

Stochastic Navier-Stokes Equations and Related Models

Luigi Amedeo Bianchi and Franco Flandoli

In honour of Giuseppe Da Prato

Abstract. Regularization by noise for certain classes of fluid dynamic equations, a theme dear to Giuseppe Da Prato [23], is reviewed focusing on 3D Navier-Stokes equations and dyadic models of turbulence.

1. Introduction

This is a review paper dealing with a specific question of stochastic fluid dynamics which occupied many years of research of Giuseppe Da Prato, prepared on the occasion of his 80th birthday. The question is whether noise may improve the theory of well posedness of certain equations of fluid dynamics, first of all the 3D incompressible Navier-Stokes equations.

As better remarked below, the deterministic theory of such equations has been frozen for many years in the following duality (up to numerous side results, including very advanced ones, which however do not change this simplified picture):

- i. Weak solutions exist, globally in time, but their uniqueness is an open problem;
- ii. More regular solutions exist uniquely locally in time, but their blow-up or persistence is an open problem (see the Millennium Prize problem described by Fefferman in [32]).

Why should the presence of a noise improve such results? Specifically for the 3D Navier-Stokes equations we do not have a precise intuition, except for the vague feeling that disorder, created intrinsically by turbulence or imposed from outside by a noise, could disgregate well prepared configurations which could otherwise blow up. Even if the intuition is poor, the question is meaningful, having in mind analogous results of *regularization by noise* holding for several classes of stochastic differential equations, ranging from classical finite-dimensional cases such as [60, 52] to infinite-dimensional ones, although the latter have been proved until now only for nonlinear systems much simpler than 3D Navier-Stokes equations, as for example [48, 24, 25, 26]. These results prove uniqueness for certain equations with

nondegenerate additive noise, in cases where the same equations without noise miss uniqueness; and, for the purpose of the upcoming discussion, let us mention that all of them (with the exception of [48]) are based on suitable regularity results for the Kolmogorov equations associated to the stochastic equations. In response to such results, the hope of proving uniqueness of weak solutions to the 3D Navier-Stokes equations by adding a nondegenerate noise with suitable covariance rose high. Giuseppe Da Prato made a tremendous contribution to answering this question, although for the time being the final question is still open: together with Arnaud Debussche, in the paper [23] he constructed a smooth solution of the infinite-dimensional Kolmogorov equation associated to the stochastic 3D Navier-Stokes equations, with a really original and highly non trivial procedure. Existence of sufficiently smooth solutions of Kolmogorov equation is usually considered the first step in proving uniqueness for the corresponding stochastic equation (at least uniqueness in law, if not pathwise uniqueness). However, even though the regularity of the solutions constructed by [23] is high in terms of differentiability, the regularity of their derivatives as functions on the infinite-dimensional space is not good enough, being defined only in subspaces where the weak solutions of Navier-Stokes equations do not live continuously in time. Thus a careful consideration of the assumptions does not allow to apply Itô formula to the composition of the solution of Kolmogorov equation and a weak solution of the Navier-Stokes equations, a basic step in the usual proof of uniqueness. Said differently, if the Kolmogorov equation is seen as the dual of the stochastic equation (precisely the dual of the associated Fokker-Planck equation), the spaces where the solutions of the two problems live, are not dual of one another, and thus any argument for uniqueness based on duality fails. In spite of this, the result of [23] can be considered the closest one to the solution of the open problem. We know by personal communication that Giuseppe Da Prato always kept in mind the open problem and continued to identify potential paths to its solution.

Below we describe some side results that may enrich the previous picture. Inspired by the results in [23, 28], a theory of Markov selections for stochastic 3D Navier-Stokes equations was developed in [44], with a special property of continuous dependence on initial conditions that is unique with respect to the deterministic case and thus worthy to be mentioned; this is Section 2 of this paper. Having touched the difficulty to advance with additive noise in proving regularization by noise, around 2010 there has been a shift to other kinds of noise. Among them, multiplicative noise of transport type occupied a relevant position (but it is not the only example; see for instance a noise multiplying the Laplacian in Schrödinger equation [29] or multiplying the nonlinear term in Hamilton Jacobi equations and conservation laws [46, 47]). Heuristically, a multiplicative transport noise is the Eulerian counterpart to an additive noise at the Lagrangian level, hence could transfer the special well posedness properties of additive noise for finite-dimensional systems to the case of PDEs. The first results have been for linear transport and advection equations [37, 41] but also special solutions (point concentrated) of 2D Euler equations and 1D Vlasov-Poisson system have been regularized by a similar noise [38, 30]. The same was proved for

a Leray- α model [3] and, finally, in [6, 12] for dyadic models of turbulence, a topic that we shall review in Section 3.

All the cited results of regularization by noise due to multiplicative noise of transport type have been for inviscid problems and that seemed to be a rule. However, it was recently understood that such noise may have a regularizing effect also on viscous problems, in particular the 3D Navier-Stokes equations since it increases dissipation [40]. This is described in Section 2.4 and it is conceptually of interest also because the regularization is not in the form of restored uniqueness by noise - as all the previously mentioned results - but in the form of suppression of blow-up, the second open question mentioned above, [32]. We hope this picture may convince young researchers that there is still space for improvements, although the research on this topic is slow and rarely based on repeated schemes.

2. The 3D Navier-Stokes equations

2.1. Deterministic case

For simplicity of exposition we assume that the fluid lives on the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. We will denote by H (resp. V) the Hilbert space of $L^2(\mathbb{T}^3, \mathbb{R}^3)$ (resp. $W^{1,2}(\mathbb{T}^3, \mathbb{R}^3)$) divergence free zero average vector fields (see [59] for more precise details about the boundary conditions). Let us recall, among others, the following basic results from [59]:

1. Given $u_0 \in H$, there exists a weak solution, namely a function of class

$$u \in L^\infty(0, T; H) \cap L^2(0, T; V)$$

weakly continuous in H , satisfying the identity

$$\langle u(t), \phi \rangle + \nu \int_0^t \langle \nabla u(s), \nabla \phi \rangle ds = \langle u_0, \phi \rangle + \int_0^t \langle u(s), u(s) \cdot \nabla \phi \rangle ds$$

for every $\phi \in V$.

2. If $u_0 \in V$, there exists a unique *maximal* solution $u \in C([0, \tau); V)$.

Two questions (remember that we are in dimension 3) remain open and represent fundamental problems in PDE theory (see once more [32]):

1. Are weak solutions unique?
2. When $u_0 \in V$, do we have $\tau = +\infty$ or

$$\tau < \infty, \lim_{t \uparrow \tau} \|u(t)\|_V = +\infty?$$

Here and in the following, we denote by $\|\cdot\|_H$ and $\|\cdot\|_V$ the usual norms in H and V respectively and by $\langle \cdot, \cdot \rangle$ either the scalar product in H or its extension to a dual pairing between spaces in duality with respect to H .

2.2. Stochastic case, additive noise

Generalization of the result of existence of weak solutions to the stochastic case, with different types of noise, are now well-known, see for instance [36], [34] and references therein. Let us mention some elements in the case of additive noise. The formal notation is

$$\begin{aligned} du + (u \cdot \nabla u + \nabla p) dt &= \Delta u dt + dW_t, \\ \operatorname{div} u &= 0. \end{aligned}$$

Since space-time white noise is particularly attractive thanks to the outstanding contributions of the theory of regularity structures and paracontrolled distributions, let us first discuss this case, also because the general results existing in the literature, for simpler nonlinearities, of regularization by noise (like [48], [24]) usually assume W to be a space-time white noise, namely a formal expression of the form

$$W_t(x) = \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \beta_t^{k,\alpha} e_{k,\alpha}(x)$$

where the series converges in mean square in a distributional space. Here \mathbb{Z}_0^3 is $\mathbb{Z}^3 \setminus \{0\}$ and $(e_{k,\alpha})_{k \in \mathbb{Z}_0^3, \alpha=1,2}$ is a complete orthonormal system of H of the form

$$e_{k,\alpha}(x) = a_{k,\alpha} e^{2\pi i k \cdot x} \quad k \in \mathbb{Z}_0^3, \quad \alpha = 1, 2$$

where $a_{k,1}, a_{k,2}$ is an orthonormal basis of the plane perpendicular to k in \mathbb{R}^3 . Finally, $(\beta^{k,\alpha})_{k \in \mathbb{Z}_0^3, \alpha=1,2}$ is a family of complex Brownian motions defined as follows: we take a family $(W^{k,\alpha})_{k \in \mathbb{Z}_0^3, \alpha=1,2}$ of real independent Brownian motions, we partition \mathbb{Z}_0^3 in two sets \mathbb{Z}_+^3 and $\mathbb{Z}_-^3 = -\mathbb{Z}_+^3$, and for all $k \in \mathbb{Z}_+^3$ we set $\beta_t^{k,\alpha} = W^{k,\alpha} + iW^{-k,\alpha}$; for $k \in \mathbb{Z}_-^3$ we set $\beta_t^{k,\alpha} = W^{-k,\alpha} - iW^{k,\alpha}$.

However, the solution of a parabolic equation with space-time white noise is expected to be a function, and not just a distribution, only when the spatial dimension is 1. In dimension 2 it is expected to be a distribution of Sobolev class $H^{-\epsilon}$. This case was successfully investigated for Navier-Stokes equations by Da Prato and Debussche in a seminal paper [23]; however, it is a 2D case, not competitive with the deterministic theory (although striking from the stochastic viewpoint for several reasons). In dimension 3 the solutions are expected to be distributions of class $H^{\frac{1}{2}-\epsilon}$. A theorem of existence in such very singular regime has been proven in [62], but its relevance in view of a full well posedness result is not clear. Thus we shall always consider more regular noises, usually satisfying at least the property that W_t itself is a stochastic process in H , an assumption achieved by requiring

$$W_t(x) = \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha} \beta_t^{k,\alpha} e_{k,\alpha}(x), \quad (1)$$

where $(\sigma_{k,\alpha})_{k \in \mathbb{Z}_0^3, \alpha=1,2}$ are real numbers satisfying

$$\sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha}^2 < \infty. \quad (2)$$

As said above, with this choice of noise, the equation was considered by several authors, see for instance [36]. One can give a weak formulation as

$$\langle u(t), \phi \rangle + \nu \int_0^t \langle \nabla u(s), \nabla \phi \rangle ds = \langle u_0, \phi \rangle + \int_0^t \langle u(s), u(s) \cdot \nabla \phi \rangle ds + \langle W_t, \phi \rangle,$$

asking that

$$\begin{aligned} u &\in L^2_{\mathcal{F}}(\Omega; \mathcal{H}), \\ \mathcal{H} &:= L^2(0, T; W^{1,2}) \cap C_w(0, T; L^2), \end{aligned}$$

namely that, on a probability space (Ω, \mathcal{A}, P) with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and Brownian motions β_t^k adapted to the filtration, u is a weakly continuous (\mathcal{F}_t) -adapted process in H , with paths also of class $L^2(0, T; V)$, with suitable square integrability properties (not needed here in detail), such that for all $\phi \in V$ the previous identity holds true uniformly in time, with probability one. When the tuple $(\Omega, \mathcal{A}, P, (\mathcal{F}_t), (\beta_t^k))$ is not prescribed a priori, we say that a weak solution is a *weak martingale solution*. The existence of weak martingale solutions, as said above, is known¹.

Several extensions of more sophisticated deterministic results have been proved in this stochastic setting. Among them, let us recall a generalization of the theory of Hausdorff dimension of the set of singular points, the theory of *Caffarelli-Kohn-Nirenberg*. In the deterministic case, it claims that the set S of singular points in time-space may have at most Hausdorff dimension 1, with 1-dimensional Hausdorff measure equal to zero. A full generalization to the stochastic case has been obtained in [43], with the following probabilistic improvement.

Theorem 1. *For stationary solutions (deterministic or stochastic case), if S_t is the random set of singularities at time t , then*

$$P(S_t = \emptyset) = 1,$$

for all $t \geq 0$.

2.2.1. Role of Kolmogorov equation for uniqueness in law. For stochastic equations, uniqueness in law is the property stating that any two solutions, possibly constructed on different probability spaces, have the same law. This property is weaker than pathwise uniqueness, which is itself weaker than path by path uniqueness. At the same time, it is stronger than the uniqueness of the associated Fokker-Planck equation. (We do not discuss these definitions here.)

How could one prove uniqueness in law by means of probabilistic arguments? Girsanov theorem is the easiest method but it cannot work for Navier-Stokes equations, as Ferrario has shown in [33]. In general it seems that the Girsanov approach has limitations that are too strong. The Kolmogorov approach, on the other hand, is more flexible. The rough “principle” is that:

¹When the tuple is arbitrarily given a priori, existence of solutions is called strong existence; strong existence is open for the 3D Navier-Stokes equations with additive noise.

- i. If we control one derivative of Kolmogorov solution, we may try to prove uniqueness in law.
- ii. If we control two derivatives, and the first one is uniformly bounded, we may try to prove pathwise uniqueness.

Let us see some details on this topic. Consider an abstract stochastic equation in Hilbert space:

$$du = (Au + B(u)) dt + dW_t,$$

where A is a negative selfadjoint operator, B satisfies suitable assumptions, and W is a Brownian motion in H with trace-class covariance Q (the noise (1) is of this form when (2) holds). Consider the infinite-dimensional backward Kolmogorov equation

$$\begin{aligned} \partial_t U + \frac{1}{2} \text{Tr} (QD^2U) + \langle Au + B(u), DU \rangle &= 0, \\ U|_{t=T} &= \phi, \end{aligned}$$

where for the time being we do not give precise definitions of the single objects². Heuristically, assume that the Kolmogorov equation has a sufficiently smooth solution and assume that $u(t)$ is a solution of the stochastic equation. By Itô formula, for $0 \leq r \leq t \leq T$,

$$\begin{aligned} U(t, u(t)) - U(r, u(r)) &= \int_r^t \langle DU(s, u_s), dW_s \rangle \\ &+ \int_r^t \left(\partial_s U + \frac{1}{2} \text{Tr} (QD^2U) + \langle A \cdot + B(\cdot), DU \rangle \right) (s, u(s)) ds, \end{aligned}$$

and thus, by the Kolmogorov equation,

$$U(t, u(t)) - U(r, u(r)) = \int_r^t \langle DU(s, u_s), dW_s \rangle.$$

If DU is good enough to have

$$\mathbb{E} \int_0^T \|DU(s, u_s)\|_H^2 ds < \infty, \tag{3}$$

then $\mathbb{E} \int_0^T \langle DU(s, u_s), dW_s \rangle = 0$ and we deduce, with $r = 0$ and $t = T$,

$$\mathbb{E} \phi(u(T)) = \mathbb{E} U(0, u_0).$$

This, by the arbitrariness of ϕ , identifies the law of $u(T)$ (and T is arbitrary). With more work, as explained for instance in [57], we identify the law of the process. Let us remark that Giuseppe Da Prato was the main investigator of Kolmogorov equations in infinite-dimensional spaces, see for instance his two books [22, 27].

²Just notice, with a certain degree of formality, that $U = U(t, u)$ is a real function defined on $[0, T] \times H$, with the notation $u \in H$; $DU(t, u)$ is its differential in the H -variable, element of H , $\langle Au + B(u), DU(t, u) \rangle$ is its scalar product in H with the vector $Au + B(u)$, $D^2U(t, u)$ is the second differential, an operator on H , and $\text{Tr} (QD^2U(t, u))$ is the trace of the operator $QD^2U(t, u)$; finally ϕ is a real function on H .

The classical idea to investigate Kolmogorov equations in infinite dimensions is by perturbation. In order to describe it, let us reverse time by setting $V(t) = U(T - t)$; now we have to study the forward equation

$$\begin{aligned}\partial_t V &= \frac{1}{2} \text{Tr} (QD^2V) + \langle Au + B(u), DV \rangle, \\ V|_{t=0} &= \phi.\end{aligned}$$

Introducing the Gaussian semigroup solving

$$\partial_t \mathcal{S}_t \phi = \frac{1}{2} \text{Tr} (QD^2 \mathcal{S}_t \phi) + \langle Au, D\mathcal{S}_t \phi \rangle,$$

with $\mathcal{S}_0 \phi = \phi$, one rewrites the equation in perturbative form

$$V(t) = \mathcal{S}_t \phi + \int_0^t \mathcal{S}_{t-s} \langle B(u), DV(s) \rangle ds.$$

In order to apply a fixed point argument to this equation in suitable spaces, it is necessary to have good gradient bounds on the Gaussian semigroup. Those usually proved, under suitable assumptions on the pair (A, Q) , have the form

$$\|D\mathcal{S}_t \phi\|_0 \leq \frac{C}{t^\gamma} \|\phi\|_0, \quad (4)$$

with $\gamma \in (0, 1)$. Here $\|\phi\|_0$ is the uniform norm of a function or a vector defined on H . Unfortunately, a great limitation of this perturbative approach is that B has to be bounded, see for instance [24, 25, 26]. Moreover, the assumptions on (A, Q) to have the gradient bound (4) are far from those satisfied by the linear part of 3D Navier-Stokes equations.

Da Prato and Debussche in [23] made a breakthrough on this topic in the direction of 3D Navier-Stokes equations: under suitable assumptions on the coefficients σ_k (the idea behind the assumptions is that the coefficients cannot go to zero too fast), they discovered a way to construct smooth solutions of the associated infinite-dimensional Kolmogorov equation. Without pretending to explain in a sentence the very elaborate procedure developed in [23], let us only mention that it starts with the very innovative idea of introducing a penalized evolution operator $\mathcal{R}(s, t)$ in place of the Gaussian semigroup:

$$U(t) = \mathcal{R}(0, t) \phi + \int_0^t \mathcal{R}(s, t) (\langle B(u), DU(s) \rangle - V(s)) ds.$$

Using this method it is possible to prove the existence of a smooth solution $U(t, u)$.

The solution U is differentiable (in fact twice differentiable), but with bounds on derivatives of the form

$$\langle h, DU(t, u) \rangle \leq C(t) \|h\|_{W^{2,2}} (1 + \|u\|_{W^{2,2}}),$$

namely depending on a Sobolev norm in the infinite-dimensional variable u , which is quite demanding from the viewpoint of the regularity of solutions of 3D Navier-Stokes equations. If we go back to the sufficient condition (3), we see that weak solutions do not have sufficient regularity. In principle there could be several weaker ways to proceed, which do not require directly (3), but no way has been found yet.

Technically, Da Prato-Debussche [23] is one of the most advanced works on stochastic 3D Navier-Stokes equations. Not only it constructs solutions to the Kolmogorov equation, but it also identifies two new properties: Markov selections and strong Feller property, discussed below.

2.3. Small times versus large times

Let $u_0 \in L^2$ be an initial condition and $u \in L^2_{\mathcal{F}}(\Omega; \mathcal{H})$ be a (possibly non-unique) weak solution. Using the properties of conditional expectation, let us decompose

$$\begin{aligned} \mathbb{E}[\phi(u(T))] &= \mathbb{E}[\mathbb{E}[\phi(u(T)) | u(t_0)]] \\ &= \int_H \mathbb{E}[\phi(u(T)) | u(t_0) = v] \mu_{t_0}(dv), \end{aligned}$$

where μ_{t_0} is the law of $u(t_0)$, and ϕ is a bounded continuous function on H , $\phi \in C_b(H)$. The hope is to propagate good properties, which hold for small times, to large times.

Assume for every initial condition u_0 we *select* a weak solution $u(\cdot; u_0) \in L^2_{\mathcal{F}}(\Omega; \mathcal{H})$. Uniqueness is not known, but we may make selections, following different criteria; the simplest one is measurable-in- u_0 selection, but a more refined one, following [57], is a Markov selection (see below). For each one of the selected solutions we have the decomposition above

$$\mathbb{E}[\phi(u(T; u_0))] = \int_{L^2} \mathbb{E}[\phi(u(T; u_0)) | u(t_0; u_0) = v] \mu_{t_0, u_0}(dv), \quad (5)$$

where μ_{t_0, u_0} is the law of $u(t_0; u_0)$. One can already notice the germ of a special property: if $u_0 \in V$ and t_0 is small enough, the law $\mu_{t_0, u_0}(dv)$ is “almost” independent of the selection, since for $u_0 \in V$ the solution is locally unique. The limitation “almost” refers to the fact that “locally”, in the stochastic case, means randomly local, hence we know uniqueness up to time t_0 only with large probability.

Assume $u_0^n, u_0 \in V$ are such that

$$u_0^n \xrightarrow{V} u_0.$$

In the deterministic case, one can find t_0 small enough that unique solutions u^n, u exist on $[0, t_0]$ with initial conditions u_0^n, u_0 and $u^n \rightarrow u$ in $C([0, t_0]; V)$. In the stochastic case, a similar result holds with large probability [44]: for every $\epsilon > 0$ there exists $t_0 > 0$ such that solutions exist, and are pathwise unique, in $C([0, t_0]; V)$ with probability greater than $1 - \epsilon$; at the same time, $u^n \rightarrow u$ in $C([0, t_0]; V)$ with probability greater than $1 - \epsilon$. Forgetting about this ϵ for the sake of simplicity of the heuristic explanation (the details are in [44]), we have

$$u(t_0; u_0^n) \xrightarrow[\text{a.s.}]{V} u(t_0; u_0),$$

and

$$\lim_{u_0^n \rightarrow u_0} \int \psi(v) \mu_{t_0, u_0^n}(dv) = \int \psi(v) \mu_{t_0, u_0}(dv) \quad (6)$$

for a large class of continuous functions ψ .

The previous result is only the stochastic analog of a deterministic property of local well posedness. But in the stochastic case it is here that we have more. Under strong assumptions on the noise (the same ones that allowed to solve the Kolmogorov equation in [23]), *strong Feller property* holds at time t_0 (again we simplify the exposition forgetting about a small probability ϵ of having a different property)

$$\lim_{u_0^n \rightarrow u_0} \mu_{t_0, u_0^n} = \mu_{t_0, u_0} \text{ in total variation.}$$

Convergence in total variation essentially means that (6) is extended to a large class of measurable functions, something impossible in the deterministic case, where μ_{t_0, u_0^n} and μ_{t_0, u_0} are Dirac delta masses! Using the decomposition property (5), one can prove:

Theorem 2. *Assume $\mathbb{E}[\phi(u(T; u_0)) | u(t_0; u_0) = v]$ is independent of u_0 , for every $\phi \in C_b(H)$, and there exists a function $g_\phi(T, t_0, v)$, measurable in v , such that*

$$g_\phi(T, t_0, v) = \mathbb{E}[\phi(u(T; u_0)) | u(t_0; u_0) = v].$$

Then $\mathbb{E}[\phi(u(T; u_0^n))] \rightarrow \mathbb{E}[\phi(u(T; u_0))]$, namely continuous dependence propagates to large times.

See [44] for related details. The assumption of the theorem, the existence of $g_\phi(T, t_0, v)$, is essentially the Markov property. The question is: can we make a selection which satisfies the Markov property?

Yes, following [23, 28, 44] we know:

Theorem 3. *For 3D Navier-Stokes, Markov selections exist. If the noise is strong enough, they are strong Feller, hence solutions depend continuously on the initial conditions, also for large times, in the topology of V .*

The previous theorem can be considered the most advanced innovative result of the stochastic theory with respect to the deterministic one. Nothing like this theorem is known in the deterministic case.

Can we do more? The following trick in semigroup theory is well known: if $A : D(A) \subset H \rightarrow H$ generates a strongly continuous semigroup S_t , $t \geq 0$, and $u(t)$ solves $u'(t) = Au(t)$, then

$$u(t) = S_t u(0).$$

Indeed,

$$\frac{d}{ds} S_{t-s} u(s) = -A S_{t-s} u(s) + S_{t-s} A u(s) = 0.$$

In other words: when we have a strongly continuous flow, all solutions coincide with those of the flow. Such uniqueness result, however, holds in the framework of semigroup theory; it is only heuristically a general principle. In the case described above, we have something similar concerning the assumptions: we have a Markov, strong Feller, selection. But, in spite of many attempts, we have not found a rigorous way to deduce that it “incorporates” every weak solution.

The Markov strong Feller selection is a priori not unique and, based on results proved in [57] in an easier context than the Navier-Stokes equations, we should

expect uniqueness of Markov selections if and only if there is uniqueness of individual solutions. It is however possible that some Markov selection carries more specific information and may be elevated to a special role. Sufficient conditions for uniqueness of Markov selections are given in [44, 56].

2.4. Multiplicative transport noise

Another noise received increasing attention in fluid mechanics problems. It is inspired by the transport term $u \cdot \nabla u$ and has the form (compare with (1))

$$\nabla u \circ dW = \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha} (e_{k,\alpha} \cdot \nabla u) \circ d\beta_t^{k,\alpha}.$$

The multiplication is understood in the Stratonovich sense, recognized to be the right one throughout the literature on this subject (e.g. [49, 55, 53]). A short introduction to this detail can be found in [41].

In a sense, the velocity field u which transports other quantities (like u itself in $u \cdot \nabla u$, or terms like $u \cdot \nabla T$ in heat transport) is replaced by $u + W$. The resulting stochastic Navier-Stokes equations are

$$\begin{aligned} du + (u \cdot \nabla u + \nabla p) dt &= \Delta u dt + \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha} (e_{k,\alpha} \cdot \nabla u) \circ d\beta_t^{k,\alpha}, \\ \operatorname{div} u &= 0. \end{aligned}$$

There is another, non-equivalent, way to introduce transport noise; it is at the level of the equation for the vorticity $\xi = \operatorname{curl} u$, which in the case of 3D deterministic Navier-Stokes equations is

$$\partial_t \xi + u \cdot \nabla \xi = \Delta \xi + \xi \cdot \nabla u,$$

also written, using the Lie derivative $\mathcal{L}_u \xi = u \cdot \nabla \xi - \xi \cdot \nabla u$, as

$$\partial_t \xi + \mathcal{L}_u \xi = \Delta \xi.$$

The natural perturbation of this equation is

$$\begin{aligned} d\xi + u \cdot \nabla \xi dt &= \Delta \xi dt + \xi \cdot \nabla u dt + \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha} (e_{k,\alpha} \cdot \nabla \xi) \circ d\beta_t^{k,\alpha} \\ &\quad - \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha} (\xi \cdot \nabla e_{k,\alpha}) \circ d\beta_t^{k,\alpha} \end{aligned}$$

considered in [49, 21]: it corresponds to the replacement of u with $u + W$ in the Lie derivative (which corresponds to the same replacement at the Lagrangian level)

$$\begin{aligned} \mathcal{L}_u \xi dt &\rightarrow \mathcal{L}_u \xi dt + \mathcal{L}_{\circ dW} \xi \\ &:= \mathcal{L}_u \xi dt + \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \mathcal{L}_{\sigma_{k,\alpha} e_{k,\alpha} \circ d\beta_{tt}^{k,\alpha}} \xi. \end{aligned}$$

The nonlinearity is composed, at the vorticity level, of two terms: the transport of vorticity $u \cdot \nabla \xi$ and the vortex stretching $\xi \cdot \nabla u$. Accordingly, in the previous equation there is an additional stochastic transport and stochastic stretching. When vorticity

is replaced by magnetic moment, this stochastic perturbation was considered in the framework of the dynamo theory in [55]. Notice that in the 2D case stretching cannot occur, since the vorticity is orthogonal to the plane of fluid motion, hence the equation reduces to (see for instance [15])

$$d\xi + u \cdot \nabla \xi dt = \Delta \xi dt + \sum_{k \in \mathbb{Z}_0^3} \sigma_k (e_k \cdot \nabla \xi) \circ d\beta_t^k.$$

It is worth noticing that it is not necessary anymore to sum over the index $\alpha = 1, 2$ because the linear space orthogonal to k is now a line). Concerning motivations for the model with transport noise, let us mention model reduction, see [53], in addition to other motivations like [14], [54], and more recently [49].

Starting from 2010, several simpler models proved to be regularized by transport noise, as already remarked in the Introduction: linear transport and advection equations [37, 41, 42], special solutions of 2D Euler equations and 1D Vlasov-Poisson system [38, 30], Leray- α model [3] and, as more extensively discussed below in Section 3, dyadic models of turbulence [6], [12]. In all these cases the PDE is inviscid. But it was recently understood that such a noise may have a regularizing effect also on viscous problems, in particular the 3D Navier-Stokes equations because it increases dissipation. Let us briefly summarize this result, from [40].

The first important remark is that it holds for a sort of artificial modification of the noise above: we consider only the stochastic transport term - as in the 2D case -, neglecting the stochastic stretching term, but maintaining the 3-dimensionality of the equation. The precise model is

$$d\xi + u \cdot \nabla \xi dt = \Delta \xi dt + \xi \cdot \nabla u dt + \sum_{\substack{k \in \mathbb{Z}_0^3 \\ \alpha=1,2}} \sigma_{k,\alpha} \Pi (e_{k,\alpha} \cdot \nabla \xi) \circ d\beta_t^{k,\alpha}, \quad (7)$$

where Π is the projection on divergence free fields, necessary since the sum of all other terms is divergence free (notice that, on the contrary, the full noise $\mathcal{L}_{odW}\xi$ does not require projection since it is already divergence free). In [40] there is an attempt to motivate this choice of noise, but it remains true that the full noise $\mathcal{L}_{odW}\xi$ is much more natural, while at the same time the latter spoils the result of regularization by noise, as shown in [40]. This discrepancy will be the object of future investigation.

In order to understand the result in [40], let us recall the second open problem presented in Section 2.1, restated here as follows:

when $\xi_0 \in H$, is the probability

$$P \left(\tau < \infty, \lim_{t \uparrow \tau} \|\xi(t)\|_H = +\infty \right) \quad (8)$$

equal to one, or at least large? We have discovered that transport noise may improve the control of $\|\xi\|_H$. In the deterministic case, the norm $\|\xi\|_H^2$ can be controlled *locally* from

$$\partial_t \xi + u \cdot \nabla \xi - \xi \cdot \nabla u = \Delta \xi$$

by energy-type estimates:

$$\frac{1}{2} \frac{d}{dt} \|\xi\|_H^2 + \|\nabla \xi\|_H^2 = \langle \xi \cdot \nabla u, \xi \rangle.$$

The term $\langle \xi \cdot \nabla u, \xi \rangle$ describes the *stretching* of vorticity ξ produced by the deformation tensor ∇u . This is the potential source of unboundedness of $\|\xi\|_H^2$. Sobolev and interpolation inequalities give us (up to constants):

$$\langle \xi \cdot \nabla u, \xi \rangle \leq \|\xi\|_{L^3}^3 \leq \|\xi\|_{W^{1,2}}^3 \leq \|\xi\|_{L^2}^{3/2} \|\xi\|_{W^{1,2}}^{3/2} \leq \|\xi\|_{W^{1,2}}^2 + \|\xi\|_{L^2}^6$$

and this leads to

$$\frac{d}{dt} \|\xi\|_H^2 \leq C \|\xi\|_H^6$$

which provides only a local control. However the interval of existence depends on the *viscosity coefficient* ν : if we consider

$$\partial_t \xi + u \cdot \nabla \xi - \xi \cdot \nabla u = \nu \Delta \xi,$$

the energy estimate becomes

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\xi\|_H^2 + \nu \|\nabla \xi\|_H^2 &= \langle \xi \cdot \nabla u, \xi \rangle \\ &\leq \|\xi\|_{L^2}^{3/2} \|\xi\|_{W^{1,2}}^{3/2} \\ &\leq \nu \|\nabla \xi\|_H^2 + \frac{C}{\nu^3} \|\xi\|_H^6 \end{aligned}$$

leading in this case to

$$\frac{d}{dt} \|\xi\|_H^2 \leq \frac{C}{\nu^3} \|\xi\|_H^6.$$

The explosion is delayed for large ν . Not only that: beyond a threshold the solution is global. This is the key for a regularization by noise: transport noise improves dissipation, hence it delays blow-up.

Let us rewrite equation (7) In Itô form (see [41] for an easy introduction to this operation):

$$\begin{aligned} d\xi + u \cdot \nabla \xi dt &= \Delta \xi dt + \xi \cdot \nabla u dt + \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha} \Pi(e_{k,\alpha} \cdot \nabla \xi) d\beta_t^{k,\alpha} \\ &+ \frac{1}{2} \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha}^2 \Pi(e_{k,\alpha} \cdot \nabla \Pi(e_{k,\alpha} \cdot \nabla \xi)) dt, \end{aligned}$$

where the stochastic term is now understood in Itô sense. The corrector is a pseudo-differential operator of second order, quite complicated algebraically by the presence of the projector Π . Under suitable technical conditions on the family of coefficients $\sigma = (\sigma_{k,\alpha})_{k \in \mathbb{Z}_0^3, \alpha=1,2}$ (still quite general), the corrector turns out to be of the form

$$\frac{1}{2} \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha}^2 \Pi(e_{k,\alpha} \cdot \nabla \Pi(e_{k,\alpha} \cdot \nabla \xi)) = \nu_\sigma \Delta \xi + R_\sigma(\xi),$$

where $\nu_\sigma > 0$ is a coefficient depending on σ and $R_\sigma(\xi)$ is a quite complicated non-local second order differential operator. The decomposition of the RHS as $\nu_\sigma \Delta \xi +$

$R_\sigma(\xi)$ is not purely artificial: the same corrector without the two projections Π would be simply equal to $\nu_\sigma \Delta \xi$; the remainder $R_\sigma(\xi)$ is what is left due to the presence of the projections.

Now the key point is to parametrize σ by a scaling parameter N :

$$\sigma^N = (\sigma_{k,\alpha}^N)_{k \in \mathbb{Z}_0^3, \alpha=1,2}$$

in such a way that the corresponding coefficient ν_{σ^N} is independent of N

$$\nu_{\sigma^N} = \nu$$

and (this is the most difficult technical part of the work [40])

$$\lim_{N \rightarrow \infty} R_{\sigma^N}(\xi) = -\frac{2}{5} \nu \Delta \xi.$$

The solutions ξ^N of the corresponding equation

$$d\xi^N + u^N \cdot \nabla \xi^N dt = \Delta \xi^N dt + \xi^N \cdot \nabla u^N dt + \sum_{k \in \mathbb{Z}_0^3, \alpha=1,2} \sigma_{k,\alpha}^N \Pi(e_{k,\alpha} \cdot \nabla \xi^N) \circ d\beta_t^{k,\alpha} \quad (9)$$

will have the following properties, which are the main results of [40].

Theorem 4. *Let $\xi_0 \in H$ and $[0, T]$ be given. In a suitable scaling limit $N \rightarrow \infty$ corresponding to a sequence σ^N , ξ^N converges in probability to the solution of*

$$\partial_t \xi + \mathcal{L}_u \xi = \left(1 + \frac{5}{3} \nu\right) \Delta \xi.$$

It follows that for large N the norm $\|\xi^N(t)\|_H^2$ is bounded on $[0, T]$, with high probability (implying well posedness of ξ^N).

Theorem 5. *Given $R_0, \epsilon > 0$, there exists N with the following property: for every initial condition $\xi_0 \in H$ with $\|\xi_0\|_H \leq R_0$, the stochastic 3D Navier-Stokes equations (9) have a global unique solution, up to probability ϵ : if τ is the maximal time of existence,*

$$P(\tau = \infty) > 1 - \epsilon.$$

The theorem does not exclude that the probability (8) is positive, but at least it states it is small, under a proper noise perturbation. This result is a regularization by noise result because the viscosity in equation (9) is 1 and, as discussed above for the deterministic equations, with such viscosity only very small initial conditions lead to global existence.

The previous results are inspired by several sources, among which we quote [1, 2, 3, 20, 39, 35, 45].

3. Regularization by noise in dyadic models

Even though the regularization by noise techniques did not work for 3D Navier-Stokes equation, there are other equations that proved to be more accessible with this tool. One special case, still in the area of fluid-dynamics, is that of the dyadic models of turbulence.

3.1. Dyadic models

Shell models were introduced by the Russian school in the 1970s, as a theoretical and computational tool to study the cascade phenomenon in turbulent fluid dynamics. This is a mechanism (not yet completely understood) that moves energy from one lengthscale to another, thus sustaining turbulence. Richardson's cascade, also called direct energy cascade, moves the energy from larger scales to smaller ones, whereas the inverse cascade moves energy from smaller to larger scales, and seems to appear only in 2D turbulence.

The phenomenological idea behind the tree model proposed by Katz and Pavlović [50], and called KP model in the next pages, is the following: larger eddies in the turbulent fluid split into smaller ones because of dynamical instabilities, and the kinetic energy moves from the larger scales to the smaller ones. We simplify the picture by assuming that eddies appear only at certain discrete scales, each the half of the previous one. We also assume that the eddies fill the space, so that each eddy contains 2^d eddies of the next scale.

In this way we have a tree structure, where each node is an eddy. Following the notation introduced in [4], if we denote by J the set of nodes, each node j has a set of children \mathcal{O}_j , representing the smaller eddies generated by instability from j . We call generations the discrete scales where the eddies are, and denote the generation of an eddy j by $|j|$. At level (or generation) 0 we have the single largest eddy, denoted with \emptyset , at generation 1 the 2^d eddies generated by the eddy at level 0 and so on. Also, we denote the parent of a node j by \bar{j} .

Every node has a scalar quantity X_j attached to it, the intensity of the velocity field, with the square of this intensity being the kinetic energy. In other words, the energy is the square of the l^2 -norm: $\mathcal{E}(t) := \sum_j X_j^2(t)$. The intensities are coupled by the following differential rules:

$$\dot{X}_j = -\nu\tilde{c}_j X_j + c_j X_{\bar{j}}^2 - X_j \sum_{k \in \mathcal{O}_j} c_k X_k, \quad (10)$$

where we consider the coefficients $c_j = d_j 2^{\alpha|j|}$, with $\alpha > 0$ and $d_j > 0$ for all $j \in J$ (and similarly for the $\tilde{c}_j = \tilde{d}_j 2^{\gamma|j|}$), $d_\emptyset = 1$, and $X_\emptyset(t) \equiv f$, that is the forcing acts only on the largest eddy. Most results are independent of the choice of α , however there are heuristic arguments that suggest $\alpha = \frac{d}{2} + 1$, which is the value usually considered in the literature (see for example [50, 4]). In [10] it was proven that $\alpha \leq \frac{5}{2}$ for a Littlewood-Paley decomposition of 3D Euler dynamics.

In [50] and in [4], $d_j = 1$ for all $j \in J$, but in [13], restricted to the inviscid case (i.e. $\nu = 0$) the coefficients d_j are allowed to vary for different nodes, with the assumption that $|\log d_j|$ is bounded. Moreover a particular choice is introduced, the

repeated coefficients models (or RCM), in which the same fixed 2^d coefficients δ_ω appear in every set of siblings $\{d_k : k \in \mathcal{O}_j\}$. For the RCM it is possible to state and prove more interesting and deep results, due to its simpler form.

Heuristically, we can think of this model as a (simplified) wavelet decomposition of Navier-Stokes equations, see for example [50, 16, 13]. However, this is not a rigorous derivation, as pointed out in [61]: the KP model is constructed in such a way that it mimics Navier-Stokes (in particular with respect to the energy cascade phenomenon).

If we choose to have only one intensity per shell, that is we consider all nodes in a generation as collapsed into a single element, we get a “linear” dyadic model, that turns out to be one of the first shell models, the one introduced by Desnianskii and Novikov in 1974 [31]. For this reason we will call it DN model. Also in this case we can give a heuristical interpretation of the model as a Littlewood-Paley decomposition: see for example [50, 16, 51]). The step from the KP model to the DN one was first done by Waleffe [61]. In the same paper, he also discussed a different model, the aforementioned Obukhov model. All three of KP, DN and Obukhov models were investigated by Kiselev and Zlatos [51], with particular focus on the question of regularity and blow-up.

Let us now see the DN mode in some more detail: the differential rule coupling the intensities associated to the different shells takes the following form,

$$\dot{X}_j = -\nu l_j^2 X_j + c_{j-1} X_{j-1}^2 - c_j X_j X_{j+1},$$

with $c_j = 2^{\alpha j}$ and $l_j = 2^{\gamma j}$, with j taking value in $J = \mathbb{N}$, so that $\bar{j} = j - 1$, $\mathcal{O}_j = \{j + 1\}$, and $|j| \equiv j$.

This model, though physically less appealing than the KP one, has a much simpler structure. For this reason many results, in particular regarding uniqueness and regularity of solutions, haven been proven first for the DN model and extended to the KP model only later.

In order to talk about existence and uniqueness of solutions, we need to state what notion of solution are we considering for such models. A *componentwise solution* of the KP model is a family $(X_j)_{j \in J}$ of differentiable functions such that (10) is satisfied. If a componentwise solution is in $L^\infty(\mathbb{R}_+, l^2(J))$, it is called a *Leray solution*. Analogous definitions hold for the DN model (actually, we just have to consider $J = \mathbb{N}$, and the other conventions written above).

Theorem 6. *For the KP model (10), for any initial condition in l^2 , there exists a Leray solution.*

The argument for the proof is quite standard, using Galerkin approximations, and can be found in [4, 11, 13]. A similar result holds for the DN model, and actually, with some assumptions on the coefficients, solutions of the DN model can be lifted to the KP model.

A natural question that can be raised at this point is the following: what about more regular solutions? This question is in fact strongly tied to another interesting

property of dyadic models, that of anomalous dissipation. As a matter of fact, ignoring the dissipative term, one can show that for both KP and DN energy is formally preserved. However, if we approach the issue rigorously, we see that this is only true for solutions that are regular enough. However, it is possible to show that energy actually dissipates, hence solutions cannot be that regular. This kind of argument is presented in the aforementioned [8] for the DN model. The same is true also for the KP model and the RCM, as it is shown in [4] and [13].

3.2. Uniqueness and Regularization by noise for dyadic models

For the DN model there is uniqueness if we restrict ourselves to non-negative solutions but we lose it if we allow for solutions that change sign [5, 9]. For the KP model, uniqueness for non-negative solutions is an open problem, but counterexamples to uniqueness can be shown for solutions that are allowed to change sign.

In both cases, counterexamples can be constructed through self-similar solutions, that is solutions of the form $X_j(t) = \frac{a_j}{t-t_0}$, for some $t_0 < 0$, for all $j \in J$ and $t \geq 0$, with the coefficient a_j satisfying some coupling conditions. Once we have such solutions, we can use time reversal to have solutions that blow up in finite time, that are in particular non Leray, hence showing non-uniqueness of componentwise solutions.

For the RCM it is hard to prove results for general solutions, because they are quite complicated to deal with. However if we focus on constant solutions, we not only have an existence and uniqueness result of a (finite energy) forced solution that dissipates energy, but we can also write such solution explicitly. In the case of the RCM, this allows us to obtain some interesting results regarding the structure function and the geometry of the anomalous dissipation. Existence and uniqueness of constant solutions hold for the KP and the DN models as well [4, 18]. In the case of the inviscid DN model, the constant solution is particularly interesting, because it has been proven to be a global attractor [19]. A similar result holds for the viscous model, too [17]. For the KP and the RCM the existence of such a global attractor is still a conjecture.

For constant solutions we have a uniqueness result. However this is not the case if we consider generic solution, as mentioned above. In order to recover some kind of uniqueness, we resort to regularization by noise techniques.

It is true that we started with a PDE, but the model that we are considering is now made of an infinite system of coupled ODEs. So it should not be surprising that we can obtain regularization results by adding noise. Let us see some more details. Notice that we focus only on the inviscid case.

In order to recover uniqueness of the solution, we want to define a stochastic perturbation of the deterministic KP model: among the several options possible, we choose a multiplicative term (so that the perturbation “scales” with the solution, being neither irrelevant nor dominant) such that the total energy is (formally) \mathbb{P} -a.s. preserved

$$dX_j = \left(c_j X_j^2 - X_j \sum_{k \in \mathcal{O}_j} c_k X_k \right) dt + c_j X_{\bar{j}} \circ dW_j - \sum_{k \in \mathcal{O}_j} c_k X_k \circ dW_k, \quad (11)$$

with $(W_j)_{j \in J}$ a family of independent Brownian motions, together with deterministic initial conditions $X(0) = x = (x_j)_{j \in J} \in l^2$.

For this model (which we can also write in Itô formulation) we consider solutions that are weak in the probabilistic sense. Of particular interest, for obvious physical reasons, are energy controlled solutions, that is weak solutions that satisfy

$$\mathbb{P} \left(\sum_{j \in J} X_j^2(t) \leq \sum_{j \in J} x_j^2 \right) = 1 \text{ for all } t \geq 0,$$

that is, the energy is almost surely bounded by the initial one.

Theorem 7. *There exists an energy controlled solution to (11) in $L^\infty(\Omega \times [0, T]; l^2)$ for initial conditions $X(0) = x = (x_j)_{j \in J} \in l^2$.*

Moreover, there is uniqueness in law in the same class of energy controlled solutions.

Both weak existence and weak uniqueness are achieved through Girsanov theorem, transforming the nonlinear SDEs in linear ones. However the first step is to translate our model from the Stratonovich formulation (11) into Ito formulation, which is easier to manipulate:

$$dX_j = \left(c_j X_j^2 - X_j \sum_{k \in \mathcal{O}_j} c_k X_k \right) dt + c_j X_{\bar{j}} dW_j - \sum_{k \in \mathcal{O}_j} c_k X_k dW_k - \frac{1}{2} \left(c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2 \right) X_j dt. \quad (12)$$

Moreover, since we want to use Girsanov theorem, it makes sense to rewrite (12) in the following form:

$$dX_j = c_j X_{\bar{j}} (X_{\bar{j}} dt + c_j dW_j) - \sum_{k \in \mathcal{O}_j} c_k X_k (X_j dt + dW_k) - \frac{1}{2} \left(c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2 \right) X_j dt,$$

where we isolated the terms $X_{\bar{j}} dt + c_j dW_j$, which are (for all $j \in J$) Brownian motions with respect to a new measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_\infty)$. More precisely we can state:

Proposition 8. *Given an energy controlled solution $(\Omega, (\mathcal{F}_t)_t, \mathbb{P}, W, X)$ of (12) (or equivalently (11)), we can define a measure $\tilde{\mathbb{P}}$ as follows:*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Bigg|_{\mathcal{F}_t} = \exp \left(- \sum_{j \in J} \int_0^t X_{\bar{j}}(s) dW_j(s) - \frac{1}{2} \int_0^t \sum_{j \in J} X_j^2(s) ds \right).$$

Then the processes

$$B_j(t) = W_j(t) + \int_0^t X_j(s) ds$$

are a J -indexed family of independent Brownian motions on $(\Omega, (\mathcal{F}_t)_t, \tilde{\mathbb{P}})$, and $(\Omega, (\mathcal{F}_t)_t, \tilde{\mathbb{P}}, B, X)$ satisfies the linear equations

$$dX_j = c_j X_j dB_j(t) - \sum_{k \in \mathcal{O}_j} c_k X_k dB_k(t) - \frac{1}{2} \left(c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2 \right) X_j dt.$$

For this linear system we can easily prove, by Galerkin approximations, that there exists a strong solution. The next step is to prove strong uniqueness for the linear system.

To do so, we consider the second $\tilde{\mathbb{P}}$ -moments of the X_j s: for every solution X of the (nonlinear) system (12), for every $j \in J$ and $t \geq 0$, $\tilde{\mathbb{E}}[X_j^2(t)] < \infty$ and satisfies the differential equation:

$$\frac{d}{dt} \tilde{\mathbb{E}}[X_j^2(t)] = - \left(c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2 \right) \tilde{\mathbb{E}}[X_j^2(t)] + c_j^2 \tilde{\mathbb{E}}[X_j^2(t)] + \sum_{k \in \mathcal{O}_j} c_k^2 \tilde{\mathbb{E}}[X_k^2(t)].$$

Now we have obtained a system of closed equations, with a very nice structure: if we write it in matricial form, it is strongly reminiscent of the forward equations of a Markov chain (even though it actually is not). Thanks to this link, we can show uniqueness for the second moments and, hence, for the solution of the linear system. This strong uniqueness result translates into uniqueness in law for the nonlinear system, as the two measures \mathbb{P} and $\tilde{\mathbb{P}}$ are not equivalent on \mathcal{F}_∞ . More precise statements, as well as detailed proofs, can be found in [12] and [11].

A similar result holds for the DN linear dyadic model, and was obtained earlier in [6]. In this case the model has the following form:

$$dX_j = (c_j X_{j-1}^2 - c_{j+1} X_j X_{j+1}) dt + c_j X_{j-1} \circ dW_{j-1} - c_{j+1} X_{j+1} \circ dW_j,$$

with $(W_j)_{j \in J = \mathbb{N}}$ a sequence of independent Brownian motions, and the form of the noise chosen to be formally energy preserving (almost surely). In this case, anomalous dissipation has been shown in [7].

Of course one can deduce weak existence and uniqueness for DN from Theorem 7 for the KP model. It is interesting to notice that the different behaviour seen in the deterministic case for non-negative and mixed-sign solutions is now absent, even though this is not surprising, because the noise is causing sign changes.

In the end, regularization by noise techniques had at least a partial success in the area of fluid dynamics. Even though the techniques used for dyadic models did not immediately translate back to Navier-Stokes equations, there are also ideas born in the study of shell models that trickled back to Navier-Stokes. In particular, in [58] some ideas from previous works on dyadic models were used to show blow-up of an averaged version of 3D Navier-Stokes, proving a meta-theorem: no technique that does not distinguish the DN model from Navier-Stokes can show regularity for NSE.

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Luigi Amedeo Bianchi
Università di Trento
Dipartimento di Matematica
Via Sommarive, 14
38123 Povo (Trento)
Italy
e-mail: luigiamedeo.bianchi@unitn.it

Franco Flandoli
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa
Italy
e-mail: franco.flandoli@sns.it

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