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Abstract. We extend the work of A. Beauville on rank 2 vector bundles on a smooth curve in several directions. We give families of examples with large dimension, add new existence and non-existence results and prove the existence of indecomposable limits with arbitrary rank. To construct the large dimensional families we use the examples of limits of rank 2 trivial bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$ due to C. Banica. We also consider a more flexible notion: limits of trivial bundles on nearby curves.

1. Introduction

Let X be a smooth and connected curve defined over an algebraically closed field $\mathbb{K}$ with characteristic 0. For any vector bundle F on X let $\Sigma(F)$ denote the set of all rank 2 vector bundles on X which are a flat limit of a family of vector bundles whose general fiber is isomorphic to F, i.e., the set of all vector bundles E on X such that there are a connected and affine curve B, $o \in B$ and a vector bundle G on $X \times B$ such that $G|X \times \{o\} \cong E$ and $G|X \times \{t\} \cong F$ for a general $t \in B$ (up to the identification of X and $X \times \{a\}$, $a \in B$). If $F = \mathcal{O}_X^{\oplus r}$ for some r we say that E is a limit of trivial bundles. Almost always in this paper we take $\mathrm{rank}(E) = 2$ and $F = \mathcal{O}_X^{\oplus 2}$, i.e., the case considered by A. Beauville in [8]. For curves with positive genera we also consider bundles which are limits of trivial bundles on different curves in the following sense.

We fix $X \in \mathcal{M}_g$ (with $\mathcal{M}_1$ just a short-hand for a reasonable moduli space or moduli stack of elliptic curves) and ask which are the rank $r \geq 2$ vector bundles on X which are limits of trivial vector bundles as vector bundles over a subset of $\mathcal{M}_g$, i.e., the vector bundles E on X for which there exists $(T, \pi, \mathcal{C}, \mathcal{E}, o)$ with T an integral quasi-projective variety, $o \in T$, $\pi : \mathcal{C} \to T$ is a smooth morphism with as fibers genus g curves, $\mathcal{E}$ is a vector bundle on $\mathcal{C}$, $\mathcal{E}_{\pi^{-1}(t)} \cong \mathcal{O}_{\pi^{-1}(t)}^{\oplus r}$ for all $t \in T \setminus \{o\}$, $\pi^{-1}(o) \cong X$ and the latter isomorphism identify E and $\mathcal{E}_{|\pi^{-1}(o)}$. In this case we say that E is *a limit of trivial bundles on nearby curves.*

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For any smooth curve X let $\Sigma'(X,r)$ (resp. $\Sigma''(X,r)$) denote the set of all isomorphism classes of rank r indecomposable vector bundles on X which are limits of trivial bundles (resp. limit of trivial bundles on nearby curves).

Remark 1.1. As in [8, Remark 1] we see that each $E \ne \mathcal{O}_X^{\oplus 2}$ which is a limit of trivial bundles on nearby curves fits in an exact sequence

$$0 \to L \to E \to L^\vee \to 0, \tag{1}$$

where L is a line bundle of positive degree and $h^0(L) \ge 2$. Thus a necessary condition for the existence of an indecomposable rank 2 vector bundle E which is a limit of trivial bundles in this weaker sense is the existence of a line bundle L on X such that $h^1(L^{\otimes 2}) > 0$ and $h^0(L) \ge 2$. No such bundle exists on a general $X \in \mathcal{M}_g$ ([3, Ch. 21, Proposition 6.7]). In a similar way we exclude all $X \in \mathcal{M}_2$ and all curves of genus ≤ 1. For all cases with $g \le 4$ see Remark 5.1 and Proposition 5.2. With this weaker definition of limits of trivial bundles the non-existence results [8, Proposition 3], Remark 5.1 and Propositions 5.2, 5.4 and 5.9 are true and their proofs require no modifications. A non-trivial vector bundle which is a limit of trivial bundles on nearby curves is not semistable. Thus Atiyah's classification of vector bundles on an elliptic curve ([5]) shows that on an elliptic curve the trivial line bundle is the only indecomposable vector bundle limit of trivial bundles on nearby curves.

Theorem 1.2. *Take $L \in \mathrm{Pic}(X)$ with $\deg(L) > 0$. The vector bundle $L \oplus L^\vee$ is a limit of trivial bundles on nearby curves if and only if $h^0(L) \ge 2$.*

For each $E \in \Sigma''(X,2)$ the main invariant of E is the integer $\delta(E) := \deg(L)$ with L as in (1). For any rank 2 vector bundle F on a smooth curve set $s(F) := \deg(F) - 2\deg(L)$, where L is a rank 1 subsheaf of F with maximal degree ([19–21]). The integer $s(F)$ is often called the Segre invariant of F. If $E \in \Sigma''(X,2)$ we have $s(E) = -2\delta(E)$. Using [8, Remark 1] it is easy to see that for each $g \ge 3$ and each $X \in \mathcal{M}_g$ the sets $\Sigma'(X,2)$ and $\Sigma''(X,2)$ are contained in the union of finitely many algebraic varieties. Let $\gamma'(g)$ (resp. $\gamma''(g)$) the maximal integer x such that there is $X \in \mathcal{M}_g$, an integral variety T with $\dim(T) = x$ and a vector bundle $\mathcal{E}$ on $X \times T$ such that $\mathcal{E}_{|X\times\{t\}} \in \Sigma'(X,2)$ (resp. $\mathcal{E}_{|X\times\{t\}} \in \Sigma''(X),2)$) for all $t \in T$ and for every $a \in T$ there is a finite set $S_a \subset T$ such that $\mathcal{E}_{|X\times\{t\}} \ne \mathcal{E}_{|X\times\{a\}}$ for all $t \in T \setminus S_a$. Obviously $\gamma'(g) \le \gamma''(g)$.

We prove the following result.

Theorem 1.3.

$$\lim_{g\to+\infty} \gamma'(g) = +\infty.$$

In the example needed to prove Theorem 1.3 we find T such that $\mathcal{E}_{|X\times\{t\}} \ne \mathcal{E}_{|X\times\{a\}}$ for all $a, t \in T$ such that $a \ne t$ and an explicit value for a lower bound for $\gamma'(g)$ which grows linearly with g. Thus Theorem 1.3 holds even if we restrict to perfectly parametrized families of vector bundles, a very unusual situation when (as here) we are looking at unstable vector bundles.

It would be nice to study the possible triples (g, k, δ) such that there is $E \in \Sigma'(X, 2)$ (or $E \in \Sigma''(X, 2)$) with $\delta(E) = \delta$ for some smooth curve X with genus g and with gonality k.

See Remark 4.2 and Proposition 4.4 for the construction on $\mathbb{P}^2$ and $\mathbb{P}^3$ of indecomposable rank r bundles, $r > 2$.

Question 1.4. Fix integers $g \geq 2$ and $r \geq 2$. Is it true that a general $X \in \mathcal{M}_g$ has no indecomposable rank r vector bundle limit of trivial bundles? (or limits of trivial bundles on nearby curves)? If this is true can we find an example working for all $r \geq 2$ and defined over $\mathbb{Q}$?

For rank 2 vector bundles Question 1.4 is true. Indeed, for the first part use [8, Proposition 5], for the second part, see [1] and use [8, Remark 1].

In the last section we explore two stronger notions of limits (or limits for nearby curves) of trivial bundles: limits with constant cohomology or limits, E, such that the image $\mathrm{ev}(E)$ of the evaluation map $H^0(E) \otimes \mathcal{O}_X \to E$ is a subbundle of E, i.e., $E/\mathrm{ev}(E)$ is locally free. We consider the following topics:

(1) Find criteria of existence and/or non-existence for $E \in \Sigma'(X, r)$ (or $E \in \Sigma''(X, r)$) such that $h^0(E) = r$.
(2) Find criteria of existence and/or non-existence for $E \in \Sigma'(X, r)$ (or $E \in \Sigma''(X, r)$) such that $E/\mathrm{ev}(E)$ is locally free.

We conclude the introduction with the following questions.

Question 1.5. Is it true that for $n \gg 0$, $\mathcal{O}_{\mathbb{P}^n}^{\oplus 2}$ is the only rank 2 vector bundle on $\mathbb{P}^n$ which is a limit of the trivial bundle?

Question 1.6. Fix an integer $n \geq 4$. Is it true that, for $r \gg n$, we can construct an indecomposable rank r vector bundle on $\mathbb{P}^n$ which is a limit of the trivial bundle?

Question 1.7. Is

$$\lim_{g \to +\infty} \gamma'(g)/g = +\infty?$$

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2. Preliminaries

Let X be a smooth and connected projective curve of genus $g \geq 0$. For any vector bundle F on an integral projective variety Y such that F is a limit of trivial vector bundles on varieties near Y (and in particular if F is a limit of trivial vector bundles on Y) we have $\det(F) \cong \mathcal{O}_Y$. Thus $F \cong F^\vee$ if F has rank 2. In particular $E \cong E^\vee$ for all $E \in \Sigma''(X, 2)$.

For any rank $r > 0$ vector bundle E on X let $\mu(E) := \deg(E)/r$ denote the slope of E. If E is semistable set $\mu_+(E) = \mu_-(E) := \mu(E)$. If E is not semistable let $E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$, $s \geq 2$, be its Harder-Narasimhan filtration; set $\mu_+(E) := \mu(E_1)$ and $\mu_-(E) := \mu(E_s/E_{s-1})$. If E is not semistable we have $\mu_-(E) < \mu(E) < \mu_+(E)$.

Lemma 2.1. *Consider an exact sequence*

$$0 \to F \to E \to G \to 0 \tag{2}$$

of vector bundles on X. If $\mu_+(G) + 2g - 2 < \mu_-(F)$, then (2) splits.

Proof. It is sufficient to prove that $H^1(F \otimes G^\vee) = 0$, i.e., by duality it is sufficient to prove that there are no non-zero maps $F \to G \otimes \omega_X$. We have $\mu_+(G \otimes \omega_X) = \mu_+(G) + 2g - 2$. Use that $h^0(A \otimes B^\vee) = 0$ if A, B are semistable vector bundles and $\mu(B) > \mu(A)$. □

Corollary 2.2. *Assume $g > 0$. Let E be an indecomposable rank r vector bundle on X. Then $\mu_+(E) - \mu_-(E) \leq (r-1)(2g-2)$.*

Proof. Since $g > 0$, the corollary is true if E is semistable. Now assume that E is not semistable. Let $E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$, $s \geq 2$, be the Harder-Narasimhan filtration of E. Since E is indecomposable, no exact sequence

$$0 \to E_i \to E \to E/E_i \to 0,$$

$i = 1, \ldots, s-1$, splits. By Lemma 2.1 we have $\mu_-(E_i) \leq \mu_+(E/E_i) + 2g - 2$. We have $\mu_-(E_i) = \mu(E_i/E_{i-1})$, with the convention $E_0 = 0$ and $\mu_+(E/E_i) = \mu(E_{i+1}/E_i)$. Since $\mu(E_1) = \mu_+(E)$ and $\mu_-(E) = \mu(E_s/E_{s-1})$, we get $\mu_+(E) \leq \mu_-(E) + (s-1)(2g-2)$. Use that $s \leq r$. □

Corollary 2.3. *We have $\mu_+(E) \leq \lceil r/2 \rceil (2g-2)$, $\mu_-(E) \geq -\lceil r/2 \rceil (2g-2)$ and $\mu_+(E) - \mu_-(E) \leq (r-1)(2g-2)$ for every $E \in \Sigma''(X,r)$.*

Proof. Since $E \in \Sigma''(X,r)$, we have $\deg(E) = 0$. If $E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$ is the Harder-Narasimhan filtration of E we have $s \leq r$. Apply Corollary 2.2. □

Remark 2.4. Corollary 2.3 implies that for all X and r the set $\Sigma''(X,r)$ is contained in finitely many algebraic varieties. This is not true for any projective manifold of dimension 2 or 3 (see [7] for the case $\mathbb{P}^2$ and $\mathbb{P}^3$ and Remark 3.6 for the case of surfaces and threefolds). One cannot easily extend Remark 3.6 to the case of all projective varieties since conjecturally for $m\mathcal{G}0$ every rank 2 vector bundle on $\mathbb{P}^m$ splits.

Remark 2.5. Let Y be a smooth curve such that there is a rank 2 indecomposable vector bundle F on Y which is a limit of trivial bundles. Let $f : X \to Y$ be a finite covering with X a smooth curve. Then $f^*(F)$ is indecomposable and it is a limit of trivial bundles ([8, Remark 3]).

Remark 2.6. Let E, F and G be vector bundles on the curve X. If $E \in \Sigma(F)$ and $F \in \Sigma(G)$, then $E \in \Sigma(G)$.

Lemma 2.7. *Let Y be an integral projective curve, L a line bundle on Y with positive degree and E, F vector bundles on Y which are extensions of $L^\vee$ by L. If $E \cong F$, then the extension giving E is proportional to the extension giving F.*

Proof. Assume the existence of an isomorphism $f : E \to F$. Since $\deg(L) > 0$, (1) is the Harder-Narasimhan filtration of E (and similarly for F). Thus f sends the line subbundle L' of E isomorphic to L isomorphically onto the line subbundle L'' of F isomorphic to L. Thus f induces an isomorphism between $E/L' \cong L^\vee$ and $F/L'' \cong L^\vee$. Use that $h^0(\mathrm{End}(L)) = h^0(\mathrm{End}(L^\vee)) = 1$. □

3. Limits coming from $\mathbb{P}^2$ and $\mathbb{P}^3$

We use the indecomposable rank 2 vector bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$ constructed in [7]. See also [24] for an earlier example ($k = 1$ for $\mathbb{P}^2$) and [25] for the deformation theory of these vector bundles on the plane.

3.1. Limits in $\mathbb{P}^2$

Fix an indecomposable rank 2 vector bundle F on $\mathbb{P}^2$, which is a limit of trivial vector bundles and it is indecomposable ([7,24,25]). Let k be the unique positive integer such that F fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k) \to F \to \mathcal{I}_Z(-k) \to 0 \tag{3}$$

with Z a complete intersection zero-dimensional scheme of 2 curves of degree k and $\deg(Z) = k^2$ ([7, 25]; see in particular [7, Lemma 4] to see that Z is always the complete intersection of 2 curves of degree k). We call $N(k)$ the set of all indecomposable rank 2 vector bundles on $\mathbb{P}^2$ which are limits of trivial bundles and with maximal degree line subbundle isomorphic to $\mathcal{O}_{\mathbb{P}^2}(k)$. Thus each element of $N(k)$ fits in (3) for some Z.

Proposition 3.1. *Fix integers $k > 0$ and $d \geq 2k+1$. Let $Y \subset \mathbb{P}^2$ be an integral projective curve of degree d. Let $N_Y(k)$ be the set of all $F \in N(k)$ fitting in* (3) *for some Z with $Z \cap Y = \emptyset$. Then*

(1) *$F_{|Y}$ is indecomposable for each $F \in N_Y(k)$.*
(2) *If $F, G \in N_Y(k)$ and $G \neq F$, then $G_{|Y} \neq F_{|Y}$.*

Proof. Take $F \in N_Y(k)$ and set $E := F_{|Y}$. Since $F \in N_Y(k)$, E fits in an exact sequence (3) with $Z \cap Y = \emptyset$. Thus restricting (3) to Y we get that E is an extension of $\mathcal{O}_Y(-k)$ by $\mathcal{O}_Y(k)$. Thus to prove that E is indecomposable it is sufficient to prove that $h^0(E(-k)) = 0$. Since F is indecomposable, we have $h^0(F(-k)) = 0$. Thus to get $h^0(E(-k)) = 0$ it is sufficient to prove that $h^1(F(-k-d)) = 0$. Since $F \cong F^\vee$, duality gives $h^1(F(-k-d)) = h^1(F(k+d-3))$. By (3) we have $h^1(F(k+d-3)) = 0$ if $h^1(\mathcal{I}_Z(d-3)) = 0$. Since Z is the complete intersection of 2 plane curves of degree k, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2k) \to \mathcal{O}_{\mathbb{P}^2}(-k)^{\oplus 2} \to \mathcal{I}_Z \to 0. \tag{4}$$

From (4) we get $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \geq 2k-2$. From (4) and (3) we get the following remark.

Observation 1. We have $h^1(E(t)) = 0$ for all $E \in N(k)$ and all $t \geq 3k-2$.

Since Z has codimension 2, we have $h^2(\mathcal{I}_Z(x)) = h^2(\mathcal{O}_{\mathbb{P}^2}(x))$ for every integer x. Thus we have the following remark.

Observation 2. We have $h^2(E(x)) = 0$ for all $E \in N(k)$ and all $x \geq k-2$.

Now we take $F, G \in N_Y(k)$ such that $F \neq G$. Since F and G are not semistable, every map $f : F \to G$ sends the Harder-Narasimhan filtration (3) of F into the Harder-Narasimhan filtration of G. Since $G \cong G^\vee$, we have $F \otimes G^\vee \cong F \otimes G$. Since $F \neq G$ we see that f is the composition of the surjection $F \to \mathcal{I}_Z(-k)$ and a map $j : \mathcal{I}_Z(-k) \to \mathcal{O}_{\mathbb{P}^2}(k)$. Since Z has codimension 2 the set of all j's is the vector space $H^0(\mathcal{O}_{\mathbb{P}^2}(2k))$. Thus it is sufficient to prove the surjectivity of the restriction map $H^0(F \otimes G) \to H^0(F_{|Y} \otimes G_{|Y})$. Thus it is sufficient to prove that $h^1(F \otimes G(-d)) = 0$. By duality it is sufficient to prove $h^1(F \otimes G(d-3)) = 0$. By (3) it is sufficient to prove that $h^1(G(k+d-3)) = h^1(\mathcal{I}_Z \otimes G(d-k-3)) = 0$. By Observation 1 we have $h^1(G(k+d-3)) = 0$, because $d \geq 2k+1$. Tensoring (4) by $G(d-k-3)$ and using Observations 1 and 2 we get $h^1(\mathcal{I}_Z \otimes G(d-k-3)) = 0$ if $d \geq 3k-2$. □

Remark 3.2. Let $Y \subset \mathbb{P}^2$ be an integral plane curve. The set $N_Y(k)$ defined in the statement of Proposition 3.1 is a non-empty open subset of $N(k)$. Now we fix $E \in N(k)$. There are many Y such that $E \in N_Y(k)$. If we are interested in a pair (E, Y) with $E \in N(k) \setminus N_Y(k)$ at least we may say that $g^*(E) \in N_Y(k)$ for a general $g \in \mathrm{Aut}(\mathbb{P}^2)$.

3.2. Limits in $\mathbb{P}^3$

Following [7] for any rank 2 vector bundle F on $\mathbb{P}^3$ which is a limit of trivial bundles there is a unique integer $d \geq 0$ such that F fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(k) \to F \to \mathcal{I}_Z(-k) \to 0 \tag{5}$$

with $Z \subset \mathbb{P}^3$ a complete intersection curve of degree k^2 and with $\omega_Z \cong \mathcal{O}_Z(-2k-4)$ ([7, Lemma 4]). Call $M(k)$ the set of all isomorphism classes of rank 2 vector bundles on $\mathbb{P}^3$ fitting in (5) for some Z and which are limits of trivial bundles. By [7, Proposition 2] each $M(k)$ has a natural structure of algebraic variety, $M(0) = \{\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}\}$, $M(1) = \emptyset$ and $M(k) \neq \emptyset$ for each $k \geq 2$. C. Bănică gave a complete description of $M(2)$ ([7, §3.2]). Fix any $k \geq 2$ and any smooth curve $X \subset \mathbb{P}^3$. Take $F \in M(k)$ fitting in (5) and set $E := F_{|X}$. Obviously E is a limit of trivial bundles. If $Z \cap X = \emptyset$ (and for any $F' \in M(k)$ this is the case for $h^*(F')$ with h general in $\mathrm{Aut}(\mathbb{P}^3)$), then E fits in an exact sequence (1) with $L = \mathcal{O}_X(k)$. The aim is to find conditions on X which assure that E is indecomposable. In many cases E cannot be indecomposable, e.g. by [8, Propositions 2 and 3] E is decomposable if either X is hyperelliptic or if it has general moduli. As a corollary we will get that E is indecomposable if $X \cap Z = \emptyset$ and X is the complete intersection of 2 surfaces, one of degree $a \geq 2$ and the other one of degree $b \geq 3k+a-3$ (Proposition 3.5), but we will get many more X with this property (Proposition 3.4).

Lemma 3.3. *Fix an integer $a > 0$ and take an integral degree a surface $T \subset \mathbb{P}^3$ such that $\dim(Z \cap T) = 0$. Then $G := F_{|T}$ is indecomposable and $h^0(G(k)) = h^0(\mathcal{O}_T(2k))$. If $a \geq 4k+1$ and $F' \in M(k)$ is associated to some Z' with $\dim Z' \cap T = 0$ and $F' \neq F$, then $F'_{|T} \neq F'$.*

Proof. Since the scheme $T \cap Z$ has dimension 0, the equation of T is not contained in any prime associated to the primary decomposition of the $\mathcal{I}_Z$. Thus the equation of T is a not zero-divisor of the sheaf $\mathcal{I}_Z$ (use that $\mathcal{I}_Z$ has depth 2, since Z is a (locally) complete intersection and hence the sheaves $\mathcal{I}_Z$ and $\mathcal{O}_Z$ have no embedded components). Thus restricting (5) to T we get an exact sequence

$$0 \to \mathcal{O}_T(k) \to G \to \mathcal{I}_{Z\cap T}(-k) \to 0. \tag{6}$$

Since $k > 0$, G is not semistable and (6) is the Harder-Narasimhan filtration of E. Thus $G \cong \mathcal{O}_T(k) \oplus \mathcal{I}_{Z\cap T}(-k)$ if G is decomposable. Since G is locally free and $Z\cap T$ is a non-empty codimension 2 subscheme of T, we get a contradiction. Since $Z \cap T \neq \emptyset$, (6) implies $h^0(G(k)) = h^0(\mathcal{O}_T(2k))$.

Now we take $F' \neq F$ and prove that $F'_{|T} \neq G$. As in the proof of Proposition 3.1 using the Harder-Narasimhan filtration of F, F', G and G' we see that it is sufficient to prove that $h^1(F \otimes F'(-a)) = 0$. By duality it is sufficient to prove $h^2(F \otimes F'(a-4)) = 0$. By (5) it is sufficient to prove that $h^2(F'(k+a-4)) = h^2(\mathcal{I}_Z \otimes F'(-k+a-4)) = 0$. The sheaf $\mathcal{I}_Z$ has the minimal free resolution (4) with $\mathbb{P}^3$ instead of $\mathbb{P}^2$. By (5) for F' we have $h^2(F'(k+a-4)) = 0$ if $h^2(\mathcal{I}_Z(a-4)) = 0$ and this is true by (4) if $h^3(\mathcal{O}_{\mathbb{P}^3}(-2k+a-4)) = 0$, i.e., $a \geq 2k+1$. Tensoring (4) with $F'(-k+a-2)$ we see that $h^2(\mathcal{I}_Z \otimes F'(-k+a-4)) = 0$ if $h^3(F'(-3k+a-4)) = 0$ and $h^2(F'(-k+a-4)) = 0$. We saw that to get the latter vanishing it is sufficient to assume $a \geq 4k+1$. Since Z is a curve, the exact sequence

$$0 \to \mathcal{I}_Z(x) \to \mathcal{O}_{\mathbb{P}^3}(x) \to \mathcal{O}_Z(x) \to 0$$

shows that $h^3(\mathcal{I}_Z(x)) = h^3(\mathcal{O}_{\mathbb{P}^3}(x))$ for all $x \in \mathbb{Z}$. Thus (5) gives $h^3(F'(-3k+a-4)) = 0$ for all $a \geq 3k+1$. □

Proposition 3.4. *Fix an integer $d > 0$ and take an integral degree d surface $T \subset \mathbb{P}^3$ such that $\dim(Z \cap T) = 0$ and T has only finitely many singular points. Set $G := F_{|T}$. Let $X \subset T$ be a smooth curve such that $X \cap \mathrm{Sing}(T) = X \cap Z = \emptyset$ and set $E := F_{|X}$. If $h^1(G(-X)) = 0$, then E is an indecomposable limit of trivial bundles. Take another $F' \in M(k)$ associated to some Z' with $\dim Z' \cap T = 0$ and set $G' := F'_{|T}$ and $E' := F'_{|X}$. If $d \geq 4k+1$, $F' \neq F$ and $h^1(G \otimes G'(-X)) = 0$, then $E' \neq E$.*

Proof. Since $X \cap \mathrm{Sing}(T) = \emptyset$, $\mathcal{O}_T(-X)$ is a line bundle. Since $X \cap Z = \emptyset$, E fits in an exact sequence (1) with $L = \mathcal{O}_X(k)$. By [8, Lemma 2] to prove that E is indecomposable, it is sufficient to prove that $h^0(E(k)) = h^0(\mathcal{O}_X(2k))$. By Lemma 3.3 it is sufficient to use the assumption $h^1(F(k)(-X)) = 0$ and a standard exact sequence of sheaves on T. Now we check the last assertion. Since $d \ge 4k+1$, Lemma 3.3 gives $F'_{|T} \ne F_{|T}$. We conclude as in the proof of Lemma 3.3. □

Proposition 3.5. *Fix $F \in M(k)$, $k \ge 2$, fitting in* (5) *and let $X \subset \mathbb{P}^3$ be a complete intersection of a surface of degree b and a surface of degree a with $b \ge 3k+a-3$. Then $E := F_{|X}$ is an indecomposable limit of trivial bundles. If $F' \in M(k)$ with associated Z' such that $Z' \cap X = \emptyset$, $F' \ne F$ and $a \ge 4k+1$, then $F'_{|X} \ne E$.*

Proof. We only need to prove that E is indecomposable. By [8, Lemma 2], it is sufficient to prove that $h^0(E(k)) = h^0(\mathcal{O}_X(2k))$. Take a degree a surface $T \subset \mathbb{P}^3$ containing X. Set $G := F_{|T}$. Since $X \in |\mathcal{O}_T(b)|$, X is an ample Cartier divisor of X. Since $X \cap Z = \emptyset$, T contains no irreducible component of $Z \cap T$. Thus we may apply Lemma 3.3 to T and G. By Lemma 3.3 G is indecomposable and $h^0(G(k)) = h^0(\mathcal{O}_X(2k))$. Since $h^1(\mathcal{O}_X(2k-b)) = 0$, the restriction map $H^0(\mathcal{O}_T(2k)) \to H^0(\mathcal{O}_X(2k))$ is surjective. Thus it is sufficient to prove that the restriction map $H^0(G(k)) \to H^0(E(k))$ is surjective. Hence it is sufficient to prove that $h^1(G(k-b)) = 0$. Look at (5). Since Z is a complete intersection of 2 surfaces of degree k ([7, Lemma 4]), Z has the following minimal free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2k) \to \mathcal{O}_{\mathbb{P}^3}(-k)^{\oplus 2} \to \mathcal{I}_Z \to 0. \tag{7}$$

Since T contains no irreducible component of Z, we have $\mathrm{Tor}_1(\mathcal{I}_Z, \mathcal{O}_T) = 0$. Thus restricting (7) to T we get the exact sequence

$$0 \to \mathcal{O}_T(-2k) \to \mathcal{O}_T(-k)^{\oplus 2} \to \mathcal{I}_{Z\cap T,T} \to 0. \tag{8}$$

From (8) we get $h^1(T, \mathcal{I}_{Z\cap T,T}(x)) \le h^2(\mathcal{O}_T(x-2k))$ for all $x \in \mathbb{N}$. Since $\omega_X \cong \mathcal{O}_T(a-4)$, duality gives $h^2(\mathcal{O}_T(x-2k)) = h^0(\mathcal{O}_T(2k+a-4-x))$. Thus $h^1(T, \mathcal{I}_{Z\cap T,T}(x)) = 0$ for all $x \ge 2k+a-3$. From (6) we get $h^1(G(y)) = 0$ for all $y \ge 3k+a-3$.

Now we fix F' and prove that $F'_{|X} \ne F_{|X}$ under the stated assumptions on a and b. Set $G' := F'_{|T}$. Since $a \ge 4k+1$, Lemma 3.3 gives $G' \ne G$. We see that it is sufficient to prove that $h^1(G \otimes G'(-b)) = 0$. Since $\omega_T \cong \mathcal{O}_T(a-4)$, duality gives $h^1(G \otimes G'(-b)) = h^1(G \otimes G'(b+a-4))$. Tensoring with the locally free sheaf $G'(b+a-4)$ the restriction of (5) to T we see that it is sufficient to prove that $h^1(G'(k+b+a-4)) = h^1(\mathcal{I}_{Z\cap T} \otimes G'(-k+b+a-4)) = 0$. The restriction to T of the exact sequence (4) for $\mathbb{P}^3$ instead of $\mathbb{P}^2$ is exact, because $\dim Z \cap T = 0$. Thus from (5) restricted to T we see that $h^1(G'(k+b+a-4)) = 0$ (since $k+b+a-4 > 2k-4$ and $-k+b+a-4 \ge 2k-1$). □

3.3. Pull-backs

Now we use pull-backs of bundles from $\mathbb{P}^m$, $m = 2, 3$, by a finite covering $f : W \to \mathbb{P}^m$ and then restrict the bundle to many smooth curves $X \subset W$ (most of these curves will not be finite covering of a curve in $\mathbb{P}^m$ (case $m = 2$ or for

$m = 3$ at least $f_{|X}$ will be an embedding)). Thus we get existence not covered by other examples plus the quotation of [8, Remark 3]. We only give the details for the case $m = 2$. Let W be a smooth and connected projective surface and M an ample and spanned vector bundle on W. Take a 3-dimensional linear subspace $V \subseteq H^0(M)$ spanning M (e.g. take as V a general 3-dimensional linear subspace of $H^0(M)$). By the universal property of projective spaces the pair $(\mathcal{L}, V)$ induces a morphism $u_{M,V} : W \to \mathbb{P}^2$ such that $M = u_{M,V}^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $V = u_{M,V}^*(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))$. Set $u := u_{M,V}$. Fix an indecomposable rank 2 vector bundle F on $\mathbb{P}^2$, which is a limit of trivial vector bundles and it is indecomposable ([7, 24, 25]) and set $G := u^*(F)$. Let k be the unique positive integer such that F fits in an exact sequence Since M is ample, u has finite fibers. Thus u is surjective and $u : W \to \mathbb{P}^2$ is a finite covering. Set $e := \deg(u)$. From (3) we get an exact sequence

$$0 \to M^{\otimes k} \to G \to \mathcal{I}_{Z'} \otimes (M^\vee)^{\otimes k} \to 0 \tag{9}$$

with Z' a degree ek^2 zero-dimensional scheme on W. Since M is ample, (9) shows that G is unstable with respect to the polarization M and hence that (9) is the Harder-Narasimhan filtration of G. In particular the inclusion in (9) gives the only rank 1 subsheaf of G with positive degree with respect to the polarization M.

Claim 1. G is indecomposable and it is a flat limit of a family of trivial bundles on W.

Proof Claim 1. The pull-back by u of the flat family of trivial bundles on $\mathbb{P}^2$ with F as its limit shows that G is a flat limit of a family of trivial vector bundles on W. Assume that G is decomposable. Since $\det(G) \cong \mathcal{O}_W$, $G \cong R \oplus R^\vee$ for some line bundle R on W. By (9) and the M-unstability of G we get $M^{\otimes k} \in \{R, R^\vee\}$. Thus $c_2(G) = ek^2 \neq 0$. Since $c_2(F) = 0$, we get a contradiction. □

Fix a very ample line bundle $\mathcal{R}$ on W such that $h^1(G \otimes L^{\otimes k} \otimes \mathcal{R}^\vee) = h^1(L^{\otimes 2k} \otimes \mathcal{R}^\vee) = 0$ and take any smooth $X \in |\mathcal{R}|$ such that $Z' \cap X = \emptyset$. Set $E := G_{|X}$ and $M := L_{|X}^{\otimes k}$. Since $Z' \cap X = \emptyset$, (9) gives an exact sequence (1).

Claim 2. E is indecomposable and it is a flat limit of a family of trivial bundles on X.

Proof Claim 2. By Claim 1, E is a flat limit of trivial bundles on X. Assume that E is decomposable. Since L is ample, (1) implies $E \cong L \oplus L^\vee$. By [8, Lemma 2] to get a contradiction it is sufficient to prove that $h^0(E \otimes L) = h^0(L^{\otimes 2})$. Use that $h^1(G \otimes L^{\otimes k}(-X)) = h^1(L^{\otimes 2k}(-X)) = 0$ and that $h^0(G \otimes L^{\otimes k}) = h^0(L^{\otimes 2k})$. □

Claim 3. Take $F' \in N(k)$ such that $F' \neq F$ and set $G' := f^*(F')$. We have $G' \neq G$.

Proof Claim 3. Any isomorphism $G \to G'$ sends the Harder-Narasimhan filtration of G onto the Harder-Narasimhan filtration of G'. Thus (since G and G' are isomorphic to their duals) it is sufficient to prove that $h^0(G \otimes G) = h^0(G \otimes M^{\otimes k})$. This is true, because $Z \neq \emptyset$. □

Remark 3.6. By [7] the set of all rank 2 vector bundles on $\mathbb{P}^m$, $m = 2, 3$, which are limits of trivial bundles is not contained in a union of finitely many algebraic varieties (it is the union of countably many irreducible algebraic varieties V_k, $k \geq 1$, with $\lim_{k \to +\infty} \dim V_k = +\infty$). For every projective m-fold, $m = 2, 3$, there is a finite morphism $f : W \to \mathbb{P}^m$. By Claim 3 the set of all rank 2 vector bundles on W which are limits of the trivial rank 2 vector bundle (on W) is not contained in finitely many algebraic varieties (it contains the union of countably many algebraic varieties V_k, $k \geq 1$, with $\lim_{k \to +\infty} \dim V_k = +\infty$).

4. Higher rank vector bundles

In this section we discuss the existence of indecomposable rank $r > 2$ which are limits of trivial bundles. We only look at vector bundles $S^{r-1}(F)$ with F indecomposable and limit of trivial bundles, where S^{r-1} denote the $(r-1)$-th symmetric product. If the bundle F is on a smooth curve X and it fits in an exact sequence (1), then $E : S^{r-1}(F)$ has a filtration $\{E_i\}_{0 \leq i \leq r}$ with $E_0 = 0$, $E_r = E$, E_i a rank i subbundle of E, $E_1 \cong L^{\otimes(r-1)}$ and $E_i/E_{i-1} \cong L^{\otimes r+1-2i}$ for $2 \leq i \leq r-1$ (use the filtration of $S^{r-1}(F(L))$ induced by the twist of (1) and that $S^{r-1}(F(L)) \cong E \otimes L^{\otimes(r-1)}$). This is the Harder-Narasimhan filtration of E. On $\mathbb{P}^m$, $m = 2, 3$, we have a similar filtration over the complement of Z_{red} with $L := \mathcal{O}_{\mathbb{P}^m}(k)$ and hence in this case E is unstable with Harder-Narasimhan filtration of maximal length, i.e., with subquotients of rank 1.

Proposition 4.1. *E is indecomposable.*

Proof. Call $\epsilon \in H^1(L^{\otimes 2})$ the extension class of (1). Each bundle E_i/E_{i-2}, $2 \leq i \leq r$, is an extension of $L^{\otimes j}$, $j = 2r-1-2i$, by $L^{\otimes j+2}$ and this extension class is induced by ϵ and hence it is non-zero. Thus each E_i/E_{i-2} is indecomposable. Suppose that E is decomposable, say $E = D_1 \oplus \cdots \oplus D_t$ with $t \geq 2$ and each D_i indecomposable. Since the Harder-Narasimhan filtration of E has rank 1 subquotients, each of them isomorphic to some power of L, the same is true for each D_i. By induction on r we may also assume that $S^{r-2}(F)$ is indecomposable. Twisting by suitable line bundles we get that E_{r-1} and E/E_1 are indecomposable. Since E_{r-1} (resp. E/E_1) is indecomposable, its Harder-Narasimhan filtration and the one of E, plus the decomposability of E give $E \cong E_{r-1} \oplus E/E_{r-1}$ (resp. $E \cong E_1 \oplus E/E_1$). Since E_1 and E/E_1 are different powers of L and $r \geq 3$, we get a contradiction. □

Now we consider $\mathbb{P}^2$.

Remark 4.2. Take $F \in N(k)$, an integer $r \geq 3$ and a large degree plane curve $X \subset \mathbb{P}^2$. Since $F_{|Y}$ is indecomposable (Proposition 3.4), Proposition 4.1 shows that $S^{r-1}(F)$ is indecomposable. Thus we get infinitely many indecomposable limits of trivial bundles for any rank $r \geq 3$. Using $M(k)$ and Proposition 3.5 we get the same for $\mathbb{P}^3$.

Fix an integer $k > 0$ and take $E_1, \dots, E_s \in N(k)$, $s \geq 2$, with E_i sitting in an exact sequence (3) with $(E, Z) = (E_i, Z_i)$. Assume $Z_i \cap Z_j = \emptyset$ for all $i \neq j$. Set $E := E_1 \otimes \cdots \otimes E_s$. E has rank 2^s and it is a limit of trivial bundles on $\mathbb{P}^2$. Each E_i satisfies $E_i \cong E_i^\vee$. Thus $E \cong E^\vee$. Let $E = D_1 \oplus \cdots \oplus D_t$, $t \geq 1$, be a decomposition of E into indecomposable factors. The uniqueness, up to isomorphisms, of the indecomposable factors of any vector bundles on any projective variety ([4]) shows that any isomorphism between E and $E^\vee$ induces a permutation $\sigma : \{1, \dots, t\} \to \{1, \dots, t\}$ such that $D_{\sigma(i)} \cong D_i$ for all $i \in \{1, \dots, t\}$.

Question 4.3. Is $E_1 \otimes \cdots \otimes E_s$ indecomposable?

We can prove this question when $s = 2$.

Proposition 4.4. *$E_1 \otimes E_2$ is indecomposable.*

Proof. The Harder-Narasimhan filtration of $E := E_1 \otimes E_2$ has the first rank 1 subsheaf with slope $2k$, one rank 1 subquotient with slope $-2k$ while the other subquotients (with total rank 2) have slope 0. Assume E decomposable. Tensoring (3) for E_1 (so with $Z = Z_1$) with E_2 we get an exact sequence

$$0 \to E_2(k) \to E \to \mathcal{I}_{Z_1} \otimes E_2(-k) \to 0. \tag{10}$$

Since $Z_1 \neq \emptyset$ and $Z_2 \neq \emptyset$, from (10) and (3) for $(E, Z) = (E_2, Z_2)$ we get $h^0(E) = h^0(\mathcal{O}_{\mathbb{P}^2}(2k)) = \binom{2k+2}{2}$.

(a) Assume that E has a factor $\mathcal{O}_{\mathbb{P}^2}(2k)$ (resp. $\mathcal{O}_{\mathbb{P}^2}(-2k)$). In this case E has a factor $\mathcal{O}_{\mathbb{P}^2}(-2k)$ (resp. $\mathcal{O}_{\mathbb{P}^2}(2k)$) because $E \cong E^\vee$. Call D the direct sum of the other factors of E. Since $\mu(D) = 0$, we get that there is no non-zero map $E \to \mathcal{I}_{Z_1 \cup Z_2}(-2k)$, a contradiction.

(b) Assume that $\mathcal{O}_{\mathbb{P}^2}$ is a factor of E. There is no non-zero map $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_{Z_1} \otimes E_2(-k)$ by (3) for (E_2, Z_2), because $Z_1 \neq \emptyset$. Thus (10) gives an injective map $j : \mathcal{O}_{\mathbb{P}^2} \to E_2(k)$ with locally free quotient, contradicting the fact that $E_2(k)$ is indecomposable.

(c) By steps (a) and (b) we have $E \cong D_1 \oplus D_2$ with D_1 and D_2 indecomposable and of rank 2. First assume that $D_1 \cong D_1^\vee$. In this case we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2k) \to D_1 \to \mathcal{I}_{Z_1 \cup Z_2}(-2k) \to 0. \tag{11}$$

Since $h^0(D_1) = h^0(E)$ by (11) and Claim 1, we get $h^0(D_2) = 0$. Since $c_1(D_2) = 0$, we get that D_2 is stable. Thus there is no non-zero map $D_2 \to E_2(-k)$. By (10) we get $D_2 \cong E_2(k)$, contradicting the stability of D_2. $\square$

Remark 4.5. Restricting to a general plane we get that if E_1 and E_2 are elements of $M(k)$ with disjoint Z's, then $E_1 \otimes E_2$ is indecomposable.

5. Curves with a prescribed gonality or low genus

For all integers $k \geq 2$ and $g \geq 2k-2$ let $\mathcal{M}^1{}_{g,k}$ denote the set of all smooth curves of genus g with gonality k. If $g \in \{2k-2, 2k-1\}$, then $\mathcal{M}^1{}_{g,k}$ is a non-empty open subset of the moduli space $\mathcal{M}_g$ of all genus g smooth curves, while if $2k+1 < g$, then $\mathcal{M}^1{}_{g,k}$ is an irreducible locally closed subset of $\mathcal{M}_g$ with dimension $2g-2+2k$ and a general $X \in \mathcal{M}^1{}_{g,k}$ has a unique g^1_k ([2, Theorem 2.6]).

Remark 5.1. Let X be a genus g curve having an indecomposable rank 2 vector bundle E which is a limit of trivial bundles. By [8, Remark 1] there is a line bundle L on X such that $h^1(L^{\otimes 2}) > 0$ and $h^0(L) \geq 2$. Since $\deg(L) \leq g-1$, we get $g \geq 3$. Since X is not hyperelliptic ([8, Proposition 3]) we get $g \geq 4$.

See Proposition 5.2 for a classification of all (X, E) with $E \in \Sigma'(X, 2)$ when $g = 4$. In this case it is easy to check that $\Sigma'(X, 2) = \Sigma''(X, 2)$.

Proposition 5.2. *Let X be a smooth curve of genus 4. X has an indecomposable vector bundle E which is a limit of trivial bundles if and only if it has a unique g^1_3 and in this case E is unique and fits in (1) with L the unique g^1_3 of X.*

Proof. Since X is not hyperelliptic ([8, Proposition 3]) and it has genus 4, it is trigonal. Since $\deg(\omega_X) = 6$, E lies in (1) with L a g^1_3 on X. Since $h^1(L^{\otimes 2}) > 0$ and $\deg(\omega_X) = 6$, we have $L^{\otimes 2} \cong \omega_X$. Let $\phi : X \to \mathbb{P}^3$ the canonical map. Since X is not hyperelliptic, ϕ is an embedding and $\phi(X)$ is the complete intersection of a quadric surface Q and a cubic surface. We have $L^{\otimes 2} \cong \omega_X$ if and only if Q is a quadric cone (in the other case it has two g^1_3, say L_1, L_2 with $L_1 \otimes L_2 \cong \omega_X$ and hence $L_i^{\otimes 2} \ncong \omega_X$ for all i) ([17, Ex. IV.5.3]). The existence of canonical models lying on a quadric cone is obvious (take the intersection of a quadric cone and a general cubic surface or use [17, Ex. V.2.9]). If one is interested in singular curves, too, one can look at [9, 13, 18, 23].

The existence part is a particular case of [8, Proposition 5], because X has a unique g^1_3 if and only if its canonical model lies on a quadric cone and in this case the g^1_3 is a theta-characteristic of X. Of course, this part of the study of theta-characteristic on low genus curves is well described in [15, 26, 27]. □

Remark 5.3. Fix a general $X \in \mathcal{M}^1{}_{g,k}$, $g > 2k \geq 6$. There is a unique $R \in \mathrm{Pic}^k(X)$ such that $h^0(R) = 2$ ([2]) and the first integer t such that $h^0(R^{\otimes t}) > t+1$ (call it ρ_R), it the first integer t such that $t+2 \leq kt+1-g$, i.e., $\rho_R = \lceil (g+1)/(k-1) \rceil$ ([6], [11, 1.1.3], [10, 1.1]).

Adapting [8, Proposition 3] we get the next propositions.

Proposition 5.4. *Fix $X \in \mathcal{M}^1_{g,k}$ and take $R \in \mathrm{Pic}^k(X)$ such that $h^0(R) = 2$. Let ρ_R be the maximal positive integer t such that $h^0(R^{\otimes t}) = t+1$. Fix an integer t such that $1 \leq t \leq \rho_R/2$ and take any rank 2 vector bundle E on X*

which is a limit of trivial bundles and which fits in (1) *with* $L = R^{\otimes t}(D)$ *with* D *an effective divisor with* $h^0(R^{\otimes 2t}(D)) = 2t+1$. *Then* E *splits.*

Proof. Twisting with $R^{\otimes t}$ the family of trivial bundles with E as a limit, we get $h^0(E \otimes R^{\otimes t}) \geq 2t+2$. Twisting (1) with $R^{\otimes t}$ and using that $h^0(R^{\otimes 2t}(D)) = 2t+1$ we first get $D = 0$ and then that the extension class in (1) splits. Thus E is decomposable. □

Proposition 5.5. *Fix* $X \in \mathcal{M}_g$ *and a rank* 2 *bundle on* X *which is a limit of trivial bundles on nearby curves. Write* $L = M(B)$ *with* B *an effective divisor and* $h^0(L) = h^0(M)$. *If* $h^0(M^{\otimes 2}(B)) < 2h^0(M)$, *then* E *is decomposable.*

Corollary 5.6. *Take a general* $X \in \mathcal{M}^1_{g,4}$, $g \geq 7$. *Let* E *be a rank* 2 *vector bundle limit of trivial bundles on nearby curves and fitting in* (1) *for some* $L \in \mathrm{Pic}^d(X)$. X *is decomposable if* $2d < \lfloor (g+8)/2 \rfloor$.

Proof. Let $R \in \mathrm{Pic}^4(X)$ be the only line bundle with $h^0(R) = 2$ (it is unique by the generality of X). Write $L = M(B)$ with D effective and $h^0(M) = h^0(L)$. By [11, Theorem 2.3.2] and the assumption $d < \lfloor (g+8)/2 \rfloor$, the morphism induced by $|M|$ is not birational onto its image. Since X is a general 4-gonal curve, it is not a multiple covering of a curve of genus > 0. Thus it is easy to check that $M \cong R^{\otimes t}$ for some positive integer t (or see [11, Claim 1.1.6]). We have $h^0(R^{\otimes 2t}) = 2t+1$ by Remark 5.3, because $2d+1-g \leq 2t+1$. Assume $h^0(R^{\otimes 2t}(D)) \geq 2t+2$ and take $D' \subseteq D$ minimal with $h^0(R^{\otimes 2t}(D')) = 2t+2$. Note that $R^{\otimes 2t}(D')$ is spanned and not composed with $|R|$, because $h^0(\mathcal{O}_X(D) \otimes R^{\vee}) = 0$. Thus $|R^{\otimes 2t}(D')|$ is simple (i.e., induces a morphism birational onto its image), contradicting the inequality $2d < \lfloor (g+8)/2 \rfloor$. □

Corollary 5.7. *Fix an integer* $g \geq 9$ *and a general* $X \in \mathcal{M}^1_{g,5}$. *Let* E *be a rank* 2 *vector bundle limit of trivial bundles on nearby curves and fitting in* (1) *for some* $L \in \mathrm{Pic}^d(X)$. X *is decomposable if* $2d < \lfloor (g+8)/2 \rfloor$.

Proof. Let $R \in \mathrm{Pic}^5(X)$ be the only line bundle with $h^0(R) = 2$. Write $L = M(B)$ with D effective and $h^0(M) = h^0(L)$. By [11, Proposition 2.3.7] and the assumption $d < \lfloor (g+8)/2 \rfloor$, the morphism induced by $|M|$ is not birational onto its image. Since 5 is a prime number, this implies the existence of an integer $t > 0$ such that $M \cong R^{\otimes t}$. By Proposition 5.4 it is sufficient to prove that $h^0(R^{\otimes 2t}(D)) = 2t+1$. We have $h^0(R^{\otimes 2t}) = 2t+1$ by Remark 5.3, because $2d+1-g \leq 2t+1$. Assume $h^0(R^{\otimes 2t}(D)) \geq 2t+2$ and take $D' \subseteq D$ minimal with $h^0(R^{\otimes 2t}(D')) = 2t+2$. Note that $R^{\otimes 2t}(D')$ is spanned and not composed with $|R|$, because $h^0(\mathcal{O}_X(D) \otimes R^{\vee}) = 0$. Thus $|R^{\otimes 2t}(D')|$ is simple (i.e., induces a morphism birational onto its image), contradicting the inequality $2d < \lfloor (g+8)/2 \rfloor$. □

Remark 5.8. Let X be a trigonal curve of genus $g \geq 5$. There is a unique $R \in \mathrm{Pic}^3(X)$ such that $h^0(R) = 2$. Let $m = m(X)$ the Maroni invariant of X, i.e., let $m+2$ be the first positive integer t such that $h^0(R^{\otimes t}) > t+1$ ([22, §1]).

We have $\lceil (g-4)/3 \rceil \le m \le \lfloor (g-2)/2 \rfloor$. In the set-up of Proposition 5.4 we have $\rho_R = m+1$.

Proposition 5.9. *Let X be a smooth curve of genus g with an indecomposable rank 2 vector bundle limit of trivial bundles and with $\delta(E) = 3$. Then $g = 4$ and X, L and E are as in Proposition 5.2.*

Proof. By Proposition 5.2 it is sufficient to prove that $g = 4$. By Remark 5.1 we have $g \ge 4$. Assume $g \ge 5$. Take $L \in \mathrm{Pic}^3(X)$ such that E fits in (1) with $\deg(L) = 3$ and $h^0(L) \ge 2$. Since X is not hyperelliptic ([8, Proposition 3]), it is trigonal, $h^0(L) = 2$ and L has no base points. Let m be the Maroni invariant of X (Remark 5.8). Since $m \ge \lceil (g-4)/3 \rceil \ge 1$, we have $h^0(L^{\otimes 2}) = 3$. Proposition 5.4 with $t = 1$, $D = 0$ and $R = L$ gives a contradiction. Thus $g = 4$ and X, L and E are described in Proposition 5.2. □

Proposition 5.10. *Take $X \in \mathcal{M}^1_{g,k}$, $k \ge 3$, $g \ge k^2 - 2k + 2$. If k is not a prime assume that X is not a multiple covering of a curve of positive genus. Then $\delta(E) > k$ for all $E \in \Sigma''(X,2)$.*

Proof. Assume the existence of $E \in \Sigma''(X,2)$ such that $\delta(E) \le k$. We have $\delta(E) = k$ by Remark 1.1 and there is $L \in \mathrm{Pic}^k(X)$ such that $|L|$ is a g^1_k on X and E is an extension of $L^\vee$ by L. By Proposition 5.4 we have $h^0(L^{\otimes 2}) \ge 4$. Since L is spanned, $L^{\otimes 2}$ is spanned. Thus $|L^{\otimes 2}|$ induces a morphism $\phi : X \to \mathbb{P}^3$ such that $\deg(\phi) \cdot \deg(\phi(X)) = 2k$. Since any integral and non-degenerate space curve has arithmetic genus at most $k^2 - 2k + 1$ and $g \ge k^2 - 2k + 2$, $\deg(\phi) > 0$. Since X is not a multiple covering of a smooth curve of positive genus, the normalization of $\phi(X)$ is $\mathbb{P}^1$. Since $\phi(X)$ is non-degenerate, we have $\deg(\phi(X)) \ge 3$. Thus X has gonality $\le 2k/3$, a contradiction. □

Proposition 5.11. *Let X be a trigonal curve with genus g and Maroni invariant m. If $E \in \Sigma''(X,2)$, then $2\delta(E) \ge g+2$.*

Proof. Set $\delta := \delta(E)$. Assume $2\delta \le g+1$. Take $L \in \mathrm{Pic}^\delta(X)$ such that $h^0(L) \ge 2$ and E fits in (1). Since $g \ge 5$, there is a unique $R \in \mathrm{Pic}^3(X)$ such that $h^0(R) = 2$. By [22, definition of V^r_n at page 173 and Proposition 1] there is an integer $k > 0$ and an effective divisor B such that $L = R^{\otimes k}(B)$ and $h^0(L) = h^0(R^{\otimes k})$. By Proposition 5.4 to get a contradiction it is sufficient to prove that $h^0(R^{\otimes 2k}(B)) = 2k+1$. Set $b := \deg(B) \ge 0$. We have $\delta = 3k + b$. Since $k > 0$, we have $6k + 3b = 2\delta + b \le 3\delta - 3 = 3(\delta - 1)$. Since $|R|$ covers X, we have $h^0(R^{\otimes 2k}(B)) \le h^0(R^{\otimes(\delta-1)})$. Since $2k \le m+1$ we have $h^0(R^{\otimes 2k}) = 2k+1$. Assume $h^0(R^{\otimes 2k}(B)) > k+1$. Since $2(6k+b) \le g+1$, $R^{\otimes 2k}(B)$ is of type I ([22, definition of V^r_n at page 173 and Proposition 1]), i.e., $R^{\otimes 2k}(B) \cong R^{\otimes x}(B')$ with $B' \ge 0$ and B' the base locus of $|R^{\otimes 2k}(B)|$. Since $h^0(R^{\otimes 2k}) = 2k+1$, we get $x > 2k+1$, i.e., $h^0(\mathcal{O}_X(G) \otimes R^\vee) > 0$. Thus $h^0(L) \ge h^0(R^{\otimes(2k+1)}) \ge 2k+2$, a contradiction. □

6. Proofs of Theorems 1.2 and 1.3

Remark 6.1. Assume the existence of $E \in \Sigma''(X,2)$ fitting in (1) and call $\epsilon \in H^1(L^{\otimes 2})$ the associated extension class. For each $\lambda \in \mathbb{K}$ let E_λ be the extension of $L^\vee$ by L associated $\lambda\epsilon$. We have $E_0 \cong L \oplus L^\vee$ and $E_\lambda \cong E$ for all $\lambda \neq 0$. Thus $L \oplus L^\vee$ is a flat limit of a family of elements of $\Sigma''(X,2)$. Thus Theorem 1.2 may be rephrased as a criterion for the existence of an element of $\Sigma''(X,2)$ fitting in (1).

Proof of Theorem 1.2. The "only if" part is true by [8, Remark 1] and Remarks 1.1 and 6.1.

Now assume $h^0(L) \geq 2$. By the proof of [8, Proposition 1] it is sufficient to prove that L is a limit of spanned line bundles on nearby curves, i.e., it is sufficient to find an integral quasi-projective variety T, $o \in T$, a smooth morphism $f : W \to T$ with as fibers smooth curves of genus g, an isomorphism $\phi : X \to f^{-1}(o)$ and a line bundle $\mathcal{L}$ on W such that $\mathcal{L}_{|f^{-1}(t)}$ is globally generated for each $t \in T \setminus \{o\}$ and $\phi^*(\mathcal{L}_{|f^{-1}(o)}) \cong L$. Set $d := \deg(L)$. If either $d \geq g+1$, or L is globally generated, then $L \oplus L^\vee \in \Sigma'(X,2)$ ([8, Proposition 1]). Now assume that L is not globally generated, say $L \cong M(B)$ with $B > 0$ and $h^0(L) = h^0(M)$. Write $b := \deg(B)$. L is a limit of spanned line bundles on nearby curves by the theory of admissible covering due to D. Mumford and J. Harris ([16]) or any other compactification and/or generalization of the Hurwitz scheme described in [14], e.g. the theory of stable maps with $\mathbb{P}^1$ as target. In the set-up of [16] it is almost exactly [16, Theorem 5], except that in that theorem $g = 2d-1$; the proof given in [16] works for any g with minimal modifications, as clear by the pictures (see [16, Fig. at page 68]; all the theory of admissible coverings was generalized and put on stronger and more general foundations in the study of other compactifications of the Hurwitz scheme and in the study of stable maps. One can also use the theory of limit linear series due to D. Eisenbud and J. Harris ([12]) or adapt [10, Proposition A.3]. □

Remark 6.2. Theorem 1.2 shows that $L \oplus \mathcal{O}_X \oplus L^\vee$ is a limit of trivial bundles on nearby curves if L is a line bundle with $h^0(L) \geq 2$.

Lemma 6.3. *Let Y be an integral projective curve and $u : X \to Y$ its normalization. Fix a line bundle L on Y such that* $\deg(L) > 0$. *Assume the existence of an algebraic family $\mathcal{V}$ of dimension $t \geq \kappa$ of pairwise non-isomorphic vector bundles on Y limit of trivial bundles and fitting in* (1). *Then there is an algebraic family $\mathcal{G}$ of dimension $t \geq \kappa$ of pairwise non-isomorphic vector bundles on X limit of trivial bundles and fitting in* (1) *with $u^*(L)$ instead of L and hence with $\delta(E) = \deg(L)$ for all $E \in \mathcal{G}$. If $t = \kappa$, we claim that $\mathcal{G} \neq \emptyset$.*

Proof. Set $M := u^*(L)$. By Lemma 2.7 $\mathcal{V}$ corresponds to an algebraic subset $V \subset H^1(Y, L^{\otimes 2}) \setminus \{0\}$ such that no 2 of them are collinear. Hence the union $V' \subseteq H^1(Y, L^{\otimes 2})$ of the lines though 0 meeting $\mathcal{V}$ has dimension $t+1$. Since $h^0(\mathcal{O}_Y) = h^0(\mathcal{O}_X) = 1$, and $u_*(M^{\otimes 2}) \cong L^{\otimes 2} \otimes u_*(\mathcal{O}_X)$, the natural map

$H^1(Y, L^{\otimes 2}) \to H^1(M^{\otimes 2})$ has kernel of dimension at most κ. Apply Lemma 2.7 to X and M. $\square$

Remark 6.4. Proposition 3.1, Remark 3.2 and Lemma 6.3 give that $\Sigma'(X, 2) \neq \emptyset$ and that $\Sigma'(X, 2)$ is large for all curves X with a singular plane model such that the integer $p_a(Y) - p_a(X)$ is small with respect to the integer $\deg(Y)$. Our proof does not show if stronger statements are true when we impose more on the singularities of Y.

Proof of Theorem 1.3. Fix a positive integer x. To prove the theorem it is sufficient to find $g_0 \in \mathbb{N}$ such that $\gamma'(g) \geq x$ for all $g \geq g_0$. Set $g_0 := 16x$. Fix an integer $g \geq g_0$. Let d be the minimal positive integer such that $(d-1)(d-2)/2 \geq g$. Set $k := \lfloor (d-1)/4 \rfloor$. We have $4k + 4 \geq d \geq 4k + 1$ and $k > 0$. Set $\kappa := (d-1)(d-2)/2 - g$. The minimality of d gives $0 \leq \kappa \leq d - 3$. Let $Y \subset \mathbb{P}^2$ be an integral plane curve of degree d with geometric genus g (e.g. with κ nodes as its only singularities). By Proposition 3.1 and Remark 3.2 there is an algebraic family $\mathcal{V}$, $\dim \mathcal{V} = 3k^2 - 1$, of pairwise non-isomorphic indecomposable rank 2 vector bundles, which are limits of trivial bundles on Y and that fits in (1) with $L = \mathcal{O}_Y(k)$. Let $u : X \to Y$ be the normalization. By Lemma 6.3 X has a family of dimension $3k^2 - 1 - \kappa$ of pairwise non-isomorphic indecomposable rank 2 vector bundles which are limits of trivial bundles. Since $\kappa \leq d - 3 \leq 4k + 1$, it is sufficient to check that $3k^2 - 4k - 2 \geq x$. This is true, because $(d-1)(d-2)/2 \geq g \geq g_0$ and $d \leq 4k + 4$. $\square$

7. Limits of non-trivial bundles

In this section we take limits of rank 2 decomposable vector bundles on a smooth curve C of genus $g \geq 1$. Up to a twist by a line bundle it is sufficient to describe all $\Sigma(\mathcal{O}_C \oplus M)$ with M a fixed line bundle on C.

Remark 7.1. Let A, B, L line bundles on C such that $A \neq B$. There is a surjection $A \oplus B \to L$ if and only if either $L \cong A$ or $L \cong B$ or there are effective divisors D, D' on C such that $D \cap D' = \emptyset$ and $L \cong A(D) \cong B(D')$.

Lemma 7.2. *Let F be a rank 2 vector bundle on C. Take $E \in \Sigma(F)$.*

(1) *$s(E) \leq s(F)$;*
(2) *If $s(E) = s(F)$ and F is decomposable, then $E \cong F$.*

Proof. (1) follows from the semicontinuity of the Segre invariant.

(2) Now assume $s(E) = s(F)$ and F decomposable. Up to a twist we may assume $F \cong \mathcal{O}_C \oplus M$ with $\deg(M) \leq 0$. Note that $s(F) = -\deg(E)$. By the semicontinuity theorem for cohomology we get $h^0(E \otimes M^\vee) > 0$. Since $h^0(E \otimes M^\vee) > 0$ and $s(M) = -\deg(M)$, E fits in an exact sequence

$$0 \to M \to E \to \mathcal{O}_C \to 0. \tag{12}$$

If $M \cong \mathcal{O}_C$, then we also have $h^0(E) \geq 2$ and as observed in [8, Remark 1] together with the condition $s(E) = 0$ we get $E \cong \mathcal{O}_C^{\oplus 2}$. Now assume $M \neq \mathcal{O}_C$.

Since $\deg(M) \le 0$, (12) gives $h^0(E) \le 1$. The semicontinuity theorem for cohomology gives $h^0(E) > 0$. Thus a non-zero element of $H^0(E)$ induces a non-zero element of $\mathcal{O}_C$ and hence a splitting of (12). $\square$

Remark 7.3. Every $E \in \Sigma(\mathcal{O}_X \oplus M)$ is not semistable and $s(E) \le s(F) = -\deg(M)$. By Lemma 7.2 if $\deg(M) \le 3-2g$, then every $E \in \Sigma(\mathcal{O}_C \oplus M)$ is decomposable. The decomposable $E \in \Sigma(\mathcal{O}_C \otimes M)$ are exactly the bundle $\mathcal{O}_C \oplus M$ and the bundles $L \oplus L^\vee \otimes M$ with $\deg(L) > 0$, $L \cong \mathcal{O}_C(D) \cong M(D')$ with D, D' effective divisors on C with disjoint support. In particular if $\deg(L) \ge 2g + \max\{0, \deg(M)\}$, then $L \oplus L^\vee \otimes M \in \Sigma(\mathcal{O}_C \oplus M)$. For certain M we obviously may get many other L such that $L \oplus L^\vee \otimes M \in \Sigma(\mathcal{O}_C \oplus M)$. For instance, if $M \cong \mathcal{O}_C(p-q)$ with $p, q \in C$ and $p \ne q$ we may take $L = \mathcal{O}_C(p)$.

Corollary 7.4. *Assume $g \le 1$ and take any decomposable rank 2 vector bundle F on C. Then every element of $\Sigma(F)$ is decomposable.*

Proof. Fix $E \in \Sigma(F)$. Since F is decomposable, it is unstable, i.e., $s(F) \le 0$. By Lemma 7.2 either $E \cong F$ or $s(E) < 0$. If $s(E) < 0$, then E is decomposable by Atiyah's classification of vector bundles on an elliptic curve ([4]). $\square$

Proposition 7.5. *Fix a smooth curve C of genus g. For any integer x and any general $M \in \mathrm{Pic}^x(C)$ every $E \in \Sigma(\mathcal{O}_C \oplus M)$ is decomposable.*

Proof. For any integer $k > 0$ let $C^{(k)}$ denote the k-symmetric power of C and let $j_{C,k} : C^{(k)} \to \mathrm{Pic}^x(C)$ denote the natural map $D \mapsto \mathcal{O}_C(D)$.

Up to a twist by the line bundle $M^\vee$ it is sufficient to do the case $x \le 0$.

Fix an integer $x \le 0$ and any $M \in \mathrm{Pic}^x(C)$. Assume the existence of an indecomposable $E \in \Sigma(\mathcal{O}_C \oplus M)$ and let L be a maximal degree rank 1 subsheaf of E. We have $\deg(L) \ge \deg(\mathcal{O}_C) = 0$. We have $s(E) = \deg(E) - 2\deg(L) = -k + 2\deg(L)$ and E/L is a line bundle of degree $-k - \deg(L)$ isomorphic to $M \otimes L^\vee$. Since E is indecomposable and an extension of E/L by L, we have $h^1(L^{\otimes 2} \otimes M^\vee) > 0$. Thus $2\deg(L) - x \le 2g-2$. Thus to prove the proposition we only need to check finitely many x, say $x \ge 2g-2$.

Take any $G \in \Sigma(\mathcal{O}_C \oplus M)$ with $G \ne \mathcal{O}_C \oplus M$ and let L be a maximal degree line subbundle of E. Set $y := \deg(L)$. Lemma 7.2 gives $y > 0$. Since L has maximal degree among the rank 1 subsheaves of E, E/L has no torsion. Since $\det(E) \cong M$, we have $E/L \cong M \otimes L^\vee$. As in Remark 6.1 we see that $L \oplus L^\vee \otimes M \in \Sigma(\mathcal{O}_C \oplus M)$. Hence there is a surjection $\mathcal{O}_C \oplus M \to L$, i.e., there are effective divisors D, D' on C such that $\deg(D) = y$, $\deg(D') = y - x$, $L \cong \mathcal{O}_C(D) \cong M(D')$ and D, D' have disjoint support (Remark 7.1). Since $M \cong \mathcal{O}_C(D - D')$, M is uniquely determined by the pair (D, D'). Since E is indecomposable, we have $h^1(L^{\otimes 2} \otimes M^\vee) > 0$. We have $L^{\otimes 2} \otimes M^\vee \cong \mathcal{O}_C(D + D')$. Since $\dim |\omega_C| = g-1$, the effective divisor $D + D'$ depends on at most $g-1$ parameters. Since C is a smooth curve, for any effective divisor $A \subset C$ there are only finitely many pairs (D, D') such that $D + D' = A$. The line bundle M is uniquely determined by the pair (D, D'). Thus the possible M of degree x depends on at most $g-1$ parameters. $\square$

8. Limits with constant cohomology and/or good global sections

Remark 8.1. Fix $L \in \mathrm{Pic}(X)$ such that $\deg(L) > 0$. By Theorem 1.2 $L \oplus L^\vee \in \Sigma''(X, 2)$ with constant cohomology if and only if $h^0(L) = 2$.

Remark 8.2. Fix $L \in \mathrm{Pic}(X)$ such that $\deg(L) > 0$. By Theorem 1.2 $E := L \oplus L^\vee \in \Sigma''(X, 2)$ with $E/\mathrm{ev}(E)$ locally free if and only if L is globally generated. Take $F \in \Sigma''(X, 2)$ fitting in (1). The sheaf $F/\mathrm{ev}(F)$ is locally free, if and only if L is globally generated.

Proposition 8.3. *Fix an integer $g \geq 5$. Let X be a trigonal curve of genus g. Let m be the Maroni invariant of X. There is no indecomposable $E \in \Sigma''(X, 2)$ such that $E/\mathrm{ev}(E)$ is locally free and with $\delta(E) \leq \min\{(3m + 3)/2, g - 1 - m\}$.*

Proof. Fix an indecomposable $E \in \Sigma''(X, 2)$ such that $E/\mathrm{ev}(E)$ is locally free and set $d := \delta(E)$, i.e., assume that E fits in (1) with $\deg(L) = d$. By assumption L is globally generated (since $\mathrm{ev}(E) \cong L(-B)$, where B is the base divisor of $|L|$). Let $R \in \mathrm{Pic}^3(X)$ be the trigonal line bundle (it is unique, because $g \geq 5$). We use Maroni's theory of linear series on trigonal curves ([22, §1]). Since L has no base locus and $d \leq g - 1 - m$, there is an integer $t > 0$ such that $L \cong R^{\otimes t}$ ([22, Definition of V_n^r at p. 173 and Proposition 1]). We have $d \equiv 0 \pmod 3$ and $t = d/3$. Since $2t \leq m + 1$ by assumption, we have $h^0(L^{\otimes 2}) = 2t + 1$. Apply Proposition 5.4. □

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