Three-Dimensional Representation of Electrical Circuit Quantities

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***Abstract*—This paper proposes a new representation of electrical circuit quantities based on three-phase dimensional space, in contrast to the conventional two-dimensional representation. It contributes to an alternative mathematical approach to display the voltage and current waveforms in three-dimensional frame, in which an unbalance displacement angle becomes evident, and can be used as an unbalance metric. It is outside the scope of this paper to propose a new power theory, or state new definitions for the power terms. In fact, such electric power representation can be applied to different power theories. Finally, this paper exemplifies the three-dimensional representation through numerical and illustrative case studies.**

***Index Terms*—meta-theory, power theory, electrical circuit quantity, power property, unbalance.**

I. Introduction

Power theories have been subject of investigation for almost a century [1], pursuing to precisely describe the power properties of electrical circuits relating them to the physical phenomena, and mathematical expressions [2]. However, no power theory has fully succeeded, because there is always a proof of inconsistency. In view of this complex subject of study, this paper contributes with an alternative mathematical representation of voltage and current waveforms which may assist in this arduous task involving the power theories in three-phase circuits.

Then, this paper proposes a novel representation of electrical circuit quantities based on three-dimensional space, in contrast to the conventional two-dimensional one. It contributes to display the voltage and current waveforms in a three-dimensional frame, in which a novel unbalance displacement angle becomes as evident as the well-known displacement angle [3] caused by reactive power circulation in the two-dimensional space.

The unbalance displacement angle is used to define an unbalance displacement factor (UDF) that quantifies the currents unbalance with respect to the three-phase voltage frame. Hence, it is a relative measure of unbalance between three-phase current and voltage signals, and it is not primarily intended to assess the inherently unbalanced nature of a single set of voltages or currents, as other unbalance metrics detailed in Annex I. Nevertheless, UDF could also capture the inherent degree of unbalance of a set of currents (or voltages) if calculated against a corresponding ideal (i.e., sinusoidal, symmetrical) set of voltages (or sinusoidal, balanced currents, respectively).

Furthermore, as none of the existing metrics visually show the current unbalance with respect to voltage frame, because they are all based on a two-dimensional representation and normally expressed in percentage values, they are less suitable than UDF for a visual Cartesian representation.

Overall, the current paper does not aim at proposing a new electric power theory or set new definitions for power terms. It rather offers an alternative representation for electrical quantities. Actually, such representation can be applied to different power theories published in the literature, like the *Conservative Power Theory* (CPT) [4],[5], and the *Current Physical Components* (CPC) [6],[7],[8].

II. Conventional Electric Power Representation

Periodic voltage and current quantities are commonly represented in two-dimensional space, as:

$$v(t) = V_0 + \sum_{k=1}^{n} V_k \cdot \sin(\omega_k . t + \theta_{vk})$$
$$i(t) = I_0 + \sum_{k=1}^{n} I_k \cdot \sin(\omega_k . t + \theta_{ik}) \tag{1}$$

such that k is the n^{th}-harmonic order, ω is the angular frequency and θ_{vk} and θ_{ik} represent the phase of each harmonic voltage and current, respectively. The voltage quantity is usually considered as the frame of reference, so that $\theta_{v1} = 0$.

To simplify the presentation, all voltage and current waveforms are assumed to have zero mean values ($V_0 = I_0 = 0$). Moreover, there is substantial confusion on power definitions with unbalanced loads even under sinusoidal voltage conditions, and it is expected that investigations increase gradually in complexity to avoid inconsistent results, so this paper is restricted just to circuits with sinusoidal voltages. It is worth noting that, under such assumption, θ_{i1}, hereafter simply indicated as θ, represents the relative displacement angle between the (fundamental) current and the corresponding voltage, which is also known as *reactive displacement (power) angle* [9].

Let us define the bold variables as vectors, considering the three-phase quantities, as represented in (2).

$$\boldsymbol{v}(t) = \begin{bmatrix} v_a(t) \\ v_b(t) \\ v_c(t) \end{bmatrix} \text{ and } \boldsymbol{i}(t) = \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix}^T \tag{2}$$

such that the superscript T means transpose. Then, the current vector ($\boldsymbol{i}$) is decomposed into in-phase ($\boldsymbol{i}_{\parallel}$) and quadrature ($\boldsymbol{i}_{\perp}$) current terms, and thereupon in positive (+) and negative (-) sequence components:

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$$\begin{aligned} \boldsymbol{i}(t) &= \boldsymbol{i}_{\parallel}(t) + \boldsymbol{i}_{\perp}(t) \\ \boldsymbol{i}_{\parallel}(t) &= \boldsymbol{i}_{\parallel}^{+}(t) + \boldsymbol{i}_{\parallel}^{-}(t) \\ \boldsymbol{i}_{\perp}(t) &= \boldsymbol{i}_{\perp}^{+}(t) + \boldsymbol{i}_{\perp}^{-}(t) \end{aligned} \tag{3}$$

the instantaneous active (p) and reactive (q) power are calculated through the internal product as:

$$\begin{aligned} p(t) &= (\boldsymbol{v}^{+} + \boldsymbol{v}^{-}) \circ \left(\boldsymbol{i}_{\parallel}^{+} + \boldsymbol{i}_{\parallel}^{-} + \boldsymbol{i}_{\perp}^{+} + \boldsymbol{i}_{\perp}^{-}\right) = \\ &= \underbrace{\boldsymbol{v}^{+} \circ \boldsymbol{i}_{\parallel}^{+} + \boldsymbol{v}^{-} \circ \boldsymbol{i}_{\parallel}^{-}}_{P} + \underbrace{\boldsymbol{v}^{+} \circ \boldsymbol{i}_{\parallel}^{-} + \boldsymbol{v}^{+} \circ \boldsymbol{i}_{\perp}^{-} + \boldsymbol{v}^{-} \circ \boldsymbol{i}_{\parallel}^{+} + \boldsymbol{v}^{-} \circ \boldsymbol{i}_{\perp}^{+}}_{\tilde{p}} \\ q(t) &= (\widehat{\boldsymbol{v}}^{+} + \widehat{\boldsymbol{v}}^{-}) \circ \left(\boldsymbol{i}_{\parallel}^{+} + \boldsymbol{i}_{\parallel}^{-} + \boldsymbol{i}_{\perp}^{+} + \boldsymbol{i}_{\perp}^{-}\right) = \\ &= \underbrace{\widehat{\boldsymbol{v}}^{+} \circ \boldsymbol{i}_{\perp}^{+} + \widehat{\boldsymbol{v}}^{-} \circ \boldsymbol{i}_{\perp}^{-}}_{Q} + \underbrace{\widehat{\boldsymbol{v}}^{+} \circ \boldsymbol{i}_{\perp}^{-} + \widehat{\boldsymbol{v}}^{+} \circ \boldsymbol{i}_{\parallel}^{-} + \widehat{\boldsymbol{v}}^{-} \circ \boldsymbol{i}_{\perp}^{+} + \widehat{\boldsymbol{v}}^{-} \circ \boldsymbol{i}_{\parallel}^{+}}_{\tilde{q}} \end{aligned} \tag{4}$$

The variable $\widehat{\boldsymbol{v}}$ is the phase shifted by 90º with respect to vector $\boldsymbol{v}$ on the xy-plane, and can be defined as the (unbiased) homo-integral of voltage [4]. Hence, the association terms $\boldsymbol{v}^{+} \circ \boldsymbol{i}_{\perp}^{+} + \boldsymbol{v}^{-} \circ \boldsymbol{i}_{\perp}^{-}$ and $\widehat{\boldsymbol{v}}^{+} \circ \boldsymbol{i}_{\parallel}^{+} + \widehat{\boldsymbol{v}}^{-} \circ \boldsymbol{i}_{\parallel}^{-}$ are zero because of orthogonality properties.

According to (4), the projection of $\boldsymbol{i}$ on $\boldsymbol{v}$ results in the average active power (P), while the projection of $\boldsymbol{i}$ on $\widehat{\boldsymbol{v}}$ is the average reactive power (Q). However, the interaction of voltages and currents from different sequences results in power oscillations, which can be split into in-phase and quadrature terms in relation to the voltage frame of reference, as:

$$\begin{aligned} \tilde{p}_{\parallel}(t) &= \boldsymbol{v}^{+} \circ \boldsymbol{i}_{\perp}^{-} + \boldsymbol{v}^{-} \circ \boldsymbol{i}_{\perp}^{+} + \widehat{\boldsymbol{v}}^{+} \circ \boldsymbol{i}_{\parallel}^{-} + \widehat{\boldsymbol{v}}^{-} \circ \boldsymbol{i}_{\parallel}^{+} = \|\tilde{p}_{\parallel}\| \sin(2\omega t) \\ \tilde{p}_{\perp}(t) &= \boldsymbol{v}^{-} \circ \boldsymbol{i}_{\parallel}^{+} + \boldsymbol{v}^{+} \circ \boldsymbol{i}_{\parallel}^{-} + \widehat{\boldsymbol{v}}^{-} \circ \boldsymbol{i}_{\perp}^{+} + \widehat{\boldsymbol{v}}^{+} \circ \boldsymbol{i}_{\perp}^{-} = \|\tilde{p}_{\perp}\| \cos(2\omega t) \end{aligned} \tag{5}$$

where $\|\tilde{p}_{\parallel}\|$ and $\|\tilde{p}_{\perp}\|$ are the power oscillation magnitudes.

III. Three-Dimensional Representation of Electrical Circuit Quantities

Firstly, let us consider the three-dimensional space as (x,y,z) such that x-axis is the horizontal, y-axis is the vertical and z-axis is the depth. The proposed representation of electrical circuit quantities in three-dimensional space also considers the voltage as the frame of reference, which is plotted in the xy-plane. So, the voltage and current expressions are defined as:

$$\begin{aligned} v(t) &= V \cdot \sin(\omega t) \\ i(t) &= I^{y} \sin(\omega t + \theta) \cos(\phi) + j I^{z} \sin(\omega t + \theta) \sin(\phi) \end{aligned} \tag{6}$$

in which θ is the reactive displacement angle visible in the xy-plane. Finally, I^y and I^z are the projections of the current magnitude onto the xy-plane and xz-plane, respectively, and ϕ is defined as the displacement angle between the voltage frame (i.e., xy-plane) and the current plane. Fig. 1 illustrates a half cycle waveform of voltage and current based on the proposed mathematical approach, in which the variable ϕ corresponds to the newly defined *unbalance displacement angle* and $\theta = 0$ for simplicity. Observe that the unbalance displacement angle provides a global information on the degree of load unbalance in the three-phase system, hence for a single-phase system, the unbalance displacement angle is zero, and then it is identical to the conventional electric voltage and current representation.

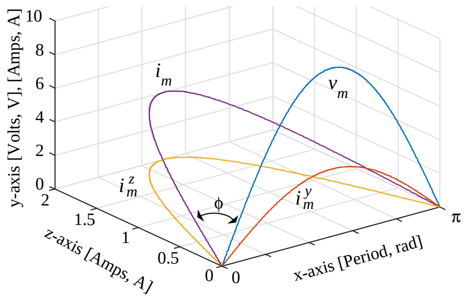

Fig. 1. Three-dimensional representation of instantaneous voltage and current.

Then, on the basis of this representation, the m-phase instantaneous currents may be mathematically expressed in the three-dimensional frame as:

$$i_m(t) = i_m^y(t) + j i_m^z(t) = \overbrace{i_{\parallel m}^{y}(t) + i_{\perp m}^{y}(t)}^{xy-plane} + j\overbrace{\left(i_{\parallel m}^{z}(t) + i_{\perp m}^{z}(t)\right)}^{xz-plane} \tag{7}$$

such that the i_m^y and i_m^z are the projection of the m-phase instantaneous currents onto the xy- and xz-plane, respectively.

The current terms can be defined considering the power terms of (4) and (5) as:

$$i_{\parallel m}^{y}(t) = \frac{P}{\|\boldsymbol{v}\|^2} v_m \qquad i_{\perp m}^{y}(t) = \frac{Q}{\|\widehat{\boldsymbol{v}}\|^2} \widehat{v}_m \tag{8.a}$$

$$i_{\parallel m}^{z}(t) = \frac{\|\tilde{p}_{\parallel}\|}{\|\boldsymbol{v}\|^2} v_m \qquad i_{\perp m}^{z}(t) = \frac{\|\tilde{p}_{\perp}\|}{\|\widehat{\boldsymbol{v}}\|^2} \widehat{v}_m \tag{8.b}$$

or in terms of θ_m and ϕ with respect to the peak value of measured currents, I_m.

$$\begin{aligned} i_{\parallel m}^{y}(t) &= I_m \cdot \cos(\theta_m) \cdot \cos(\phi) \cdot \sin(\omega t + \varphi_m) \\ i_{\perp m}^{y}(t) &= I_m \cdot \sin(\theta_m) \cdot \cos(\phi) \cdot \cos(\omega t + \varphi_m) \\ i_{\parallel m}^{z}(t) &= I_m \cdot \cos(\theta_m) \cdot \sin(\phi) \cdot \sin(\omega t + \varphi_m) \\ i_{\perp m}^{z}(t) &= I_m \cdot \sin(\theta_m) \cdot \sin(\phi) \cdot \cos(\omega t + \varphi_m) \end{aligned} \tag{9}$$

in which φ_m is the three-phase displacement angle, i.e., following the positive voltage sequence: 0º for phase a, -120º for phase b, and 120º for phase c, if voltages are symmetrically shifted.

Then, the power terms could be defined on the basis of the displacement angle caused by reactive power (θ) and the proposed displacement angle caused by load unbalance (ϕ).

The apparent power is calculated as usual:

$$A = \boldsymbol{V} \cdot \boldsymbol{I} \tag{10}$$

and the other four power terms as:

$$\begin{aligned} P &= \boldsymbol{V} \cdot \boldsymbol{I} \cdot \cos(\phi) \cdot \cos(\theta) \\ Q &= \boldsymbol{V} \cdot \boldsymbol{I} \cdot \cos(\phi) \cdot \sin(\theta) \\ \|\tilde{p}_{\parallel}\| &= \boldsymbol{V} \cdot \boldsymbol{I} \cdot \sin(\phi) \cdot \cos(\theta) \\ \|\tilde{p}_{\perp}\| &= \boldsymbol{V} \cdot \boldsymbol{I} \cdot \sin(\phi) \cdot \sin(\theta) \end{aligned} \tag{11}$$

such that $\boldsymbol{V}$ and $\boldsymbol{I}$ are the collective rms values of voltage and current, P is the average active power, while Q is the average reactive power. $\|\tilde{p}_{\parallel}\|$ and $\|\tilde{p}_{\perp}\|$ are the power terms related to in-phase and quadrature unbalance, respectively. Thus, an unbalanced power term could be used as $N_{\tilde{p}} = \sqrt{\|\tilde{p}_{\parallel}\|^2 + \|\tilde{p}_{\perp}\|^2}$.

A. Proposed Unbalance Displacement Factor

On the basis of the three-dimensional representation, the *reactive displacement factor* (RDF) could be calculated as $RDF = \cos(\theta)$. Note that this corresponds to the traditional displacement (power) factor and also coincides with the power factor, PF, under sinusoidal conditions. Here the *unbalance displacement factor* is introduced, which can be calculated as $UDF = \cos(\phi)$. Overall, the unbalance angle ϕ ranges from 0º to 90º, and consequently, UDF ranges from 1 (i.e., balanced system) to 0 (i.e., load currents with different sequence components from the voltages).

In the current literature, the definition of unbalanced power is not univocally defined [10], and then based on how the different power theories compute the unbalanced (N) power, in addition to active (P), reactive (Q) power terms, one could calculate the reactive (θ) and unbalance (ϕ) displacement angles as in (12) and (13), respectively. Equations (12) and (13) are generic and any power theory that identifies unbalanced power (N) can be used to identify N (some possible alternatives are recalled in Annex II). This would correspondingly change also the current projections on the xz-plane, given in (8.b).

Note that the reactive displacement angle can be applied phase by phase or based on the equivalent three-phase quantities, whereas the unbalance displacement angle is strictly a three-phase quantity and must be calculated based on the three-phase power terms.

$$\theta = \tan^{-1}\left(\frac{Q}{P}\right) \quad or \quad \theta_m = \tan^{-1}\left(\frac{Q_m}{P_m}\right), \tag{12}$$

$$\phi = \tan^{-1}\left(\frac{N}{\sqrt{P^2+Q^2}}\right). \tag{13}$$

IV. Illustrative Case Studies

Herein, some case studies are presented to supplement the best understanding of the three-dimensional representation of electrical circuit quantities. The graphic representation of the voltage and current waveforms is a tool that can be used as an unbalance quantifier irrespectively of the specific power theory used to deal with unbalanced systems. So, for the sake of explanation, two of the most recent power theories that define the required power terms are selected: *1*) the CPT [4] that defines active (P_{cpt}), reactive (Q_{cpt}), active unbalanced (N_a) and reactive unbalanced (N_r) powers. The unbalanced power is calculated as $N_{cpt} = \sqrt{N_a^2 + N_r^2}$; *2*) the CPC [7] that defines active (P_{cpc}), reactive (Q_{cpc}), negative-sequence unbalanced (D_u^n) and zero-sequence unbalanced (D_u^z) powers. The unbalanced power is calculated as $D_u = \sqrt{{D_u^n}^2 + {D_u^z}^2}$.

A. Asymmetrical Voltages with Balanced Resistive Load

Independently of the voltage condition, the frame of reference is the instantaneous voltages. Then, the sinusoidal and asymmetrical voltages ($V_a = 139.7\angle 0°$, $V_b = 127\angle -120°$, and $V_c = 114.3\angle 120°$) are applied to the (balanced, resistive) circuit of Fig. 2. The three-dimensional representation of the instantaneous three-phase voltage and current waveforms is shown in Fig. 3. As can be seen, the *unbalance displacement angle* is zero ($\phi = 0°$, UDF = 1), and the current waveforms are proportional to the voltage ones and lay on the same *xy*-plane. Moreover, both power theories result in the same portray.

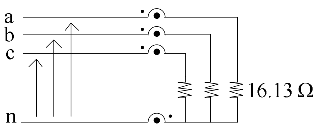

Fig. 2. Asymmetrical voltages with balanced resistive load (case study #1).

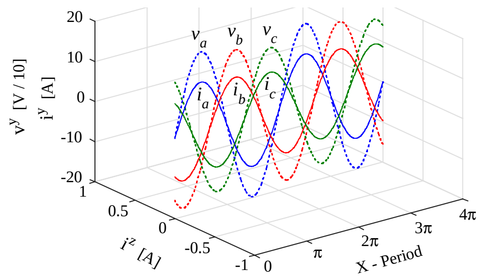

Fig. 3. Three-dimensional representation of the asymmetrical voltages with balanced resistive load (case study #1).

B. Positive-Sequence Voltages Supplying Negative-Sequence Currents

To highlight the meaning of the proposed *unbalance displacement angle* and the metric UDF, in Fig. 4 the loads (i.e., ideal current sources) draw only negative-sequence currents ($I_a = 6.82\angle 0°$, $I_b = 6.82\angle 120°$, and $I_c = 6.82\angle -120°$) from the supply voltage that has only positive-sequence components ($V_a = 127\angle 0°$, $V_b = 127\angle -120°$, and $V_c = 127\angle 120°$). According to (11-13) such condition represents a circuit with only unbalanced current terms circulating in the three-phase system, while active and reactive power are null. Fig. 5 shows the voltage and current waveforms represented in the three-dimensional frame, where the voltage waveforms are in the *xy*-plane and currents in the *xz*-plane. The UDF is zero as $\phi = 90°$.

C. Symmetrical Voltages with Unbalanced Load – Four-Wire and Three-Wire Circuits

In this section only the CPT was used. The instantaneous voltage and current waveforms of the circuit shown in Fig. 6 are displayed in the three-dimensional frame, as shown in Fig. 7. Note that the reactive displacement angle is zero, $\theta_m = 0$, and the unbalance displacement angle is $\phi = 35.28°$. This result indicates an unbalanced circuit but without reactive power circulation, as expected for three-phase four-wire circuits with resistive loads. On the other hand, if the load is purely inductive, it is expected zero active power, and only reactive and unbalanced power terms. The electric circuit and the waveforms are shown in Figs. 8 and 9, respectively. The values of angles (θ and ϕ), and RDF and UDF are shown in Table I.

If the same circuit of Fig. 6 is therefore re-drawn with three wires, as shown in Fig. 10, the phase voltage and line current waveforms in the three-dimensional representation are shown in Fig. 11. Note that the *m*-phase currents are not in-phase with their corresponding *m*-phase voltages ($\theta_a = -30°$ and $\theta_b = 30°$), despite the absence of energy storage elements. Such *m*-phase shift is caused by the reference point of voltage measurement in three-phase three-wire circuit, and it is quantitatively analyzed in Table I.

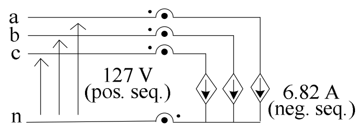

Fig. 4. Symmetrical voltages with negative-sequence currents (case study #2).

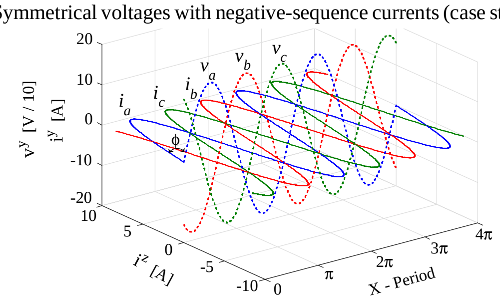

Fig. 5. Three-dimensional representation of the symmetrical voltages with negative sequence currents, four-wire circuit (case study #2).

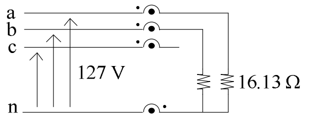

Fig. 6. Symmetrical voltages with unbalanced resistive load, four-wire circuit (case study #3).

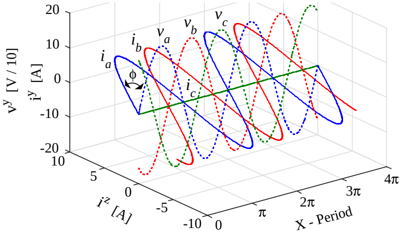

Fig. 7. Three-dimensional representation of the symmetrical voltages with unbalanced resistive load, four-wire circuit (case study #3).

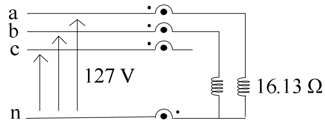

Fig. 8. Symmetrical voltages with unbalanced inductive load, four-wire circuit (case study #4).

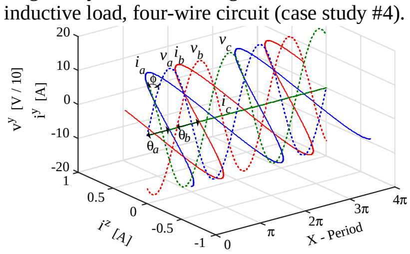

Fig. 9. Three-dimensional representation of the symmetrical voltages with unbalanced inductive load, four-wire circuit (case study #4).

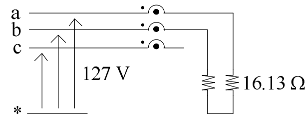

Fig. 10. Symmetrical voltages with unbalanced resistive load, three-wire circuit (case study #5).

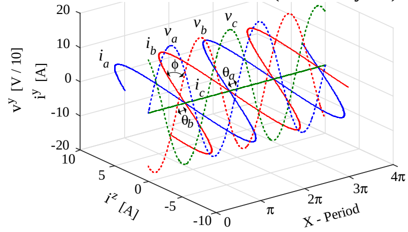

Fig. 11. Three-dimensional representation of the symmetrical voltages with unbalanced resistive load, three-wire circuit (case study #5).

TABLE I
Power terms values based on CPT [4] and power quality metrics.

Quantities	Case #1	Case #2	Case #3 (4-wire)	Case #4 (4-wire)	Case #5 (3-wire)
A_{cpt} [kVA]	3.02	2.60	2.45	2.45	2.12
P_{cpt} [kW]	3.02	0.00	2.00	0.00	1.50
Q_{cpt} [kVAr]	0.00	0.00	0.00	2.00	0.00
Na [kVA]	0.00	1.837	1.415	0.00	1.061
Nr [kVA]	0.00	1.837	0.00	1.415	1.061
N [kVA]	0.00	2.60	1.415	1.415	1.50
Pa_{cpt} [kW]	1.21	0.866	1.00	0.00	0.75
Pb_{cpt} [kW]	1.00	-0.433	1.00	0.00	0.75
Pc_{cpt} [kW]	0.81	-0.433	0.00	0.00	0.00
Qa_{cpt} [kVAr]	0.00	0.00	0.00	1.00	-0.433
Qb_{cpt} [kVAr]	0.00	0.750	0.00	1.00	0.433
Qc_{cpt} [kVAr]	0.00	-0.750	0.00	0.00	0.00
θ	0º	0º	0º	90º	0º
RDF	1.00	1.00	1.00	0.00	1.00
Proposed unbalance metrics					
ϕ	0º	90º	35.28º	35.28º	45º
UDF	1.00	0.00	0.816	0.816	0.707

Table I shows the values of power based on the CPT. Then, the proposed unbalance displacement angle, ϕ, and the conventional reactive displacement angle, θ, can be computed for the circuits of Figs. 2, 4, 6, 8 and 10. The values for the four-wire circuit are: $\phi = 35.26°$ and $\theta = 0°$, while for the three-wire circuit are: $\phi = 45°$ and $\theta = 0°$. These numbers result in RDF equals to zero, which means null reactive power circulation in both circuits; and UDF equals to 0.816 and 0.707, respectively, indicating that the three-wire circuit is more unbalanced than the four-wire one.

D. Symmetrical Voltages with Unbalanced RL Load

This case study is the same circuit used as example in [7]. Considering the circuit of Fig. 12, the corresponding values of power terms are shown in Table II applying both the selected power theories: CPT – [4] and CPC – [7]. Therefore, Fig. 13 shows the three-dimensional representation of the instantaneous three-phase symmetrical voltages and currents.

On the basis of Table II, the conventional reactive displacement angle, θ, and the proposed unbalance displacement angle, ϕ, can be calculated through the power theories using (12) and (13), respectively. Considering [4]: $\phi = 46.90°$ and $\theta = 19.06°$; while considering [7]: $\phi = 45.55°$ and $\theta = 18.43°$. Despite the difference in the numerical values, this proves that the mathematical tool proposed can be applied to different power theories.

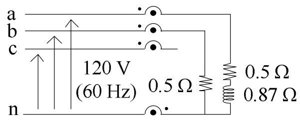

Fig. 12. Symmetrical voltages with unbalanced RL load (case study #6).

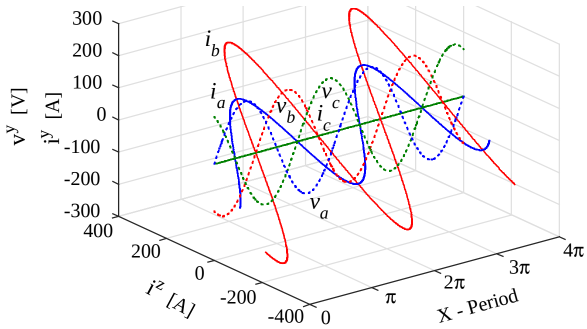

Fig. 13. Three-dimensional representation of the symmetrical voltages with unbalanced RL load (case study #6).

TABLE II
Power terms based on [4] and [7] for Fig. 12 under symmetrical voltages.

CPT – [4]		CPC – [7]	
A_{cpt} [kVA]	55.73	S_{cpc} [kVA]	54.18
P_{cpt} [kW]	36.0	P_{cpc} [kW]	36.0
Q_{cpt} [kVAr]	12.44	Q_{cpc} [kVAr]	12.0
N_a [kVA]	36.70	D^z_u [kVA]	38.0
N_r [kVA]	17.6	D^n_u [kVA]	7.2
N [kVA]	40.70	D_u [kVA]	38.68
θ	19.06º	θ	18.43º
RDF	0.945	*RDF*	0.949
Proposed unbalance metrics			
ϕ	46.90º	ϕ	45.54º
UDF	0.68	*UDF*	0.70

V. Conclusions

This paper proposed a mathematical expression of electrical circuit quantities in three-dimensional frame, (6), and a voltage and current waveforms representation in three-dimensional space, Fig. 1. Finally, on the basis of the three-dimensional representation, the unbalance displacement angle, ϕ, becomes visually evident, which can be used to define an unbalance displacement factor, UDF. The mathematical and graphic representations, as well as the unbalance displacement factor, were exemplified through numerical and illustrative case studies considering different power theories, i.e., CPT and CPC.

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ANNEX I

Herein the most common unbalance metrics used are shown. They are typically applied to voltages; however, the mathematical tool applies also to currents. Table A shows the main recommendation about unbalance, in which X represents the input variable, i.e., voltages or currents.

TABLE A
Main recommendation expression about unbalance.

recommendation	**expression**
IEC (*k*-factor)	$FD\% = \frac{X^-}{X^+}.100$
IEEE	$FD\% = \frac{3.(X^{max} - X^{min})}{X_a + X_b + X_c}.100$
NEMA -MG1	$FD\% = \frac{\Delta X}{X_{ave}}.100$
CIGRÉ-C04	$FD\% = \sqrt{\frac{1-\sqrt{3-6\beta}}{1+\sqrt{3-6\beta}}}.100$

Such that:

- X^- and X^+ are the negative- and positive-sequence components;
- X^{max} and X^{min} are the maximum and minimum values among the three-phases, X_a, X_b and X_c.
- ΔX and X_{ave} are the maximum deviation and average value, respectively.
- $\beta = \frac{|X_{ab}|^4+|X_{bc}|^4+|X_{ca}|^4}{(|X_{ab}|^2+|X_{bc}|^2+|X_{ca}|^2)^2}$

ANNEX II

Possible, alternative definitions of unbalanced power according to different power theories that can be used for corresponding quantification of the unbalance displacement angle, as shown in Table B.

TABLE B
Alternatives for calculating the unbalance displacement angle.

Based on CPT	$\phi = \tan^{-1}\left(\frac{N_{cpt}}{\sqrt{P^2+Q^2}}\right) = \tan^{-1}\left(\frac{\sqrt{N_a^2+N_r^2}}{\sqrt{P^2+Q^2}}\right)$
Based on CPC	$\phi = \tan^{-1}\left(\frac{D_u}{\sqrt{P^2+Q^2}}\right) = \tan^{-1}\left(\frac{\sqrt{D_u^{n\,2}+D_u^{z\,2}}}{\sqrt{P^2+Q^2}}\right)$
Based on (5)	$\phi = \tan^{-1}\left(\frac{N_{\tilde{p}}}{\sqrt{P^2+Q^2}}\right) = \tan^{-1}\left(\frac{\sqrt{\|\tilde{p}_{\parallel}\|^2+\|\tilde{p}_{\perp}\|^2}}{\sqrt{P^2+Q^2}}\right)$