If the primes are finite, then all of them divide the number one

We propose a novel proof of the infinitude of the primes based on elementary considerations of Legendre's function ϕ, defined in [1, p. 153] as

$$\phi(x, y) = |\{1 \leq n \leq x : \text{integer } n \text{ has no prime factors} \leq y\}|,$$

where x and y are positive integers. The reader can see that

$$\pi(x) = \pi(\sqrt{x}) + \phi(x, \sqrt{x}) - 1,$$

where $\pi(\cdot)$ is the prime-counting function. Let $p_1, ..., p_s$ be the prime numbers less than or equal to y. Using the inclusion-exclusion principle, it can be proved that

$$\phi(x, y) = x - \sum_{1 \leq i \leq s} \left[\frac{x}{p_i}\right] + \sum_{1 \leq i,j \leq s} \left[\frac{x}{p_i p_j}\right] + ... + (-1)^s \left[\frac{x}{p_1 \cdots p_s}\right],$$

where $[\cdot]$ is the floor function. Pinasco [2] also used this principle for proving the infinitude of the primes, but his proof is remarkably different from ours.

Suppose that $\{p_1, ..., p_s\}$ is the set of all prime numbers. Consider $N = p_1 \cdots p_s$. Then $\phi(N^2, N) = 1$. On the other hand, we have

$$\phi(N^2, N) = N^2 - \sum_{1 \leq i \leq s} \left[\frac{N^2}{p_i}\right] + \sum_{1 \leq i,j \leq s} \left[\frac{N^2}{p_i p_j}\right] + ... + (-1)^s N.$$

Hence, $\phi(N^2, N) = mN$ for some integer m, i.e., every prime number divides 1. This means that all primes, and, consequently, all non–zero natural numbers, are invertible in $\mathbb{N}$, i.e., we find that $\mathbb{N}$ is a field. This completes the proof.

References

[1] R. Crandall, C. Pomerance, *Prime numbers. A computational perspective*, Springer–Verlag, New York, 2nd Edition, 2005.

[2] J. P. Pinasco, New proofs of Euclid's and Euler's theorems, *Amer. Math. Monthly* **116** (2009) 172–174.

—Submitted by Umberto Cerruti and Nadir Murru

MSC: Primary 11A41, Secondary 11A99