Improved estimate of the singular set of Dir-minimizing Q-valued functions via an abstract regularity result

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Abstract

In this note we prove an abstract version of a recent quantitative stratification priciple introduced by Cheeger and Naber (*Invent. Math.*, 191 (2013), no. 2, 321–339; *Comm. Pure Appl. Math.*, 66 (2013), no. 6, 965–990). Using this general regularity result paired with an ε-regularity theorem we provide a new estimate of the Minkowski dimension of the set of higher multiplicity points of a Dir-minimizing Q-valued function. The abstract priciple is applicable to several other problems: we recover recent results in the literature and we obtain also some improvements in more classical contexts.

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1. Introduction

An abstract regularity result. We propose an abstraction of a quantitative stratification principle introduced and developed in a series of papers by Cheeger and Naber [**? ?**], Cheeger, Haslhofer and Naber [**? ?**] and Cheeger, Naber and Valtorta [**?**].

The interest in finding general formulations of this kind of regularity results is driven by a number of important applications in geometric analysis. Apart from those contained in the papers quoted above, we mention the cases of Dir-minimizing Q-valued maps according to Almgren, of varifold with bounded mean curvature and of almost minimizers of the perimeter. The former is treated in

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details in § 4 and § 5, the latters in § 6. We explicitly remark that the papers [? ?] deal also with parabolic examples, a case that is not covered by our results.

To our knowledge the first example in this direction of abstraction is the general regularity theorem proven by Simon [? , Appendix A] based on the so called *dimension reduction argument* introduced by Federer in his pioneering work [?]. Similarly, the paper by White [?] generalizes the refinement of Federer's reduction argument made by Almgren in his big regularity paper [?].

The basic principle and the main ingredients of our abstract formulation can be explained roughly as follows.

> Abstract stratification: *the set of points where a solution to a geometric problem is faraway at every scale from being homogeneous with $k+1$ indipendent invariant directions has Minkowski dimension less than or equal to k.*

The main sets of quantities we consider are:

(a) a family of density functions Θ_s, increasing w.r.t. $s \geq 0$;

(b) a family of distance functions d_k, $k \in \{0, \ldots, m\}$, measuring the distance from k-invariant homogeneous solutions.

In addition, we assume suitable compatibility conditions, namely

(i) a quantitative differentiation principle that allows to quantify the number of those scales for which closeness to homogeneous solutions fails, and that tipically follows in the applications from monotonicity type formulas;

(ii) a consistency relation between the distances d_k: if a solution is close to a k-invariant one and additionally is 0-invariant with respect to another point away from the invariant k-dimensional space, then it is actually close to a $(k+1)$-invariant solution (see § 2 for the detailed formulation).

This set of hypotheses is common to many problems in geometric analysis such as the multiple valued functions dealt with below, harmonic maps, almost minimizing currents and several others (see [?]-[?] for other applications). Indeed, the stratification result and the estimate on the Minkowski dimension in the settings quoted above only depend on these assumptions (i) and (ii), thus making the common aspects of all previous results clear.

It turns out that there is a simple connection between White's approach to Almgren's stratification and the one outlined above. In § 3.3 we show how to recast the result by White in our framework. In this respect, we stress that the stratification in [?] and in our Theorem 4 can be applied to some cases not covered by the ideas in [?], such as stationary harmonic maps (cp. [? , Corollary 2.6], § 2.4.2 and [? , Section 6]).

Our main application of the abstract stratification principle is outlined in the following subsection.

Application to Q-valued functions. In the regularity theory for higher codimension minimal surfaces (in the sense of *mass minimizing integer rectifiable currents*) a fundamental role is played by the multiple valued functions introduced by Almgren in [**?**], which turn out to be the correct blowup limits for the analysis of singularities (see also [**? ? ? ? ?**] for a simplified new proof of the result in [**?**]).

Following [**?**], a Q-valued function u is a measurable map from a bounded open subset $\Omega \subset \mathbb{R}^n$ (for simplicity we always assume that the boundary of Ω is smooth) taking values in the space of positive atomic measures in $\mathbb{R}^m$ with mass Q, namely

$$\Omega \ni x \mapsto u(x) \in \mathcal{A}_Q(\mathbb{R}^m) := \left\{ \sum_{i=1}^{Q} [\![p_i]\!] \;:\; p_i \in \mathbb{R}^m \right\},$$

where $[\![p]\!]$ denotes the Dirac delta at p. Almgren proves in [**?**] (cp. also [**?**]) that the blowups of higher codimension mass minimizing integral currents are actually graphs of Q-valued functions u in a suitable Sobolev class $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ minimizing a generalized Dirichlet energy (cp. [**?** , Definition 0.5]):

$$\int_\Omega |Du|^2 \le \int_\Omega |Dv|^2 \quad \forall\, v \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m)),\ v|_{\partial\Omega} = u|_{\partial\Omega},$$

(explicit examples of Dir-minimizing Q-valued functions are given in [**?**]).

In order to estimate the size of the singular set of a minimizing current it is essential to bound the dimension of the set of points where the graph of a Dir-minimizing Q-valued function has higher multiplicity. Almgren's main result in the analysis of multiple valued functions is in fact an estimate of the Hausdorff dimension of the set Δ_Q of multiplicity Q points of a Dirichlet minimizing Q-valued function u, i.e. the set of points $x \in \Omega$ such that $u(x) = Q\,[\![p]\!]$ for some $p \in \mathbb{R}^m$, which turns out not to exceed $n-2$ in the case it does not coincide with Ω (cp. [**?** , Proposition 3.22]).

In this paper we improve Almgren's result by showing an estimate of the Minkowski dimension of Δ_Q. To this aim we denote by $\mathcal{T}_r(E) := \{z \in \mathbb{R}^n : \operatorname{dist}(z, E) < r\}$ the tubular neighborhood of radius r of a given set $E \subset \mathbb{R}^n$.

Theorem 1. *Let $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$ be a Dir-minimizing function, where $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. Then either $\Delta_Q = \Omega$, or for every $\Omega' \subset\subset \Omega$ the Minkowski dimension of $\Delta_Q \cap \Omega'$ is less than or equal to $n-2$, i.e. for every $\Omega' \subset\subset \Omega$ and for every $\kappa_0 > 0$ there exists a constant $C > 0$ such that*

$$|\mathcal{T}_r(\Delta_Q \cap \Omega')| \le C\, r^{2-\kappa_0} \quad \forall\, 0 < r < \operatorname{dist}(\Omega', \partial\Omega). \tag{1.1}$$

We also obtain a stratification result for the whole set of singular points of multiple valued functions that, even if known to the experts, we were not able to find in the literature. To this aim we introduce the following notation. Given a Q-valued function $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$, we denote by $\mathrm{Sing}_u \subset \Omega$ its singular set, i.e. $x_0 \notin \mathrm{Sing}_u$ if and only if there exists $r > 0$ such that

$$\mathrm{graph}(u|_{B_r(x_0)}) := \{(x, y) \in \mathbb{R}^{n\times m} \ : \ |x - x_0| < r,\, y \in \mathrm{supp}\,(u(x))\}$$

is a smooth n-dimensional embedded submanifold (not necessarily connected). For every $k \in \{0, \dots, n\}$, we define the subset Sing_u^k of the singular set Sing_u made of those points having all tangent functions with at most k independent directions of invariance (we refer to § 5.3 for the precise definition).

Theorem 2. *Let $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$ be a Dir-minimizing function, where $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, and let Sing_u^k be the singular strata defined in § 5.3. Then, $\mathrm{Sing}_u = \mathrm{Sing}_u^{n-2}$ and*

$$\mathrm{Sing}_u^0 \quad \textit{is countable} \tag{1.2}$$

$$\dim_{\mathcal{H}}(\mathrm{Sing}_u^k) \le k \qquad \forall\, k \in \{1, \dots, n-2\}. \tag{1.3}$$

In the case $Q = 2$ a more refined analysis by Krummel and Wickramasekera [**?**] shows the rectifiability of the singular set, remarkably improving Almgren's work.

We prove Theorems 1 and 2 as a consequence of our abstract stratification principle. More precisely, Theorem 2 is a direct consequence of it, while Theorem A requires a further stability property deduced by an ε-regularity result (see Proposition 5.4).

Applications to generalized submanifolds. In the final section § 6 we apply the abstract stratification principle to varifolds with bounded mean curvature and almost minimizers of the perimeter, two relevant cases for applications that are not covered by the results in [**?**]. Also in these cases we derive some improvements of well-known estimates for the singular set. Stratification for the singular set of stationary varifolds with bounded mean curvature is addressed in § 6.1. Eventually, in Theorem 18 we give a bound on the Minkowski dimension of the singular set of an almost minimizer of the perimeter rather than the classical Hausdorff dimension estimate, and in Theorem 19 we show higher integrability for its generalized second fundamental form.

On the organization of the paper. A few words are worthwhile concerning the structure of the paper. The first two sections of the paper are devoted to the abstract regularity results. In particular, § 2 contains the estimate of the volume of the tubular neighborhood of the singular strata given in Theorem 3 (which is proved in the first part of § 3) and the abstract stratification in Theorem 4. In order to make our statements and hypotheses recognizable and "natural" to the readers, we illustrate them in § 2.4 for the model examples of area minimizing currents and harmonic maps. The last part of § 3 is devoted to the comparison with the results by White in [**?**]. Then, we specialize our results to the case of Q-valued functions in §5, the needed preliminaries are collected in § 4. We finally focus on varifolds with suitable hypotheses on their mean curvature and on almost minimizers of the perimeter in § 6.

2. Abstract Stratification

The general abstract approach we propose is based on two main sets of quantities: namely, a family of density functions Θ_s and an increasing family of distance functions d_k.

Densities and distance functions. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and for every $s \geq 0$ set $\Omega^s := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq 2s\}$. We assume the following.

(a) For every s such that $\Omega^s \neq \emptyset$, there exist functions $\Theta_s \in L^\infty(\Omega^s)$ such that

$$0 \leq \Theta_s(x) \leq \Theta_{s'}(x),$$

for all $0 \leq s < s'$ and for all $x \in \Omega^{s'}$. Moreover, for every $s_0 > 0$ there exists $\Lambda_0 = \Lambda_0(s_0) > 0$ such that

$$\Theta_s(x) \leq \Lambda_0,$$

for every $0 \leq s \leq s_0$ and for every $x \in \Omega^{s_0}$.

(b) Setting $U := \{(x, s) : x \in \Omega^s, \Theta_0(x) > 0\}$, there exist a positive integer $m \leq n$ and control functions $\mathrm{d}_k : U \to [0, +\infty)$ for $k \in \{0, \ldots, m\}$ such that

$$\mathrm{d}_0 \leq \mathrm{d}_1 \leq \cdots \leq \mathrm{d}_m.$$

Structural hypotheses. These two sets of quantities are then related by the following structural hypotheses.

(i) For every $s_0 > 0$, $\varepsilon_1 > 0$ there exist $0 < \lambda_1(s_0, \varepsilon_1), \eta_1(s_0, \varepsilon_1) < 1/4$ such that if $(x, s) \in U$, with $x \in \Omega^{s_0}$ and $s < s_0$, then

$$\Theta_s(x) - \Theta_{\lambda_1 s}(x) \leq \eta_1 \quad \Longrightarrow \quad \mathrm{d}_0(x, s) \leq \varepsilon_1.$$

(ii) For every $s_0 > 0$, for every $\varepsilon_2, \tau \in (0, 1)$ there exists $0 < \eta_2(s_0, \varepsilon_2, \tau) \leq \varepsilon_2$ such that if $(x, 5s) \in U$, with $x \in \Omega^{s_0}$ and $5s < s_0$, satisfies for some $k \in \{0, \ldots, m-1\}$

$$\mathrm{d}_k(x, 4s) \leq \eta_2 \quad \text{and} \quad \mathrm{d}_{k+1}(x, 4s) \geq \varepsilon_2,$$

then there exists a k-dimensional linear subspace V for which

$$\mathrm{d}_0(y, 4s) > \eta_2 \quad \forall\, y \in B_s(x) \setminus \mathcal{T}_{\tau s}(x + V),$$

where $\mathcal{T}_{\tau s}(x+V) := \{z : \mathrm{dist}(z, x+V) < \tau s\}$ is the tubular neighborhood of $x + V$ of radius τs.

2.1. Volume of the neighborhoods of singular strata

The sets we consider in our estimates are the following.

Definition 2.1.1 (Singular Strata). *For every* $0 < \delta < 1$, $0 < r \le r_0$ *and for every* $k \in \{0, \ldots, m-1\}$ *we set*

$$\mathcal{S}^k_{r,r_0,\delta} := \big\{ x \in \Omega^{r_0} : \Theta_0(x) > 0 \quad \textit{and} \quad \mathrm{d}_{k+1}(x,s) \ge \delta \quad \forall\, r \le s \le r_0 \big\} \tag{2.1}$$

and

$$\mathcal{S}^k_{r_0,\delta} := \bigcap_{0<r\le r_0} \mathcal{S}^k_{r,r_0,\delta} \quad \textit{and} \quad \mathcal{S}^k_{r_0} := \bigcup_{0<\delta<1} \mathcal{S}^k_{r_0,\delta}. \tag{2.2}$$

Note that, by the monotonicity of the control functions, $\mathcal{S}^k_{r,\delta} \subset \mathcal{S}^{k'}_{r',\delta'}$ if $\delta' \le \delta$, $r \le r'$ and $k \le k'$.

Our abstract stratification result relies on the following estimate for the tubular neighborhoods of the singular strata. Its proof is postponed to §3.

Theorem 3. *Under the Structural Hypotheses 2, for every* $\kappa_0, \delta \in (0,1)$ *and* $r_0 > 0$ *there exists* $C = C(\kappa_0, \delta, r_0, n, \Omega) > 0$ *such that*

$$|\mathcal{T}_r(\mathcal{S}^k_{r,r_0,\delta})| \le C\, r^{n-k-\kappa_0} \quad \forall\, 0 < r < r_0 \quad \forall\, k \in \{1, \ldots, m-1\} \tag{2.3}$$

$$\mathcal{S}^0_{r_0,\delta} \textit{ is countable.} \tag{2.4}$$

2.2. Hausdorff dimension of the singular strata

It is now an immediate consequence of Theorem 3 the following stratification result.

Theorem 4. *Under the Structural Hypotheses 2 for every* $r_0 > 0$ *the estimate* $\dim_{\mathcal{H}}(S^k_{r_0}) \le k$ *holds for* $k \in \{1, \ldots, m-1\}$*. Moreover,* $S^0_{r_0}$ *is countable.*

Proof. Indeed Theorem 3 implies that $\dim_{\mathcal{M}}(S^k_{r_0,\delta}) \le k$, where $\dim_{\mathcal{M}}$ is the Minkowski dimension. Since the Hausdorff dimension of a set is always less than or equal to the Minkowski dimension, we also infer that

$$\dim_{\mathcal{H}}(S^k_{r_0}) \le \dim_{\mathcal{H}} \Big(\bigcup_{\delta>0} S^k_{r_0,\delta} \Big) \le k$$

because, being the union monotone, it is enough to consider a countable set of parameters.

2.3. Minkowski dimension of the singular strata

The dependence of the constant C in (2.3) on δ prevents the derivation of an estimate on the Minkowski dimension of the singular strata $S^k_{r_0}$. Nevertheless, if such dependence drops, then Theorem 3 turns actually into an estimate on the Minkowski dimension of the singular strata which is not implied by the Almgren's stratification principle.

Theorem 5. *Under the hypotheses of Theorem 3, if there exist* $k \in \{0, \ldots, m-1\}$ *and* $\delta_0 > 0$ *such that*

$$\mathcal{S}^k_{r_0,\delta} = \mathcal{S}^k_{r_0} \quad \forall\, \delta \in (0, \delta_0), \tag{2.5}$$

then for every $0 < \kappa_0 < 1$ *and* $r_0 > 0$ *there exists* $C = C(\kappa_0, \delta_0, r_0, n, \Omega) > 0$ *such that*

$$|\mathcal{T}_r(\mathcal{S}^k_{r_0})| \leq C\, r^{n-k-\kappa_0} \quad \forall\, 0 < r < r_0. \tag{2.6}$$

In particular $\dim_{\mathcal{M}}(\mathcal{S}^k_{r_0}) \leq k$.

2.4. Examples

The meaning of the Structural Hypotheses 2 is very well illustrated by the two familiar examples of area minimizing currents and stationary harmonic maps treated in [**?**] for which Theorem 3 and 4 hold. Moreover for area minimizing currents of codimension one in $\mathbb{R}^n$ Theorem 5 can be also applied for $k = n - 8$.

2.4.1. Area minimizing currents

Let T be an m-dimensional area minimizing integral current in Ω. Then we can set

$$\Theta_s(x) := \frac{\|T\|(B_s(x))}{w_m s^m} \quad \text{for} \quad s > 0 \quad \text{and} \quad \Theta_0(x) := \lim_{r \downarrow 0^+} \Theta_r(x)$$

and for $k \in \{0, \ldots, m\}$

$$\mathrm{d}_k(x,s) := \inf \big\{ \mathbb{F}\big((T_{x,s} - C) \llcorner B_1\big) : C \text{ is } k\text{-conical \& area minimizing} \big\},$$

where

- $T_{x,r}$ is the rescaling of the current around any point $x \in \mathbb{R}^n$ at scale $r > 0$:

$$T_{x,r} := \big(\eta_{x,r}\big)_{\#} T \tag{2.7}$$

  and the push-forward is done via the proper map $\eta_{x,r}$ given by $y \mapsto {}^{(y-x)}/_{r}$;

- $\mathbb{F}$ is the flat norm (see [**?** , § 31]);

- an m-dimensional current C in $\mathbb{R}^n$ is *k-conical* for $k \in \{0, \dots, m\}$, if there exists a linear subspace $V \subset \mathbb{R}^n$ of dimension bigger than or equal to k such that
$$T_{x,r} = T \text{ for all } r > 0 \text{ and } x \in V.$$
Note that a 0-conical current is simply a cone with respect to the origin.

One can choose $\Lambda_0(r_0) := {}^{\mathbb{M}(T)}/_{\omega_m r_0^m}$. Then (a) is a consequence of the Monotonicity Formula (see [**?** , Theorem 17.6]) and (b) follows from the inclusion of k-conical currents in the k'-conical ones when $k' \leq k$. With this choice, the structural hypoteses in 2 are satisfied, indeed (i) is an other consequence of the Monotonicity Formula and (ii) follows from a rigidity property of cones sometimes called "cylindrical blowup" (see [**?** , Lemma 35.5]).

Then the quantitative stratification principle in Theorem 3 recovers the corresponding result in [**?**]:

> *the set of points that are faraway from $(k+1)$-conical area minimizing currents, at every scale in $[r, r_0]$, has Minkowski dimension less than or equal to k.*

2.4.2. Stationary harmonic maps

Similarly let $u \in W^{1,2}(\Omega, \mathscr{N})$ be a stationary harmonic map from an open set $\Omega \subset \mathbb{R}^n$ to a Riemannian manifold $(\mathscr{N}^m, h)$ isometrically embedded in some Euclidean space $\mathbb{R}^p$ (see, e.g., [**?**]). We can set

$$\Theta_s(x) := s^{2-n} \int_{B_s(x)} |\nabla u|^2 dy, \quad s \in \big(0, \operatorname{dist}(x, \partial\Omega)\big),$$

and for every $k \in \{0, \dots, n\}$

$$\mathrm{d}_k(x, r) := \inf_{v \in \mathscr{C}_k} \fint_{B_1} \operatorname{dist}^2_{\mathscr{N}}\big(u_{x,r}, v\big) dy,$$

with

- $u_{x,r}(y) := u(x + ry)$ for $x \in \Omega$ and $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$;
- a measurable map v is said to be *k-conical* if there exists a vector space V with $\dim V \geq k$ that leaves v invariant, i.e.
$$v(x) = v(y + x) \quad \forall\, x \in \mathbb{R}^n,\, y \in V, \tag{2.8}$$
and such that v is 0-homogeneous with respect to the points in V, i.e.
$$v(y + x) = v(y + \lambda\, x) \quad \forall\, x \in \mathbb{R}^n,\, y \in V \text{ and } \lambda > 0; \tag{2.9}$$
- $\mathscr{C}_k := \{v : B_1 \to \mathscr{N} \ k\text{-conical}\}$.

Assumption (a) in § 2 is easily verified and the monotonicity formula

$$\Theta_r(x) - \Theta_s(x) = \int_s^r \int_{\partial B_t(x)} t^{2-n} \left|\frac{\partial u}{\partial t}\right|^2 d\mathcal{H}^{n-1} dt$$

together with an elementary contradiction argument show that the Structural hypothesis (i) in § 2 is satisfied. Moreover the structural hypothesis (ii) follows similarly to the one for minimizing currents (cp. [**?**] for more details), thus leading to the stratification of Theorem 3.

In Section 6 we give other applications of this abstract regularity result to the case of varifolds with bounded variation and almost minimizers of the mass in codimension one.

3. Proof of the Abstract Stratification and comparison with Almgren's Stratification

3.1. Preliminary results

To begin with, we state a simple consequence of the Structural Hypothesis 2 (ii) in the following

Lemma 6. *For every $s_0 > 0$, for every $\varepsilon, \tau \in (0,1)$ there exists $0 < \gamma_0 \leq \varepsilon$ such that if $(x, 5s) \in U$, with $x \in \Omega^{s_0}$ and $5s < s_0$, satisfies for some $k \in \{0, \dots, m-1\}$*

$$\mathrm{d}_0(x, 4s) \leq \gamma_0 \quad \textit{and} \quad \mathrm{d}_{k+1}(x, 4s) \geq \varepsilon,$$

then there exists a linear subspace V with $\dim(V) \leq k$ such that

$$y \in B_s(x) \;\; \& \;\; \mathrm{d}_0(y, 4s) \leq \gamma_0 \quad \Longrightarrow \quad y \in \mathcal{T}_{\tau s}(x + V). \tag{3.1}$$

Proof. Let $\gamma_0 \leq \gamma_1 \leq \ldots \leq \gamma_{k+1}$ be set as $\gamma_{k+1} = \varepsilon$ and $\gamma_{j-1} = \eta_2(s_0, \gamma_j, \tau)$ with η_2 the constant in the Structural Hypothesis (ii). Let $i \in \{0, \dots, k\}$ be the smallest index such that $\mathrm{d}_{i+1}(x, 4s) \geq \gamma_{i+1}$ (which exists because of the assumption $\mathrm{d}_{k+1}(x, 4s) \geq \varepsilon = \gamma_{k+1}$). Then, applying the Structural Hypothesis (ii) we deduce that there exists an i-dimensional linear subspace V such that every point $y \in B_s(x)$ with $\mathrm{d}_0(y, 4s) \leq \gamma_0 \leq \gamma_i$ belongs to the tubular neighborhood $\mathcal{T}_{\tau s}(x + V)$.

In the proof of Theorem 3 we shall repeatedly use the following elementary covering argument.

Lemma 7. *For every measurable set $E \subset \mathbb{R}^n$ with finite measure and for every $\rho > 0$, there exists a finite covering $\{B_\rho(x_i)\}_{i \in I}$ of $\mathcal{T}_{\rho/5}(E)$ with $x_i \in E$ and*

$$\mathcal{H}^0(I) \leq \frac{5^n \, |\mathcal{T}_{\rho/5}(E)|}{\omega_n \, \rho^n}. \tag{3.2}$$

Proof. Consider the family of balls $\{B_{\rho/5}(x)\}_{x \in E}$. By the Vitali $5r$-covering lemma, there exists a finite subfamily $\{B_{\rho/5}(x_i)\}_{i \in I}$ of disjoint balls such that $\mathcal{T}_{\rho/5}(E) \subset \cup_{i \in I} B_\rho(x_i)$. By a simple volume comparison we conclude (3.2).

3.2. Proof of Theorem 3

Proof (of Theorem 3). We start fixing a parameter $\tau = \tau(n, \kappa_0) > 0$ such that

$$\omega_n \, \tau^{\frac{\kappa_0}{2}} \leq 20^{-n}. \tag{3.3}$$

We choose the other constants involved in the Structural Hypotheses in the following way:

1. let $\gamma_0 \leq \gamma_1 \leq \ldots \leq \gamma_k$ be such that $\gamma_k = \delta$ and $\gamma_{j-1} = \eta_2(r_0, \gamma_j, \tau)$ for every $j \in \{1, \ldots, k\}$ as in the Structural Hypothesis (ii);
2. let $\lambda_1 = \lambda_1(r_0, \gamma_0)$ and $\eta_1 = \eta_1(r_0, \gamma_0)$ be as in the Structural Hypothesis (i);
3. fix $q \in \mathbb{N}$ such that $\tau^q \leq \lambda_1$.

We divide the proof into four steps.

Step 1: reduction to dyadic radii. Let $\Lambda_0 = \Lambda_0(r_0)$ given in § 2. It suffices to prove (2.3) for every r of the form $r = \frac{r_0 \tau^p}{5}$ with $p \in \mathbb{N}$ such that $p \geq p_0 := q + M + 1$ and $M := \lfloor q \, \Lambda_0 / \eta_1 \rfloor$. Indeed for $\frac{r_0 \tau^{p_0}}{5} < s < r_0$ we simply have

$$\begin{aligned}
|\mathcal{T}_s(\mathcal{S}^k_{s,r_0,\delta})| \leq |\Omega| &\leq \frac{|\Omega|}{\left(\frac{r_0 \tau^{p_0}}{5}\right)^{n-k-\kappa_0}} \, s^{n-k-\kappa_0} \\
&= C_2(\kappa_0, \delta, r_0, n, \Omega) \, s^{n-k-\kappa_0}.
\end{aligned}$$

On the other hand, if we assume that (2.3) holds with a constant $C_1 > 0$ for every r of the form $r = \frac{r_0 \tau^p}{5}$ with $p \geq p_0$, we conclude that for $r\tau < s < r$ it holds

$$|\mathcal{T}_s(\mathcal{S}^k_{s,r_0,\delta})| \leq |\mathcal{T}_r(\mathcal{S}^k_{r,r_0,\delta})| \leq C_1 \, r^{n-k-\kappa_0} \leq C_1 \, \tau^{k+\kappa_0-n} \, s^{n-k-\kappa_0}.$$

Therefore setting $C := \max\{\tau^{k+\kappa_0-n} \, C_1, \, C_2\}$ we deduce that (2.3) holds for every $r \in (0, r_0)$.

Step 2: selection of good scales. Fix a value $p \in \mathbb{N}$ with $p \geq p_0$ as above and set $r = {}^{r_0 \tau^p}/_5$. For all $(x, r_0) \in U$ we have

$$\begin{aligned}
\sum_{l=q}^{p} \Theta_{4\tau^l \, r_0}(x) - \Theta_{4\tau^{l+q} \, r_0}(x) &= \sum_{l=q}^{p} \sum_{i=l}^{l+q-1} \Theta_{4\tau^i \, r_0}(x) - \Theta_{4\tau^{i+1} \, r_0}(x) \\
&\leq q \sum_{h=q}^{p+q-1} \left(\Theta_{4\tau^h \, r_0}(x) - \Theta_{4\tau^{h+1} \, r_0}(x)\right) \\
&= q \left(\Theta_{4\tau^q r_0}(x) - \Theta_{4\tau^{p+q} \, r_0}(x)\right) \leq q \, \Lambda_0.
\end{aligned}$$

Therefore, there exist at most M indices $l \in \{q, \ldots, p\}$ for which it does not hold that

$$\Theta_{4\tau^l \, r_0}(x) - \Theta_{4\tau^{l+q} \, r_0}(x) \leq \eta_1. \tag{3.4}$$

For any subset $A \subset \{q, \ldots, p\}$ with cardinality M we consider

$$S_A := \left\{x \in S^k_{r,r_0,\delta} : (3.4) \ \text{ holds } \ \forall\, l \notin A\right\}.$$

We prove in the next step that

$$|\mathcal{T}_r(S_A)| \le C\, r^{n-k-\frac{\kappa_0}{2}} \tag{3.5}$$

for some $C = C(\kappa_0, \delta, r_0, n, \Omega) > 0$. From (3.5) one concludes because the number of subsets A as above is estimated by

$$\binom{p-q+1}{M} \le (p-q+1)^M \le C\,|\log r|^M$$

for some $C(\kappa_0, \delta, r_0, n) > 0$, and

$$|\mathcal{T}_r(S^k_{r,r_0,\delta})| \le \sum_A |\mathcal{T}_r(S_A)| \le C\,|\log r|^M\, r^{n-k-\frac{\kappa_0}{2}} \le C\, r^{n-k-\kappa_0}$$

for some $C(\kappa_0, \delta, r_0, n, \Omega) > 0$.

Step 3: proof of (3.5). We estimate the volume of $\mathcal{T}_r(S_A)$ by covering it iteratively with families of balls centered in S_A and with radii $\tau^j r_0$ for $j \in \{q, \ldots, p\}$. We can then proceed as follows. Firstly we consider a cover of $\mathcal{T}_{\tau^q r_0/5}(S_A)$ made of balls $\{B_{\tau^q r_0}(x_i)\}_{i \in I_q}$ with $x_i \in S_A$ and by a straightforward use of Lemma 7

$$\mathcal{H}^0(I_q) \le 5^n \tau^{-nq} r_0^{-n} \big(\mathrm{diam}(\Omega) + 1\big)^n.$$

Iteratively, for every $j \in \{q+1, \ldots, p\}$, we assume to be given the cover $\{B_{\tau^{j-1} r_0}(x_i)\}_{i \in I_{j-1}}$ of $\mathcal{T}_{\tau^{j-1} r_0/5}(S_A)$, and we select a new cover of $\mathcal{T}_{\tau^j r_0/5}(S_A)$ which is made of balls of radii $\tau^j r_0$ centered in S_A according to the following two cases:

(a) $j - 1 \in A$,

(b) $j - 1 \notin A$.

Case (a). For every x_i in the family at level $j-1$, using Lemma 7 we cover $S_A \cap B_{\tau^{j-1} r_0}(x_i)$ with finitely many balls $B_{\tau^j r_0/2}(y_l)$ with $y_l \in S_A \cap B_{\tau^{j-1} r_0}(x_i)$ and the cardinality of the cover is bounded by

$$\frac{5^n\, |B_{(\tau^{j-1} + \tau^j/10)\, r_0}(x_i)|}{\omega_n\, \left(\tau^j r_0/2\right)^n} \le 20^n\, \tau^{-n}$$

(note that $\mathcal{T}_{\tau^j r_0/10}(S_A \cap B_{\tau^{j-1} r_0}(x_i)) \subset B_{(\tau^{j-1} + \tau^j/10)\, r_0}(x_i)$). We claim next that the union of $B_{\tau^j r_0}(y_l)$ covers the tubular neighborhood

$$\mathcal{T}_{\frac{\tau^j r_0}{5}}(S_A \cap B_{\tau^{j-1} r_0}(x_i)).$$

Indeed for every $z \in \mathcal{T}_{\tau^j r_0/5}(S_A \cap B_{\tau^{j-1} r_0}(x_i))$ there exists $z' \in S_A \cap B_{\tau^{j-1} r_0}(x_i)$ such that $|z - z'| < \tau^j r_0/5$. Since $z' \in B_{\tau^j r_0/2}(y_l)$ for some y_l, then $z \in B_{\tau^j r_0}(y_l)$.

Therefore, collecting all such balls, the cardinality of the new covering is estimated by

$$\mathcal{H}^0(I_j) \le 20^n\, \tau^{-n}\, \mathcal{H}^0(I_{j-1}). \tag{3.6}$$

Case (b). If $j-1 \notin A$, then (3.4) holds with $l = j-1$. By the Structural Hypothesis (i) and the choice of λ_1, η_1 in (2) and τ in (3) at the beginning of the proof, we have that $\mathrm{d}_0(x, 4\tau^{j-1} r_0) \le \gamma_0$ for every $x \in S_A$. Since $x_i \in S_A \subset S^k_{r,r_0,\delta}$ we have also $\mathrm{d}_{k+1}(x_i, 4\tau^{j-1} r_0) \ge \delta$. We can then apply Lemma 6 and conclude that

$$S_A \cap B_{\tau^{j-1} r_0}(x_i) \subset \mathcal{T}_{\tau^j r_0}(x_i + V) \tag{3.7}$$

for some linear subspace V of dimension less than or equal to k. Note that

$$|\mathcal{T}_{\tau^j r_0}((x_i + V) \cap B_{\tau^{j-1} r_0}(x_i))| \le \omega_n\, \tau^{n-k}\, |B_{\tau^{j-1} r_0}(x_i)|. \tag{3.8}$$

Thus applying Lemma 7 we find a covering of $\mathcal{T}_{\tau^j r_0/5}(S_A)$ with balls $B_{r_0 \tau^j}(y_l)$ such that $y_l \in S_A$ and using (3.8) the cardinality of the covering is bounded by

$$\mathcal{H}^0(I_j) \le 10^n \omega_n\, \mathcal{H}^0(I_{j-1})\, \tau^{-k}. \tag{3.9}$$

In any case the procedure ends at $j = p$ with a covering of $\mathcal{T}_{\tau^p r_0/5}(S_A)$ which is made of balls $\{B_{\tau^p r_0}(x_i)\}_{i \in I_p}$ such that $x_i \in S_A$ and

$$\begin{aligned}
\mathcal{H}^0(I_p) &\le 5^n \tau^{-nq} r_0^{-n} \big(\mathrm{diam}(\Omega) + 1\big)^n \big(20^n\, \tau^{-n}\big)^M \,\big(10^n \omega_n\, \tau^{-k}\big)^{p-q-M} \\
&\le C\, \tau^{-kp} (20^n \omega_n)^p \le C\, \tau^{-p\left(k + \frac{\kappa_0}{2}\right)}
\end{aligned} \tag{3.10}$$

with $C = C(\kappa_0, \delta, r_0, n, \Omega) > 0$ and where we used (3.3) in the last inequality. Estimate (3.5) follows at once

$$|\mathcal{T}_r(S_A)| \le \mathcal{H}^0(I_p)\, |B_{\tau^p r_0}| \overset{(3.10)}{\le} C\, r^{n-k-\frac{\kappa_0}{2}},$$

for some $C = C(\kappa_0, \delta, r_0, n, \Omega) > 0$.

Step 4: proof of (2.4). Let j_x be the smallest index such that (3.4) holds for every $j \ge j_x$, and for every $i \in \mathbb{N}$ set

$$A_i := \{x \in \mathcal{S}^0_{r_0,\delta} \;:\; j_x = i\}.$$

We will prove that A_i is discrete, and hence $\mathcal{S}^0_{r_0,\delta}$ is at most countable. Fix $x \in A_i$. By the choice of the parameters applying the Structural Hypothesis (i) it follows that $\mathrm{d}_0(x, 4r_0\tau^j) \le \gamma_0$ for every $j \ge i$. Since $x \in \mathcal{S}^0_{r_0,\delta}$, we can apply Lemma 6 and infer that the points $y \in B_{r_0\tau^j}(x)$ satisfying $\mathrm{d}_0(y, 4r_0\tau^j) \le \gamma_0$ are contained in $B_{r_0\tau^{j+1}}(x)$. Therefore $A_i \cap B_{r_0\tau^j}(x) \subset B_{r_0\tau^{j+1}}(x)$ for every $j \ge i$, which implies that A_i is discrete.

3.3. Almgren's stratification principle

In this section we make the connection to the approach to Almgren's stratification principle by White [**?**]. Indeed under very natural assumptions the results by White for the time independent case follow from ours.

. White's stratification criterion in its simplest formulation is based on:

(a′) an upper semi-continuous function $f : \Omega \to [0, \infty)$ defined on a bounded open set $\Omega \subset \mathbb{R}^n$;

(b′) for every $x \in \Omega$ a compact class of conical functions $\mathcal{G}(x)$ according to the following definition.

Definition 3.3.1. *(1) An upper semi-continuous map $g : \mathbb{R}^n \to [0, \infty)$ is* conical *if $g(z) = g(0)$ implies that g is positively 0-homogeneous with respect to z, i.e.,*

$$g(z + \lambda x) = g(z + x) \ \text{ for all } x \in \mathbb{R}^n \text{ and } \lambda > 0.$$

(2) A class $\mathscr{G}$ of conical functions is compact *if for all sequences $(g_i)_{i\in\mathbb{N}} \subseteq \mathscr{G}$ there exist a subsequence $(g_{i_j})_{j\in\mathbb{N}}$ and an element $g \in \mathscr{G}$ such that*

$$\limsup_{j\to\infty} g_{i_j}(y_{i_j}) \le g(y) \quad \forall\, y \in \mathbb{R}^n,\ (y_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n \ \text{ with } \ y_i \to y.$$

In particular a conical function is 0-homogeneous with respect to 0.

White's Structural Hypotheses. The stratification theorem by White is then based on the following two structural hypotheses:

(i′) $g(0) = f(x)$ for all $g \in \mathscr{G}(x)$;

(ii′) for all $r_i \downarrow 0$ there exist a subsequence $r_{i_j} \downarrow 0$ and $g \in \mathscr{G}(x)$ such that

$$\limsup_{j\to+\infty} f(x + r_{i_j} y_j) \le g(y) \quad \text{for all } y, y_j \in B_1 \text{ with } y_j \to y.$$

White's result. By the upper semi-continuity of any conical function g, the closed set

$$S_g := \{z \in \mathbb{R}^n : g(z) = g(0)\}$$

is in fact the set of the maximum points of g. S_g is called the *spine* of g. Moreover S_g is the largest vector space that leaves g invariant, i.e.,

$$S_g = \{z \in \mathbb{R}^n : g(y) = g(z + y) \ \text{for all } y \in \mathbb{R}^n\} \tag{3.11}$$

(cp. [**?** , Theorem 3.1]). We set $d(x) := \sup\{\dim S_g : g \in \mathscr{G}(x)\}$, and

$$\Sigma_\ell := \{x \in \Omega : f(x) > 0,\, d(x) \le \ell\}.$$

The stratification criterion in [**?** , Theorem 3.2] is the following.

Theorem 8 (White). *Under the Structural Hypotheses 3.3,*

$$\Sigma_0 \ \textit{ is countable}; \tag{3.12}$$

$$\dim_{\mathcal{H}}(\Sigma_\ell) \le \ell \quad \forall\, \ell \in \{1, \dots, n\}, \tag{3.13}$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

The reader who is interested in the application of this criterion to the model cases of area minimizing currents and harmonic maps is referred to [**?**].

Comparing the two approaches. Theorem 8 can be recovered from our Theorem 4 if we assume the following relations between the Structural Hypotheses 2 and 3.3:

(1) $f = \Theta_0$;

(2) for every $x \in \Omega$, if

$$\lim_j \mathrm{d}_k(x, r_j) = 0 \quad \text{for some } (r_j)_{j\in\mathbb{N}} \subset (0, \operatorname{dist}(x, \partial\Omega)),$$

then $x \notin \Sigma_{k-1}$.

Note that (1) and (2) are always satisfied in the relevant examples considered in the literature.

Theorem 8 follows from Theorem 4. To prove that the conclusions of Theorem 8 are implied by Theorem 4 it is enough to show that

$$\Sigma_\ell \subset \bigcup_{r_0>0} \mathcal{S}^\ell_{r_0}. \tag{3.14}$$

This means that for every $r_0 > 0$ and for every $x \in \Sigma_\ell \cap \Omega^{r_0}$ there exists $\delta > 0$ such that

$$\mathrm{d}_{\ell+1}(x, r) \geq \delta \quad \forall\, 0 < r \leq r_0. \tag{3.15}$$

Assume by contradiction that (3.15) does not hold, we find r_0 and x as above such that for a sequence $r_j \in (0, r_0]$ we have $\mathrm{d}_{\ell+1}(x, r_j) \downarrow 0$. Then by § 3.3 (2) x cannot belong to Σ_ℓ.

4. Preliminary results on Dir-minimizing Q-valued functions

We follow [**?**] for the notation and the terminology, which we briefly recall in the following subsections.

The space of Q-points of $\mathbb{R}^m$ is the subspace of positive atomic measures in $\mathbb{R}^m$ with mass Q, i.e.

$$\mathcal{A}_Q(\mathbb{R}^m) := \left\{ \sum_{i=1}^{Q} [\![p_i]\!] \;:\; p_i \in \mathbb{R}^m \right\}$$

where $[\![p_i]\!]$ denotes the Dirac delta at p_i. $\mathcal{A}_Q$ is endowed with the complete metric $\mathcal{G}$ given by: for every $T = \sum_i [\![p_i]\!]$ and $S = \sum_i [\![p'_i]\!] \in \mathcal{A}_Q(\mathbb{R}^m)$

$$\mathcal{G}(T, S) := \min_{\sigma \in \mathscr{P}_Q} \left(\sum_{i=1}^{Q} |p_i - p'_{\sigma(i)}|^2 \right)^{1/2}$$

where $\mathscr{P}_Q$ is the symmetric group of Q elements.

A Q-valued function is a measurable map $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$ from a bounded open set $\Omega \subset \mathbb{R}^n$ (with smooth boundary $\partial\Omega$ for simplicity). It is always possible to find measurable functions $u_i : \Omega \to \mathbb{R}^m$ for $i \in \{1, \dots, Q\}$ such that $u(x) = \sum_i [\![u_i(x)]\!]$ for a.e. $x \in \Omega$. Note that the u_i's are not uniquely determined: nevertheless, we often use the notation $u = \sum_i [\![u_i]\!]$ meaning an admissible choice of the functions u_i's has been fixed. We set

$$|u|(x) := \mathcal{G}(u(x), Q\,[\![0]\!]) = \left(\sum_i |u_i(x)|^2\right)^{1/2}.$$

The definition of the Sobolev space $W^{1,2}(\Omega, \mathcal{A}_Q)$ is given in [**?** , Definition 0.5] and leads to the notion of approximate differential $Du = \sum_i [\![Du_i]\!]$ (cp. [**?** , Definitions 1.9 & 2.6]. We set

$$|Du|(x) := \left(\sum_i |Du_i(x)|^2\right)^{1/2}$$

and say that a function $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ is *Dir-minimizing* if

$$\int_\Omega |Du|^2 \le \int_\Omega |Dv|^2 \quad \forall\, v \in W^{1,2}(\Omega),\ v|_{\partial\Omega} = u|_{\partial\Omega}$$

where the last equality is meant in the sense of traces (cp. [**?** , Definition 0.7]). By [**?** , Theorem 0.9] Dir-minimizing Q-valued functions are locally Hölder continuous with exponent $\beta = \beta(n, Q) > 0$.

In what follows we shall always assume that $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ is a nontrivial Dir-minimizing function, i.e. $u \not\equiv Q\,[\![0]\!]$, with

$$\boldsymbol{\eta} \circ u := \frac{1}{Q} \sum_{i=1}^{Q} u_i \equiv 0. \tag{4.1}$$

As explained in [**?** , Lemma 3.23] the mean value condition in (4.1) does not introduce any substantial restriction on the space of Dir-minimizing functions. Moreover, in this case Δ_Q reduces to the set $\{x \in \Omega : u(x) = Q\,[\![0]\!]\}$. Note that, if $u \not\equiv Q\,[\![0]\!]$, then $\Delta_Q \subset \mathrm{Sing}_u$ by [**?** , Theorem 0.11].

4.1. Frequency function

We start by introducing the following quantities: for every $x \in \Omega$ and $s > 0$ such that $B_s(x) \subset \Omega$ we set

$$D_u(x,s) := \int_{B_s(x)} |Du|^2$$

$$H_u(x,s) := \int_{\partial B_s(x)} |u|^2$$

$$I_u(x,s) := \frac{s\, D_u(x,s)}{H_u(x,s)}.$$

I_u is called the *frequency function* of u. Since u is Dir-minimizing and nontrivial, it holds that $H_u(x,s) > 0$ for every $s \in (0, \mathrm{dist}(x, \partial\Omega))$ (cp. [**?** , Remark 3.14]), from which I_u is well-defined.

We recall that the functions $s \mapsto D_u(x,s)$, $s \mapsto H_u(x,s)$, and $s \mapsto I_u(x,s)$ are absolutely continuous on $(0, \mathrm{dist}(x, \partial\Omega))$. Similarly for fixed $s \in (0, \mathrm{dist}(x, \partial\Omega))$ one can prove the continuity of $x \mapsto D_u(x,s)$, $x \mapsto H_u(x,s)$ and $x \mapsto I_u(x,s)$ for $x \in \{y : \mathrm{dist}(y, \partial\Omega) > s\}$. The former follows by the absolute continuity of Lebesgue integral; while for the remaining two it suffices the following estimate:

$$
\begin{aligned}
\left|\sqrt{H_u(x,s)} - \sqrt{H_u(y,s)}\right| &\le \left(\int_{\partial B_s(y)} ||u|(z) - |u|(z+x-y)|^2\, dz\right)^{\frac{1}{2}} \\
&\le |x-y| \left(\int_{\partial B_s(y)} \int_0^1 |\nabla |u|(z + t\,(x-y))|^2\, dt\, dz\right)^{\frac{1}{2}} \\
&\le |x-y| \left(\int_{B_{s+|x-y|}(y) \setminus B_{s-|x-y|}(y)} |Du|^2\right)^{\frac{1}{2}} \qquad (4.2)
\end{aligned}
$$

where we use the fact that $|u| \in W^{1,2}(\Omega)$ with $|\nabla |u|| \le |Du|$ (cp. [**?** , Definition 0.5]).

T. he following monotonicity formula discovered by Almgren in [**?**] is the main estimate about Dir-minimizing functions (cp. [**?** , Theorem 3.15 & (3.48)]): for all $0 \le r_1 \le r_2 < \mathrm{dist}(x, \partial\Omega)$ it holds

$$
\begin{aligned}
&I_u(x,r_2) - I_u(x,r_1) \\
&\quad = \int_{r_1}^{r_2} \frac{t}{H_u(t)} \Big(\int_{\partial B_t(x)} |\partial_\nu u|^2 \int_{\partial B_t(x)} |u|^2 - \Big(\int_{\partial B_t(x)} \langle \partial_\nu u, u\rangle\Big)^2\Big)\, dt. \qquad (4.3)
\end{aligned}
$$

We finally recall that from [**?** , Corollary 3.18] we also deduce that

$$
H_u(z,r) = O(r^{n+2\,I_u(z,0^+)-1}) \qquad (4.4)
$$

where $I_u(z,0^+) = \lim_{r\downarrow 0} I_u(z,r)$.

4.2. Compactness

From [**?** , Proposition 2.11 & Theorem 3.20], if $(u_j)_{j\in\mathbb{N}}$ is a sequence of Dir-minimizinig functions in Ω such that

$$
\sup_j \|u_j\|_{L^2(\Omega)} + \sup_j \int_\Omega |Du_j|^2 < +\infty,
$$

then there exists $u \in W^{1,2}(\Omega, \mathcal{A}_Q)$ such that u is Dir-minimizing, and up to passing to a subsequence (not relabeled in the sequel) $\mathcal{G}(u_j, u) \to 0$ in $L^2(\Omega)$, and for every $\Omega' \subset\subset \Omega$

$$
\|\mathcal{G}(u_j, u)\|_{L^\infty(\Omega')} \to 0 \quad \text{and} \quad \int_{\Omega'} |Du_j|^2 \to \int_{\Omega'} |Du|^2.
$$

In particular this implies that $(|Du_j|^2)_{j\in\mathbb{N}}$ are equi-integrable in Ω', and

$$\lim_{j\to+\infty} I_{u_j}(x,s) = I_u(x,s) \quad \forall\, x \in \Omega,\ \forall\, 0 < 2s < \operatorname{dist}(x,\partial\Omega). \tag{4.5}$$

4.3. Homogeneous Q-valued functions

We discuss next some properties of the class of homogeneous Q-valued functions: $w \in W^{1,2}_{\rm loc}(\mathbb{R}^n, \mathcal{A}_Q(\mathbb{R}^m))$ satisfying

(1) w is locally Dir-minimizing with $\boldsymbol{\eta}\circ w \equiv 0$,

(2) w is α-homogeneous, in the sense that

$$w(x) = |x|^\alpha\, w\left(\frac{x}{|x|}\right) \quad \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

for some $\alpha \in (0, \Lambda_0]$, where Λ_0 is a constant to be specified later.

We denote this class by $\mathcal{H}_{\Lambda_0}$. Note that $I_w(x,0^+) = 0$ if $w(x) \neq Q\,[\![0]\!]$. The following lemma is an elementary consequence of the definitions.

Lemma 9. *Let $w \in \mathcal{H}_{\Lambda_0}$. Then $I_w(\cdot,0^+)$ is conical in the sense of Definition 3.3.1 (1).*

Proof. Firstly $I_w(\cdot,0^+)$ is upper semi-continuous. Indeed since w is Dir-minimizing, we can use (4.3) and deduce that $I_w(x,0^+) = \inf_{s>0} I_w(x,s)$, i.e. $I_w(\cdot,0^+)$ is the infimum of continuous (by (4.2)) functions $x \mapsto I_w(x,s)$ and hence upper semi-continuous.

We need only to show that $I_w(\cdot,0^+)$ is 0-homogeneous at every point z such that $I_w(z,0^+) = I_w(0,0^+)$. We can assume without loss of generality that w is nontrivial, i.e. $w \not\equiv Q\,[\![0]\!]$. We start noticing that if $I_w(z,0^+) = I_w(0,0^+)$ then

$$I_w(z,0^+) = I_w(0,0^+) = I_w(0,1) > 0$$

where in the last equality we used the homogeneity of w. Therefore in particular $w(z) = Q\,[\![0]\!]$. Next we show that $I_w(z,r) = I_w(0,0^+)$ for all $r > 0$. By a simple estimate we get

$$I_w(z,r) = \frac{r\,D_w(z,r)}{H_w(z,r)} \le I_w(0,r+|z|)\ \frac{H_w(0,r+|z|)}{H_w(0,r)}\ \frac{H_w(0,r)}{H_w(z,r)}. \tag{4.6}$$

Since w is homogeneous with respect to the origin and the frequency of w at 0 is exactly α (cp. [**?** , Corollary 3.16]), we have also

$$H_w(0,r) = H_w(0,1)\, r^{n+2\,\alpha-1}$$
$$D_w(0,r) = D_w(0,1)\, r^{n+2\,\alpha-2}.$$

In particular

$$I_w(0,r+|z|) = \alpha = I_w(0,0^+) = I_w(z,0^+)$$
$$\frac{H_w(0,r+|z|)}{H_w(0,r)} \to 1 \quad \text{as} \quad r \uparrow +\infty.$$

For what concerns the third factor in (4.6)

$$\frac{H_w(0,r)}{H_w(z,r)} = 1 + \frac{H_w(0,r) - H_w(z,r)}{H_w(z,r)} \tag{4.7}$$

and from (4.4) and (4.2) we infer that

$$\begin{aligned}
|H_w(0,r) - H_w(z,r)| &= \big(\sqrt{H_w(0,r)} + \sqrt{H_w(z,r)}\big)|\sqrt{H_w(0,r)} - \sqrt{H_w(z,r)}| \\
&\leq C\, r^{\frac{n+2\, I_u(0,0^+)-1}{2}}\, |z|\big(D_w(0,r+|z|) - D_w(0,r-|z|)\big)^{\frac{1}{2}} \\
&\leq C\, |z|\, r^{\frac{n+2\, I_u(0,0^+)-1}{2}}\, \big((r+|z|)^{n+2\,\alpha-2} - (r-|z|)^{n+2\,\alpha-2}\big)^{\frac{1}{2}} \\
&\leq C\, |z|^{\frac{3}{2}}\, r^{n+2\,\alpha-2}.
\end{aligned} \tag{4.8}$$

This in turn implies
$$\frac{H_w(0,r)}{H_w(z,r)} \to 1 \quad \text{as} \;\; r \uparrow +\infty$$
and from (4.6)
$$\lim_{r\to+\infty} I_w(z,r) \leq \lim_{r\downarrow 0^+} I_w(z,r),$$
i.e. by Almgren's monotonicity estimate (4.3) we deduce that $I_w(z,r) = I_w(z,0^+)$ for all $r > 0$. As a consequence (cp. [**?** , Corollary 3.16]) w is α-homogeneous at z which straightforwardly implies that $I_w(\cdot,0^+)$ is 0-homogeneous at z.

Spines. We can then define the spine of a homogeneous Q-valued function $w \in \mathcal{H}_{\Lambda_0}$:
$$S_w := \{x \in \mathbb{R}^n \;:\; I_w(x,0^+) = I_w(0,0^+)\}.$$
By the proof of Lemma 9 it follows that w is α-homogeneous at every point $x \in S_w$. Similarly it is simple to verify that S_w is the largest vector space which leaves w invariant, as well as $I_w(\cdot,0^+)$:

$$S_w = \big\{z \in \mathbb{R}^n \;:\; w(y) = w(z+y) \quad \forall\, y \in \mathbb{R}^n\big\}. \tag{4.9}$$

Indeed it is enough to prove that every $z \in S_w$ leaves w invariant (the other inclusion is obvious). To show this, note that by the α-homogeneity of w at z and 0 it follows that for every $y \in \mathbb{R}^n$

$$\begin{aligned}
w(y) &= w\,(z+y-z) = 2^\alpha\, w\left(z + \frac{y-z}{2}\right) = 2^\alpha\, w\left(\frac{y+z}{2}\right) \\
&= w\,(z+y)\,.
\end{aligned}$$

W. e denote by $\mathcal{C}_k$ for $k \in \{0,\ldots,n\}$ the set of k-invariant homogeneous Q-functions
$$\mathcal{C}_k := \{w \in \mathcal{H}_{\Lambda_0} \;:\; \dim(S_w) \geq k\}. \tag{4.10}$$
Note that $\mathcal{C}_n = \mathcal{C}_{n-1} = \{Q\,[\![0]\!]\}$, i.e. these sets are singleton consisting of the constant function $w \equiv Q\,[\![0]\!]$. For $\mathcal{C}_n$ this is follows straightforwardly from the

definition and (4.9). While for $\mathcal{C}_{n-1}$ one can argue via the cylindrical blowup in [**?** , Lemma 3.24]. Assume without loss of generality that

$$w \in \mathcal{C}_{n-1}, \quad w \not\equiv Q [\![0]\!] \quad \text{and} \quad S_w = \mathbb{R}^{n-1} \times \{0\}.$$

Then by the invariance of w along S_w it follows that w is a function of one variable. By [**?** , Lemma 3.24] it follows that $\tilde{w} : \mathbb{R} \to \mathcal{A}_Q(\mathbb{R}^m)$ is locally Dir-minimizing and

$$\tilde{w} \not\equiv Q [\![0]\!], \quad \boldsymbol{\eta} \circ \tilde{w} \equiv 0.$$

This is clearly a contradiction because the only Dir-minimizing function of one variable are non-intersecting linear functions (cp. [**?** , 3.6.2]).

Finally, a simple consequence of (4.9) is that $\{w|_{B_1} : w \in \mathcal{C}_k\}$ is a closed subset of $L^2(B_1, \mathcal{A}_Q(\mathbb{R}^m))$.

Lemma 10. *Let $(w_j)_{j\in\mathbb{N}} \subset \mathcal{C}_k$ and $w \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{A}_Q(\mathbb{R}^m))$ be such that $w_j \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{A}_Q(\mathbb{R}^m))$. Then $w \in \mathcal{C}_k$.*

Proof. Let α_j be the homogeneity exponent of w_j. Since for Dir-minimizing α-homogeneous Q-valued functions w it holds that $D_w(1) = \alpha\, H_w(1)$, we deduce from $\alpha_j \leq \Lambda_0$ and $w_j \to w$ that the functions w_j have equi-bounded energies in any compact set of $\mathbb{R}^n$. Therefore by the compactness in § 4.2 it follows that $w_j \to w$ locally uniformly and $w \in \mathcal{H}_{\Lambda_0}$.

For every $j \in \mathbb{N}$ let now V_j be a k-dimensional linear subspace of $\mathbb{R}^n$ contained in S_{w_j}. By the compactness of the Grassmannian $\mathrm{Gr}(k, n)$, we can assume that up to passing to a subsequence (not relabeled) V_j converges to a k-dimensional subspace V. Using the uniform convergence of w_j to w we then conclude that for every $z \in V$ and $y \in \mathbb{R}^n$

$$w(z + y) = \lim_j w_j(z_j + y) = \lim_j w_j(y) = w(y)$$

where $z_j \in V_j$ is any sequence such that $z_j \to z$. This shows that $V \subset S_w$, thus implying that $\dim(S_w) \geq k$.

4.4. Blowups

Let u be a Dir-minimizing Q-valued function, $\boldsymbol{\eta} \circ u \equiv 0$ and $u \not\equiv Q [\![0]\!]$. Fix any $r_0 > 0$. For every $y \in \Delta_Q \cap \Omega^{r_0}$, i.e. for every y such that $u(y) = Q [\![0]\!]$ and $\mathrm{dist}(y, \partial\Omega) \geq 2r_0$, we define the rescaled functions of u at y as

$$u_{y,s}(x) := \frac{s^{\frac{m-2}{2}} u(y + sx)}{D_u^{1/2}(y, s)} \qquad \forall\, 0 < s < r_0,\ \forall\, x \in B_{\frac{r_0}{s}}(0).$$

From [**?** , Theorem 3.20] we deduce that for every $s_k \downarrow 0$ there exists a subsequence $s'_k \downarrow 0$ such that u_{y,s'_k} converges locally uniformly in $\mathbb{R}^n$ to a function $w : \mathbb{R}^n \to \mathcal{A}_Q(\mathbb{R}^m)$ such that $w \in \mathcal{H}_{\Lambda_0}$ with

$$\Lambda_0 = \Lambda_0(r_0) := \frac{r_0 \int_\Omega |Du|^2}{\min_{x\in\Omega^{r_0}} H_u(x, r_0)}. \tag{4.11}$$

Note that $\min_{x\in\Omega^{r_0}} H_u(x,r_0) > 0$. Indeed, by the continuity of $x \mapsto H_u(x,r_0)$ and the closure of Ω^{r_0}, the minimum is achieved and cannot be 0 because of the condition $u \not\equiv 0$. In particular, $\Lambda_0 \in \mathbb{R}$.

5. Stratification for Dir-minimizing Q-valued functions

In this section we apply Theorems 3, 4 and 5 to the case of Almgren's Dir-minimizing Q-valued functions. Keeping the notation Ω^s and U as in § 2, we set

(1) $\Theta_s : \Omega^s \to [0,+\infty)$ given by

$$\Theta_0(x) := \lim_{r\downarrow 0^+} I_u(x,r) \quad \text{and} \quad \Theta_s(x) := I_u(x,s) \quad \text{for} \quad s>0, x\in\Omega^s\ ,$$

(2) for every $k\in\{0,\ldots,n\}$, $\mathrm{d}_k : U \to [0,+\infty)$ is given by

$$\mathrm{d}_k(x,s) := \min\Big\{\|\mathcal{G}(u_{x,s},w)\|_{L^2(\partial B_1)} : w\in\mathcal{C}_k\Big\}.$$

Note that since $\{w|_{B_1} : w\in\mathcal{C}_k\}$ is a closed subset of $L^2(B_1)$ the minimum in the definition of d_k is achieved.

It follows from Almgren's monotonicity formula (4.3) that conditions (a) and (b) of § 2 are satisfied.

We verify next that the Structural Hypotheses 2 are fulfilled. For simplicity we write the corresponding statements for fixed r_0. The corresponding $\Lambda_0 > 0$ is defined as in (4.11) above. Therefore, the sets $\mathcal{H}_{\Lambda_0}$ and $\mathcal{C}_k$, introduced respectively in § 4.3 and (4.10), are defined in terms of $\Lambda_0 = \Lambda_0(r_0)$.

Lemma 11. *For every $\varepsilon_1 > 0$ there exist $0 < \lambda_1(\varepsilon_1), \eta_1(\varepsilon_1) < 1/4$ such that, for all $(x,s)\in U$ with $x\in\Omega^{r_0}$ and $s<r_0$, it holds*

$$I_u(x,s) - I_u(x,\lambda_1 s) \le \eta_1 \quad\Longrightarrow\quad \exists\, w\in\mathcal{C}_0\ :\ \|\mathcal{G}(u_{x,s},w)\|_{L^2(\partial B_1)} \le \varepsilon_1.$$

Proof. We argue by contradiction and assume there exist points (x_j,s_j) with $x_j\in\Omega^{r_0}$ and $s_j<r_0$ such that

$$I_u(x_j,s_j) - I_u(x_j,\tfrac{s_j}{2^j}) \le 2^{-j} \quad \text{and} \quad \|\mathcal{G}(u_{x_j,s_j},w)\|_{L^2(\partial B_1)} \ge \varepsilon_1 \quad \forall\, w\in\mathcal{C}_0$$

or equivalently, setting $u_j := u_{x_j,s_j}$,

$$I_{u_j}(0,1) - I_{u_j}(0,2^{-j}) \le 2^{-j} \quad \text{and} \quad \|\mathcal{G}(u_j,w)\|_{L^2(\partial B_1)} \ge \varepsilon_1 \quad \forall\, w\in\mathcal{C}_0. \quad (5.1)$$

From [**?** , Corollary 3.18] it follows that

$$\sup_j D_{u_j}(0,2) \le 2^{n-2+2\,I_{u_j}(0,2)}\,\frac{I_{u_j}(0,2)}{I_{u_j}(0,1)} \le C \qquad (5.2)$$

where $C = C(\Lambda_0)$ because $I_{u_j}(0,2) \leq \Lambda_0$ by definition of Λ_0. We can then use the compactness for Dir-minimizing functions in § 4.3 to infer the existence of a Dir-minimizing w such that (up to subsequences) $u_j \to w$ locally strongly in $W^{1,2}(B_2)$ and uniformly. We then can pass into the limit in (4.3) and using (5.1) we obtain

$$\int_0^1 \frac{t}{H_w(t)} \left(\int_{\partial B_t} |\partial_\nu w|^2 \int_{\partial B_t} |w|^2 - \left(\int_{\partial B_t} \langle \partial_\nu w, w \rangle \right)^2 \right) dt = 0.$$

This implies that w is α-homogeneous (cp. [? , Corollary 3.16]) with $\alpha = \lim_j I_{u_j}(0,1) \leq \Lambda_0$ because of § 4.2. This contradicts $\|\mathcal{G}(u_j, w)\|_{L^2(\partial B_1)} \geq \varepsilon_1$ for all $w \in \mathcal{C}_0$ in (5.1) and proves the lemma.

Remark 1. Using the regularity theory of Dir-minimizing functions proven in [?] it is in fact possible to prove a stronger claim then Lemma 11, namely that for every $\varepsilon_1 > 0$ there exists $0 < \eta_1(\varepsilon_1) < 1/4$ such that for all $(x,s) \in U$ with $x \in \Omega^{r_0}$ and $s < r_0$

$$I_u(x,s) - I_u(x, s/2) \leq \eta_1 \quad \Longrightarrow \quad \exists\, w \in \mathcal{C}_0 \,:\, \|\mathcal{G}(u_{x,s}, w)\|_{L^2(\partial B_1)} \leq \varepsilon_1. \tag{5.3}$$

Since (5.3) is not needed in the sequel, we leave the details of the proof to the reader.

For what concerns (ii) we argue similarly using a rigidity property of homogeneous Dir-minimizing functions.

Lemma 12. *For every $0 < \varepsilon_2, \tau < 1$ there exists $0 < \eta_2(\varepsilon_2, \tau) \leq \varepsilon_2$ such that if $(x, 5s) \in U$, with $x \in \Omega^{r_0}$ and $5s < r_0$, $\mathrm{d}_k(x, 4s) \leq \eta_2$ and $\mathrm{d}_{k+1}(x, 4s) \geq \varepsilon_2$ for some $k \in \{0, \ldots, n-1\}$ then there exists a k-dimensional affine space V such that*

$$\mathrm{d}_0(y, 4s) > \eta_2 \quad \forall\, y \in B_s(x) \setminus \mathcal{T}_{\tau s}(V).$$

Proof. We prove the statement for $V = S_w$ with $w \in \mathcal{C}_k$ such that $\|\mathcal{G}(u, w)\|_{L^2(\partial B_{4s}(x))} = \mathrm{d}_k(x, 4s)$. We argue by contradiction. Reasoning as above with the rescalings of u (eventually composing with a rotation of the domain to achieve (4) below for a fixed space V), we find a sequence of functions $u_j \in W^{1,2}(B_5, \mathcal{A}_Q(\mathbb{R}^k)$ such that

(1) $\sup_j D_{u_j}(0,5) < +\infty$;

(2) there exists $w_j \in \mathcal{C}_k$ such that $\|\mathcal{G}(u_j, w_j)\|_{L^\infty(B_4)} \downarrow 0$;

(3) $\|\mathcal{G}(u_j, w)\|_{L^2(B_4)} \geq \varepsilon_2$ for every $w \in \mathcal{C}_{k+1}$;

(4) there exists $y_j \in B_1 \setminus \mathcal{T}_\tau(V)$ such that $\mathrm{d}_0(y_j, 4) \downarrow 0$ and $V = S_{w_j}$ is the k-dimensional spine of w_j (note that by (2) & (3) the dimension of the spine of w_j cannot be higher than k).

Possibly passing to subsequences (as usual not relabeled), we can assume that $u_j \to w$, $w_j \to w$ locally in $L^2(\mathbb{R}^n, \mathcal{A}_Q(\mathbb{R}^m))$ and $y_j \to y$ for some $w \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{A}_Q(\mathbb{R}^m))$ and $y \in \bar{B}_1 \setminus \mathcal{T}_\tau(V)$. By Lemma 10 we deduce that $w \in \mathcal{C}_k$ with $S_w \supset V$; since by (3) $w \notin \mathcal{C}_{k+1}$, we conclude $S_w = V$.

It follows from (4) that $w_{y,s} = w_{y,1}$ for every $s \in (0,1]$. Indeed there exist $z_j \in \mathcal{C}_0$ such that $\|\mathcal{G}((u_j)_{y_j,1}, z_j)\|_{L^2(\partial B_4)} \downarrow 0$ and by continuity $(u_j)_{y_j,1} \to w_{y,1} \in \mathcal{C}_0$. In particular $w(y) = 0$ and by the upper semi-continuity of $x \mapsto I_w(x, 0^+)$ we deduce also that $I_w(y, 0^+) = I_w(0, 0^+)$, i.e. $y \in S_w$ which is the desired contradiction.

We can then infer that Theorem 3 holds for Q-valued functions.

Theorem 13. *Let $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$ be a nontrivial Dir-minimizing function with average $\boldsymbol{\eta} \circ u \equiv 0$.*

For every $0 < \kappa_0, \delta < 1$ and $r_0 > 0$, there exists a constant $C = C(\kappa_0, \delta, r_0, n) > 0$ such that

$$|\mathcal{T}_r(\Delta_Q \cap \mathcal{S}^k_{r,r_0,\delta})| \leq C\, r^{n-k-\kappa_0} \quad \forall\, k \in \{1, \ldots, n-1\}$$
$$\text{and} \quad \mathcal{S}^0_{r_0,\delta} \text{ is countable.}$$

In particular, Theorem 4 applies and we conclude that $\dim_{\mathcal{H}}(\mathcal{S}^k_{r_0}) \leq k$ and that $\mathcal{S}^0_{r_0}$ is at most countable. We shall improve upon the latter estimate on the stratum $\mathcal{S}^{n-1}_{r_0}$ in the next paragraph.

5.1. Minkowski dimension

We can actually give an estimate on the Minkowski dimension of the set of maximal multiplicity points Δ_Q by means of Theorem 5. An ε-regularity result is the key tool to prove this.

Proposition 5.1.1. *There exists a constant $\delta_0 = \delta_0(r_0) > 0$ such that*

$$\mathcal{S}^{n-1}_r = \mathcal{S}^{n-2}_r = \mathcal{S}^{n-2}_{r,\delta_0} \quad \forall\, r \in (0, r_0). \tag{5.4}$$

Proof. The first equality is an easy consequence of $\mathcal{C}_n = \mathcal{C}_{n-1} = \{Q\, [\![0]\!]\}$ that gives $\mathrm{d}_n \equiv \mathrm{d}_{n-1}$.

Set $\delta_0 := (\Lambda_0 + 1)^{-1/2}$, we show that $\mathcal{S}^{n-2}_{r,\delta} \subset \mathcal{S}^{n-2}_{r,\delta_0}$ for every $\delta \in (0, \delta_0)$. Assume by contradiction that there exists $x \in \mathcal{S}^{n-2}_{r,\delta} \setminus \mathcal{S}^{n-2}_{r,\delta_0}$ for some δ as above. From $\mathcal{C}_{n-1} = \{Q\, [\![0]\!]\}$ we deduce the existence of $s \in (0, r)$ such that

$$0 < \delta \leq \|u_{x,s}\|_{L^2(\partial B_1)} < \delta_0.$$

In particular, the condition $\int_{B_1} |Du_{x,s}|^2 = 1$ gives

$$I_{u_{x,s}}(0,1) = \frac{\int_{B_1} |Du_{x,s}|^2}{\int_{\partial B_1} |u_{x,s}|^2} \geq \frac{1}{\delta_0^2} > \Lambda_0.$$

By recalling that $I_u(x,s) = I_{u_{x,s}}(0,1)$, the desired contradiction follows from Almgren's monotonicity formula (4.3) and the very definition of Λ_0 in (4.11).

In particular Theorem 1 follows from Theorem 5.

Proof (of Theorem 1). It is a direct consequence of Proposition 5.1.1 and Theorem 5. Given $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$ a nontrivial Dir-minimizing function (i.e. $\Delta_Q \neq \Omega$), we can consider the function

$$v(x) := \sum_i [\![u_i(x) - \boldsymbol{\eta} \circ u(x)]\!] .$$

Then by [**?** , Lemma 3.23] v is Dir-minimizing with $\boldsymbol{\eta} \circ v \equiv 0$. Moreover, the set of Q-multiplicity points of u in Ω^{r_0} corresponds to the set $\mathcal{S}^{n-2}_{r_0}$ for the function v and the conclusion follows straightforwardly.

5.2. White's stratification

In this section we show that Theorem 8 applies in the case of Q-valued functions, as well. In particular, this implies that the singular strata for Dir-minimizing Q-valued functions can also be characterized by the spines of the blowup maps, thus leading to the proof of Theorem 2 in the introduction.

By following the notation in § 3.3 (1), we set

$$f(x) := I_u(x, 0^+) \quad \forall\, x \in \Omega.$$

For every $x \in \Omega$ such that $f(x) = 0$ (or, equivalently, $u(x) \neq Q[\![0]\!]$) we define $\mathscr{G}(x)$ to be the singleton made of the constant function 0, i.e. $\mathscr{G}(x) = \{Q[\![0]\!]\}$; otherwise

$$\mathscr{G}(x) := \big\{ I_w(\cdot, 0^+) : w \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{A}_Q(\mathbb{R}^m)) \text{ blowup of } u \text{ at } x \big\}. \tag{5.5}$$

As explained in § 4.3 $\mathscr{G}(x)$ is never empty because there always exist (possibly non-unique) blowup of u at any multiplicity Q point.

Since every blowup of u is a nontrivial homogeneous Dir-minimizing function, it follows from Lemma 9 that every function $g \in \mathscr{G}(x)$ is conical in the sense of Definition 3.3.1 (1). We need then to show the following.

Lemma 14. *For every $x \in \Omega$ the class $\mathscr{G}(x)$ is compact in the sense of Definition 3.3.1 (2).*

Proof. If x is not a multiplicity Q point, then there is nothing to prove. Otherwise consider a sequence of maps $g_j = I_{w_j}(\cdot, 0^+) \in \mathscr{G}(x)$, with w_j blowup of u at x. By § 4.3 w_j is Dir-minimizing α-homogeneous with $\alpha = I_u(x, 0^+)$ and $D_{w_j}(1) = 1$. Then by the compactness in § 4.2, there exists w such that $w_j \to w$ locally in L^2 up to subsequences (not relabeled) with $D_w(1) = 1$. By a simple diagonal argument it follows that w is as well a blowup of u at x, i.e. $g = I_w(\cdot, 0^+) \in \mathcal{G}(x)$. For every $y_j \in B_1$ with $y_j \to y \in B_1$ and for every $s > 0$, we then deduce

$$\begin{aligned}
\limsup_{j\uparrow+\infty} g_j(y_j) \le \limsup_{j\uparrow+\infty} I_{w_j}(y_j, s) \\
&= \limsup_{j\uparrow+\infty} \left(\frac{s\, D_{w_j}(y, s)}{H_{w_j}(y, s)} \, \frac{D_{w_j}(y_j, s)}{D_{w_j}(y, s)} \, \frac{H_{w_j}(y, s)}{H_{w_j}(y_j, s)} \right) \\
&= I_w(y, s)
\end{aligned}$$

where we used

- the monotonicity of $I_{w_j}(y_j,\cdot)$ in the first line,
- the continuity of $x \mapsto D_{w_j}(x,s)$ and $x \mapsto H_{w_j}(x,s)$,
- and the convergence of the frequency functions $I_{w_j}(y,s) \to I_w(y,s)$ (cp. 4.2).

Sending s to 0 provides the conclusion.

F. inally we prove that the Structure Hypothes1s 3.3 (ii) of White's theorem holds as well:

$$
\begin{aligned}
\limsup_{j\uparrow+\infty} f(x+r_{i_j}y_j) &= \limsup_{j\uparrow+\infty} I_u(x+r_{i_j}y_j,0^+) \\
&\leq \limsup_{j\uparrow+\infty} I_u(x+r_{i_j}y_j,r_{i_j}s) \\
&= \limsup_{j\uparrow+\infty} I_{u_{x,r_{i_j}}}(y_j,s) = I_w(y,s)
\end{aligned}
$$

where we used the strong convergence of the frequency of § 4.2.

In particular, Theorem 8 holds true, which in turn leads to the proof of Theorem 2 by a simple induction argument.

5.3. Stratification: Theorem 2

We define now the singular strata Sing_u^k for a Dir-minimizing multiple valued function $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^m)$. Consider any point $x_0 \in \mathrm{Sing}_u$, and let

$$
u(x_0) = \sum_{i=1}^{J} \kappa_i \, [\![p_i]\!]
$$

with $\kappa_i \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^{J} \kappa_i = Q$ and $p_i \neq p_j$ for $i \neq j$. Then by the uniform continuity of u there exist $r > 0$ and Dir-minimizing multiple valued functions $u_i : B_r(x_0) \to \mathcal{A}_{\kappa_i}(\mathbb{R}^m)$ for $i \in \{1,\ldots,J\}$ such that

$$
u|_{B_r(x_0)} = \sum_{i=1}^{J} [\![u_i]\!]\,,
$$

where by a little abuse of notation the last equality is meant in the sense $u(x) = \sum_i u_i(x)$ as measures. For every $i \in \{1,\ldots,J\}$ let $v_i : B_r(x_0) \to \mathcal{A}_{\kappa_i}(\mathbb{R}^m)$ be given by

$$
v_i(x) := \sum_{l=1}^{\kappa_i} [\![(u_i(x))_l - \boldsymbol{\eta} \circ u_i(x)]\!]\,.
$$

Then we say that a point $x_0 \in \mathrm{Sing}_u$ belongs to Sing_u^k, $k \in \{0,\ldots,n\}$, if the spine of every blowup of v_i at x_0, for every $i \in \{1,\ldots,J\}$, is at most k-dimensional.

We can then prove Theorem 2 by a simple induction argument on the number of values Q.

Proof (of Theorem 2). Clearly if $Q = 1$ there is nothing to prove because every harmonic function is regular and $\mathrm{Sing}_u = \emptyset$. Now assume we have proven the theorem for every $Q^* < Q$ and we prove it for Q.

We can assume without loss of generality that $\Delta_Q \neq \Omega$. Then, as noticed, $\Delta_Q = \mathrm{Sing}_u \cap \Delta_Q$ by [? , Theorem 0.11]. Moreover $\mathrm{Sing}^k_u \cap \Delta_Q = \Sigma_k$, where Σ_k is that of Theorem 8. Indeed $x_0 \in \Sigma_k$ if and only if the maximal dimension of the spine of any $g \in \mathcal{G}(x_0)$ is at most k. By (5.5) $g \in \mathcal{G}(x_0)$ if and only if $g = I_w(\cdot, 0^+)$ for some blowup w of u at x_0. Hence by (4.9) $x_0 \in \Sigma_k$ if and only if the dimension of the spines of the blowups of u at x_0 is at most k. Note that $\mathrm{Sing}^{n-2}_u \cap \Delta_Q = \Delta_Q$ since $\mathcal{C}_n = \mathcal{C}_{n-1} = \{Q\,[\![0]\!]\}$ (we use here the notation in § 4.3) and u is not trivial. Therefore we deduce that

$$\mathrm{Sing}^0_u \cap \Delta_Q \quad \text{is countable}$$
$$\dim_{\mathcal{H}}(\mathrm{Sing}^k_u \cap \Delta_Q) \le k \quad \forall\, k \in \{1, \dots, n-2\}.$$

Next we consider the relatively open set $\Omega \setminus \Delta_Q$ (recall that both Sing_u and Δ_Q are relatively closed sets). Thanks to the continuity of u we can find a cover of $\Omega \setminus (\mathrm{Sing}_u \cap \Delta_Q)$ made of countably many open balls $B_i \subset \Omega \setminus (\mathrm{Sing}_u \cap \Delta_Q)$ such that $u|_{B_i} = [\![u^1_i]\!] + [\![u^2_i]\!]$ with u^1_i and u^2_i Dir-minimizing multiple valued functions taking strictly less than Q values. Since $\mathrm{Sing}^k_u \cap B_i = \mathrm{Sing}^k_{u^1_i} \cup \mathrm{Sing}^k_{u^2_i}$ by the very definition, using the inductive hypotheses for u^1_i and u^2_i we deduce that

$$\mathrm{Sing}^0_u \cap B_i \quad \text{is countable}$$
$$\mathrm{Sing}^{n-2}_u \cap B_i = \mathrm{Sing}_u \cap B_i$$
$$\dim_{\mathcal{H}}(\mathrm{Sing}^k_u \cap B_i) \le k \quad \forall\, k \in \{1, \dots, n-2\},$$

thus leading to (1.2) and (1.3).

6. Applications to generalized submanifolds

In the present section we apply the abstract stratification results in § 2 to integral varifolds with mean curvature in L^∞ and to almost minimizers in codimension one (both frameworks are not covered by the results in [?] although they can be considered as slight variants of those). This case is relevant in several variational problems (see the examples in [? , § 4]) most remarkably the case of stationary varifolds or area minimizing currents in a Riemannian manifold. For a more complete account on the theory of varifolds and almost minimizing currents we refer to [?], [?] and the lecture notes [?].

6.1. Tubular neighborhood estimate

In what follows we consider integer rectifiable varifolds $\mathscr{V} = (\Gamma, f)$, where Γ is an m-dimensional rectifiable set in the bounded open subset $\Omega \subset \mathbb{R}^n$, and $f : \Gamma \to \mathbb{N} \setminus \{0\}$ is locally $\mathcal{H}^m$-integrable. We assume that $\mathscr{V}$ has bounded generalized mean curvature, i.e. there exists a vector field $H_{\mathscr{V}} : \Omega \to \mathbb{R}^n$ such that $\|H_{\mathscr{V}}\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq H_0$ for some $H_0 > 0$ and

$$\int_\Gamma \mathrm{div}_{T_y\Gamma} X \, d\mu_{\mathscr{V}} = -\int X \cdot H_{\mathscr{V}} \, d\mu_{\mathscr{V}} \qquad \forall\, X \in C^1_c(\Omega, \mathbb{R}^n)$$

where $\mu_{\mathscr{V}} := f\,\mathcal{H}^m \llcorner \Gamma$. It is then well-known (cp., for example, [? , Theorem 17.6]) that the quantity

$$\Theta_{\mathscr{V}}(x,\rho) := e^{H_0\,\rho}\,\frac{\mu_{\mathscr{V}}(B_\rho(x))}{\omega_m \rho^m}$$

is monotone and the following inequality holds for all $0 < \sigma < \rho < \mathrm{dist}(x, \partial\Omega)$

$$\Theta_{\mathscr{V}}(x,\rho) - \Theta_{\mathscr{V}}(x,\sigma) \geq \int_{B_\rho \setminus B_\sigma(x)} \frac{|(y-x)^\perp|^2}{|y-x|^{m+2}} d\mu_{\mathscr{V}}(y) \tag{6.1}$$

where $(y-x)^\perp$ is the orthogonal projection of $y - x$ on the orthogonal complement $(T_y\Gamma)^\perp$. In particular the family $(\Theta(\cdot,s))_{s\in[0,r_0]}$ (with the obvious extended notation $\Theta(\cdot, 0^+) := \lim_{r\downarrow 0} \Theta(\cdot, r)$) satisfies assumption (a) in Paragraph 2 for every fixed $r_0 > 0$ with

$$\Lambda_0(r_0) := e^{H_0\,\mathrm{diam}(\Omega)}\,\frac{\mu_{\mathscr{V}}(\Omega)}{\omega_m r_0^m}. \tag{6.2}$$

In order to introduce the control functions d_k we recall next the definition of *cone.*

Definition 6.1.1. *An integer rectifiable m-varifold $\mathscr{C} = (R, g)$ in $\mathbb{R}^n$ is a cone if the m-dimensional rectifiable set R is invariant under dilations i.e.*

$$\lambda\, y \in R \qquad \forall\, y \in R,\ \forall\, \lambda > 0$$

and g is 0-homogeneous, i.e.

$$g(\lambda\, y) = g(y) \qquad \forall\, y \in R,\ \forall \lambda > 0.$$

An integer rectifiable m-varifold $\mathscr{C} = (R, g)$ in B_ρ, $\rho > 0$ is a cone if it is the restriction to B_ρ of a cone in $\mathbb{R}^n$.

The spine of a cone $\mathscr{C} = (R, g)$ in $\mathbb{R}^n$ is the biggest subspace $V \subset \mathbb{R}^n$ such that $R = R' \times V$ up to $\mathcal{H}^m$-null sets.

The class of cones whose spine is at least k-dimensional is denoted by $\mathcal{C}_k$ and its elements are called k-conical.

If d_* is a distance inducing the weak $*$ topology of varifolds with bounded mass in B_1 (cp., for instance, [? , Theorem 3.16] for the general case of dual spaces), the control function d_k is then defined as

$$\mathrm{d}_k(x,s) := \inf\left\{\mathrm{d}_*\big(\mathscr{V}_{x,s}, \mathscr{C}\big) : \mathscr{C} \in \mathcal{C}_k,\ \|H_{\mathscr{C}}\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq H_0\right\} \tag{6.3}$$

where $\mathscr{V}_{x,s} := (\eta_{x,s}(\Gamma), f \circ \eta_{x,s}^{-1})$ with $\eta_{x,s}(y) := {}^{(y-x)}/_{s}$.

By very definition, then (b) in Paragraph 2 is satisfied. We are now ready to check that the conditions in the Structural Hypotheses are satisfied. As usual, we write the corresponding statements for fixed r_0 and $\Lambda_0 := \Lambda_0(r_0)$, for simplicity.

Lemma 15. *For every $\varepsilon_1 > 0$ there exist $0 < \lambda_1(\varepsilon_1), \eta_1(\varepsilon_1) < {}^1/_4$ such that for all $(x, \rho) \in U$, with $x \in \Omega^{r_0}$ and $\rho < r_0$,*

$$\Theta_{\mathscr{V}}(x, \rho) - \Theta_{\mathscr{V}}(x, \lambda_1 \rho) \leq \eta_1 \quad \Longrightarrow \quad \mathrm{d}_0(x, \rho) \leq \varepsilon_1.$$

Proof. Assume by contradiction that for some $\varepsilon_1 > 0$ there exists $(x_j, \rho_j) \in U$, with $x_j \in \Omega^{r_0}$ and $\rho_j < r_0$, such that

$$\Theta_{\mathscr{V}}(x_j, \rho_j) - \Theta_{\mathscr{V}}(x_j, j^{-1} \rho_j) \leq j^{-1} \quad \text{and} \quad \mathrm{d}_0(x_j, \rho_j) \geq \varepsilon_1. \tag{6.4}$$

We consider the sequence $(\mathscr{V}_j)_{j \in \mathbb{N}}$ with $\mathscr{V}_j := \mathscr{V}_{x_j, \rho_j}$, and note that for all $t > 0$ there is an index $\bar{j}$ such that $t \rho_j < r_0$ if $j \geq \bar{j}$, so that

$$\mu_{\mathscr{V}_j}(B_t) \leq \omega_m t^m \Theta_{\mathscr{V}_j}(x_j, t \rho_j) \leq \omega_m t^m \Lambda_0 \quad \forall\, j \geq \bar{j}.$$

Therefore, up to the extraction of subsequences and a diagonal argument, Allard's rectifiability criterion (cp., for instance, [**?** , Theorem 42.7, Remark 42.8]) yields a limiting m-dimensional integer varifold $\mathscr{V}_j \to \mathscr{C} = (R, g)$ with $\|H_{\mathscr{C}}\|_{L^\infty(\Omega} \leq H_0$. Since $\Theta_{\mathscr{V}}(x_j, s\rho_j) = \Theta_{\mathscr{V}_j}(0, s) \to \Theta_{\mathscr{C}}(0, s)$ except at most for countable values of s, by monotonicity and (6.4) for all $j^{-1} < r < s < 1$ we have $\Theta_{\mathscr{C}}(0, s) = \Theta_{\mathscr{C}}(0, 0^+)$ for every $s \geq 0$. The monotonicity formula (6.1) applied to $\mathscr{C}$ implies that $\mathscr{C}$ is actually a cone, thus contradicting $\mathrm{d}_0(x_j, \rho_j) \leq \varepsilon_1$.

Lemma 16. *For every $\varepsilon_2, \tau \in (0, 1)$, there exists $0 < \eta_2(\varepsilon_2, \tau) < \varepsilon_2$ such that, for every $(x, 5s) \in U$, with $x \in \Omega^{r_0}$ and $5s < r_0$, if for some $k \in \{0, \ldots, m-1\}$*

$$\mathrm{d}_k(x, 4s) \leq \eta_2 \quad \text{and} \quad \mathrm{d}_{k+1}(x, 4s) \geq \varepsilon_2,$$

then there exists a k-dimensional affine space $x + V$ such that

$$\mathrm{d}_0(y, 4s) > \eta_2 \quad \forall\, y \in B_s(x) \setminus \mathcal{T}_{\tau s}(x + V).$$

Proof. The proof is by contradiction. Assume that there exist $0 < \varepsilon_2, \tau < 1$, $k \in \{0, \ldots, m-1\}$ and a sequence of points $(x_j, 5s_j) \in U$, with $x_j \in \Omega^{r_0}$ and $5s_j < r_0$, for $2j \geq \varepsilon_2^{-1}$ such that

$$\mathrm{d}_k(x_j, 4s_j) \leq j^{-1} \quad \text{and} \quad \mathrm{d}_{k+1}(x_j, 4s_j) \geq \varepsilon_2, \tag{6.5}$$

and such that the conclusion of the lemma fails, in particular, for V_j given by the spine of $\mathscr{C}_j$ with

$$\mathrm{d}_*\big(\mathscr{V}_{x_j, 4s_j}, \mathscr{C}_j\big) \leq 2j^{-1} \tag{6.6}$$

(note that by $2j \geq \varepsilon_2^{-1}$ necessarily $\dim(V_j) = k$). Without loss of generality (up to a rotation) we can assume that $V_j = V$ a given vector subspace for every j. This means that there exist $y_j \in B_{s_j}(x_j) \setminus \mathcal{T}_{\tau s_j}(x_j + V)$ such that

$$\mathrm{d}_0(y_j, 4s_j) \leq j^{-1}. \tag{6.7}$$

Using the compactness for varifolds with bounded generalized mean curvature, (up to passing to subsequences) we can assume that

1. $s_j \to s_\infty \in [0, r_0/5]$;
2. $\mathscr{C}_j \to \mathscr{C}_\infty$ in the sense of varifolds, where $\mathscr{C}_\infty$ is a cone with $\|H_{\mathscr{C}_\infty}\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq H_0$;
3. $(y_j - x_j)/s_j \to z \in \overline{B}_1 \setminus \mathcal{T}_\tau(V)$;
4. $\mathscr{V}_{x_j,s_j} \to \mathscr{W}_\infty$ and $\mathscr{V}_{y_j,s_j} \to \mathscr{Z}_\infty$ in the ball B_4 in the sense of varifolds, where $\mathscr{W}_\infty$ and $\mathscr{Z}_\infty$ are cones thanks to (6.5) and (6.7), respectively.

Note that by (6.6) it follows that $\mathscr{C}_j \to \mathscr{W}_\infty$ and therefore $\mathscr{W}_\infty \in \mathcal{C}_k$ because all the $\mathscr{C}_j$ are invariant under translations in the directions of V. Moreover, arguing as above it also follows from $\mathrm{d}_{k+1}(x_j, 4s_j) \geq \varepsilon_2$ that the spine of $\mathscr{W}_\infty$ is exactly V.

Note that $\eta_{(y_j - x_j)/s_j,1}$ corresponds to the translation of vector $(y_j - x_j)/s_j$. By the equality of $(\eta_{(y_j - x_j)/s_j,1})_\sharp \mathscr{V}_{x_j,s_j}$ and $\mathscr{V}_{y_j,s_j}$ in B_3, we deduce that $(\eta_{(y_j - x_j)/s_j,1})_\sharp \mathscr{W}_\infty = \mathscr{Z}_\infty$ as varifolds in B_3, i.e. $\mathscr{W}_\infty$ is a cone around z too. We claim that this implies that $\mathscr{W}_\infty$ is invariant along the directions of $\mathrm{Span}\{z, V\}$, thus contradiction the fact that the spine of $\mathscr{W}_\infty$ equals V. To prove the claim, let $\mathscr{W}_\infty = (R_\infty, g)$ with R_∞ cone around the origin and z. It suffices to show that $y + z \in R_\infty$ for all $y \in R_\infty$. Indeed $(z + y)/2 = z + y - z/2 \in R_\infty$ being R_∞ a cone with respect to z; and then $y + z \in R_\infty$ being R_∞ a cone with respect to 0.

In particular we deduce that Theorem 3 and Theorem 4 hold in the case of varifolds with generalized mean curvature in L^∞.

6.2. Almost minimizer in codimension one

It is well-known by the classical examples by Federer [**?**] that no Allard's type ε-regularity results can hold for higher codimension generalized submanifolds without any extra-hypotheses on the densities. Vice versa for generalized hypersurfaces one can strengthen the results of the previous subsection giving estimates on the Minkowski dimension of the singular set. The arguments in this part resemble very closely those in [**?**], therefore we keep them to the minimum.

In what follows we consider sets of finite perimeter, i.e. borel subsets $E \in \Omega$ such that the distributional derivative of corresponding characteristic function has bounded variation: $D\chi_E \in BV_\Omega$. Following [**? ?**], a set of finite perimeter is almost minimizing in Ω if for all $A \subset\subset \Omega$ open there exist $T \in \big(0, \mathrm{dist}(A, \partial\Omega)\big)$ and $\alpha : (0, T) \to [0, +\infty)$ non-decreasing and infinitesimal in 0 such that whenever $E \triangle F \subset\subset B_r(x) \subset A$

$$\mathrm{Per}(E, B_r(x)) \leq \mathrm{Per}(F, B_r(x)) + \alpha(r)\, r^{n-1} \qquad \forall\, r \in (0, T) \tag{6.8}$$

and

$$(0,T) \ni t \mapsto \frac{\alpha(t)}{t} \text{ is non-increasing, and } \int_0^T \frac{\alpha^{1/2}(t)}{t} dt < \infty. \tag{6.9}$$

Examples of almost minimizing sets not only include minimal boundaries on Riemannian manifolds, but also boundaries with generalized mean curvature in L^∞, minimal boundaries with volume constraint, and minimal boundaries with obstacles (cp. [**?** , § 1.14]).

We use here again the control functions introduced in Section 2.4.1 in terms of flat distance: given a set of finite perimeter E, we denote by ∂E its boundary (in the sense of currents) and set

$$\mathrm{d}_k(x,s) := \inf \left\{ \mathbb{F}\big((\partial E_{x,s} - C) \llcorner B_1\big) \ : \ C \ \ k\text{-conical \& area minimizing} \right\}$$

where the dimension of the cones C is always $n-1$, and $E_{x,s}$ is the push-forward of E via the rescaling map $\eta_{x,s}$. In particular d_{n-1} denotes the distance of the rescaled boundary $\partial E_{x,s}$ rescaling of the from flat $(n-1)$-dimensional vector spaces.

The main ε-regularity result for almost minimizing sets can be stated as follows (cp. [**?** , Theorem 1.9], [**?** , Lemma 17] and [**?** , Theorem B.2]).

Theorem 17. *Suppose that E is a perimeter almost minimizer in Ω satisfying* (6.8) *and* (6.9) *for a given function α. Then, there exists $\varepsilon > 0$ and $\omega : [0,+\infty) \to [0,+\infty)$ continuous, non-decreasing and satisfying $\omega(0) = 0$ with the following property: if*

$$\rho + \mathrm{d}_{n-1}(x,\rho) + \int_0^\rho \frac{\alpha^{1/2}(t)}{t} dt \le \varepsilon,$$

then $\partial E \cap B_{\rho/2}(x)$ is the graph of a C^1 function f satisfying

$$|\nabla f(x) - \nabla f(y)| \le \omega(|x-y|). \tag{6.10}$$

Moreover, there are no singular area minimizing cones with dimension of the singular set bigger than $n-8$, i.e. equivalently

$$\mathrm{d}_{n-7} = \mathrm{d}_{n-6} = \ldots = \mathrm{d}_{n-1}. \tag{6.11}$$

Remark 2. The smallness condition $\mathrm{d}_{n-1} \le \varepsilon$, together with the almost minimizing property, implies the more familiar smallness condition on the *Excess*, i.e.

$$\mathrm{Exc}(E, B_r(x)) := r^{1-n} \, \|D\chi_E\|(B_r(x)) - r^{1-n} \, |D\chi_E(B_r(x))| \le \varepsilon'$$

for some $\varepsilon' = \varepsilon'(\varepsilon) > 0$ infinitesimal as ε goes to 0 because of the continuity of the mass for converging uniform almost minimizing currents. Therefore (6.10) readily follows from [**?** , Theorem 1.9].

By a simple use of Theorem 17 we can the prove the following.

Corollary 6.2.1. *Under the hypotheses of Theorem 17 there exist constants* $\delta_0 = \delta_0(\Lambda_0, n, \alpha) > 0$ *and* $\rho_0 = \rho_0(\Lambda_0, n, \alpha) > 0$ *such that*

$$\mathcal{S}^{n-8}_{r_0,\delta_0} = \mathcal{S}^{n-8}_{r_0} = \mathcal{S}^{n-7}_{r_0} = \ldots = \mathcal{S}^{n-2}_{r_0} \qquad \forall\, r_0 \in (0, \rho_0].$$

Proof. Set $\delta_0 = \varepsilon/2$ and let ρ_0 be sufficiently small to have

$$\rho_0 + \int_0^{\rho_0} \frac{\alpha^{1/2}(t)}{t} dt \leq \varepsilon/2.$$

If $x \notin \mathcal{S}^{n-2}_{r_0,\delta_0}$, $r_0 \in (0, \rho_0]$, then there exists $0 < z_0 \leq r_0$ such that $\mathrm{d}_{n-1}(x, z_0) < \delta_0$. In particular, by the choices of δ_0 and of ρ_0 the assumptions of Theorem 17 are satisfied at s_0. Therefore, it turns out that x is a regular point of ∂E and that $B_{z_0/2}(x) \cap \partial E$ can be written as a graph of a function f satisfying (6.10). In particular, $\lim_{s\downarrow 0} \mathrm{d}_{n-1}(x, s) = 0$. Therefore, given any $\delta' < \delta_0$, we have that $x \notin \mathcal{S}^{n-2}_{r_0,\delta'}$, thus implying that $\mathcal{S}^{n-2}_{r_0} = \mathcal{S}^{n-2}_{r_0,\delta_0}$. By taking into account (6.11) we conclude the corollary straightforwardly.

In particular, Theorem 5 holds and we deduce the following refinement of the Hausdorff dimension estimate of the singular set.

Theorem 18. *Let* $E \subset \Omega$ *be a almost minimizing set of finite perimeter in a bounded open set* $\Omega \subset \mathbb{R}^n$ *according to* (6.8) *and* (6.9). *Then there exists a closed subset* $\Sigma \subset \partial E \cap \Omega$ *such that* $\partial E \cap \Omega \setminus \Sigma$ *is a* C^1 *regular* $(n-1)$*-dimensional submanifold of* $\mathbb{R}^n$ *and* $\dim_{\mathcal{M}}(\Sigma) \leq n - 8$.

Proof. Let $\Omega' \subset\subset \Omega$ be compactly supported and set $r_0 := \mathrm{dist}(\Omega', \partial\Omega)$. By the regularity Theorem 17, a point $x \in \Omega$ is regular if and only if there exists $r > 0$ sufficiently small such that $\mathrm{d}_{n-1}(x, r) \leq \varepsilon/2$. In particular, the set of singular points Σ coincides with $\mathcal{S}^{n-2}_{r_0,\varepsilon/2}$ and the conclusion follows combining Theorem 5 with Corollary 6.2.1.

In addition, a higher integrability estimate for almost minimizers with bounded generalized mean curvature can be also derived. Given a set of finite perimeter $E \subset \Omega$, one can associate to ∂E a varifold in a canonical way (cp. [**?**]). One can then talk about sets of finite perimeter with bounded generalized mean curvature. Important examples of such an instance are:

1. the minimizers of the area functional in a Riemannian manifold;
2. the minimizers of the prescribed curvature functional in $\Omega \subset \mathbb{R}^n$

   $$\mathcal{F}(E) := \|D\chi_E\|(\Omega) + \int_{\Omega\cap E} H$$

   with $H \in L^\infty(\Omega)$;
3. minimizers of the area functional with volume constraint;

4. more general Λ-minimizers for some $\Lambda > 0$, i.e. sets E such that

$$\|D\chi_E\|(\Omega) \leq \|D\chi_F\|(\Omega) + \Lambda\,|E \setminus F| \quad \forall\, F \subset \Omega.$$

Given a point $x \in \partial E$ such that $B_r(x) \cap \partial E$ is the graph of a C^1 function f, if the generalized mean curvature H of ∂E is bounded then we can also talk about generalized second fundamental form A in $B_{r/2}(x)$, because in a suitable chosen system of coordinates f solves in a weak sense the prescribed mean curvature equation

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = H \in L^\infty. \tag{6.12}$$

Note that, since in this case f satisfies (6.10), we can choose a suitable system of coordinates and use the L^p theory for uniformly elliptic equations to deduce that actually $A \in L^p(B_{r/4}(x), \mathcal{H}^{n-1} \llcorner \partial E)$ for every $p < +\infty$ with uniform estimate

$$\int_{B_{\frac{r}{4}}(x)\cap\partial E} |A|^p\,\mathcal{H}^{n-1} \leq C\,r^{n-p-1} \tag{6.13}$$

for some dimensional constant $C > 0$. For convenience we set $A \equiv +\infty$ on the singular set $\Sigma \subset \partial E$.

Theorem 19. *Let $E \subset \Omega$ be as in Theorem 18 and assume moreover that the varifold induced by ∂E has bounded generalized mean curvature. Then, for every $p < 7$ there exists a constant $C > 0$ such that*

$$\int_{\partial E\cap\Omega} |A|^p\,\mathrm{d}\mathcal{H}^{n-1} \leq C. \tag{6.14}$$

Proof. Let $\rho_0 > 0$ be the constant in Corollary 6.2.1 and ε that of Theorem 17. Then $\Sigma = \mathcal{S}^{n-8}_{\rho_0,\varepsilon/2}$. In then follows that for a fixed $\bar{k} > \log_2(\rho_0/10)$

$$\big(\operatorname{supp}(\partial E) \setminus \Sigma\big) \cap \Omega = \bigcup_{k\geq\bar{k}} \mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2} \setminus \mathcal{S}^{n-8}_{2^{-k-1},\rho_0,\varepsilon/2}.$$

Applying Theorem 3 we infer that for every $\eta > 0$ there exists $C > 0$ such that

$$\big|\mathcal{T}_{2^{-k}}(\mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2})\big| \leq C\,2^{-k(8-\eta)}. \tag{6.15}$$

By Lemma 7 there exists a cover of $\mathcal{T}_{2^{-k-2}/5}(\mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2} \setminus \mathcal{S}^{n-8}_{2^{-k-1},\rho_0,\varepsilon/2})$ by balls $\{B_{2^{-k-3}}(x_i^k)\}_{i\in I_k}$ with $x_i^k \in \mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2} \setminus \mathcal{S}^{n-8}_{2^{-k-1},\rho_0,\varepsilon/2}$ whose cardinality is estimated by (3.2) as

$$\mathcal{H}^0(I_k) \leq C\,2^{-k(8-\eta-n)} \tag{6.16}$$

where $C > 0$ is a dimensional constant.

We start estimating the integral in (6.14) as follows:

$$\begin{aligned}\int_{\partial E\cap\Omega} |A|^p \,\mathrm{d}\mathcal{H}^{n-1} &= \sum_{k\geq\bar{k}} \int_{\mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2}\setminus\mathcal{S}^{n-8}_{2^{-k-1},\rho_0,\varepsilon/2}} |A|^p \,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq \sum_{k\geq\bar{k}} \sum_{i\in I_k} \int_{\partial E\cap B_{2^{-k-3}}(x_i^k)} |A|^p \,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

Since $x_i^k \in \mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2} \setminus \mathcal{S}^{n-8}_{2^{-k},\rho_0,\varepsilon/2}$ it follows that there exists $r_i^k \in [2^{-k-1}, 2^{-k})$ such that $d_n(x_i^k, r_i^k) < \varepsilon/2$. In particular by Theorem 17 $\partial E \cap B_{2^{-k-2}}(x_i^k)$ is a graph of a C^1 function satisfying (6.10). From (6.13) we conclude that

$$\int_{\partial E\cap\Omega} |A|^p \,\mathrm{d}\mathcal{H}^{n-1} \leq C \sum_{k\geq\bar{k}} \mathcal{H}^0(I_k)\, 2^{-k(n-p-1)} \leq C \sum_{k\geq\bar{k}} 2^{-k(7-\eta-p)} < C$$

as soon as $\eta < 7 - p$.

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References

References