On the finiteness and periodicity of the p–adic Jacobi–Perron algorithm

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Abstract

Multidimensional continued fractions (MCFs) were introduced by Jacobi and Perron in order to obtain periodic representations for algebraic irrationals, as it is for continued fractions and quadratic irrationals. Since continued fractions have been studied in the field of p–adic numbers $\mathbb{Q}_p$, also MCFs have been recently introduced in $\mathbb{Q}_p$ together to a p–adic Jacobi–Perron algorithm. In this paper, we address the study of two main features of this algorithm, i.e., finiteness and periodicity. In particular, regarding the finiteness of the p–adic Jacobi–Perron algorithm our results are obtained by exploiting properties of some auxiliary integer sequences. Moreover, it is known that a finite p–adic MCF represents $\mathbb{Q}$–linearly dependent numbers. We see that the viceversa is not always true and we prove that in this case infinite partial quotients of the MCF have p–adic valuations equal to -1. Finally, we show that a periodic MCF of dimension m converges to algebraic irrationals of degree less or equal than $m + 1$ and for the case $m = 2$ we are able to give some more detailed results.

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1   Introduction

Multidimensional continued fractions (MCFs) were introduced by Jacobi [20] in order to answer to a question posed by Hermite [19], namely the existence of an algorithm defined over the real numbers that becomes eventually periodic when it processes algebraic irrationalities. In other words, Hermite asked for a generalization of the classical continued fraction algorithm that produces a periodic expansion if and only if the input is a quadratic irrational. The Jacobi algorithm deals with cubic irrationals and it was generalized to higher dimensions by Perron [28]. However, the Jacobi–Perron algorithm does not solve the Hermite problem, since it has never been proved that it becomes eventually periodic when processing algebraic irrationals. Many studies have been conducted on MCFs and their modifications, see, e.g., [1], [2], [5], [13], [14], [18], [21], [24], [25], [32], [33], [34].

In the 1970s, some authors started to study one–dimensional continued fractions over the p–adic numbers [7], [30], [31]. From these studies, it appeared difficult to find an algorithm working on the p–adic numbers that produces continued fractions having the same properties holding true over the real numbers (regarding approximation, finiteness and periodicity). In particular, no algorithm which provides a periodic expansion for all quadratic irrationalities has been found. Continued fractions over the p–adic numbers have been also studied recently in several works, like [3], [4], [8], [9], [17], [22], [23], [27], [29], [35], [36].

Motivated by the above researches, in [26], the authors started the study of MCFs in $\mathbb{Q}_p$, providing some results about convergence and finiteness. In particular, they gave a sufficient condition on the partial quotients of a MCF that ensures the convergence in $\mathbb{Q}_p$. Moreover, they presented an algorithm that terminates in a finite number of steps when rational numbers are processed. The scope of that work was introducing the subject and providing some general properties; the terminating input of this algorithm was not fully characterized and the periodicity properties were not studied at all. This paper represents a continuation of the previous work, extending the investigation in these two directions. In particular, in Section 2, we fix the notation

and we show some properties that can also be of general interest for MCFs. Section 3 is devoted to the finiteness of the p-adic Jacobi–Perron algorithm, providing some results that improves a previous work [26] and showing also some differences with the real case. In $\mathbb{R}$, it is known that the Jacobi–Perron of dimension 2 detects rational dependence, i.e., it terminates in a finite number of steps if and only if it processes rational linearly dependent inputs. On the contrary we show that this is not always true in $\mathbb{Q}_p$ and we also prove that in this case infinite partial quotients of the MCF have p–adic valuations equal to -1. Moreover, we give a condition that ensures the finiteness of the p–adic Jacobi–Perron algorithm in any dimension in terms of the p–adic valuation of the partial quotient. In Section 4, we study the periodicity of MCF in $\mathbb{Q}_p$. Specifically, we introduce the characteristic polynomial related to a purely periodic p–adic MCF and we see that, as in the real case, it admits a p–adic dominant root which generates a field containing the limits of the MCF. Consequently, we see that a periodic MCF of dimension m converges to algebraic irrationalities of degree less or equal than $m + 1$ as in the real case. A further investigation on the characteristic polynomial allows to characterize some cases where the degree is maximum. We conclude our work with a conjecture which, if proved to be true, would give a characterization of the MCFs arising by applying the p-adic Jacobi-Perron algorithm to m-tuples consisting of $\mathbb{Q}$-linear dependent numbers.

2 Preliminaries and notation

The classical Jacobi–Perron algorithm processes a m–tuple of real numbers $\boldsymbol{\alpha}_0 = (\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$ and represents them by means of a m–tuple of integer sequences $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) = ((a_n^{(1)})_{n \geq 0}, \ldots, (a_n^{(m)})_{n \geq 0})$ (finite or infinite) determined by the following iterative equations:

$$\begin{cases} a_n^{(i)} = [\alpha_n^{(i)}], \quad i = 1, \ldots, m, \\ \alpha_{n+1}^{(1)} = \dfrac{1}{\alpha_n^{(m)} - a_n^{(m)}}, \\ \alpha_{n+1}^{(i)} = \dfrac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}}, \quad i = 2, \ldots, m, \end{cases} \qquad n = 0, 1, 2, \ldots$$

The integer numbers $a_n^{(i)}$ and the real numbers $\alpha_n^{(i)}$, for $i = 1, \ldots m$ and $n = 0, 1, \ldots$, are called *partial quotients* and *complete quotients*, respectively. The sequences of the partial quotients represent the starting vector $\boldsymbol{\alpha}_0$ by means of the equations

$$\begin{cases} \alpha_n^{(i-1)} = a_n^{(i-1)} + \dfrac{\alpha_{n+1}^{(i)}}{\alpha_{n+1}^{(1)}}, \quad i = 2, \ldots, m \\ \alpha_n^{(m)} = a_n^{(m)} + \dfrac{1}{\alpha_{n+1}^{(1)}} \end{cases} \qquad n = 0, 1, 2, \ldots \qquad (1)$$

which produce objects that generalize the classical continued fractions and are usually called *multidimensional continued fractions* (MCFs).

The Jacobi–Perron algorithm has been translated into the p–adic field in [26], using the function s defined below that will play the role of the floor function. We define the set

$$\mathcal{Y} = \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right).$$

Definition 1. *The* Browkin s-function $s : \mathbb{Q}_p \longrightarrow \mathcal{Y}$ *is defined by*

$$s(\alpha) = \sum_{j=k}^{0} x_j p^j,$$

for every $\alpha \in \mathbb{Q}_p$ written as $\alpha = \sum_{j=k}^{\infty} x_j p^j$, with $k, x_j \in \mathbb{Z}$ and $x_j \in \left(-\frac{p}{2}, \frac{p}{2}\right)$.

Hence, the p–adic Jacobi–Perron algorithm processes a m–tuple of p–adic numbers $\boldsymbol{\alpha}_0 =$

$(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$ by the following iterative equations

$$
\begin{cases}
a_n^{(i)} = s(\alpha_n^{(i)}) \\
\alpha_{n+1}^{(1)} = \dfrac{1}{\alpha_n^{(m)} - a_n^{(m)}} \\
\alpha_{n+1}^{(i)} = \alpha_{n+1}^{(1)} \cdot (\alpha_n^{(i-1)} - a_n^{(i-1)}) = \dfrac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}}, \quad i = 2, \ldots, m
\end{cases}
\tag{2}
$$

for $n = 0, 1, 2, \ldots$, which define a p–adic MCF $[(a_0^{(1)}, a_1^{(1)}, \ldots), \ldots, (a_0^{(m)}, a_1^{(m)}, \ldots)]$ representing the starting m–tuple $\boldsymbol{\alpha}_0$ in the following way:

$$
\alpha_n^{(i)} = a_n^{(i)} + \frac{\alpha_{n+1}^{(i+1)}}{\alpha_{n+1}^{(1)}}, \quad \alpha_n^{(m)} = a_n^{(m)} + \frac{1}{\alpha_{n+1}^{(1)}}
$$

for $i = 1, \ldots, m - 1$ and any $n \geq 0$. The partial quotients satisfy the following conditions:

$$
\begin{cases}
|a_n^{(1)}| > 1 \\
|a_n^{(i)}| < |a_n^{(1)}|, \quad i = 2, \ldots, m
\end{cases}
\tag{3}
$$

for any $n \geq 1$, where in the following $|\cdot|$ will always denote the p–adic norm. Moreover, for any $n \geq 1$, we have

$$
|a_n^{(1)}| = |\alpha_n^{(1)}|, \text{ and for } i = 2, \ldots, m
$$
$$
|a_n^{(i)}| =
\begin{cases}
|\alpha_n^{(i)}| \text{ if } |\alpha_n^{(i)}| \geq 1 \\
0 \text{ if } |\alpha_n^{(i)}| < 1
\end{cases}
\tag{4}
$$
$$
|\alpha_n^{(i)}| < |\alpha_n^{(1)}| \quad .
$$

Remark 1. *In [26], the authors showed that equations (3) ensure the convergence of a MCF in $\mathbb{Q}_p$, i.e., given a sequence of partial quotients satisfying (3) (even if they are not obtained by a specific algorithm), then the corresponding MCF converges to a m–tuple of p–adic numbers.*

Similarly to the real case, we have the *n–th convergents* of a multidimensional continued fraction defined by

$$
Q_n^{(i)} = \frac{A_n^{(i)}}{A_n^{(m+1)}},
$$

for $i = 1, \ldots, m$ and $n \in \mathbb{N}$, where

$$
A_{-j}^{(i)} = \delta_{ij}, \quad A_0^{(i)} = a_0^{(i)}, \quad A_n^{(i)} = \sum_{j=1}^{m} a_n^{(j)} A_{n-j}^{(i)} + A_{n-m-1}^{(i)}
\tag{5}
$$

for $i = 1, \ldots, m+1$, $j = 1, \ldots, m$ and any $n \geq 1$, where δ_{ij} is the Kronecker delta. It can be proved by induction that for every $n \geq 1$ and $i = 1, \ldots, m$, we have

$$
\alpha_0^{(i)} = \frac{\alpha_n^{(1)} A_{n-1}^{(i)} + \alpha_n^{(2)} A_{n-2}^{(i)} + \ldots + \alpha_n^{(m+1)} A_{n-m-1}^{(i)}}{\alpha_n^{(1)} A_{n-1}^{(m+1)} + \alpha_n^{(2)} A_{n-2}^{(m+1)} + \ldots + \alpha_n^{(m+1)} A_{n-m-1}^{(m+1)}}
\tag{6}
$$

We can also use the following matrices for evaluating numerators and denominators of the convergents:

$$
\mathcal{A}_n =
\begin{pmatrix}
a_n^{(1)} & 1 & 0 & \ldots & 0 \\
a_n^{(2)} & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_n^{(m)} & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\tag{7}
$$

for any $n \geq 0$. Indeed, if we put

$$
\mathcal{B}_n = \begin{pmatrix} A_n^{(1)} & A_{n-1}^{(1)} & \cdots & A_{n-m}^{(1)} \\ A_n^{(2)} & A_{n-1}^{(2)} & \cdots & A_{n-m}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ A_n^{(m+1)} & A_{n-1}^{(m+1)} & \cdots & A_{n-m}^{(m+1)} \end{pmatrix}
$$

we have

$$
\mathcal{B}_n = \mathcal{B}_{n-1}\mathcal{A}_n = \mathcal{A}_0\mathcal{A}_1 \ldots \mathcal{A}_n, \quad \det \mathcal{B}_n = (-1)^{m(n+1)}.
$$

We also recall some properties proved in [26].

Proposition 1. *With the notation above, we have*

$$
|A_n^{(m+1)}| = \prod_{h=1}^{n} |a_h^{(1)}|.
$$

for any $n \geq 1$.

Proposition 2. *Given the sequences $(V_n)_{n \geq -m}$, $i = 1, \ldots, m$, defined by*

$$
V_n^{(i)} = A_n^{(i)} - \alpha_0^{(i)} A_n^{(m+1)}
$$

we have

1. $\displaystyle\lim_{n \to +\infty} |V_n^{(i)}| = 0$

2. $V_n^{(i)} = \sum_{j=1}^{m+1} a_n^{(j)} V_{n-j}^{(i)}$

3. $\sum_{j=1}^{m+1} \alpha_n^{(j)} V_{n-j}^{(i)} = 0.$

Finally, we prove the following propositions that will be useful in the next sections.

Proposition 3. *For $n \geq 1$, we have*

$$
\sum_{i=1}^{m+1} \alpha_n^{(i)} A_{n-i}^{(m+1)} = \prod_{j=1}^{n} \alpha_j^{(1)}.
$$

Proof. We proceed by induction on n. If $n = 1$ the left-hand side is equal to $\alpha_1^{(1)} A_0^{(m+1)} = \alpha_1^{(1)}$. For $n > 1$ we can use the inductive hypothesis and write

$$
\begin{aligned}
\prod_{j=1}^{n-1} \alpha_j^{(1)} &= \sum_{i=1}^{m+1} \alpha_{n-1}^{(i)} A_{n-1-i}^{(m+1)} \\
&= A_{n-m-2}^{(m+1)} + \sum_{i=1}^{m} \alpha_{n-1}^{(i)} A_{n-1-i}^{(m+1)} \\
&= A_{n-m-2}^{(m+1)} + \sum_{i=1}^{m} \left(a_{n-1}^{(i)} + \frac{\alpha_n^{(i+1)}}{\alpha_n^{(1)}} \right) A_{n-1-i}^{(m+1)} \\
&= \left(\sum_{i=1}^{m} a_{n-1}^{(i)} A_{n-1-i}^{(m+1)} + A_{n-m-2}^{(m+1)} \right) + \sum_{i=1}^{m} \left(\frac{\alpha_n^{(i+1)}}{\alpha_n^{(1)}} \right) A_{n-1-i}^{(m+1)} \\
&= A_{n-1}^{(m+1)} + \sum_{i=1}^{m} \left(\frac{\alpha_n^{(i+1)}}{\alpha_n^{(1)}} \right) A_{n-1-i}^{(m+1)} \\
&= \frac{1}{\alpha_n^{(1)}} \left(\alpha_n^{(1)} A_{n-1}^{(m+1)} + \sum_{i=1}^{m} \alpha_n^{(i+1)} A_{n-1-i}^{(m+1)} \right) \\
&= \frac{1}{\alpha_n^{(1)}} \sum_{i=1}^{m+1} \alpha_n^{(i)} A_{n-i}^{(m+1)}.
\end{aligned}
$$

proving the claim. □

Proposition 4. *For $i = 1, \ldots, m$ and $n \in \mathbb{N}$ we have*

$$|A_n^{(i)}|_\infty < \frac{p^{n+1}}{2},$$

where $|\cdot|_\infty$ denotes the Euclidean norm.

Proof. We prove the thesis by induction. For $n = 0$,

$$|A_0^{(i)}|_\infty = |a_0^{(i)}|_\infty < \frac{p}{2};$$

for $n \leq m$

$$A_n^{(i)} = a_n^{(1)} A_{n-1}^{(i)} + \ldots + a_n^{(n)} A_0^{(i)} + a^{(n-i)}$$

By induction hypothesis, and since $|a_k^{(i)}|_\infty < \frac{p}{2}$ for every k,

$$|A_n^{(i)}|_\infty < \frac{p^{n+1}}{4} + \frac{p^n}{4} + \ldots + \frac{p^2}{4} + \frac{p}{2} = \frac{p^2}{4}\left(\frac{p^n - 1}{p - 1}\right) + \frac{p}{2} < \frac{p^{n+1}}{2}.$$

For $n > m$, we have

$$A_n^{(i)} = a_n^{(1)} A_{n-1}^{(i)} + \ldots + a_n^{(m)} A_{n-m}^{(i)} + A_{n-m-1}^{(i)}.$$

Again by induction hypothesis, and since $|a_k^{(i)}|_\infty < \frac{p}{2}$ for every k,

$$|A_n^{(i)}|_\infty < \frac{p^{n+1}}{4} + \frac{p^n}{4} + \ldots + \frac{p^{n-m+2}}{4} + \frac{p^{n-m}}{2} = \frac{p^{n-m+2}}{4}\left(\frac{p^m - 1}{p - 1}\right) + \frac{p^{n-m}}{2} < \frac{p^{n+1}}{2}.$$

$\square$

Proposition 5. *Given the MCF $[(a_0^{(1)}, a_1^{(1)}, \ldots), \ldots, (a_0^{(m)}, a_1^{(m)}, \ldots)]$*

a) *every minor of $\mathcal{B}_n$ is a polynomial in $\mathbb{Z}[a_j^{(i)}, i = 1, \ldots, m, \ j = 0, \ldots n]$ and each monomial has the form*

$$\lambda c_0 c_1 \ldots c_n$$

where $\lambda \in \mathbb{Z}$ and $c_j = 1$ or $c_j = a_j^{(i)}$ for some $i = 1, \ldots m$.

b) *The summand $\lambda a_0^{(1)} \ldots a_n^{(1)}$ does not appear in any principal minor of $\mathcal{B}_n$ except for the 1×1 minor obtained by removing all rows and columns indexed by $2, \ldots, m+1$; in this case $\lambda = \pm 1$.*

Proof.
a) We prove the thesis by induction on n. For $n = 0$, we have $\mathcal{B}_n = \mathcal{A}_0$ and the thesis immediately follows. Suppose now that the statement holds for n and consider $\mathcal{B}_{n+1}$. Let M be a square submatrix of $\mathcal{B}_{n+1}$. If $M = \mathcal{B}_{n+1}$ then $\det(M) = \pm 1$ and we are done. So we suppose that some rows and columns miss in M. If M does not contain the first column, then M is a square submatrix of $\mathcal{B}_n$ and the result holds by inductive hypothesis. Therefore we suppose that M contains the first column of $\mathcal{B}_{n+1}$ which is
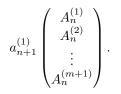
$$\begin{pmatrix} A_{n+1}^{(1)} \\ A_{n+1}^{(2)} \\ \vdots \\ A_{n+1}^{(m+1)} \end{pmatrix} = \begin{pmatrix} a_{n+1}^{(1)} A_n^{(1)} + a_{n+1}^{(2)} A_{n-1}^{(1)} + \ldots + a_{n+1}^{(m)} A_{n-m+1}^{(1)} + A_{n-m}^{(1)} \\ a_{n+1}^{(1)} A_n^{(2)} + a_{n+1}^{(2)} A_{n-1}^{(2)} + \ldots + a_{n+1}^{(m)} A_{n-m+1}^{(2)} + A_{n-m}^{(2)} \\ \vdots \\ a_{n+1}^{(1)} A_n^{(m+1)} + a_{n+1}^{(2)} A_{n-1}^{(m+1)} + \ldots + a_{n+1}^{(m)} A_{n-m+1}^{(m+1)} + A_{n-m}^{(m+1)} \end{pmatrix} \quad (8)$$

By the properties of the determinant, $\det(M)$ is the sum for $i = 1, \ldots, m + 1$ of the determinants of all matrices M_i where M_i is obtained from M by replacing the first column by a subvector of

$$a_{n+1}^{(i)} \begin{pmatrix} A_{n+1-i}^{(1)} \\ A_{n+1-i}^{(2)} \\ \vdots \\ A_{n+1-i}^{(m+1)} \end{pmatrix}$$

(to get an uniform notation, we put $a_k^{(m+1)} = 1$, for every $k \in \mathbb{N}$). Then we see that either two columns of M_i are proportional, so that $\det(M_i) = 0$, or $\det(M_i) = \pm a_{n+1}^{(i)} \det(M_i')$ where M_i' is a submatrix of $\mathcal{B}_n$. Then the claim holds by inductive hypothesis.

b) Let M be the square submatrix obtained from $\mathcal{B}_n$ by removing all rows and columns indexed by $I \subseteq \{1, \ldots, m+1\}$, and suppose that the summand $\lambda a_0^{(1)} \ldots a_n^{(1)}$ appears in M. Then by a), M must contain the first column of $\mathcal{B}_n$, so that it must contain also the first row. Moreover, since $\det(\mathcal{B}_n) = \pm 1$, at least one row and the corresponding column are missing. We argue again by induction on n. If $n = 0$, then the last row must miss, (otherwise $\det(M) \in \{1, 0\}$) so that the last column too must miss; then the row indexed by m has the form $(a_0^{(m)}, 0, \ldots, 0)$ and this implied that it must miss, unless $m = 1$, so that column m is missing and so on. It follows that $I = 2, \ldots, m+1$, $\lambda = 1$. Now suppose that the result holds for $\mathcal{B}_n$. The first column of $\mathcal{B}_{n+1}$ being as in (8), wee deduce by a) that $\lambda a_0^{(1)} \ldots a_n^{(1)}$ must be a summand of $\det(M_1)$, where M_1 is obtained from M by replacing the first column by a subvector of

$$
a_{n+1}^{(1)} \begin{pmatrix} A_n^{(1)} \\ A_n^{(2)} \\ \vdots \\ A_n^{(m+1)} \end{pmatrix}.
$$

Then we see that the second column (and the second row) must miss in M (otherwise $\det(M) = 0$). Therefore $\det(M_1) = a_{n+1}^{(1)} \det(M_1')$ where M_1' is a square submatrix of B_n giving rise to a principal minor. Since $\lambda a_0^{(1)} \ldots a_n^{(1)}$ is a summand in $\det(M_1')$, by inductive hypothesis $I = \{2, \ldots, m+1\}$ and $\lambda = 1$.

$\square$

3   On the finiteness of the p–adic Jacobi–Perron algorithm

In [26], the authors gave some results about the finiteness of the p-adic Jacobi–Perron algorithm. We recall these results below.

Proposition 6. *If the p–adic Jacobi–Perron algorithm stops in a finite number of steps when processing the m–tuple $(\alpha^{(1)}, \ldots, \alpha^{(m)}) \in \mathbb{Q}_p^m$, then $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent.*

Proposition 7. *For an input $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \in \mathbb{Q}^m$, the p–adic Jacobi–Perron algorithm terminates in a finite number of steps.*

Thus, a full characterization of the input vectors which lead to a finite Jacobi–Perron expansion is still missing in the p–adic case. On the other hand, in the real field it is known that the Jacobi–Perron algorithm stops in a finite number of steps if and only if $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent for $m = 2$, whereas this is not true for $m \geq 3$, see [32, Theorem 44] and [11, 12]. Counterexamples in the latter case are provided by m-tuples of algebraic numbers belonging to a finite extension of $\mathbb{Q}$ of degree $< m + 1$ and giving rise to a periodic MCF. This shows that the finiteness an the periodicity of the Jacobi–Perron algorithm are in some way interrelated.

In this section we shall assume that $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are linearly dependent over $\mathbb{Q}$, and associate to every linear dependence relation a sequence of integers $(S_n)_{n \geq 0}$, which will be useful in the investigation of the finiteness of the p–adic Jacobi–Perron algorithm. In particular, in the case $m = 2$, we shall provide a condition that must be satisfied by the partial quotients of an infinite MCF obtained by the p-adic Jacobi–Perron algorithm processing a couple (α, β), where $1, \alpha, \beta$ are $\mathbb{Q}$-linearly dependent. We shall show in next section that, unlike the real case, even for $m = 2$ there exist some input vectors $\boldsymbol{\alpha}$ such that $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent but their p-adic Jacobi–Perron expansion is periodic (and hence not finite).

Let us consider $\boldsymbol{\alpha}_0 = (\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \in \mathbb{Q}_p^m$ and assume that there is a linear dependence relation

$$
x_1 \alpha_0^{(1)} + \ldots + x_m \alpha_0^{(m)} + x_{m+1} = 0 \tag{9}
$$

with $x_1, \ldots, x_{m+1} \in \mathbb{Z}$ coprime. Then we can associate to it the sequence

$$
S_n = x_1 A_{n-1}^{(1)} + \ldots + x_m A_{n-1}^{(m)} + x_{m+1} A_{n-1}^{(m+1)} \tag{10}
$$

6

for any $n \geq -m$, where $A_n^{(i)}$ are, as usual, the numerators and denominators of the convergents of the MCF of $\boldsymbol{\alpha}_0$ defined by (5). It is straightforward to see that the following identities hold:

$$S_n \alpha_n^{(1)} + \ldots + S_{n-m+1}\alpha_n^{(m)} + S_{n-m} = 0, \text{ for any } n \geq 0; \tag{11}$$

$$S_n = a_{n-1}^{(1)}S_{n-1} + \ldots + a_{n-1}^{(m)}S_{n-m} + S_{n-m-1}, \text{ for any } n \geq 1; \tag{12}$$

$$S_n = (a_{n-1}^{(1)} - \alpha_{n-1}^{(1)})S_{n-1} + \ldots + (a_{n-1}^{(m)} - \alpha_{n-1}^{(m)})S_{n-m}, \text{ for any } n \geq 1; \tag{13}$$

$$S_n = x_1 V_{n-1}^{(1)} + \ldots + x_m V_{n-1}^{(m)}, \text{ for any } n \geq -m + 1. \tag{14}$$

$$\begin{pmatrix} S_n \\ S_{n-1} \\ \vdots \\ S_{n-m} \end{pmatrix} = \mathcal{B}_{n-1}^T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \end{pmatrix} \tag{15}$$

where the superscript T denotes transposition.

Proposition 8. *Given the sequence $(S_n)_{n \geq -m}$ defined by (10), we have that $S_n \in \mathbb{Z}$, for any $n \geq -m$ and the gcd of $S_n, \ldots, S_{n-m}$ is a power of p. Moreover,*

$$|S_n| < \max_{1 \leq i \leq m}\{|S_{n-i}|\}$$

so that if the MCF for $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$ is infinite, then

$$\lim_{n \to +\infty} S_n = 0 \text{ in } \mathbb{Q}_p.$$

Proof. By definition $S_n \in \mathbb{Z}\left[\dfrac{1}{p}\right]$, for any $n \geq -m$, and $S_{-m+1}, \ldots S_0 \in \mathbb{Z}$. Then, using formula (13), and observing that $v_p(a_{n-1}^{(i)} - \alpha_{n-1}^{(i)}) > 0$, for $i = 1, \ldots, m$, where $v_p(\cdot)$ is the p-adic valuation, we get $S_n \in \mathbb{Z}$. The assertion about the gcd is easily proved by induction, using formula (12).

Since $|a_n^{(i)} - \alpha_n^{(i)}| < 1$, from (13), we have

$$|S_n| \leq \max_{1 \leq i \leq m}\{|a_n^{(i)} - \alpha_n^{(i)}||S_{n-1}|\} < \max_{1 \leq j \leq m}\{|S_{n-i}|\}.$$

Finally, by Proposition 2 and formula (14) we see that $\lim_{n \to +\infty} S_n = 0$ in $\mathbb{Q}_p$. $\qquad\square$

An immediate consequence of Proposition 8 is the following

Corollary 1. *For $n \geq 0$, write $n = qm + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < m$; then $v_p(S_n) > q$. In particular $v_p(S_n) > \left[\frac{n}{m}\right]$ for every $n \geq 0$.*

Proposition 8 and Corollary 1 describe the behaviour of the sequence (S_n) with respect to the p-adic norm. We study now its behaviour with respect to the euclidean norm. We start by a general result.

Proposition 9. *Let $(T_n)_{n \geq -m}$ be any sequence in $\mathbb{R}$ satisfying*

$$T_n = y_n^{(1)}T_{n-1} + \ldots + y_n^{(m)}T_{n-m} + T_{n-m-1}, \quad n \geq 1$$

where $(y_n^{(1)})_{n \geq 1}, \ldots (y_n^{(m)})_{n \geq 1}$ are sequences of elements in $\mathcal{Y}$; then

$$\lim_{n \to +\infty} \frac{T_n}{p^n} = 0$$

in $\mathbb{R}$.

Proof. In the following $|\cdot|_\infty$ stands for the Euclidean norm. We have

$$\left|\frac{T_n}{p^n}\right|_\infty < \frac{1}{2}\left|\frac{T_{n-1}}{p^{n-1}}\right|_\infty + \frac{1}{2p}\left|\frac{T_{n-2}}{p^{n-2}}\right|_\infty + \ldots + \frac{1}{2p^{m-1}}\left|\frac{T_{n-m}}{p^{n-m}}\right|_\infty + \frac{1}{p^{m+1}}\left|\frac{T_{n-m-1}}{p^{n-m-1}}\right|_\infty$$

$$\leq K_p \max\left\{\left|\frac{T_{n-1}}{p^{n-1}}\right|_\infty, \left|\frac{T_{n-2}}{p^{n-2}}\right|_\infty, \ldots, \left|\frac{T_{n-m}}{p^{n-m}}\right|_\infty, \left|\frac{T_{n-m-1}}{p^{n-m-1}}\right|_\infty\right\},$$

where $K_p = \dfrac{1}{p^{m+1}} + \dfrac{1}{2}\sum_{k=0}^{m-1}\dfrac{1}{p^k} < 1$. Therefore

$$\left|\frac{T_n}{p^n}\right|_\infty < K_p^{n-2}\max\left\{\left|\frac{T_m}{p^m}\right|_\infty, \left|\frac{T_{m-1}}{p^{m-1}}\right|_\infty, \ldots, \left|\frac{T_1}{p}\right|_\infty, |T_0|_\infty\right\}$$

and the claim follows. □

Corollary 2. *For the sequence $(S_n)_{n \geq -m+1}$, we have*

$$\lim_{n \to +\infty}\frac{S_n}{p^n} = 0$$

in $\mathbb{R}$.

Proof. By formula (12), the sequence $(S_n)_{n \geq -m}$ satisfies the hypothesis of Proposition 9. □

We shall use the properties stated above to establish some partial converse of Proposition 6. The case $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \in \mathbb{Q}^m$ is dealt by Proposition 7, so that we can assume that $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \in \mathbb{Q}_p^m \setminus \mathbb{Q}^m$. Notice that in the case $m = 2$ under this hypothesis there can be only one linear dependence relation (9), so that the sequence S_n depends only on the sequence $(\alpha_0^{(1)}, \alpha_0^{(2)}) \in \mathbb{Q}_p^2$.

Proposition 10. *Assume that the sequence* $\left(\dfrac{S_n}{p^n}\right)$ *has bounded denominator, i.e. there exist $k \in \mathbb{Z}$ such that $v_p(S_n) \geq n + k$, for every n. Then the Jacobi–Perron algorithm stops in finitely many steps when processing the input $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$.*

Proof. Assume that the Jacobi–Perron algorithm does not stop. Put $z_n = p^k \dfrac{S_n}{p^n}$; then $z_n \in \mathbb{Z}$ and the sequence (z_n) tends to 0 in the euclidean norm, by Corollary 2. It follows that z_n (and hence S_n) is 0 for $n \gg 0$, and this is impossible by formula (11). □

The following theorem is the main result of this section. To get an uniform notation, we shall put $\alpha_n^{(m+1)} = a_n^{(m+1)} = 1$ for every n.

Theorem 1. *Assume that $1, \alpha_0^{(1)}, \ldots, \alpha_0^{(m)}$ are $\mathbb{Q}$-linearly dependent and*

$$v_p(a_n^{(j)}) - v_p(a_n^{(1)}) \geq j - 1 \quad \text{for } j = 3, \ldots, m+1 \text{ and any } n \text{ sufficiently large.} \tag{16}$$

Then the Jacobi–Perron algorithm stops in finitely many steps when processing the input $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$.

Notice that the condition $v_p(a_n^{(2)}) - v_p(a_n^{(1)}) \geq 1$ is always true by conditions (3).

Proof. By (11) we get

$$\frac{S_n}{p^n} = -\frac{S_{n-1}}{p^{n-1}}\gamma_n^{(1)} - \ldots - \frac{S_{n-m}}{p^{n-m}}\gamma_n^{(m)},$$

where for $j = 1, \ldots, m$

$$\gamma_n^{(j)} = \frac{\alpha_n^{(j+1)}}{p^j \alpha_n^{(1)}}.$$

By equations (3), (4) and hypotheses (16) we have $v_p(\gamma_n^{(j)}) \geq 0$ for n sufficiently large. Therefore $v_p\left(\dfrac{S_n}{p^n}\right) \geq \min\left\{v_p\left(\dfrac{S_{n-1}}{p^{n-1}}\right), \ldots, v_p\left(\dfrac{S_{n-m}}{p^{n-m}}\right)\right\}$ for n sufficiently large, so that $v_p\left(\dfrac{S_n}{p^n}\right) \geq K$ for some $K \in \mathbb{Z}$. Then we conclude by Proposition 10. □

In the case $m = 2$, Theorem 1 assumes the following simple form.

Corollary 3. *For $m = 2$, if $1, \alpha_0^{(1)}, \alpha_0^{(2)}$ are linearly dependent over $\mathbb{Q}$ and the p-adic Jacobi–Perron algorithm does not stop then $v_p(a_n^{(1)}) = -1$ for infinitely many $n \in \mathbb{N}$.*

In the next section we shall present some examples where the hypotheses of Corollary 3 are satisfied.

Remark 2. *In the classical real case, for $m = 2$, it is possible to prove that the Jacobi–Perron algorithm detects rational dependence because the sequences $(V_n^{(1)})$ and $(V_n^{(2)})$ are bounded with respect to the euclidean norm. In fact, this implies that the set of triples (S_n, S_{n-1}, S_{n-2}) is finite and consequently the corresponding MCF is finite or periodic. Moreover it is possible to show that a periodic expansion can not occur and consequently the Jacobi–Perron algorithm stops when processes two real numbers α, β such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly dependent, see [32] for details. In the p-adic case, the sequences $(V_n^{(i)})$ are bounded (because they approach zero in $\mathbb{Q}_p$, see Proposition 2); but the argument above does not apply, because the p-adic norm is non-archimedean. However, considering that $v_p(S_n) > \frac{n}{2}$ by Corollary 1, it could be interesting to focus on the sequence of integers $\left(\dfrac{S_n}{p^{n/2}} \right)$. When this sequence is bounded with respect to the euclidean norm, it is possible to argue similarly to the real case and deduce the finiteness of the p-adic Jacobi–Perron algorithm on the given input.*

4 On the characteristic polynomial of periodic multidimensional continued fractions

The classical Jacobi–Perron algorithm was introduced over the real numbers with the aim of providing periodic representations for algebraic irrationalities. However, the problem regarding the periodicity of MCFs is still open, since it is not known if every algebraic irrational of degree $m+1$ belongs to a real input vector of lenght m for which the Jacobi–Perron algorithm is eventually periodic. On the contrary, periodic MCFs have been fully studied over the real numbers. Indeed, it is known that a periodic MCF represents real numbers belonging to an algebraic number field of degree less or equal than $m + 1$, see [6] for a survey on this topic. Moreover, for $m = 2$, Coleman [10] gave also a criterion for establishing when the periodic MCF converges to cubic irrationalities. In this section, we start the study of the periodicity of MCFs over $\mathbb{Q}_p$. In particular, we shall see that, analogously to the real case, a periodic p-adic m-dimesional MCF represents algebraic irrationalities of degree less or equal than $m + 1$.

Let us consider a purely periodic MCF of period N:

$$(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)}, \ldots, a_{N-1}^{(1)} \right)}, \ldots, \overline{\left(a_0^{(m)}, \ldots, a_{N-1}^{(m)} \right)} \right], \tag{17}$$

i.e., $a_{k+N}^{(i)} = a_k^{(i)}$ for every $k \in \mathbb{N}$ and $i = 1, \ldots, m$. By (1), we also have $\alpha_{k+N}^{(i)} = \alpha_k^{(i)}$ for every $k \in \mathbb{N}$ and $i = 1, \ldots, m$, from which, follows

$$\alpha_0^{(i)} = \frac{\alpha_0^{(1)} A_{N-1}^{(i)} + \ldots + \alpha_0^{(m)} A_{N-m}^{(i)} + A_{N-m-1}^{(i)}}{\alpha_0^{(1)} A_{n-1}^{(m+1)} + \ldots + \alpha_0^{(m)} A_{n-m}^{(m+1)} + A_{N-m-1}^{(m+1)}} \tag{18}$$

using (6). We define the matrix

$$\mathcal{M} := \mathcal{B}_{N-1} = \prod_{j=0}^{N-1} \mathcal{A}_j = \begin{pmatrix} A_{N-1}^{(1)} & A_{N-2}^{(1)} & \cdots & A_{N-m-1}^{(1)} \\ A_{N-1}^{(2)} & A_{N-2}^{(2)} & \cdots & A_{N-m-1}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N-1}^{(m+1)} & A_{N-2}^{(m+1)} & \cdots & A_{N-m-1}^{(m+1)} \end{pmatrix}$$

whose characteristic polynomial $P(X)$ will be also called the characteristic polynomial of the periodic MCF (17). From equation (18), we have

$$\mathcal{M} \begin{pmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{pmatrix} = \left(\alpha_0^{(1)} A_{N-1}^{(m+1)} + \alpha_0^{(2)} A_{N-2}^{(m+1)} + \ldots + \alpha_0^{(m)} A_{N-m}^{(m+1)} + A_{N-m-1}^{(m+1)} \right) \begin{pmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{pmatrix}.$$

Moreover, by Proposition 3 we know that $\sum_{i=1}^{m+1} \alpha_N^{(i)} A_{N-i}^{(m+1)} = \alpha_1^{(1)} \cdots \alpha_N^{(1)}$ and, since $\alpha_0^{(1)} = \alpha_N^{(1)}$,

9

we have

$$
\mathcal{M} \begin{pmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{pmatrix} = \alpha_0^{(1)} \ldots \alpha_{N-1}^{(1)} \begin{pmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(m)} \\ 1 \end{pmatrix}.
$$

Therefore $\mu := \alpha_0^{(1)} \ldots \alpha_{N-1}^{(1)}$ is an eigenvalue of $\mathcal{M}$ and a root of the characteristic polynomial $P(X)$. In the next theorems, we shall see that μ is the p-adic dominant eigenvalue, that is the root greatest in p-adic norm of $P(X)$ and that the limits of the periodic MCF (17) are strictly related to μ. Note that it is not a loss of generality to consider purely periodic MCFs, since the algebraic properties of the complete quotients of a MCF coincide with those of the input vector.

Theorem 2. *Given the purely periodic MCF*

$$
(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)}, \ldots, a_{N-1}^{(1)} \right)}, \ldots, \overline{\left(a_0^{(m)}, \ldots, a_{N-1}^{(m)} \right)} \right]
$$

and its characteristic polynomial $P(X)$, then $\mu = \alpha_0^{(1)} \ldots \alpha_{N-1}^{(1)}$ is the greatest root in p-adic norm.

Proof. We consider $a_n^{(1)} = \dfrac{\tilde{a}_n^{(1)}}{p^{k_n}}$, for any $n \geq 0$, where $k_n \geq 0$ ($k_n > 0$, for $n > 0$). We define the quantity $k = k_0 + \ldots + k_{N-1}$ and the matrix

$$
\mathcal{M}' := p^k \mathcal{M} = \mathcal{A}'_0 \ldots \mathcal{A}'_{N-1}
$$

where

$$
\mathcal{A}'_i = p^{k_i} \mathcal{A}_i = \begin{pmatrix} \tilde{a}_i^{(1)} & p^{k_i} & 0 & \ldots & 0 \\ p^{k_i} a_i^{(2)} & 0 & p^{k_i} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p^{k_i} a_i^{(m)} & 0 & 0 & \ldots & p^{k_i} \\ p^{k_i} & 0 & 0 & \ldots & 0 \end{pmatrix} \equiv \begin{pmatrix} \tilde{a}_i^{(1)} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \pmod{p}.
$$

Therefore

$$
\mathcal{M}' \equiv \begin{pmatrix} \tilde{a}_0^{(1)} \ldots \tilde{a}_{N-1}^{(1)} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \pmod{p}. \tag{19}
$$

Let $Q(X)$ be the characteristic polynomial of $\mathcal{M}'$. Then, λ is an eigenvalue of $\mathcal{M}$ if and only if $p^k \lambda$ is an eigenvalue of $\mathcal{M}'$. If $\lambda_1, \ldots, \lambda_{m+1}$ are the eigenvalues of $\mathcal{M}$, then

$$
Q(X) = \prod_{i=1}^{m+1} (X - p^k \lambda_i) = p^{k(m+1)} \prod_{i=1}^{m+1} \left(\frac{x}{p^k} - \lambda_i \right) = p^{k(m+1)} P\left(\frac{X}{p^k} \right)
$$

so that

$$
P(X) = \frac{1}{p^{k(m+1)}} Q(p^k X).
$$

From (19) we have

$$
Q(X) \equiv X^m (X - \tilde{a}_0^{(1)} \ldots \tilde{a}_{N-1}^{(1)}) \pmod{p}.
$$

Thus

$$
Q(X) = X^{m+1} + \delta_m X^m + \ldots + \delta_0
$$

with

$$
\delta_m \equiv \tilde{a}_0^{(1)} \ldots \tilde{a}_{N-1}^{(1)} \pmod{p}, \quad \delta_i \equiv 0 \pmod{p} \text{ for } i = 0, \ldots, m-1, \quad \delta_0 = \pm p^{k(m+1)}.
$$

It follows

$$P_\mu(X) = \frac{1}{p^{k(m+1)}} Q(p^k X)$$

$$= \frac{1}{p^{k(m+1)}} (p^{k(m+1)} X^{m+1} + \delta_m p^{km} X^m + \ldots + \delta_i p^{ki} X^i + \ldots + \delta_0)$$

$$= X^{(m+1)} + \frac{\delta_m}{p^k} X^m + \ldots + \frac{\delta_i}{p^{k(m+1-i)}} X^i + \ldots \pm 1$$

$$= X^{m+1} + \gamma_m X^m + \ldots + \gamma_0,$$

where

$$\gamma_i = \frac{\delta_i}{p^{k(m+1-i)}} \quad \text{for } i = 0, \ldots, m, \quad (\gamma_0 = \pm 1).$$

Now, we put $\mu_i = v_p(\delta_i)$ for $i = 1, \ldots, m$ and observe that

$$\mu_m = 0, \quad \mu_i > 0 \text{ for } i = 1, \ldots, m-1, \quad \mu_0 = k(m+1).$$

We can see that

$$v_p(\gamma_m) = v_p(a_0^{(1)} \ldots a_{N-1}^{(1)}) = \sum_{i=0}^{N-1} v_p(a_i^{(1)}) = -k$$

$$v_p(\gamma_i) = v_p(\delta_i) - k(m+1-i)$$
$$= \mu_i + ik - (m+1)k \quad \text{for } i = 0, \ldots, m.$$

Now we want to study the Newton polygon (see [16]) of $P(X)$ for proving that μ is the root greatest in p-adic norm.

The line, in the real plane, passing through the points $(i, v_p(\gamma_i))$ and $(m+1, 0)$ has equation

$$y = \frac{v_p(\gamma_i)}{m+1-i}(-x+m+1), \tag{20}$$

for any $i = 1, \ldots, m-1$. We will denote with s_i the slope of this line. From the fact that

$$v_p(\gamma_i) = \mu_i - k(m+1-i) \text{ and } \mu_i = v_p(\delta_i) > 0,$$

we get

$$\frac{v_p(\gamma_i)}{m+1-i} = \frac{\mu_i}{m+1-i} - k > -k,$$

i.e., the point in the real plane with coordinates $(m, v_p(\gamma_m)) = (m, -k)$ lies strictly under the line (20), for any $i = 1, \ldots, m-1$.

Thus, the Newton polygon associated to the polynomial $P(X)$ has slopes $(s_1, \ldots, s_m)$, which is a strictly increasing sequence, where the last slope s_m is equal to k. Hence, the claim of the theorem follows from [16, Theorem 6.4.7] and the fact that the sequence $(s_1, \ldots, s_m)$ of slopes is strictly increasing.

$\square$

Theorem 3. *Given the purely periodic MCF*

$$(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)}, \ldots, a_{N-1}^{(1)}\right)}, \ldots, \overline{\left(a_0^{(m)}, \ldots, a_{N-1}^{(m)}\right)} \right]$$

and its characteristic polynomial $P(X)$, then

a) $\mathbb{Q}(\mu) = \mathbb{Q}(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$

b) $\mu \notin \mathbb{Q}$

where μ is the greatest root in p-adic norm of $P(X)$.

Proof.

a) Since $\mu = \alpha_0^{(1)} \ldots \alpha_{N-1}^{(1)}$, obviously $\mu \in \mathbb{Q}(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$; conversely by Theorem 2 the nullspace of $\mathcal{B}_{N-1} - \mu I_{m+1} \in M_{m+1}(\mathbb{Q}(\mu))$ is 1-dimensional (where I_{m+1} is the $(m+1) \times (m+1)$ identity matrix and $M_{m+1}(\mathbb{Q}(\mu))$ denotes the set of $(m+1) \times (m+1)$ matrices with entries in $\mathbb{Q}(\mu)$). Therefore it is generated by a vector $\beta = (\beta_1, \ldots, \beta_{m+1})$ with entries in $\mathbb{Q}(\mu)$, which must be proportional to $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}, 1)$. It follows that $\alpha^{(i)} = \frac{\beta_i}{\beta_{m+1}} \in \mathbb{Q}(\mu)$ for $i = 1, \ldots, m$.

b) Assume that $\mu \in \mathbb{Q}$, then $\alpha_0^{(1)}, \ldots, \alpha_0^{(m)} \in \mathbb{Q}$. But in this case the MCF corresponding to $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$ is finite (see [26]), so that it cannot be periodic. $\square$

From the previous theorems, we have that a periodic MCF converges to a m–tuple of algebraic irrationalities of degree less or equal than $m + 1$, belonging to the field generated over $\mathbb{Q}$ by the the root greatest in p-adic norm of the characteristic polynomial. In the characteristic polynomial is irreducible, then the algebraic irrationalities are of the maximum degree. In the following, we see some further properties of the roots of the characteristic polynomial and then we focus on the case $m = 2$ for some specific considerations.

Lemma 1. *Let $P(X) = X^{m+1} + \gamma_m X^m + \ldots + \gamma_1 X + (-1)^{m(N+1)+1}$ be the characteristic polynomial of the purely periodic MCF $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) = \left[\left(a_0^{(1)}, \ldots, a_{N-1}^{(1)} \right), \ldots, \left(a_0^{(m)}, \ldots, a_{N-1}^{(m)} \right) \right]$. We have that*

a) every γ_i is a polynomial in $\mathbb{Z}[a_j^{(i)}, i = 1, \ldots, m, \ j = 0, \ldots N - 1]$ and each monomial has the form $\lambda c_0 c_1 \ldots c_{N-1}$ where $\lambda \in \mathbb{Z}$ and $c_j = 1$ or $c_j = a_j^{(i)}$ for some $i = 1, \ldots, m$;

b) the monomial $a_0^{(1)} \cdots a_{N-1}^{(1)}$ appears only in γ_m.

Proof. Let us observe that any coefficient γ_i is the sum of the principal minors of the matrix $\mathcal{B}_{N-1}$ of order $m + 1 - i$, for $i = 1, \ldots, m$. Hence the thesis follows from Proposition 5. $\square$

Theorem 4. *Given the purely periodic MCF*

$$(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) = \left[\left(\overline{a_0^{(1)}, \ldots, a_{N-1}^{(1)}} \right), \ldots, \left(\overline{a_0^{(m)}, \ldots, a_{N-1}^{(m)}} \right) \right],$$

every root of its characteristic polynomial $P(X) = X^{m+1} + \gamma_m X^m + \ldots + \gamma_1 X + (-1)^{m(N+1)+1}$ has p-adic norm less than 1, except for the root greatest in p-adic norm $\mu = \alpha_0^{(1)} \cdots \alpha_{N-1}^{(1)}$.

Proof. By Lemma 1 and $|a_n^{(1)}| > 1$, $|a_n^{(1)}| > |a_n^{(j)}|$ for $n \in \mathbb{N}$, $j = 2, \ldots, m + 1$, we have $|\gamma_i| \leq |a_0^{(1)} \cdots a_{N-1}^{(1)}|$, for any $i = 1, \ldots, m$. Moreover, this inequality becomes equality if and only if $i = m$. If $\lambda_1 = \mu, \lambda_2, \ldots, \lambda_k$ are the roots of $P(X)$ with p-adic norm ≥ 1, then $\gamma_{m+1-k} \geq |\mu| = |a_0^{(1)} \cdots a_{N-1}^{(1)}|$. Recalling that γ_{m+1-k} is also the k-th elementary symmetric function of the roots, this implies $k = 1$ and the thesis follows. $\square$

Theorem 5. *Let z be a complex root of the characteristic polynomial $P(X)$ of the purely periodic MCF $\left[\left(a_0^{(1)}, \ldots, a_{N-1}^{(1)} \right), \ldots, \left(a_0^{(m)}, \ldots, a_{N-1}^{(m)} \right) \right]$. Then*

$$|z|_\infty < p^N.$$

Proof. By Gershgorin theorem [15] there esists a row $j = 1, \ldots, m + 1$ in $\mathcal{B}_{N-1}$ such that

$$|z - A_{N-j}^{(j)}|_\infty \leq \sum_{k=1,\ldots,m+1, \ k \neq j} |A_{N-k}^{(j)}|_\infty.$$

In particular

$$|z|_\infty \leq \sum_{k=1}^{m+1} |A_{N-k}^{(j)}|_\infty < \frac{1}{2} \sum_{k=1}^{m+1} p^{N-k+1}$$

by Proposition 4. Moreover,

$$\frac{1}{2} \sum_{k=1}^{m+1} p^{N-k+1} = \frac{1}{2} p^{N-m} \sum_{k=0}^{m} p^k = \frac{1}{2} p^{N-m} \frac{p^{m+1} - 1}{p - 1} \leq p^N.$$

$\square$

The previous theorems are useful in order to give some further information about the algebraic properties of the limits of a periodic MCF. We firstly consider the case $N = 1$:

Proposition 11. *The characteristic polynomial of a purely periodic MCF with period $N = 1$ does not have any rational root. In particular when $m = 2$ the characteristic polynomial is irreducible over $\mathbb{Q}$, and the limits of the MCF generate a cubic field.*

Proof. Let z be a rational root of the characteristic polynomial; by the rational root theorem it must be (up to a sign) a power of p. By Theorem 3 b), we know that $z \neq \mu$, and this implies that $v_p(z) \geq 1$ by Theorem 4. But $|z|_\infty < p$ by Theorem 5, a contradiction. $\square$

In general, a rational root of a MCF with period of length N must satisfy $|z|_\infty < p^N$ and $v_p(z) \geq 1$, so that for the rational root theorem it must be of the kind $\pm p^k$, with $k \leq N - 1$. The next proposition gives a necessary condition for the existence of such a root, in the case $m = 2$ and $N = 2$.

Proposition 12. *Let us consider the purely periodic MCF $\left[\left(\overline{a_0, a_1} \right), \left(\overline{b_0, b_1} \right) \right]$. Then its characteristic polynomial $P(X)$ is irreducible over $\mathbb{Q}$ unless the following condition is verified, possibly interchanging the indices 0 and 1:*

- *a_0 is of the form $\pm \frac{1}{p} + w$ with $w \in \mathbb{Z}, |w|_\infty \leq \frac{p-1}{2}, w \neq 0$; and*
- *either $v_p(a_1 p + 1) = v_p(a_1) + 1$ (which implies $v_p(b_1) = v_p(a_1) + 1, v_p(b_0) = 0$)*
  or a_1 is of the form $\pm \frac{1}{p} + u$ with $u \in \mathbb{Z}, |u| \leq \frac{p-1}{2}, u \neq 0$; $\qquad$ (21)
  in the latter case one between b_0 and b_1 is zero and the other one is equal to $-wu \pm p$.

Proof. Write
$$P(X) = X^3 + \gamma_2 X^2 + \gamma_1 X - 1,$$
then
$$\gamma_2 = -(a_0 a_1 + b_0 + b_1), \qquad \gamma_1 = b_1 b_0 - a_0 - a_1$$
so that
$$P(X) = X(X - b_0)(X - b_1) - (a_0 X + 1)(a_1 X + 1).$$
We put $k_1 = -v_p(a_1), k_2 = -v_p(a_2), k = k_1 + k_2$. By Theorems 4 and 5 the only possible rational roots of $P(X)$ are $\pm p$. So assume
$$P(\pm p) = \pm p(\pm p - b_0)(\pm p - b_1) - (\pm a_0 p + 1)(\pm a_1 p + 1) = 0. \qquad (22)$$
Notice that the valuation of the first summand is $\geq -k + 3$ and that of the second summand is $\geq -k + 2$. Therefore, the valuation of the second summand must be $\geq -k + 3$. This implies that at least one between a_0 and a_1, say a_0, must satisfy $v(\pm a_0 p + 1) > -k_0 + 1$, that is $a_0 p \equiv \mp 1 \pmod{p}$. Since $a_0 \in \mathcal{Y}$ this implies $a_0 = \mp \frac{1}{p} + w$ with $w \in \mathbb{Z}, |w|_\infty \leq \frac{p-1}{2}$ and (22) becomes
$$\pm (\pm p - b_0)(\pm p - b_1) - w(\pm a_1 p + 1) = 0. \qquad (23)$$
We show that $w \neq 0$: otherwise one between b_0 and b_1 should be equal to $\pm p$, which is a contradiction because $b_0, b_1 \in \mathcal{Y}$.
The right-hand side of (23) has valuation $\geq -k_1 + 1$; and $v_p(\pm p - b_0) \geq 0, v_p(\pm p - b_1) \geq -k_1 + 1$. If the valuation of the right side is exactly $-k_1 + 1$ then it must be $v(b_0) = 0, v(b_1) = -k_1 + 1$. On the other hand, if the valuation of the right side is $> -k_1 + 1$ then $a_1 p \equiv \mp 1 \pmod{p}$. As above this implies $a_0 = \mp \frac{1}{p} + u$ with $u \in \mathbb{Z}, |u|_\infty \leq \frac{p-1}{2}, u \neq 0$ and (23) becomes
$$\pm (\pm p - b_0)(\pm p - b_1) - wup = 0. \qquad (24)$$
This implies that one between b_0 and b_1 is 0, the other one (say b_i) has valuation 0, and satisfies $\pm p - b_i = wu$. $\square$

In order to provide numerical examples, the following proposition will be useful.

Proposition 13. *Let us consider the purely periodic 2-dimensional MCF $(\alpha, \beta) = \left[\left(\overline{a_0, \ldots, a_{N-1}} \right), \left(\overline{b_0, \ldots, b_{N-1}} \right) \right]$ and suppose that its characteristic polynomial $P(X)$ is reducible. Let $z = \pm p^k$ be the (unique) rational root of $P(X)$, then the 1-dimensional eigenspace $\mathcal{L} \subseteq \mathbb{Q}^3$ of the transpose of $\mathcal{B}_{N-1}$ associated to z coincides with the space $\mathcal{L}'$ of rational vectors (x, y, z) such that $x\alpha + y\beta + z = 0$.*

Proof. Notice firstly that the space $\mathcal{L}'$ is one-dimensional, because Theorem 3 and the reducibility of $P(X)$ imply that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = 2$. Therefore there is a linear dependence relation

$$x_1 \alpha + x_2 \beta + x_3 = 0$$

with coprime $x_1, x_2, x_3 \in \mathbb{Z}$, and (x_1, x_2, x_3) generates $\mathcal{L}'$. Since $\alpha_N = \alpha$, $\beta_N = \beta$, by property (11) of the sequence (S_n) defined by (10), the vector (S_N, S_{N-1}, S_{N-2}) must be a rational multiple of (x_1, x_2, x_3); then by (14) (x_0, y_0, z_0) is an eigenvector associated to a rational eigenvalue, so that it belongs to $\mathcal{L}$. $\square$

Example 1. *Condition* (21) *is essential. Consider the following examples.*

- *For $p = 5$, the periodic MCF $(\alpha, \beta) = \left[\left(\frac{4}{5}, \frac{11}{5} \right), \left(\overline{1, 2} \right) \right]$ has characteristic polynomial*

$$P(X) = X^3 - \frac{119}{25} X^2 - X - 1$$

  and

$$P(X) = (X - 5) \left(X^2 - \frac{6}{25} X + \frac{1}{5} \right).$$

  Moreover by using Proposition 13 we find the linear dependence relation between α, β and 1:

$$20\alpha + 5\beta + 4 = 0.$$

- *For $p = 3$, the periodic MCF $(\alpha, \beta) = \left[\left(\frac{2}{3}, \frac{5}{3} \right), \left(\overline{1, 0} \right) \right]$ has characteristic polynomial*

$$P(X) = X^3 - \frac{19}{9} X^2 - \frac{7}{3} X - 1 = (X - 3) \left(X^2 + \frac{8}{9} X + \frac{1}{3} \right).$$

  and

$$6\alpha + 3\beta + 2 = 0.$$

- *For $p = 3$, the periodic MCF $(\alpha, \beta) = \left[\left(\frac{2}{3}, \frac{13}{9} \right), \left(\overline{1, \frac{1}{3}} \right) \right]$ has characteristic polynomial*

$$P(X) = X^3 - \frac{62}{27} X^2 - \frac{16}{9} X - 1 = (X - 3) \left(X^2 + \frac{19}{27} X + \frac{1}{3} \right).$$

  The linear dependence relation between $\alpha, \beta, 1$ is the same as in the previous case:

$$6\alpha + 3\beta + 2 = 0.$$

The above examples also show $\mathbb{Q}$-linearly dependent numbers having a periodic (hence not finite) expansion by the p-adic Jacobi–Perron algorithm.

At the present time we were not able to find examples of m-tuples of $\mathbb{Q}$-linearly dependent p-adic numbers whose MCF is infinite and not periodic. Therefore, we state the following

Conjecture 1. *Let $\boldsymbol{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(m)}) \in \mathbb{Q}_p^m$ be such that $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent. Then the p-adic Jacobi-Perron algorithm for $\boldsymbol{\alpha}$ is finite or periodic.*

Acknowledgments

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