Bull. Korean Math. Soc. **57** (2020), No. 4, pp. 865–872
https://doi.org/10.4134/BKMS.b190546
pISSN: 1015-8634 / eISSN: 2234-3016

DEPENDENT SUBSETS OF EMBEDDED PROJECTIVE VARIETIES

Edoardo Ballico

Abstract. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim(X)$. Let $\rho(X)''$ be the maximal integer such that every zero-dimensional scheme $Z \subset X$ smoothable in X is linearly independent. We prove that X is linearly normal if $\rho(X)'' \ge \lceil (r+2)/2 \rceil$ and that $\rho(X)'' < 2\lceil (r+1)/(n+1) \rceil$, unless either $n = r$ or X is a rational normal curve.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field with characteristic zero. Set $n := \dim X$. We recall that a zero-dimensional scheme $Z \subset X$ is said to be *smoothable in* X if it is a flat limit of a family of finite subsets of X with cardinality $\deg(Z)$ (see [14] for a discussion of it). If X is smooth (or if Z is contained in the smooth locus of X) Z is smoothable in X if and only if it is smoothable in $\mathbb{P}^r$ and the notion of smoothability in $\mathbb{P}^r$ does not depend on the choice of the embedding of Z in a projective space ([14, Proposition 2.1]). Let $\rho(X)$ (resp. $\rho(X)'$, resp. $\rho(X)''$) denote the maximal integer $t > 0$ such that each zero-dimensional scheme (resp. each finite set, resp. each zero-dimensional scheme smoothable in X) $Z \subset X$ with $\deg(Z) = t$ is linearly independent. Obviously $\rho(X) \le \rho(X)'' \le \rho(X)'$. Since X is embedded in $\mathbb{P}^r$, we have $\rho(X) \ge 2$.

The integer $\rho(X)'$ appears in very classical projective geometry papers and books (see [22, Ch. 8,9,10,12], [23, Ch. 27] and references therein for the case of a finite field). When X is a finite set of a finite dimensional vector space over a finite field, this integer is related to the minimum distance of the dual of the code obtained evaluating linear forms at the points of X ([18]). In the set-up of the X-rank described below the integer $\rho(X)'$ gives a uniqueness result (see Remarks 2.3 for details and references).

Received May 31, 2019; Revised September 9, 2019; Accepted September 19, 2019.
2010 *Mathematics Subject Classification.* 14N05.
Key words and phrases. Secant varietys X-ranks zero-dimensional schemes variety with only one ordinary double points OADP.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

For algebraic varieties the integer $\rho(X)$ is very natural. Zero-dimensional schemes appeared in several papers concerning the additive decomposition of polynomials and the equations of embedded projective varieties ([7, 14, 15]) and they allowed the introduction of a name, the cactus rank (of a point, of a homogeneous polynomial or a tensor) ([11, 13, 25]); in [24] it was called the scheme-rank. Over a finite field combining zero-dimensional schemes (all of them smoothable) with [18] is a standard feature to get bounds on the minimum distance and number of minimal weight codewords for an evaluation code coming from certain projective curves ([3, 8–10]).

To justify the integer $\rho(X)''$ one should justify the use of smoothable zero-dimensional schemes, not just of zero-dimensional schemes. Smoothable zero-dimensional subschemes of X and the integer $\rho(X)''$ arise in the study of secant varieties and border ranks described below (see [14, Proposition 2.5] for the case of additive decompositions of homogeneous polynomials). Computing $\rho(X)''$ gives a lower bound for $\rho(X)'$ and an upper bound for $\rho(X)$. Thus one can try to compute $\rho(X)''$, when computing $\rho(X)$ fails. At least after [14] each time a lower bound for $\rho(X)'$ is computed, it seems useful to ask ourself if the same proof works (with minimal modifications) for $\rho(X)$ or at least for $\rho(X)''$. Using only smoothable zero-dimensional schemes instead of arbitrary ones allows the check of a shorter list of schemes in several proofs ([4–7, 12, 16]).

For any $q \in \mathbb{P}^r$ the X-rank $r_X(q)$ of q is the minimal positive integer t such that $q \in \langle S\rangle$ for some finite subset $S \subset X$ with $\sharp(S) = t$, where $\langle\ \rangle$ denotes the linear span. For any positive integer t the t-secant variety $\sigma_t(X)$ of X is the closure in $\mathbb{P}^r$ of the union of all $\langle S\rangle$ with S a finite subset of X with cardinality t.

The *border X-rank* $b_X(q)$ of $q \in \mathbb{P}^r$ is the minimal integer k such that $q \in \sigma_k(X)$. The *generic rank* $r_{X,\mathrm{gen}}$ is the minimal integer $k > 0$ such that $\sigma_k(X) = \mathbb{P}^r$. There is a non-empty open subset $U \subset \mathbb{P}^r$ such that $r_X(q) = r_{X,\mathrm{gen}}$ for all $q \in U$.

In this paper we prove that if $\rho(X)''$ is large, then X is linearly normal and that $\rho(X)''$ cannot be very large for $n > 1$ (Theorem 1.1).

We prove the following results.

Proposition 1.1. *Assume that X is a curve and $\rho(X)'' \geq \lceil (r+2)/2\rceil$. Then X is linearly normal.*

Proposition 1.2. *Assume $n := \dim X \geq 2$, $r_{X,\mathrm{gen}} = \lceil (r+1)/(n+1)\rceil$ and $\rho(X)'' > \lceil (r+1)/(n+1)\rceil$. Then X is linearly normal.*

Theorem 1.3. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. We have $\rho(X)'' \geq 2\lceil (r+1)/(n+1)\rceil$ if and only if either $r = n$ (i.e., $X = \mathbb{P}^r$) or $n = 1$, r is odd and X is a rational normal curve.*

If $n = r$ we have $\rho(X)' = \rho(X) = 2$. If X is a rational normal curve we have $\rho(X) = \rho(X)' = r + 1$. This is the only case with $\rho(X)' = r + 1$ (Lemma 2.4). Theorem 1.3 implies that $\rho(X)'' < 2\lceil (r+1)/(n+1)\rceil$ if $(r+1)/(n+1) \notin \mathbb{Z}$.

The example of a general linear projection in $\mathbb{P}^4$ of the Veronese surface shows that in Proposition 1.2 it is not sufficient to assume that $\rho(X)'' \ge \lceil (r+2)/(n+1) \rceil$.

We point out that to get our results we only use a small family of zero-dimensional schemes, each of them with connected components of degree 1 or 2, but that this family contains a complete family covering X: each $p \in X$ is contained in some scheme Z of the family.

I want to thank the referee for several useful suggestions.

2. Preliminaries

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. For any $q \in \mathbb{P}^r$ let $\mathcal{S}(X,q)$ be the set of all $S \subset X$ such that $|S| = r_X(q)$ and $q \in \langle S \rangle$.

Remark 2.1. Fix $q \in \sigma_k(X)$. If $k \le \rho(X)''$ there is a zero-dimensional scheme $Z \subset X$ smoothable in X and such that $\deg(Z) \le k$ and $q \in \langle Z \rangle$ ([15, Lemma 2.6, Theorem 1.18] and [14, Proposition 2.5]).

Remark 2.2. Let $X \subset \mathbb{P}^r$ be a smooth variety with $\dim X \le 2$. Every zero-dimensional subcheme of X is smoothable ([19]) and hence $\rho(X)'' = \rho(X)$. Easy examples show that we may have $\rho(X) < \rho(X)'$ for a smooth curve (Example 4.2).

Remark 2.3 ([15, Theorem 1.17]). Fix $q \in \mathbb{P}^r$ and $A, B \in \mathcal{S}(X,q)$. Set $x := r_X(q)$ and assume $\rho(X)' \ge 2x$. Since $|A \cup B| \le 2x$, $A \cup B$ is linearly independent. Thus $A = B$ ([4, Lemma 1]). Hence $|\mathcal{S}(X,q)| = 1$.

In following extremal case we are able to use only $\rho(X)'$ instead of $\rho(X)''$.

Lemma 2.4. *The following conditions are equivalent*:

(1) X *is a rational normal curve*;
(2) $\rho(X)' = r + 2 - n$;
(3) $\rho(X)' \ge r + 2 - n$.

Proof. It is sufficient to prove that (3) implies (1).

First assume $n = 1$. Let $H \subset \mathbb{P}^r$ be a general linear hyperplane. Since $X \cap H$ is formed by $\deg(X)$ points, if $\rho(X)' > r$ we have $\deg(X) = r$ and hence X is a rational normal curve.

Now assume $n > 1$. Take a general linear subspace $V \subset \mathbb{P}^r$ with codimension $n-1$. The scheme $X \cap V$ is an integral curve spanning V. If $X \cap V$ is not a rational normal curve we have $\rho(X)' \le \rho(X \cap V)' \le r - n + 1$ by the case $n = 1$ just proved. Now assume that $X \cap V$ is a rational normal curve, i.e., assume $\deg(X) = r + 1 - n$. The classification of minimal degree subvarieties of projective spaces ([21, Proposition 3.10]) show that X contains lines (and hence $\rho(X)' = 3$) unless $r = 5$, $n = 2$ and X is the Veronese surface. If X is the Veronese surface of $\mathbb{P}^5$ we have $\rho(X)' = 3$, because no 3 points of X are coplanar, but X contains plane conics. □

3. The proofs

Proof of Proposition 1.1. Assume that X is not linearly normal. Thus there is a non-degenerate curve $Y \subset \mathbb{P}^{r+1}$ such that X is an isomorphic linear projection of Y from some $o \in \mathbb{P}^{r+1} \setminus Y$. Set $b := b_Y(o)$. Each secant variety of a curve has the expected dimension ([1, Remark 1.6]). Thus $r_{Y,\mathrm{gen}} = \lceil (r+2)/2 \rceil$. Hence $b \le \lceil (r+2)/2 \rceil$. Let $\ell : \mathbb{P}^{r+1} \setminus \{o\} \longrightarrow \mathbb{P}^r$ denote the linear projection from o. By assumption $o \notin Y$ and $\ell_{|Y}$ is an embedding with $\ell(Y) = X$. Let $W \subset Y$ be any zero-dimensional scheme. Since $\ell_{|Y} : Y \longrightarrow X$ is an isomorphism, W is smoothable in Y if and only if $\ell(W)$ is smoothable in X and any degree b smoothable zero-dimensional subcheme of X is the image of a unique degree b zero-dimensional subcheme of Y. Thus $\rho(Y)'' \ge \rho(X)''$. The image in $\mathbb{P}^r$ of a linear subspace $V \subset \mathbb{P}^{r+1}$ has either dimension $\dim V$ (case $o \notin V$) or dimension $\dim V - 1$ (case $o \in V$). Since $\rho(Y)'' \ge \rho(X)'' \ge \lceil (r+2)/2 \rceil = r_{Y,\mathrm{gen}}$ and $b \le r_{Y,\mathrm{gen}}$, there is a smoothable zero-dimensional scheme $W \subset Y$ such that $o \in \langle W \rangle$ and $\deg(W) = b$ (Remark 2.1). Since $\ell(W)$ is not linearly independent, we have $\rho(X)'' \le b-1$, a contradiction. □

Proof of Proposition 1.2. Assume that X is not linearly normal. Thus there is a non-degenerate variety $Y \subset \mathbb{P}^{r+1}$ such that X is an isomorphic linear projection of Y from some $o \in \mathbb{P}^{r+1} \setminus Y$. Set $b := b_Y(o)$ and $a := r_{X,\mathrm{gen}} = \lceil (r+1)/(n+1) \rceil$. Let $\ell : \mathbb{P}^{r+1} \setminus \{o\} \longrightarrow \mathbb{P}^r$ denote the linear projection from o. By assumption $o \notin Y$ and $\ell_{|Y}$ is an embedding with $\ell(Y) = X$. As in the proof of Proposition 1.1 we have $\rho(Y)'' \ge \rho(X)''$ and to get a contradiction it is sufficient to prove that $b \le \rho(X)''$. Assume $b > \rho(X)''$, i.e., assume $b \ge a+2$. Since $b > a$, we have $o \notin \sigma_a(Y)$. Hence $\ell_{|\sigma_a(Y)} : \sigma_a(Y) \longrightarrow \mathbb{P}^r$ is a finite map. Since $\ell(\sigma_a(Y)) = \sigma_a(X)$, we get $\dim \sigma_a(Y) = r$. Since $\dim \sigma_{a+1}(Y) > \dim \sigma_a(Y)$ ([1, Proposition 1.3]), we get $\sigma_{a+1}(Y) = \mathbb{P}^{r+1}$. Thus $b \le a+1$, a contradiction. □

Lemma 3.1. *Assume $\rho(X)' \ge 2\lceil (r+1)/(n+1) \rceil$. Then X is not defective, $r+1 \equiv 0 \pmod{n+1}$ and for a general $q \in \mathbb{P}^r$ we have $|\mathcal{S}(X,q)| = 1$.*

Proof. Set $a := \lceil (r+1)/(n+1) \rceil$. Fix any $q \in \mathbb{P}^r$ such that $r_X(q) \le a$. Remark 2.3 gives $|\mathcal{S}(X,q)| = 1$ if $\rho(X)' \ge 2r_X(q)$. In particular $|\mathcal{S}(X,q)| = 1$ for a general $q \in \sigma_a(X)$. Thus a dimensional count shows that $\dim \sigma_a(X) = a(n+1) - 1$. Since $\dim \sigma_a(X) \le r$, we get $a \in \mathbb{Z}$, $\mathbb{P}^r = \sigma_a(X)$ and that X is not defective. Since $\mathbb{P}^r = \sigma_a(X)$, we have $r_X(q) = a$ for a general $q \in \mathbb{P}^r$. Hence $|\mathcal{S}(X,q)| = 1$ for a general $q \in \mathbb{P}^r$. □

The (smooth) n-dimensional varieties $X \subset \mathbb{P}^{2n+1}$ such that $\sigma_2(X) = \mathbb{P}^{2n+1}$ and $|\mathcal{S}(X,q)| = 1$ are classically called OADP (or varieties with only one apparent double point), because projecting them from a general point of X one gets a variety with a unique singular point ([17]). They are always linearly normal ([17, Remark 1.2]). In [17] there are also older references and the classification of the smooth ones with dimension up to 3 ([26], [17, Theorem 7.1]). Thus the thesis of Lemma 3.1 is a generalization of this concept to the case in which $(r+1)/(n+1)$ is an integer > 2. But the assumption " $\rho(X)' \ge 2\lceil (r+1)/(n+1)\rceil$ " of the lemma is too strong to be interesting for the classification of extremal varieties. Just assuming $\rho(X)' > 2$ excludes all X containing lines and hence all smooth OADP's of dimension 2 and 3.

Corollary 3.2. *Assume $n := \dim X \ge 2$, $\rho(X)'' > \lceil (r+1)/(n+1)\rceil$ and $\rho(X)' \ge 2\lceil (r+1)/(n+1)\rceil$. Then X is linearly normal, non-defective, $r+1 \equiv 0 \pmod{n+1}$ and $|\mathcal{S}(X,q)| = 1$ for a general $q \in \mathbb{P}^r$.*

Proof of Corollary 3.2. Apply Lemma 3.1 and Proposition 1.2. □

Proof of Theorem 1.3. Assume the existence of X with $\rho(X)'' \ge 2\lceil (r+1)/(n+1)\rceil$. We may assume $n < r$, i.e., $X \ne \mathbb{P}^r$.

First assume $n = 1$. Lemma 2.4 gives that X is a rational normal curve, that r is odd and that $\rho(X) = \rho(X)' = r+1$.

Now assume $n \ge 2$. By Lemma 3.1, $a := (r+1)/(n+1)$ is an integer. Since $r > n$, we have $a \ge 2$. Fix a general $S \subset X$ such that $|S| = a-1$. Since S is general, each $p \in S$ is a smooth point of X. We saw in the proof of Proposition 1.2 that $V := \langle \cup_{p\in S} T_pX\rangle$ has dimension $(a-1)(n+1)-1$. Fix $o \in X \setminus S$.

Claim 1. *$o \notin V$.*

Proof of Claim 1. Assume $o \in V$. We saw in the proof of Proposition 1.2 that there are connected degree 2 zero-dimensional schemes $v_p \subset X$ such that $(v_p)_{\mathrm{red}} = \{p\}$ and $o \in \langle Z\rangle$, where $Z := \cup_{p\in S} v_p$. Since $o \notin S$, we have $o \nsubseteq Z$. Thus the scheme $Z \cup \{o\}$ is linearly dependent. Since $\deg(Z \cup \{o\}) = 2a-1 < \rho(X)''$ and $Z \cup \{o\}$ is smoothable, we got a contradiction. □

Let $\ell : \mathbb{P}^r \setminus V \longrightarrow \mathbb{P}^n$ denote the linear projection from V. By Claim 1 we have $S = X \cap V$ and hence $\mu = \ell_{|X\setminus S} : X \setminus S \longrightarrow \mathbb{P}^n$ is a morphism. Fix $o \in X \setminus S$ and assume the existence of $o' \in X \setminus S$ such that $o \ne o'$ and $\mu(o) = \mu(o')$. Thus $o' \in \langle \{o\} \cup Z\rangle$. Hence $\{o, o'\} \cup Z$ is linearly dependent. The zero-dimensional scheme $\{o, o'\} \cup Z$ is smoothable and it has degree $2a \le \rho(X)''$, a contradiction.

Thus $\mu : X \setminus S \longrightarrow \mathbb{P}^n$ is an injective morphism between two quasi-projective varieties. Since $\mathbb{P}^n$ is smooth (it would be sufficient to assume that the target, X, is normal or even less (weakly normal)) and we are in characteristic zero, μ is an open map which is an isomorphism onto its image ([20]). Since X is smooth at each point of S, X is smooth. Since S is finite, the étale cohomology of $X \setminus S$ in dimension $n-1$ shows that $\mathbb{P}^n \setminus \mu(X \setminus S)$ is finite with cardinality $\sharp(S)$. Thus μ extends to an isomorphism $u : X \longrightarrow \mathbb{P}^n$. However, by the definition of linear projection from $V = \langle \cup_{p\in S} T_pX\rangle$, the linear independence of the linear spaces T_p, $x \in S$, and the smoothness of X, μ extends to some non-empty open subset of the exceptional divisor of the blowing-up of X at the points of S. Since $S \ne \emptyset$, this is absurd. □

4. Elementary examples

By Proposition 1.1 to complete the picture for curves we need to describe the linearly normal curves with very high $\rho(X)'$, $\rho(X)$ and $\rho(X)''$. We also gives examples of smooth curves X with prescribed $\rho(X)$ or prescribed $\rho(X)'$.

Remark 4.1. Let X be an integral projective curve. To compute $\rho(X)''$ we recall that every Cartier divisor of X is smoothable. Let $\mathcal{F}$ be any torsion free sheaf of X. Duality gives $h^1(\mathcal{F}) = \dim \mathrm{Hom}(\mathcal{F}, \omega_X)$, ([2, 1.1 at p. 5]). Thus for any zero-dimensional scheme $Z \subset X$ we have $h^1(\mathcal{I}_Z(1)) = \dim \mathrm{Hom}(\mathcal{I}_Z, \omega_X(1))$. If $d \geq 3g-2$ we have $\rho(X) = d-2g+2$. We have $\rho(X)' = d-2g+2$ if and only if X is Gorenstein, i.e., ω_X is locally free. For lower d the integers $\rho(X)$, $\rho(X)'$ and $\rho(X)''$ depends both from the Brill-Noether theory of the special line bundles on X and the choice of the very ample line bundle $\mathcal{O}_X(1)$, not just the integers d and g.

Example 4.2. Fix integers r, a such that $2 \leq a \leq r+1$. Here we prove the existence of a smooth and non-degenerate curve $X \subset \mathbb{P}^r$ such that $\rho(X)' = \rho(X) = a$ and, if $a \leq r-1$ and $2a \geq r+2$, another example $\tilde{X}$ with $\rho(\tilde{X})' > \rho(\tilde{X}) = a$. If $a = r+1$ we know that X is a rational normal curve. The case $g = 1$, $d = r+1$ covers the the case $a = r$. Now assume $2 \leq a \leq r-1$.

(a) We first cover the case $2a \leq r+1$. In this range we construct a smooth rational curve X with $\rho(X) = \rho(X)' = a$, but of course X is not linearly normal. Let $Y \subset \mathbb{P}^{r+1}$ be a rational normal curve. Fix a set $S \subset X$ such that $|S| = a+1$ and take any $o \in \langle S\rangle$ such that $o \notin \langle S'\rangle$ for any $S' \subsetneq S$. Let $\ell : \mathbb{P}^{r+1} \setminus \{o\} \longrightarrow \mathbb{P}^r$ denote the linear projection from o. Since $a \geq 2$ and $\rho(Y) = r+2$, we have $o \notin Y$. Hence $\ell_{|Y}$ is a morphism. Set $X := \ell(Y)$.

Claim 1. $\ell_{|Y}$ is an embedding.

Proof of Claim 1. It is sufficient to prove that for any zero-dimensional scheme $A \subset Y$ with $\deg(A) \leq 2$ we have $o \notin \langle A\rangle$. Assume the existence of a zero-dimensional scheme $A \subset Y$ with $\deg(A) \leq 2$ and $o \in \langle A\rangle$. Since $o \notin Y$, we have $\deg(A) = 2$. Since $o \notin \langle S'\rangle$ for any $S' \subsetneq S$ and $|S| = a+1 > 2$, we have $A \nsubseteq S$. Since $o \in \langle A\rangle \cap \langle S\rangle$, $A \cup S$ is linearly dependent. Since $\deg(A \cup S) \leq a+3$ and $\rho(Y) = r+1$, we get a contradiction. □

By Claim 1 X is a smooth rational curve and $\deg(X) = r+1$.

Claim 2. We have $\rho(X) = \rho(X)' = a$.

Proof of Claim 2. Since $\ell_{|X}$ is an embedding, we have $|\ell(S)| = a+1$. Since $o \in \langle S\rangle$, $\ell(S)$ is linear dependent and hence $\rho(X)' \leq a$. Assume $\rho(X)' < a$ and take a zero-dimensional scheme $Z \subset X$ such that $\deg(Z) \leq a$ and Z is linearly dependent. Let $W \subset Y$ be the only scheme such that $\ell(W) = Z$. Since Z is linearly dependent, we have $o \in \langle W\rangle$. Since $\deg(W) \leq a$ and $o \notin \langle S'\rangle$ for any $S' \subsetneq S$, we have $W \nsubseteq S$. Thus $S \cup W$ is linearly dependent. Hence $2a+1 \geq r+2$, a contradiction. □

(b) Now assume $2 \le a \le r-1$ and $2a \ge r+2$. Set $g := r+1-a$. Fix a smooth curve C of genus g and a zero-dimensional scheme $A \subset X$ such that $\deg(A) = a+1$. Since $\deg(\omega_C(A)) = 2g+a+1-2 \ge 2g+1$, $\omega_C(A)$ is very ample. By Riemann-Roch we have $h^0(\omega_C(A)) = g+a = r+1$. Let $f : C \longrightarrow \mathbb{P}^r$, be the embedding induced by $|\omega_X(A)|$. Set $X := f(C)$ and $Z := f(A)$. By Rieman-Roch Z is linearly dependent and Z is the only linearly dependent zero-dimensional scheme $W \subset X$ such that $\deg(W) \le a+1$. Thus $\rho(X) = a$. We have $\rho(X)' = a$ if and only if A is a reduced set. If A is not reduced we get the promised curve $\tilde{X}$.

References

[1] B. Adlandsvik, *Joins and higher secant varieties*, Math. Scand. **61** (1987), no. 2, 213–222. https://doi.org/10.7146/math.scand.a-12200

[2] A. Altman and S. Kleiman, *Introduction to Grothendieck Duality Theory*, Lecture Notes in Mathematics, Vol. 146, Springer-Verlag, Berlin, 1970.

[3] E. Ballico, *Generalized Hamming weights of codes obtained from smooth plane curves*, Afr. Mat. **24** (2013), no. 4, 565–569. https://doi.org/10.1007/s13370-012-0082-x

[4] E. Ballico and A. Bernardi, *Decomposition of homogeneous polynomials with low rank*, Math. Z. **271** (2012), no. 3-4, 1141–1149. https://doi.org/10.1007/s00209-011-0907-6

[5] ______, *Tensor ranks on tangent developable of Segre varieties*, Linear Multilinear Algebra **61** (2013), no. 7, 881–894. https://doi.org/10.1080/03081087.2012.716430

[6] ______, *Stratification of the fourth secant variety of Veronese varieties via the symmetric rank*, Adv. Pure Appl. Math. **4** (2013), no. 2, 215–250. https://doi.org/10.1515/apam-2013-0015

[7] E. Ballico, A. Bernardi, M. Christandl, and F. Gesmundo, *On the partially symmetric rank of tensor products of W-states and other symmetric tensors*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2019), no. 1, 93–124. https://doi.org/10.4171/RLM/837

[8] E. Ballico and A. Ravagnani, *The dual geometry of Hermitian two-point codes*, Discrete Math. **313** (2013), no. 23, 2687–2695. https://doi.org/10.1016/j.disc.2013.08.014

[9] ______, *On the geometry of Hermitian one-point codes*, J. Algebra **397** (2014), 499–514. https://doi.org/10.1016/j.jalgebra.2013.08.038

[10] ______, *A zero-dimensional approach to Hermitian codes*, J. Pure Appl. Algebra **219** (2015), no. 4, 1031–1044. https://doi.org/10.1016/j.jpaa.2014.05.031

[11] A. Bernardi, J. Brachat, and B. Mourrain, *A comparison of different notions of ranks of symmetric tensors*, Linear Algebra Appl. **460** (2014), 205–230. https://doi.org/10.1016/j.laa.2014.07.036

[12] A. Bernardi, A. Gimigliano, and M. Idà, *Computing symmetric rank for symmetric tensors*, J. Symbolic Comput. **46** (2011), no. 1, 34–53. https://doi.org/10.1016/j.jsc.2010.08.001

[13] A. Bernardi and K. Ranestad, *On the cactus rank of cubics forms*, J. Symbolic Comput. **50** (2013), 291–297. https://doi.org/10.1016/j.jsc.2012.08.001

[14] W. Buczyńska and J. Buczyński, *Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes*, J. Algebraic Geom. **23** (2014), no. 1, 63–90. https://doi.org/10.1090/S1056-3911-2013-00595-0

[15] J. Buczyński, A. Ginensky, and J. M. Landsberg, *Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture*, J. Lond. Math. Soc. (2) **88** (2013), no. 1, 1–24. https://doi.org/10.1112/jlms/jds073

[16] J. Buczyński and J. M. Landsberg, *Ranks of tensors and a generalization of secant varieties*, Linear Algebra Appl. **438** (2013), no. 2, 668–689. https://doi.org/10.1016/j.laa.2012.05.001

[17] C. Ciliberto, M. Mella, and F. Russo, *Varieties with one apparent double point*, J. Algebraic Geom. **13** (2004), no. 3, 475–512. https://doi.org/10.1090/S1056-3911-03-00355-2

[18] A. Couvreur, *The dual minimum distance of arbitrary-dimensional algebraic-geometric codes*, J. Algebra **350** (2012), 84–107. https://doi.org/10.1016/j.jalgebra.2011.09.030

[19] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math **90** (1968), 511–521. https://doi.org/10.2307/2373541

[20] S. Greco and C. Traverso, *On seminormal schemes*, Compositio Math. **40** (1980), no. 3, 325–365.

[21] J. Harris, *Curves in projective space*, Séminaire de Mathématiques Supérieures, **85**, Presses de l'Université de Montréal, Montreal, QC, 1982.

[22] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, The Clarendon Press, Oxford University Press, New York, 1979.

[23] J. W. P. Hirschfeld and J. A. Thas, *General Galois Geometries*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991.

[24] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, **1721**, Springer-Verlag, Berlin, 1999. https://doi.org/10.1007/BFb0093426

[25] K. Ranestad and F.-O. Schreyer, *On the rank of a symmetric form*, J. Algebra **346** (2011), 340–342. https://doi.org/10.1016/j.jalgebra.2011.07.032

[26] F. Russo, *On a theorem of Severi*, Math. Ann. **316** (2000), no. 1, 1–17. https://doi.org/10.1007/s002080050001

Edoardo Ballico
Department of Mathematics
University of Trento
38123 Povo (TN), Italy
Email address: ballico@science.unitn.it