$$L_x^{-1}(\eta) := C(x)^T B(x)^{-1}\eta\,.$$

Let us now consider the sequence of Euclidean integrands

$$\begin{aligned} f_{h,e}(x,\xi) &:= f_h(x, C(x)\xi) = \langle a_h(x)C(x)\xi, C(x)\xi\rangle \\ &= \langle C(x)^T a_h(x)C(x)\xi,\xi\rangle = \langle a_{h,e}(x)\xi,\xi\rangle \end{aligned}$$

and the related local functionals $F_h : L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ defined in (7). Since $F_h(u,A) = F_h^*(u,A)$ for each $u\in W^{1,1}_{\mathrm{loc}}(A)$, by using well-known

results of Γ-convergence for quadratic functionals (see [DM, Theorem 22.1] and Remark 4.5, one can easily prove that there exists a $n \times n$ symmetric matrix $a_e(x) = [a_{e,ij}(x)]$, with $a_{e,ij} \in L^\infty(\Omega)$ for each $i, j = 1, \dots, n$ such that

$$f_e(x, \xi) = \langle a_e(x)\xi, \xi\rangle \text{ a.e. } x \in \Omega, \forall\, \xi \in \mathbb{R}^n\,.$$

By (95), for each $x \in \Omega_X$,

$$\begin{aligned} f(x,\eta) &:= f_e(x, L_x^{-1}(\eta)) = \langle a_e(x)C(x)^T B(x)^{-1}\eta, C(x)^T B(x)^{-1}\eta\rangle \\ &= \langle (B(x)^{-1})^T C(x) a_e(x) C(x)^T B(x)^{-1}\eta, \eta\rangle = \langle a(x)\eta, \eta\rangle \end{aligned} \tag{96}$$

with

$$a(x) := (B(x)^{-1})^T C(x) a_e(x) C(x)^T B(x)^{-1}\,,$$

$m \times m$ symmetric matrix. Then $f(x, \cdot)$ turns out to be a quadratic form on $\mathbb{R}^m$, induced by the matrix $a(x)$ for a.e. $x \in \Omega$. Thus (94) follows.

2nd case. Let us now deal with the subclass J_2. Let $(f_h)_h \subset J_2$. Notice that $f_h : \mathbb{R}^m \to [0, \infty)$, $h \in \mathbb{N}$, is a sequence of locally bounded, convex functions. Thus, by a well-known result (see, for instance, [DM, Proposition 5.11]), we can infer that $(f_h)_h$ is also locally equi-Lipschitz continuous. From Ascoli-Arzelà' s theorem, we can assume that, up to a subsequence, there exists $f \in J_2$ such that

$$f_h \to f \text{ uniformly on bounded sets of } \mathbb{R}^n \text{ as } h \to \infty\,. \tag{97}$$

Let us define $\tilde{F} : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ as

$$\tilde{F}(u, A) := \begin{cases} \int_A f(Xu(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A) \\ \infty & \text{otherwise} \end{cases}\,.$$

Let us now prove that, for each $A \in \mathcal{A}$,

$$\lim_{h\to\infty} F^*_h(u, A) = \tilde{F}(u, A) \quad \forall\, u \in W^{1,p}_X(A)\,. \tag{98}$$

Let us fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$. Since $|Xu(x)| < \infty$ for a.e. $x \in A$, by (97), it follows that

$$\lim_{h\to\infty} f_h(Xu(x)) = f(Xu(x)) \text{ for a.e. } x \in A\,. \tag{99}$$

On the other hand, as

$$0 \le f_h(Xu(x)) \le c_1(1 + |Xu(x)|^p) \text{ for a.e. } x \in A, \text{for each } h\,,$$

by (99) and the dominated convergence theorem, (98) follows. We have only to prove that

$$F(u, A) = \tilde{F}(u, A) \quad \forall\, A \in \mathcal{A}, \forall\, u \in L^p(\Omega) \tag{100}$$

in order to get our desired conclusion. By (83), it is sufficient to prove (100) for each $A \in \mathcal{A}$ and for each $u \in W^{1,p}_X(A)$. The inequality

$$F(u, A) \le \tilde{F}(u, A) \quad \forall\, A \in \mathcal{A}, \forall\, u \in W^{1,p}_X(A)\,, \tag{101}$$

follows by noticing that, for each $u \in W^{1,p}_X(A)$, by $\Gamma - \liminf$ inequality and (98)

$$F(u,A) \leq \liminf_{h\to\infty} F^*_h(u,A) = \tilde{F}(u,A)\,.$$

Let us now prove the opposite inequality

$$F(u,A) \geq \tilde{F}(u,A) \quad \forall\, A \in \mathcal{A},\, \forall\, u \in W^{1,p}_X(A)\,. \tag{102}$$

Let us first recall that, for each $A \in \mathcal{A}$, by (98) and Proposition 4.9,

$$\tilde{F}(u,A) = (\Gamma(W^{1,p}_X(A))\text{-}\lim_{h\to\infty} F^*_h)(u) \quad \forall\, u \in W^{1,p}_X(A)\,. \tag{103}$$

Fix $A \in \mathcal{A}$ and let $u \in L^p(\Omega)$ with $u|_A \in W^{1,p}_X(A)$. By the $\Gamma - \lim$ equality, there exists a sequence $(u_h)_h \subset L^p(\Omega)$ such that

$$u_h \to u \text{ in } L^p(\Omega), \text{ as } h \to \infty \tag{104}$$

and

$$\lim_{h\to\infty} F^*_h(u_h,A) = F(u,A) < \infty\,. \tag{105}$$

By (105), we can assume that

$$(u_h|_A)_h \subset W^{1,p}_X(A)\,. \tag{106}$$

Let $A' \in \mathcal{A}$ with $A' \Subset A$. From Proposition 2.10 (ii), if $w := \overline{Xu_h} : \mathbb{R}^n \to \mathbb{R}^m$, that is, $\overline{Xu_h} = Xu_h$ on A and $\overline{Xu_h} = 0$ outside, for each $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$

$$\int_{A'} f_h(\rho_\epsilon * \overline{Xu_h})\,dx \leq \int_A f_h(\overline{Xu_h})\,dx \text{ for each } h\,. \tag{107}$$

By (104), (106) and Proposition 2.10 (i), for given $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$,

$$X(\rho_\epsilon * \bar{u}_h) \to X(\rho_\epsilon * \bar{u}) \text{ uniformly on } A' \text{ as } h \to \infty \tag{108}$$

and

$$\rho_\epsilon * \overline{Xu_h} \to \rho_\epsilon * \overline{Xu} \text{ uniformly on } A' \text{ as } h \to \infty\,. \tag{109}$$

In particular,

$$\rho_\varepsilon * \bar{u}_h \to \rho_\varepsilon * \bar{u} \text{ in } W^{1,p}_X(A') \text{ as } h \to \infty. \tag{110}$$

Observe now that, by (107), for each $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$, for each h,

$$\begin{aligned} F^*_h(\rho_\epsilon * \bar{u}_h, A') &= \int_{A'} f_h(X(\rho_\epsilon * \bar{u}_h))\,dx \\ &= \int_{A'} f_h(\rho_\epsilon * \overline{Xu_h})\,dx + \int_{A'} \big(f_h(X(\rho_\epsilon * \bar{u}_h)) - f_h(\rho_\epsilon * \overline{Xu_h})\big)\,dx \\ &\leq \int_A f_h(Xu_h)\,dx + \int_{A'} \big(f_h(X(\rho_\epsilon * \bar{u}_h)) - f_h(\rho_\epsilon * \overline{Xu_h})\big)\,dx \\ &= F^*_h(u_h,A) + R_{\epsilon,h}. \end{aligned} \tag{111}$$

From (97), (108) and (109), it follows that, for given $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$

$$\lim_{h\to\infty} R_{\epsilon,h} = R_\epsilon := \int_{A'} \big(f(X(\rho_\epsilon * \bar{u})) - f(\rho_\epsilon * \overline{Xu})\big)\, dx\,. \tag{112}$$

For given $0 < \varepsilon < \operatorname{dist}(A', \mathbb{R}^n \setminus A)$, by (103), (105), (110), and (112), passing to the limit in (111) as $h \to \infty$, it follows that

$$\begin{aligned} \tilde{F}(\rho_\varepsilon * \bar{u}, A') &\le \liminf_{h\to\infty} F_h^*(\rho_\varepsilon * \bar{u}_h, A') \\ &\le \lim_{h\to\infty} F_h^*(u_h, A) + \lim_{h\to\infty} R_{\epsilon,h} = F(u, A) + R_\epsilon. \end{aligned} \tag{113}$$

Let us now show that

$$\lim_{\epsilon\to 0^+} R_\epsilon = 0\,. \tag{114}$$

Indeed

$$X(\rho_\epsilon * \bar{u}) \to Xu \text{ and } \rho_\epsilon * \overline{Xu} \to Xu \text{ in } L^p(A'), \text{ as } \epsilon \to 0^+$$

and

$$f(X(\rho_\epsilon * \bar{u})) \le c_1(1+|X(\rho_\epsilon * \bar{u})|^p) \text{ and } f(\rho_\epsilon * \overline{Xu}) \le c_1(1+|\rho_\epsilon * \overline{Xu}|^p) \text{ a.e. in } A'\,.$$

Since f is continuous, from Vitali's convergence theorem, (114) follows. By the semicontinuity of $\tilde{F}$, with respect to the L^p-topology, and by (114), we can pass to the limit as $\varepsilon \to 0^+$ in (113) and we get

$$\tilde{F}(u, A') \le \lim_{\varepsilon\to 0^+} \tilde{F}(\rho_\varepsilon * \bar{u}, A') \le F(u, A) \text{ for each } A' \Subset A\,. \tag{115}$$

Finally, taking the supremum in (115) on all $A' \in \mathcal{A}$ with $A' \Subset A$, we get (102). □

Acknowledgements. We thank A. Braides, G. Buttazzo, G. Dal Maso, A. Defranceschi and B. Franchi for useful suggestions and discussions on this topic. We are also grateful to the anonymous referee for many helpful suggestions and corrections.

References

[ACM] F. Acanfora, G. Cardone, S. Mortola, *On the variational convergence of non-equicoercive quadratic integral functionals and semicontinuity problems*, NoDEA **10** (2003), 347–373.

[APS] L. Ambrosio, A. Pinamonti, G. Speight, *Weighted Sobolev Spaces on Metric Measure Spaces*, Journal fur die Reine und Angewandte Mathematik (Crelle's Journal) **746** (2019), 39–65.

[BFTT] A. Baldi, B. Franchi, N. Tchou, M.C. Tesi, *Compensated compactness for differential forms in Carnot groups and applications*, Adv. Math. **223** (2010), 1555–1607.

[BFT] A. Baldi, B. Franchi, M.C. Tesi, *Compensated compactness, div-curl theorem and H-convergence in general Heisenberg groups*, Lect. Notes Seminario Interdisciplinare di Matematiche **6** (2007), 33–47.

[BMT] M. Biroli, U. Mosco, N. Tchou, *Homogenization for degenerate operators with periodic coefficients with respect to the Heisenberg group*, C. R. Acad. Sci. Paris, **322** (1996)., Srie I 439–444.

[BPT1] M. Biroli, C. Picard, N. Tchou, *Homogenization of the p-Laplacian associated with the Heisenberg group*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **22** (1998), 23–42.

[BPT2] M. Biroli, C. Picard, N. Tchou, *Asymptotic behavior of some nonlinear subelliptic relaxed Dirichlet problems* Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **26** (2002), 55–113.

[BT] M. Biroli, N. Tchou, Γ-*convergence for strongly local Dirichlet forms in perforated domains with homogeneous Neumann boundary conditions*, Adv. Math. Sci. Appl. **17** (2007), no. 1, 149–179.

[BLU] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer, 2007.

[Bra] A. Braides, *A handbook of* Γ-*convergence*, Handbook of Differential Equations.Stationary Partial Differential Equations **3**, M. Chipot and P. Quittner, eds., Elsevier, 2006.

[B] G. Buttazzo, *Semicontinuity, relaxation and integral rapresentation*, Longman, Harlow, 1989.

[DM] G. Dal Maso, *An Introduction to* Γ-*convergence*, Birhkäuser, Boston, 1993.

[Dr] N. Dirr, F. Dragoni, P. Mannucci, C. Marchi Γ-*convergence and homogenisation for a class of degenerate functionals*, Nonlinear Anal. 190 (2020), 111618.

[ET] I. Ekeland, R. Téman, *Convex Analysis and Variational Problems*, Classic in Applied Mathematics **28**, SIAM , Philadelphia, 1999.

[FS] G.B. Folland, E.M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton,1982.

[FGVN] B. Franchi, C.E. Guitiérrez, T. van Nguyen, *Homogeneization and Convergence of Correctors in Carnot Groups*, Comm. PDE **30** (2005), 1817–1841.

[FSSC1] B. Franchi, R. Serapioni, F. Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston J. Math., **22** (1996), 859–889.

[FSSC2] B. Franchi, R. Serapioni, F. Serra Cassano, *Approximation and Imbedding Theorems for Weighted Sobolev Spaces Associated with Lipschitz Continuous Vector Fields*, Bolletino U.M.I, **11-B** (1997), 83–117.

[FTT] B. Franchi, N. Tchou, M.C. Tesi, *Div- Curl Type Theorem, H-convergence and Stokes Formula in the Heisenberg Group*, Comm. Contemp. Math. **81**, no. 1 (2006), 67–99.

[FT] B. Franchi, M.C. Tesi, *Two-scale convergence in the Heisenberg group*, J. Math. Pures Appl. **81** (2002), 495–532.

[Fu] M. Fukushima, *Dirichlet Forms and Symmetric Markov Processes*, North-Holland, Amsterdam, 1980.

[GN] N.Garofalo, D.M Nhieu, *Isoperimetric and Sobolev Inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math., **49** (1996), 1081–1144.

[La] P. D. Lax, *Linear Algebra and its Applications*- 2nd Ed., Wiley, Hoboken, NewJersey, 2007.

[MR] Z.-M. Ma, M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet forms*, Universitext, Springer-Verlag Berlin Heidelberg 1992.

[MPSC] A. Maione, A. Pinamonti, F. Serra Cassano, Γ-*convergence for functionals depending on vector fields. II. Convergence of minimizers*, forthcoming.

[MV] A. Maione, E. Vecchi, *Integral Representation of Local LeftInvariant Functionals in Carnot Groups*, Anal. Geom. Metr. Spaces **8** (2020), 1–14.

[MS] N.G. Meyers, J. Serrin, $H = W$, Proc. Nat. Acad. Sci. USA, **51** (1964), 1055–1056.

[Mo] U. Mosco, *Composite Media and Asymptotic Dirichlet Forms*, J. Funct. Anal. , **123** (1993) , 368–421.

[Mu] F. Murat, *H-convergence*, Séminaire d'analyse fonctionelle et numérique, Université d' Alger, 1977-78. English translation F. Murat, L. Tartar, *H-convergence. Topics in the mathematical modelling of composite materials*, 21–43, Prog. Nonlinear Differential Equations Appl **31**, Birkhäuser, Boston, MA,1997.

[Ro] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1972.

Alberto Maione: Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123, Povo (Trento) - Italy,
E-mail address: alberto.maione@unitn.it

Andrea Pinamonti: Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123, Povo (Trento) - Italy,
E-mail address: andrea.pinamonti@unitn.it

Francesco Serra Cassano : Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123, Povo (Trento) - Italy,
E-mail address: francesco.serracassano@unitn.it