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# A combinatorial description of finite O-sequences and aCM genera

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The goal of this paper is to explicitly detect all the arithmetic genera of arithmetically Cohen–Macaulay projective curves with a given degree $d$. It is well-known that the arithmetic genus $g$ of a curve $C$ can be easily deduced from the $h$-vector of the curve; in the case where $C$ is arithmetically Cohen–Macaulay of degree $d$, $g$ must belong to the range of integers $\left\{0, \dots, \binom{d-1}{2}\right\}$. We develop an algorithmic procedure that allows one to avoid constructing most of the possible $h$-vectors of $C$. The essential tools are a combinatorial description of the finite O-sequences of multiplicity $d$, and a sort of continuity result regarding the generation of the genera. The efficiency of our method is supported by computational evidence. As a consequence, we single out the minimal possible Castelnuovo–Mumford regularity of a curve with Cohen–Macaulay postulation and given degree and genus.



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## 0. Introduction

In this paper we introduce an algorithmic approach to the search of all possible arithmetic genera of an arithmetically Cohen–Macaulay (aCM for short) projective curve of given degree $d$. This problem has been studied in several instances, such as Roberts (1982, Example 4.6), and it has a role in the classification of algebraic curves, see for example (Hartshorne, 1994; Nagel, 2003) and the references therein.

The arithmetic genus $g$ of a curve appears in the constant term of the curve's Hilbert polynomial, hence it is related to the more general study of the coefficients of Hilbert polynomials (see Hartshorne, 1966 for a geometrical point of view, and Elias et al., 1996 in the context of local algebra).

In fact, not only does the $h$-vector encode a lot of information about the geometry of the curve; the arithmetic genus of the curve is also easily deduced from it (Hartshorne, 2010, Exercises 8.11 and 8.12), (Migliore, 1998, Section 1.4). For an aCM projective scheme the $h$-vector is actually the Hilbert function of its Artinian reduction. This result is mainly due to the fundamental paper of Macaulay (1926) characterizing the Hilbert functions of standard graded algebras.

We stress the fact that the work of Macaulay does not provide an algorithmic solution for the problem of deciding whether or not an aCM curve of degree $d$ and genus $g$ exists. This remark has been the starting point of our paper. By investigating the set of finite O-sequences of multiplicity $d$ and its properties we obtain our solution, both computational and theoretical, that relies on some closed formula considerably reducing the amount of real computations. We have not been able to find analogous results in literature.

As a first step, we provide a very natural combinatorial description of finite O-sequences, by means of suitable connected graphs, and we obtain an efficient search algorithm of the arithmetic genera of Cohen–Macaulay curves (see Algorithm 1 in Section 2).

Then, for every positive integer $d$, we denote by $R_d = \left[0, \binom{d-1}{2}\right] \cap \mathbb{N}$ the set of integers to which the genus of a Cohen–Macaulay curve of degree $d$ must belong, and we focus our attention on smaller ranges $R^s_d$, consisting of the genera of Cohen–Macaulay curves of degree $d$ and $h$-vector of length $s$. By introducing a convenient total ordering on the set of O-sequences of multiplicity $d$ and length $s$, we can single out each range $R^s_d$ (see Corollary 2.10, Theorem 3.2, Proposition 3.4).

The integers in $R_d$ that cannot be realized as genus of an aCM curve of degree $d$ are called *gaps*. Many of them are located outside every range $R^s_d$, some others lie *near* the maximal genus in $R^s_d$, for values of $s$ that can be exactly determined by suitable closed formulas (see Propositions 4.3 and 4.9).

Finally, we provide an algorithm to compute all the genera of aCM curves for a given degree $d$, avoiding to construct all the corresponding O-sequences (see Algorithm 2 in Section 5). The strategy supporting this algorithm combines the previous results together with a sort of *continuity* in the generation of the genera of aCM curves developed in Lemma 5.1 and applied in Theorem 5.4. Experimental computations point out that only a small percentage of integers of $R_d$ needs to be checked by the search algorithm (see Tables 1 and 2).

In Section 6, we apply our search algorithm to detect the minimal possible Castelnuovo–Mumford regularity of a curve with Cohen–Macaulay postulation and given degree and genus (Proposition 6.1). Moreover, we answer to a question posed in Cioffi and Di Gennaro (2011) about the Castelnuovo–Mumford regularity of even dimensional projective subschemes having the same Hilbert function of a Cohen–Macaulay projective scheme (Example 6.3).

## 1. Generalities on O-sequences and aCM genera

In this section, we state some notation and recall some basic results on O-sequences, referring to Bruns and Herzog (1993) and Valla (1998).

Given two positive integers $a$, $t$, the *binomial expansion of a in base t* is the unique writing

$$a = \binom{k(t)}{t} + \binom{k(t-1)}{t-1} + \cdots + \binom{k(j)}{j} \tag{1.1}$$

where $k(t) > k(t-1) > \cdots > k(j) \geq j \geq 1$ with the convention that $\binom{n}{m} = 0$ whenever $n < m$ and $\binom{n}{0} = 1$ for every $n \geq 0$. Letting

$$a^{\langle t\rangle} := \binom{k(t)+1}{t+1} + \binom{k(t-1)+1}{t} + \cdots + \binom{k(j)+1}{j+1},$$

by an easy computation, one gets $(a+1)^{\langle t\rangle} > a^{\langle t\rangle}$. A numerical function $\mathsf{h} : \mathbb{N} \to \mathbb{N}$ is *admissible* or an *O-sequence* if $\mathsf{h}(0) = 1$ and $\mathsf{h}(t+1) \leq \mathsf{h}(t)^{\langle t\rangle}$ for every $t \geq 1$.

If $\mathsf{h}$ is an admissible function and $\mathsf{h}(t) = 0$ for some $t$, then $\mathsf{h}(t+i) = 0$ for every $i > 0$, and $\mathsf{h}$ is called a *finite or Artinian O-sequence*. For a finite O-sequence $(h_0, \dots, h_{s-1})$ we assume $h_{s-1} \neq 0$. The integer $s$ is the *length* of the O-sequence and the integer $e(\mathsf{h}) := \sum_{i=0}^{s-1} h_i$ is its multiplicity.

It is well known that there is a bijective correspondence between the set of finite O-sequences of multiplicity $d$ and the set of Hilbert functions of a Cohen–Macaulay standard graded algebra of multiplicity $d$ over a field $K$ (Valla, 1998, Theorem 1.5). In fact, all these Hilbert functions can be computed from the finite O-sequences. In particular, if the graded algebra is the ring of regular functions on an aCM curve $C$ (i.e. a closed subscheme $C \subset \mathbb{P}^n_K$ of dimension 1), the Hilbert function $H_C$ of $C$ is the 2-*th integral* of a finite O-sequences $\mathsf{h} = (h_0, h_1, \dots, h_{s-1})$, i.e. letting $H_C(0) := H_Z(0) := \mathsf{h}(0) = 1$ and $H_Z(t) = H_Z(t-1) + \mathsf{h}(t)$ for every $t > 0$, we have

$$H_C(t) = H_C(t-1) + H_Z(t), \text{ for every } t > 0.$$

Hence, $\mathsf{h}$ is the so-called *h-vector* of $C$ and the Hilbert polynomial of $C$ is $p_C(z) = dz + 1 - g$ where, after an easy computation, we find that the *arithmetic genus* of $C$ is

$$g = 1 + (s-2)d - p(s-2) = \sum_{j=2}^{s-1} (j-1)h_j \geq 0. \tag{1.2}$$

In this situation, we say that $H_C$ is an *aCM function* or a *Cohen–Macaulay postulation*, $p_C(z)$ is an *aCM polynomial* and $g$ is an *aCM genus*.

**Remark 1.1.** The following facts are immediate remarks:

(i) the arithmetic genus of an aCM curve is non-negative;
(ii) every positive integer $g$ is the genus of some aCM curve: it is enough to take any O-sequence $(1, h_1, g)$, with $h_1^{\langle 1\rangle} \geq g$;
(iii) if $g$ is the arithmetic genus of some aCM curve $C_d$ of degree $d$, then there is also an aCM curve $C_{d+1}$ of degree $d+1$ with the same arithmetic genus $g$; indeed, if $\mathsf{h} = (1, h_1, h_2, \dots, h_{s-1})$ is the $h$-vector of $C_d$, then the sequence $\mathsf{h}' = (1, h_1 + 1, h_2, \dots, h_{s-1})$ is also an O-sequence and is the $h$-vector of a curve $C_{d+1}$ with Hilbert polynomial $(d+1)z + 1 - g$. Indeed, the multiplicity of the O-sequence $\mathsf{h}'$ is $d+1$ and then we apply formula (1.2), in which the integer $h_1$ does not occur. From a geometric point of view, this means that $C_{d+1}$ can be obtained as the union of $C_d$ and a line through a point of $C_d$.

## 2. A combinatorial description of finite O-sequences

In this section, we consider a natural structure on the set of all finite O-sequences. This structure will entail both our search algorithm of the arithmetic genera of Cohen–Macaulay curves, and some useful information about the aCM genera, such as the existence of minimal genera corresponding to O-sequences with given length (and multiplicity).

We let $\mathsf{e}_i$ denote any sequence, of any length, consisting entirely of 0 except 1 in the $i$-th position. Moreover, we introduce the following compact notation for some particular sequences:

$$(1^{\alpha_0}, h_{i_1}^{\alpha_1}, h_{i_2}^{\alpha_2}, \dots, h_{i_k}^{\alpha_k}) := (\underbrace{1, \dots, 1}_{\alpha_0 \text{ times}}, \underbrace{h_{i_1}, \dots, h_{i_1}}_{\alpha_1 \text{ times}}, \dots, \underbrace{h_{i_k}, \dots, h_{i_k}}_{\alpha_k \text{ times}}).$$

**Definition 2.1.** The *O-sequences graph* is the directed graph $\mathcal{G}$ such that:

- the set of vertices $V(\mathcal{G})$ consists of the finite O-sequences;

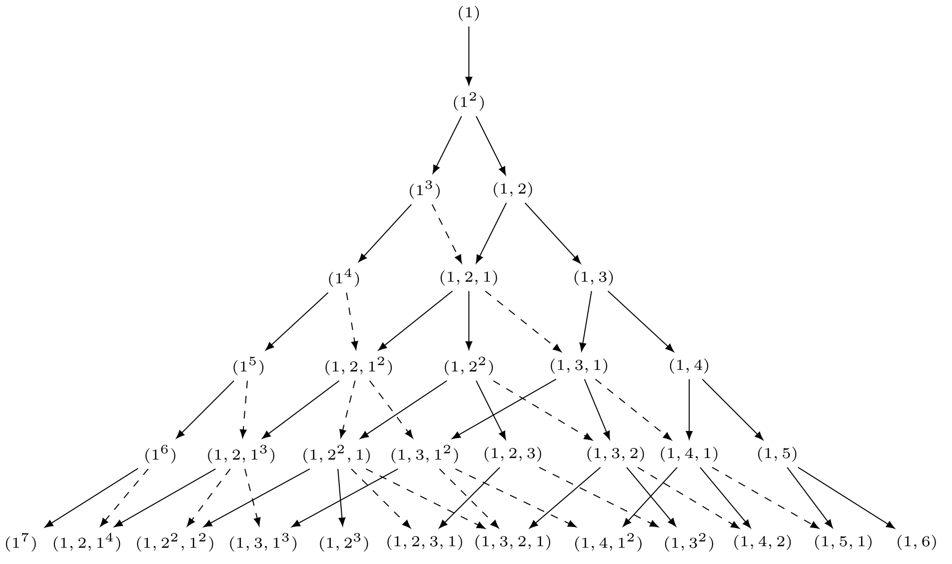



**Fig. 1.** The O-sequence graph $\mathcal{G}$ up to multiplicity 7. The dashed edges are edges of $\mathcal{G}$ that do not belong to the spanning tree $\mathcal{T}$.

- the set of edges $E(\mathcal{G})$ consists of the pairs $(\mathsf{h}, \mathsf{h}') \in V(\mathcal{G})^2$ s.t. $\mathsf{h}' - \mathsf{h} = \mathsf{e}_i$ for some $i$ (i.e. $(\mathsf{h}, \mathsf{h}') \in E(\mathcal{G})$ if $\mathsf{h}'$ can be obtained from $\mathsf{h}$ by increasing by 1 its $i$-th entry).

An edge $(\mathsf{h}, \mathsf{h}') \in E(\mathcal{G})$ from $\mathsf{h}$ to $\mathsf{h}'$ is labeled by $\mathsf{e}_i$ if $\mathsf{h}' - \mathsf{h} = \mathsf{e}_i$.

Let us consider the map $g : \mathcal{G} \to \mathbb{N}$ that associates with each O-sequence the genus of an aCM curve having this O-sequence as $h$-vector.

**Proposition 2.2.** *The O-sequences graph $\mathcal{G}$ is a rooted connected graph without loops. The root is the O-sequence of multiplicity 1.*

**Proof.** For any $\mathsf{h} = (1, h_1, \dots, h_{s-1})$, the sequence $\mathsf{h}' = \mathsf{h} - \mathsf{e}_{s-1}$ is admissible so that there is an edge going from $\mathsf{h}'$ to $\mathsf{h}$. Repeating this procedure, we get the length one O-sequence (1) which cannot be the head of any edge, proving that $\mathcal{G}$ is connected. There are no loops as each edge increases the multiplicity by 1. □

**Remark 2.3.** Denoted by $d_{\mathcal{G}}(\mathsf{h})$ the distance of the node $\mathsf{h}$ from the root, we have $d_{\mathcal{G}}(\mathsf{h}) = e(\mathsf{h}) - 1$.

We are going to define a subgraph $\mathcal{T} \subset \mathcal{G}$ which will turn out to be a spanning tree. In this way, we can design ad hoc algorithms to visit the tree in order to quickly find the O-sequences with the properties we will look for. The idea for determining $\mathcal{T}$ is the one used in the proof of Proposition 2.2. For each node of $\mathcal{G}$, we consider only the edge coming from the O-sequence obtained lowering by 1 the value with the greatest index. Indeed, notice that each O-sequence $\mathsf{h}$ (of any length $s$) has a successor in $\mathcal{T}$, as $\mathsf{h} + \mathsf{e}_s$ is always a finite O-sequence, whereas the sequence $\mathsf{h} + \mathsf{e}_{s-1}$ might not be admissible.

**Definition 2.4.** We call *O-sequences tree* the subgraph $\mathcal{T} \subset \mathcal{G}$ such that:

- $V(\mathcal{T}) = V(\mathcal{G})$;
- $E(\mathcal{T}) = \left\{(\mathsf{h}, \mathsf{h}') \in E(\mathcal{G}) \;\middle|\; \mathsf{h}' = \mathsf{h} + \mathsf{e}_s \text{ or } \mathsf{h}' = \mathsf{h} + \mathsf{e}_{s-1},\ \text{if } h_{s-2}^{\langle s-2\rangle} > h_{s-1}\right\}$.

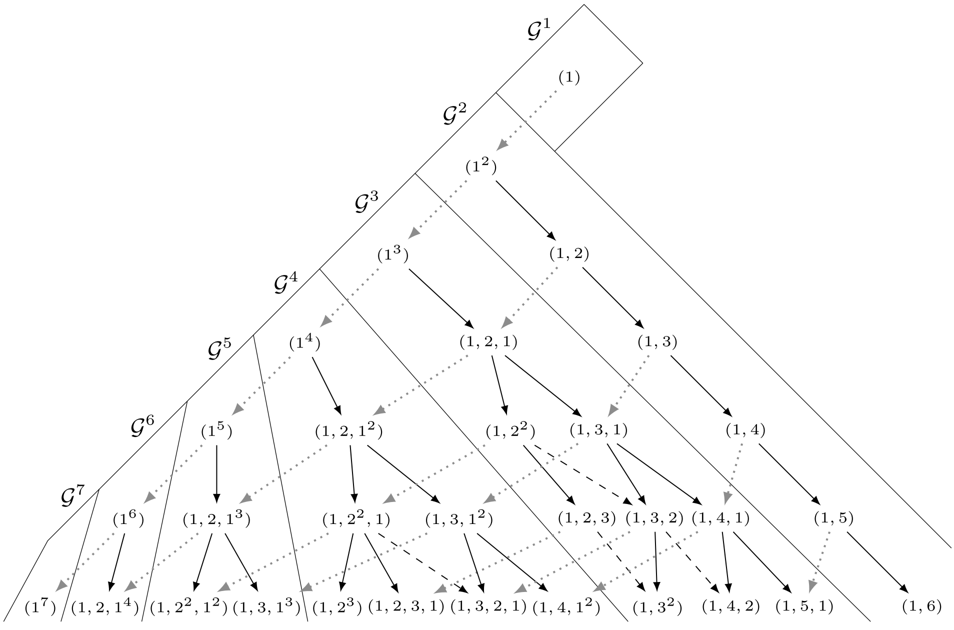



**Fig. 2.** The subgraphs $\mathcal{G}^s$ of the O-sequence graph with given length $s$. Along the grey dotted edges the length increases, so such edges of $\mathcal{G}$ do not belong to any subgraph $\mathcal{G}^s$. The dashed edges are edges of $\mathcal{G}^s$ that do not belong to the corresponding spanning tree $\mathcal{T}^s$.

In most situations, we will work with O-sequences with given multiplicity (i.e. with nodes of $\mathcal{G}$ at the same distance from the root) or with given length (see Fig. 2). We denote by $\mathcal{G}_d$ the set of O-sequences of multiplicity $d$ and by $\mathcal{G}^s$ the set of O-sequences of length $s$.

**Remark 2.5.** As in the spanning tree $\mathcal{T}$ each vertex is the tail of at most 2 edges, we have that $|\mathcal{G}_d| < 2|\mathcal{G}_{d-1}|$. Moreover, since $|\mathcal{G}_2| = 1$, by recursion $|\mathcal{G}_d| < 2^{d-2}$.

**Proposition 2.6.** *The subgraph $\mathcal{G}^s \subset \mathcal{G}$ is a rooted connected graph with root $(1^s)$ containing a spanning tree $\mathcal{T}^s$ with the same root (see Fig. 2).*

**Proof.** We need to show that, for any O-sequence $\mathsf{h} \neq (1^s)$ of length $s$, there exists another O-sequence of the same length with multiplicity $e(\mathsf{h}) - 1$. If $k = \max\{1 \leq i \leq s-1 \mid h_i > 1\}$, then $\mathsf{h} = (1, h_1, \ldots, h_k, 1^{s-k-1})$ and $\mathsf{h}' = (1, h_1, \ldots, h_k - 1, 1^{s-k-1})$ is admissible. $\square$

**Remark 2.7.** Denoted by $d^s_{\mathcal{G}}(\mathsf{h})$ the distance of the node $\mathsf{h}$ from the root of $\mathcal{G}^s$, we have $d^s_{\mathcal{G}}(\mathsf{h}) = d_{\mathcal{G}}(\mathsf{h}) - (s-1) = e(\mathsf{h}) - s$.

$\mathcal{G}_d$ is not a subgraph of $\mathcal{G}$, as there are no edges of $\mathcal{G}$ between O-sequences with the same multiplicity. But the edges of $\mathcal{G}$ induce the following natural partial order on $\mathcal{G}_d$.

**Definition 2.8.** Two O-sequences $\mathsf{h}_1$ and $\mathsf{h}_2$ in $\mathcal{G}_d$ are *directly comparable* if there exists $\mathsf{h}_0 \in \mathcal{G}_{d-1}$ such that $\mathsf{h}_1 = \mathsf{h}_0 + \mathsf{e}_i$ and $\mathsf{h}_2 = \mathsf{h}_0 + \mathsf{e}_j$, i.e. $\mathsf{h}_1 - \mathsf{h}_2 = \mathsf{e}_i - \mathsf{e}_j$. On directly comparable O-sequences we consider the order

$$\mathsf{h}_1 \prec \mathsf{h}_2 \iff i < j \tag{2.1}$$

and denote by $\prec$ also its transitive closure in $\mathcal{G}_d$.

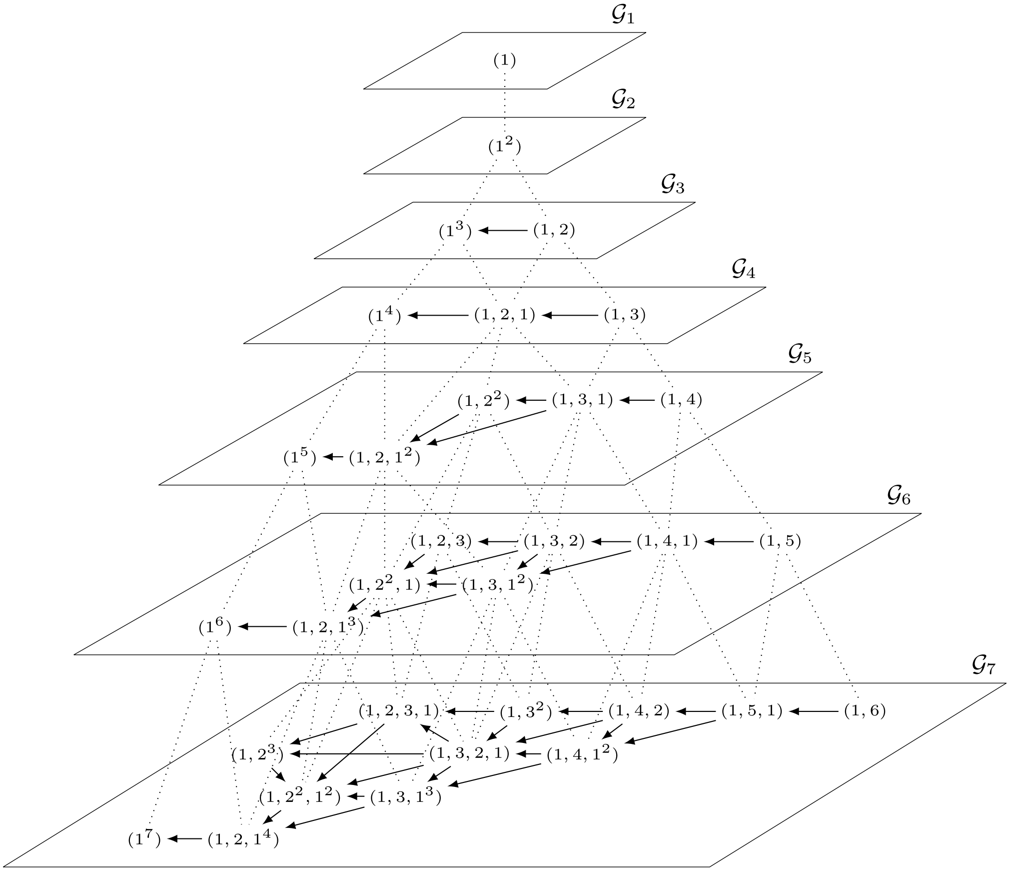



**Fig. 3.** The order relations among directly comparable elements of $\mathcal{G}_d$, $d = 1, \ldots, 7$.

The partial order $\prec$ gives a natural structure of directed graph to $\mathcal{G}_d$. The edges are all the possible pairs $(\mathsf{h}, \mathsf{h}') \in V(\mathcal{G}_d)^2$ such that $\mathsf{h} = \mathsf{h}' + \mathsf{e}_j - \mathsf{e}_i$ and $j > i$ (see Fig. 3). As before, we define a spanning tree of the graph structure of $\mathcal{G}_d$ which allows us to efficiently examine the set of O-sequences with given multiplicity. The same procedure is also extended to the set of O-sequences $\mathcal{G}_d^s$ with given multiplicity $d$ and length $s$ (see Fig. 4). Moreover, we let

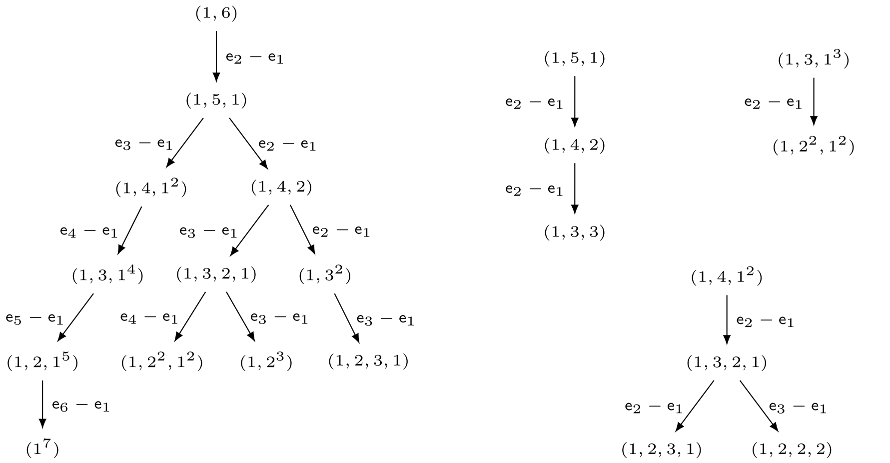
$$\mathsf{h}^s(d) := (1, d-s+1, 1^{s-2}) \quad \text{and} \quad \mathsf{g}^s(d) := g(\mathsf{h}^s(d)) = \binom{s-1}{2}. \tag{2.2}$$

**Proposition 2.9.**

(i) *The graph* $\mathcal{G}_d$ *contains a spanning tree* $\mathcal{T}_d$ *with root the O-sequence* $(1, d-1)$.
(ii) *The subgraph* $\mathcal{G}_d^s$ *contains a spanning tree* $\mathcal{T}_d^s$ *with root the O-sequence* $\mathsf{h}^s(d)$. *Thus,* $\mathcal{G}_d^s$ *is also connected.*

**Proof.** (i) For each vertex $\mathsf{h} \in \mathcal{G}_d \setminus \{(1, d-1)\}$, the spanning tree $\mathcal{T}_d$ contains the edge $\mathsf{e}_{s-1} - \mathsf{e}_1$ going from $\mathsf{h}' = \mathsf{h} - \mathsf{e}_{s-1} + \mathsf{e}_1$ to $\mathsf{h}$, where $s$ is the length of $\mathsf{h}$.

(ii) For each vertex $\mathsf{h} = (1, h_1, \ldots, h_i, 1^{d-\sum_{j=0}^{i} h_j}) \in \mathcal{G}_d^s \setminus \{(1, d-s+1, 1^{s-2})\}$ (i.e. $i > 1$), the spanning tree $\mathcal{T}_d^s$ contains the edge $\mathsf{e}_i - \mathsf{e}_1$ going from $\mathsf{h}' = \mathsf{h} - \mathsf{e}_i + \mathsf{e}_1$ to $\mathsf{h}$. $\square$




(a) The spanning tree $\mathcal{T}_7$ of $\mathcal{G}_7$.

(b) The spanning trees $\mathcal{T}_7^s$ of $\mathcal{G}_7^s$ for $s = 3, 4, 5$. For $s = 2, 6, 7$, the graph $\mathcal{G}_7^s$ has a unique vertex.

**Fig. 4.** Graph descriptions of O-sequences with given multiplicity and length.

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**Algorithm 1** The algorithm for searching aCM genera with given constraints on the multiplicity and the length of the O-sequences. A trial version of this algorithm is available at http://www.paololella.it/HSC/Finite_O-sequences_and_ACM_genus.html.

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**procedure** GENUSSEARCH($g$, $\widetilde{\mathcal{T}}$)
**Input:** $g$, a non-negative integer.
$\widetilde{\mathcal{T}}$, a spanning tree chosen among $\mathcal{T}$, $\mathcal{T}_d$, $\mathcal{T}^s$ and $\mathcal{T}_d^s$.
**Output:** an O-sequence h such that $g(\mathrm{h}) = g$ (if it exists).
    stack := {ROOT($\widetilde{\mathcal{T}}$)};
    **while** stack $\neq \emptyset$ **do**
        h := REMOVEFIRST(stack);
        **if** $g(\mathrm{h}) = g$ **then return** h;
        **else if** $g(\mathrm{h}) < g$ **then**
            ADDFIRST(stack, CHILDREN(h, $\widetilde{\mathcal{T}}$));
        **end if**
    **end while**
**end procedure**

---

**Corollary 2.10.** *The order induced on $\mathcal{G}_d$ by the total order on $\mathbb{N}$ through the map $g : \mathcal{G}_d \to \mathbb{N}$ is a refinement of the partial order $\prec$. In particular, $\mathrm{h}^s(d) = \min(\mathcal{G}_d^s)$ with respect to $\prec$, $\mathrm{g}^s(d)$ is the minimal genus corresponding to an O-sequence of length s and multiplicity d and it does not depend on d.*

**Proof.** If $\mathrm{h}_1 - \mathrm{h}_2 = \mathrm{e}_i - \mathrm{e}_j$, then $g(\mathrm{h}_1) = g(\mathrm{h}_2) + (i-1) - (j-1) = g(\mathrm{h}_2) + i - j$, by formula (1.2). Hence, we obtain

$$\mathrm{h}_1 \prec \mathrm{h}_2 \iff i < j \implies g(\mathrm{h}_1) < g(\mathrm{h}_2)$$

and the assertion about the minimum follows by Proposition 2.9. □

As the minimal genus $\mathrm{g}^s(d)$ does not depend on the value of $d$, from now on we will simply denote it by $\mathrm{g}^s$.

Now, we can state the strategy of a general algorithm for searching aCM genera. We choose the set of O-sequences corresponding to the considered constraints on multiplicity and length and, more precisely, the associated spanning tree $\widetilde{\mathcal{T}}$. Then, we perform a depth-first search on the tree using a LIFO (Last In First Out) procedure of visit of the vertices. Assume that, at some moment in the search, we stored in a list (resp. a stack) the vertices whose existence we know, having visited their parents, but that we have not yet visited. We visit the first vertex $\mathsf{h}$ in the list (resp. the top of the stack). There are three possible alternative actions:

A. if $g(\mathsf{h})$ is equal to the genus we are looking for, then we end the visit returning the O-sequence $\mathsf{h}$;
B. if $g(\mathsf{h})$ is greater than the genus we are looking for, then we can avoid to visit the tree of descendants of $\mathsf{h}$, as the genus increases along the edges (Proposition 2.2 and Corollary 2.10);
C. if $g(\mathsf{h})$ is smaller than the genus we are looking for, then we need to visit the tree of descendants of $\mathsf{h}$, so we add the children of $\mathsf{h}$ in the tree $\widetilde{\mathcal{T}}$ at the beginning of the list (resp. at the top of the stack) containing the vertices still to be visited.

## 3. Combinatorial ranges

From now on, we assume $d > 2$, as $\mathcal{G}_d$ has only one element for $d \in \{1, 2\}$.

For convenience, we let $G_d$ (resp. $G_d^s$) be the set of all the arithmetic genera of the aCM curves of degree $d$ (resp. of degree $d$ with $h$-vector of length $s$), i.e. $G_d := \{g(\mathsf{h}) \mid \mathsf{h} \in \mathcal{G}_d\}$ (resp. $G_d^s := \{g(\mathsf{h}) \mid \mathsf{h} \in \mathcal{G}_d^s\}$).

Looking at the graph $\mathcal{G}_d$, we immediately can observe the well known fact that $G_d \subseteq \left\{0, \dots, \binom{d-1}{2}\right\}$ (see Hartshorne, 1994, Theorem 3.1). Denoting by $[a, b]$ the set of integers $\{n \in \mathbb{N} \mid a \le n \le b\}$, we let $R_d := \left[0, \binom{d-1}{2}\right]$. In the range $R_d$ we single out smaller suitable ranges, taking into account also the length of the O-sequences.

Recall that, by the partial order $\prec$ introduced in Definition 2.8 and by Corollary 2.10, we have $\min(G_d^s) = g(\min(\mathcal{G}_d^s)) = \mathsf{g}^s = \binom{s-1}{2}$, thus $\mathsf{g}^s < \mathsf{g}^{s+1}$ and $\mathsf{g}^{s+1} - \mathsf{g}^s = s - 1$. In order to obtain an analogous result about a maximum, we extend the partial order $\prec$ to the following total order on $\mathcal{G}_d^s$.

**Definition 3.1.** Given two O-sequences $\mathsf{h} = (1, h_1, \dots, h_{s-1})$ and $\mathsf{h}' = (1, h'_1, \dots, h'_{s-1})$ of $\mathcal{G}_d^s$, we denote by $<$ the total order on $\mathcal{G}_d^s$ such that $\mathsf{h} < \mathsf{h}'$ if $h_\ell < h'_\ell$, where $\ell := \max\{j : h_j \ne h'_j\}$.

Although the usual order on $\mathbb{N}$ does not induce on $\mathcal{G}_d^s$ the total order $<$ (see Example 3.3), we notice that $\min_\prec(\mathcal{G}_d^s) = \min(\mathcal{G}_d^s)$ with respect to $<$. Furthermore, we can consider also $\max(\mathcal{G}_d^s)$ with respect to $<$ and obtain the following non-obvious result.

**Theorem 3.2.** *Let $\mathsf{h} = (1, h_1, \dots, h_{s-1})$ and $\mathsf{k} = (1, k_1, \dots, k_{s-1})$ be two O-sequences of $\mathcal{G}_d^s$. If $\mathsf{k} < \mathsf{h}$ and $g(\mathsf{k}) > g(\mathsf{h})$, then there is an O-sequence $\bar{\mathsf{h}} \in \mathcal{G}_d^s$ such that $\bar{\mathsf{h}} > \mathsf{h}$ and $g(\bar{\mathsf{h}}) > g(\mathsf{k})$. Thus, $\max(G_d^s) = g(\max(\mathcal{G}_d^s))$.*

**Proof.** We can assume $s - 1 = \max\{j : h_j \ne k_j\}$, hence $h_{s-1} > k_{s-1}$ because $\mathsf{h} > \mathsf{k}$. By the hypotheses, we have

$$g(\mathsf{h}) = \sum_{j}^{s-2} (j-1)h_j + (s-2)h_{s-1} < \sum_{j}^{s-2} (j-1)k_j + (s-2)k_{s-1} = g(\mathsf{k})$$

which implies there exists the integer $t := \max\{j \in \{2, \dots, s-2\} : h_j < k_j\}$ and so

$$\begin{array}{cccccccc} (1, & h_1, & \dots, & h_t, & h_{t+1}, & \dots, & h_{s-2}, & h_{s-1}) \\ & & & \wedge & \veebar & & \veebar & \vee \\ (1, & k_1, & \dots, & k_t, & k_{t+1}, & \dots, & k_{s-2}, & k_{s-1}) \end{array} \tag{3.1}$$

that is

$$\begin{cases} h_t < k_t, \\ h_i \geq k_i, & t+1 \leq i \leq s-2, \\ h_{s-1} > k_{s-1}. \end{cases}$$

Note that $k_t^{\langle t\rangle} \geq h_t^{\langle t\rangle} \geq h_{t+1} \geq k_{t+1}$. Hence, we can consider the O-sequence $\mathsf{h}' := \mathsf{k} - b\mathsf{e}_t + \sum_{j=t+1}^{s-1} c_j \mathsf{e}_j$, where

$$b = \min\left\{k_t - h_t, \sum_{j=t+1}^{s-1} h_j - k_j\right\} \quad \text{and} \quad c_j = \min\left\{h_j - k_j, b - \sum_{i=t+1}^{j-1} c_i\right\}$$

and $h'_j \leq h_j$ for every $j > t$.

The corresponding genus of $\mathsf{h}'$ is

$$g(\mathsf{h}') = g(\mathsf{k}) - (t-1)b + \sum_{j=t+1}^{s-1} (j-1)c_j > g(\mathsf{k}) > g(\mathsf{h}).$$

If needed, replacing the O-sequence $\mathsf{k}$ by $\mathsf{h}'$ and repeating the same argument as before, we obtain an O-sequence $\mathsf{h}'$ with $h'_j = h_j$ for every $j > t$ and $g(\mathsf{h}') > g(\mathsf{h})$. If $\mathsf{h}' < \mathsf{h}$, we can repeat the same argument as before until we obtain an O-sequence $\bar{\mathsf{h}}$ with $\bar{h}_j = h_j$ for every $j > t$ and $\bar{h}_t \geq h_t + 1$. □

**Example 3.3.** (a) Consider the two O-sequences $\mathsf{h} = (1, 6, 4, 2, 1)$ and $\mathsf{k} = (1, 4, 7, 1, 1)$ of $\mathcal{G}^5_{14}$. We have $\mathsf{h} > \mathsf{k}$ and $11 = g(\mathsf{h}) < g(\mathsf{k}) = 12$ as in the hypotheses of Theorem 3.2. In this case, we obtain $t = 2$, $b = \min\{3, 1\} = 1$, $c_3 = \min\{1, 3\} = 1$ and $c_4 = \min\{0, 2\} = 0$, so that $\bar{\mathsf{h}} = \mathsf{k} - \mathsf{e}_2 + \mathsf{e}_3 = (1, 4, 6, 2, 1)$ with genus $g(\bar{\mathsf{h}}) = 13 > g(\mathsf{k})$ and $\bar{\mathsf{h}} > \mathsf{h}$.

(b) Consider the two O-sequences $\mathsf{h} = (1, 13, 3, 3, 3)$ and $\mathsf{k} = (1, 6, 13, 2, 1)$ of $\mathcal{G}^5_{23}$. We have $\mathsf{h} > \mathsf{k}$ and $18 = g(\mathsf{h}) < g(\mathsf{k}) = 20$. Applying Theorem 3.2, as $t = 2$, $b = \min\{10, 3\} = 3$, $c_3 = \min\{1, 10\} = 1$ and $c_4 = \min\{2, 9\} = 2$, we determine $\bar{\mathsf{h}} = \mathsf{k} - 3\mathsf{e}_2 + \mathsf{e}_3 + 2\mathsf{e}_4 = (1, 6, 10, 3, 3) > \mathsf{h}$ and $g(\bar{\mathsf{h}}) = 18 + 2 + 3 = 21 > g(\mathsf{k})$.

Looking again at the graph $\mathcal{G}^s$, we can find a way to detect $g(\max(\mathcal{G}^s_d))$. We first note that, if $d < s$, then $\mathcal{G}^s_d$ is empty and if $d = s$, then we have a unique O-sequence $(1^s)$ corresponding to a plane curve of degree $s$, i.e. with genus $\binom{s-1}{2}$. For $d = s+1$ we have the unique O-sequence $(1, 2, 1^{s-2})$, obtained from $(1^s)$ by increasing $h_1$ by 1 and corresponding to a curve of degree $s+1$ and genus $\binom{s-1}{2}$. In the other cases, we deduce $\max(\mathcal{G}^s_d)$ assuming to know the O-sequence $\mathsf{h} = \max(\mathcal{G}^s_{d-1})$ and consequently the genus $g(\mathsf{h}) = \sum_{j=2}^{s-1} h_j(j-1) = \max(G^s_{d-1})$ (Theorem 3.2). Next result shows how to find $\max(\mathcal{G}^s_d)$ and then $g(\max(\mathcal{G}^s_d))$.

**Proposition 3.4.** *Given any $d > s \geq 3$, let $\mathsf{h} = \max(\mathcal{G}^s_{d-1})$. If $\imath$ is the highest index such that $\mathsf{h} + \mathsf{e}_\imath$ is an O-sequence in $\mathcal{G}^s_d$, then $\max(\mathcal{G}^s_d) = \mathsf{h} + \mathsf{e}_\imath$ and $g(\max(\mathcal{G}^s_d)) = g(\max(\mathcal{G}^s_{d-1})) + \imath - 1$.*

**Proof.** By the assumption, we have $h_\imath < h_{\imath-1}^{\langle \imath-1\rangle}$, so that $h_\imath + 1 \leq h_{\imath-1}^{\langle \imath-1\rangle}$ and $h_{\imath+r} = h_{\imath+r-1}^{\langle \imath+r-1\rangle}$, for every $1 \leq r \leq s-1-\imath$, that is:

$$\mathsf{h} = (1, \dots, h_\imath, h_\imath^{\langle \imath\rangle}, h_{\imath+1}^{\langle \imath+1\rangle}, \dots, h_{s-2}^{\langle s-2\rangle})$$

and

$$\mathsf{h} + \mathsf{e}_\imath = (1, \dots, h_\imath + 1, h_\imath^{\langle \imath\rangle}, h_{\imath+1}^{\langle \imath+1\rangle}, \dots, h_{s-2}^{\langle s-2\rangle}).$$

For every $\mathsf{h}' \in \mathcal{G}^s_{d-1} \setminus \{\mathsf{h}\}$, consider the integer $\ell := \max\{j : h_j \neq h'_j\}$. Then, we have $h'_\ell < h_\ell$ and $h'_{\ell+r} = h_{\ell+r}$, for every $1 \leq r \leq s-1-\ell$, because $\mathsf{h} = \max(\mathcal{G}^s_{d-1})$. Note that we have $\ell < \imath$, otherwise $h'_\ell < h_\ell$ would imply $h'_{\ell+1} \leq {h'}_\ell^{\langle \ell\rangle} < h_\ell^{\langle \ell\rangle} = h_{\ell+1}$, against the definition of $\ell$. Therefore,

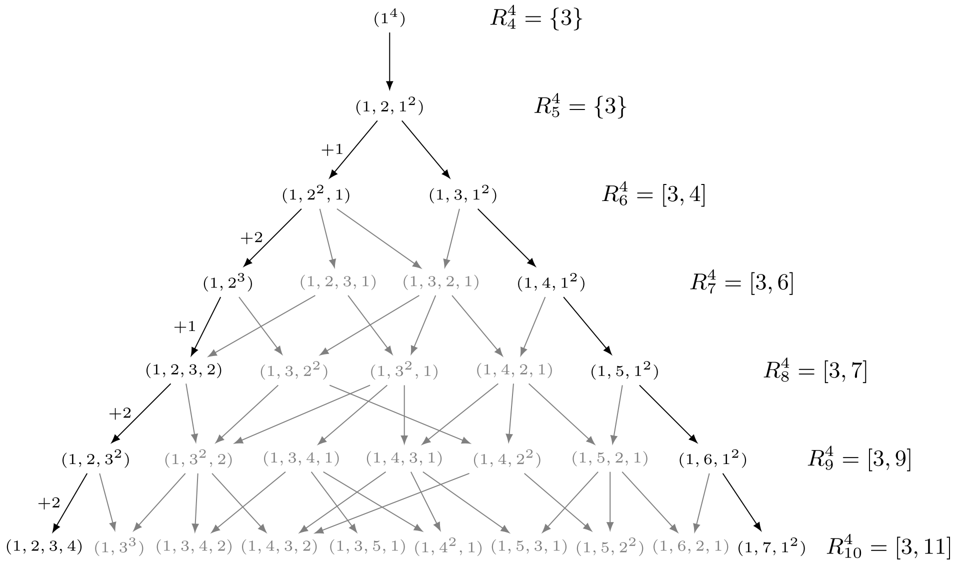


**Fig. 5.** The ranges $R_d^4$ for $d = 4, \ldots, 10$. In the picture, the edges on the left are labeled with the corresponding increase of the genus.

$$\mathsf{h}' = (1, \ldots, h'_\ell, \ldots, h_\iota, h_\iota^{\langle\iota\rangle}, \ldots, h_{s-2}^{\langle s-2\rangle}).$$

If there were an O-sequence $\mathsf{h}' \in \mathcal{G}_{d-1}^s$ such that $\mathsf{h}' + \mathsf{e}_\lambda > \mathsf{h} + \mathsf{e}_\iota$ for some index $\lambda$ such that $\mathsf{h}' + \mathsf{e}_\lambda \in \mathcal{G}_d^s$, then $\iota < \lambda$. We have seen that $\mathsf{h}$ and $\mathsf{h}'$ certainly have equal entries for indices greater than or equal to $\iota$ and $h_\iota + 1 > h_\iota = h'_\iota$. But, for indices $j > \iota$, the value $h'_j = h_j = h_{j-1}^{\langle j-1\rangle}$ cannot be increased by the definition of O-sequences. Thus, we obtain $\max(\mathcal{G}_d^s) = \mathsf{h} + \mathsf{e}_\iota$. The last assertion follows by Theorem 3.2 and formula (1.2). □

For every $d > 2$ and $s \in \{\lfloor \frac{d}{2} \rfloor + 1, \ldots, d\}$, we let

$$\mathsf{h}_s(d) := (1, 2^{d-s}, 1^{2s-d-1}) \quad \text{and} \quad \mathsf{g}_s(d) := g(\mathsf{h}_s(d)) = \binom{s-1}{2} + \binom{d-s}{2}. \tag{3.2}$$

Then, we have: $\max(\mathcal{G}_d^s) = \mathsf{h}_s(d)$, $\mathsf{g}^d = \binom{d-1}{2} = \mathsf{g}_d(d)$ and $\mathsf{g}^{d-1} = \binom{d-2}{2} = \mathsf{g}_{d-1}(d)$.

**Remark 3.5.** Another description of the maximal genus of a range $R_d^s$ could be set in terms of minimal Hilbert functions with a constant Hilbert polynomial and a given regularity (see Roberts, 1982, Examples 4.6 and 4.8 and Cioffi et al., in press). By the way, the combinatorial description we provide here arises in a very natural way and gives more information, at least from a computational point of view.

The previous results together with those of Sections 2 suggest to consider the following smaller ranges in $R_d$.

**Definition 3.6.** For every $d \geq s \geq 2$, the set of integers between $\mathsf{g}^s$ and $\max(G_d^s)$ is called $(d, s)$-*range* and denoted by $R_d^s$ (see Fig. 5), i.e. $R_d^s := \left\{\alpha \in \mathbb{N} \mid \binom{s-1}{2} = \mathsf{g}^s \leq \alpha \leq \max(G_d^s)\right\}$.

**Corollary 3.7.** *For every $d \geq s \geq 2$, the arithmetic genus of an aCM curve of degree $d$ having h-vector of length $s$ belongs to the range $R_d^s$.*

**Proof.** The statement follows by Corollary 2.10, Theorem 3.2 and Proposition 3.4. □

## 4. Unattainable aCM genera in $R_d$

Recall that we are denoting by $R_d$ the range $\left[0, \binom{d-1}{2}\right]$ and that $G_d \subseteq R_d$.

**Definition 4.1.** An integer in $R_d \setminus G_d$ is called a *gap in* $R_d$.

**Example 4.2.** The integers in the range $\left[\binom{d-2}{2}+1, \binom{d-1}{2}-1\right]$ are gaps in $R_d$. More generally, every integer of $R_d$ not contained in any $(d,s)$-range is a gap.

Next result allows us to characterize the consecutive $(d,s)$-ranges that are *separated*, i.e. ranges $R_d^s$ and $R_d^{s+1}$ such that $\mathrm{g}^{s+1} - \max(G_d^s) > 1$.

**Proposition 4.3.** *For any* $d > 2$, *we have*

$$\max(G_d^s) < \mathrm{g}^{s+1} - 1 \iff \frac{2d+1-\sqrt{8d-15}}{2} < s \leq d-1.$$

*Thus, the integers in* $[\max(G_d^s)+1, \mathrm{g}^{s+1}-1]$ *are gaps in* $R_d$, *for* $\frac{2d+1-\sqrt{8d-15}}{2} < s \leq d-1$.

**Proof.** For $s \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$, by (3.2) we have:

$$\mathrm{g}_s(d) < \mathrm{g}^{s+1} - 1 \iff \binom{s-1}{2} + \binom{d-s}{2} < \binom{s}{2} - 1.$$

Hence

$$\mathrm{g}_s(d) - \mathrm{g}^{s+1} + 1 = \frac{s^2-(2d+1)s+d^2-d+4}{2} < 0 \Rightarrow \frac{2d+1-\sqrt{8d-15}}{2} < s < \frac{2d+1+\sqrt{8d-15}}{2},$$

and thus $\mathrm{g}_s(d) < \mathrm{g}^{s+1} - 1$ if and only if $\frac{2d+1-\sqrt{8d-15}}{2} < s \leq d-1$, because $\frac{2d+1-\sqrt{8d-15}}{2} > \left\lfloor \frac{d}{2} \right\rfloor$, $\frac{2d+1+\sqrt{8d-15}}{2} > d-1$ and $\frac{2d+1-\sqrt{8d-15}}{2} > d-1$ implies $d < 3$.

To prove that there are no other pairs of separated ranges, we notice that $\mathrm{g}_s(d) \geq \mathrm{g}^{s+1} - 1$ implies $\mathrm{g}_{s-1}(d) \geq \mathrm{g}^s - 1$, for every $s$. Indeed, as $\mathrm{g}^s = \mathrm{g}^{s+1} - (s-1)$ and $\mathrm{g}_s(d) \leq \mathrm{g}_{s-1}(d) + (s-2)$ by Proposition 3.4, we have

$$\mathrm{g}_{s-1}(d) - \mathrm{g}^s + 1 \geq \mathrm{g}_s(d) - (s-2) - \mathrm{g}^{s+1} + (s-1) + 1 > \mathrm{g}_s(d) - \mathrm{g}^{s+1} + 1 \geq 0. \qquad \square$$

**Example 4.4.** For every $d \leq 11$, the gaps in $R_d$ are only those described in Proposition 4.3. For $d = 12$, in addition to the gaps described in Proposition 4.3, we find by direct computation a unique further gap $\bar{g} = 26$, belonging only to the range $R_{12}^8 = [21, 28]$.

**Example 4.5.** By a direct computation of the finite admissible O-sequences, we note that for $d = 15$ the integer $\bar{g} = 25$ belongs to the ranges $R_{15}^6$ and $R_{15}^5$. Nevertheless, whereas for each $\mathrm{h} \in R_{15}^5$ we have $g(\mathrm{h}) \neq 25$, there is $\bar{\mathrm{h}} = (1,3,3,4,2,2) \in R_{15}^6$ such that $g(\bar{\mathrm{h}}) = 25$.

Example 4.5 suggests the following definition.

**Definition 4.6.** An integer in the range $R_d^s$ is called a *hole* of the range $R_d^s$ if it is not the arithmetic genus of an aCM curve $C$ of degree $d$ with $h$-vector of length $s$.

**Remark 4.7.** Not every hole is a gap. For instance, Example 4.5 tells us that the integer 25 is not a gap in $R_{15}$, although it is a hole of $R_{15}^5$. While Example 4.4 attests that the hole 26 of $R_{12}^8$ is actually a gap in $R_{12}$.

Notice that for $s = d-1, d-2, d-3$ there are no holes in $R_d^s$. Now, we detect some values of $d$ and $s$ for which in the ranges $R_d^s$ there exist certain special gaps and we point out some particular holes which are also gaps, belonging to *parts* of different $(d,s)$-ranges not overlapping each other.

**Lemma 4.8.** *For every $d$ and $s$ such that $7 \le \left\lfloor \frac{d}{2} \right\rfloor + 1 \le s \le d-4$, the integers $\mathrm{g}_s(d) - (d-s-3), \dots, \mathrm{g}_s(d) - 1$ are holes in the range $R_d^s$.*

**Proof.** As we saw in (3.2), the maximal genus $\mathrm{g}_s(d)$ in $R_d^s$ arises from the O-sequence $\mathrm{h}_s(d) = (1, 2^{d-s}, 1^{2s-d-1})$. In the graph $\mathcal{G}_d^s$, the only edges involving this vertex are $\mathrm{e}_{d-2} - \mathrm{e}_1$ and $\mathrm{e}_{d-s} - \mathrm{e}_2$. Hence, by Corollary 2.10, for each $\mathrm{h} \in \mathcal{G}_d^s \setminus \{\mathrm{h}_s(d)\}$

$$\begin{aligned} g(\mathrm{h}) &\le \max\left\{ g\big(\mathrm{h}_s(d) - (\mathrm{e}_{d-s} - \mathrm{e}_1)\big), g\big(\mathrm{h}_s(d) - (\mathrm{e}_{d-s} - \mathrm{e}_2)\big)\right\} \\ &= \max\{\mathrm{g}_s(d) - (d-s-1), \mathrm{g}_s(d) - (d-s-2)\} = \mathrm{g}_s(d) - (d-s-2). \qquad \square \end{aligned}$$

All the holes described in the previous lemma are surely gaps if we consider $s > \frac{2d+1-\sqrt{8d-15}}{2}$ as in Proposition 4.3. Indeed, is such cases these holes do not belong to any other range.

**Proposition 4.9.** *In the hypotheses of Lemma 4.8, for every $i = 1, \dots, d-s-3$, the hole $\mathrm{g}_s(d) - i$ is a gap if $s - 1 - \binom{d-s}{2} + i > 0$. More precisely,*

(i) *the highest hole $\mathrm{g}_{d-4}(d) - 1 = \frac{d(d-11)}{2} + 20$ is always a gap;*

(ii) *every hole described in Lemma 4.8 is a gap if $s > \frac{2d-1-\sqrt{8d-31}}{2}$.*

**Proof.** The hole $\mathrm{g}_s(d) - i$ is a gap if $\mathrm{g}_s(d) - i < \mathrm{g}^{s+1}$, i.e. $\binom{s}{2} - \binom{s-1}{2} - \binom{d-s}{2} + i = s - 1 - \binom{d-s}{2} + i > 0$. The proof of (i) and (ii) is a direct computation. $\square$

**Example 4.10.** By Proposition 4.9, we find the following gaps in $R_{28}$: the gap 258 belonging only to the range $R_d^{24}$, 240 and 239 belonging only to $R_d^{23}$, 224, 223 and 222 belonging only to $R_d^{22}$ and 207, 208 and 209 belonging to $R_{28}^{21}$. Anyway, by a direct computation we find also the gap 188, actually the minimal one in $R_{28}$.

## 5. Computation of the aCM genera for curves of degree *d*

Proposition 4.9 gives a characterization of the gaps in $R_d$ belonging to the *last part* of a $(d,s)$-range. We did not find analogous conditions for gaps belonging to the *first part* of a $(d,s)$-range. In particular, it seems hard to give a characterization of the minimal gap. Hence, we will look for an algorithmic method to recognize the gaps in $R_d$, avoiding to construct all the finite O-sequences of multiplicity $d$ thanks to a sort of *continuity* in the generation of the arithmetic genera. Denote by $G_d + a$ the set of all arithmetic genera of the aCM curves of degree $d$ augmented by a non-negative integer $a$.

**Lemma 5.1.** $G_d \supseteq \bigcup_{j=1}^{d-1} \left( G_j + \binom{d-j}{2} \right)$.

**Proof.** Let $(1, h_1, \dots, h_{s-1})$ be an O-sequence of multiplicity $j < d$ corresponding to an aCM genus $g$. Assuming $h_i^{\langle i \rangle} > h_{i+1}$, for some $i \in \{1, \dots, s-2\}$, we can consider the finite O-sequence $(1, h_1, \dots, h_{i+1} + 1, \dots, h_{s-1})$ of multiplicity $j+1$, corresponding to the genus $g + i$. Then, we can take also the finite O-sequence $(1, h_1, \dots, h_{i+1} + 1, h_{i+2} + 1, \dots, h_{s-1})$ of multiplicity $j+2$, corresponding to the genus $g + i + (i+1)$, and so on. Performing this construction from $i = 1$ until $d - j$, we reach the desired conclusion. $\square$

**Remark 5.2.** By the proof of Lemma 5.1, we can observe that the arithmetic genera determined by the O-sequences $(1, h_1, \dots, h_{s-1})$ with $h_i \geq h_{i+1}$, for every $0 < i < s-1$, are included in those detected by Lemma 5.1. For example, we have:

$$\begin{aligned}
G_1 &= G_2 = \{0\}, \quad G_3 = G_2 \cup (G_1 + 1) = \{0, 1\}, \\
G_4 &= G_3 \cup (G_2 + 1) \cup (G_1 + 3) = \{0, 1, 3\}, \\
G_5 &= G_4 \cup (G_3 + 1) \cup (G_2 + 3) \cup (G_1 + 6) = \{0, 1, 2, 3, 6\}, \\
G_6 &= G_5 \cup (G_4 + 1) \cup (G_3 + 3) \cup (G_2 + 6) \cup (G_1 + 10) = \{0, 1, 2, 3, 4, 6, 10\}, \\
G_7 &\supset G_6 \cup (G_5 + 1) \cup (G_4 + 3) \cup (G_3 + 6) \cup (G_2 + 10) \cup (G_1 + 15) = \{0, 1, 2, 3, 4, 6, 7, 10, 15\}.
\end{aligned}$$

Note that for the multiplicity $d = 7$, we lose the arithmetic genus $g = 5$ which corresponds to the finite O-sequence $(1, 2, 3, 1)$.

Now, we exploit Lemma 5.1 obtaining large sets of aCM genera. To this aim, we define an increasing sequence $\{m_d\}_{d \geq 1}$ by the following procedure:

**if** $d = 1$ **then**
  $m_1 := 0$;
**else**
  $M := m_{d-1}$;
  **for** $k = 2, \dots, d-1$ **do**
    **if** $\binom{k}{2} - 1 \leq M$ **then**
      $M = \max\{M, m_{d-k} + \binom{k}{2}\}$;
    **end if**
  **end for**
  $m_d := M$;
**end if**

**Example 5.3.** In the following table, we list the values of the sequence $\{m_d\}_{d \geq 1}$ and compare them with the values of $g^{\lceil \frac{d}{2} \rceil + 2}$, for $1 \leq d \leq 45$:

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_d$ | 0 | 0 | 1 | 1 | 3 | 4 | 4 | 7 | 11 | 13 | 18 | 19 | 19 | 25 | 32 |
| $g^{\lceil \frac{d}{2} \rceil + 2}$ | 1 | 1 | 3 | 3 | 6 | 6 | 10 | 10 | 15 | 15 | 21 | 21 | 28 | 28 | 36 |

| $d$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_d$ | 40 | 43 | 52 | 62 | 73 | 85 | 89 | 102 | 116 | 118 | 133 | 149 | 166 | 184 | 203 |
| $g^{\lceil \frac{d}{2} \rceil + 2}$ | 36 | 45 | 45 | 55 | 55 | 66 | 66 | 78 | 78 | 91 | 91 | 105 | 105 | 120 | 120 |

| $d$ | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_d$ | 208 | 228 | 229 | 229 | 250 | 272 | 295 | 319 | 344 | 370 | 376 | 403 | 431 | 460 | 490 |
| $g^{\lceil \frac{d}{2} \rceil + 2}$ | 136 | 136 | 153 | 153 | 171 | 171 | 190 | 190 | 210 | 210 | 231 | 231 | 253 | 253 | 271 |

**Theorem 5.4** *(Continuity). For all $d \geq 1$, every integer in $\{0, \dots, m_d\}$ is the arithmetic genus of an aCM curve of degree $d$, i.e. $\{0, \dots, m_d\} \subseteq G_d$, and $m_d \geq g^{\lceil \frac{d}{2} \rceil + 2}$, for every $d \geq 18$.*

**Proof.** The first statement holds by Lemma 5.1 and by the definition of $m_d$. For the second affirmation, note that it is enough to consider odd degrees $d$. For $18 \leq d \leq 36$, see the tables of Example 5.3. If $d \geq 37$, let $s := \lceil \frac{d}{2} \rceil + 2$. By construction and by induction, we know that $m_d \geq m_{d-1} \geq g^{\lceil \frac{d-1}{2} \rceil + 2} = \binom{s-2}{2}$. Hence, by the definition of $m_d$ we get

**Algorithm 2** The algorithm for determining the aCM genera of curves with a given degree. A trial version of this algorithm is available at http://www.paololella.it/HSC/Finite_O-sequences_and_ACM_genus.html.

```
procedure ACMGENERA(d)
Input: d, a positive integer.
Output: the list of all possible aCM genera of a curve of degree d.
    genera := {genera determined applying recursively Lemma 5.1};
    gaps := {gaps determined applying Proposition 4.3 and Proposition 4.9};
    undecided := {0, ..., binom(d-1, 2)} \ (genera ∪ gaps);
    for s = 2, ..., d - 3 do
        g := min(undecided);
        while g ≤ UPPERBOUND(R_d^s) do
            if g < LOWERBOUND(R_d^s) then
                REMOVE(g, undecided);
                gaps = gaps ∪ {g};
            else
                searching := GENUSSEARCH(g, T_d^s);
                if searching ≠ ∅ then
                    REMOVE(g, undecided);
                    genera = genera ∪ {g};
                end if
            end if
            g = NEXT(g, undecided);
        end while
    end for
    return genera;
end procedure
```

$$m_d \geq \max\left\{m_{d-1}, m_{d-(s-2)} + \binom{s-2}{2}\right\}.$$

Being $d$ odd, we have $d-(s-2) = d - \lceil \frac{d}{2} \rceil = \lceil \frac{d}{2} \rceil - 1 = s-3 \geq 18$. Thus, by induction we obtain $m_d \geq \binom{\lceil \frac{s-3}{2} \rceil + 1}{2} + \binom{s-2}{2}$, because $m_{d-(s-2)} = m_{\lceil \frac{d}{2} \rceil - 1} = m_{s-3} \geq \mathrm{g}^{\lceil \frac{s-3}{2} \rceil + 2}$.

Note that $\binom{\lceil \frac{s-3}{2} \rceil + 1}{2} + \binom{s-2}{2} \geq \binom{s-1}{2}$ if $\binom{\lceil \frac{s-3}{2} \rceil + 1}{2} \geq s-2$, that is true for every $s \geq 10$. □

Theorem 5.4 gives a lower bound for the value assumed by $m_d$, for every $d \geq 18$. Anyway, we can obtain more information by a full application of Lemma 5.1 which, together with the algorithm GENUSSEARCH (see Algorithm 1), provides an algorithm to compute all the arithmetic genera of the aCM curves of degree $d$, avoiding to construct all the finite O-sequences. The strategy consists of the following steps:

Step 1 by Lemma 5.1, we determine recursively the set of integers $\widetilde{G}_d \subset R_d$ that are certainly aCM genera. Let $\widetilde{G}_1 = \{0\}$, we have $\widetilde{G}_d = \bigcup_i \widetilde{G}_i + \binom{d-i}{2}$;
Step 2 by results in Section 4 we determine all the integers of $R_d$ that are certainly gaps;
Step 3 using algorithm GENUSSEARCH (Algorithm 1) we investigate the remaining integers.

## 6. An application: Castelnuovo–Mumford regularity of curves with Cohen–Macaulay postulation

In this section, we show how the search algorithm of aCM genera (Algorithm 1) allows us to detect the minimal Castelnuovo–Mumford regularity $m_{d,g}^{\mathrm{aCM}}$ of a curve with Cohen–Macaulay postulation, given its degree $d$ and genus $g$. Moreover, by Example 6.3 we give a negative answer to a question posed in Cioffi and Di Gennaro (2011, Remark 2.5). A complete solution to the problem of

**Table 1**
In this table, we report some numerical information about the integers in $G_d$ up to degree 250. The first column contains the number and the percentage of values in $R_d$ which are aCM genera by an application of Lemma 5.1 (without computing the O-sequences); in the second column, the number and the percentage of gaps determined applying Proposition 4.3 and Proposition 4.9; in the third column, the number and the percentage of values of $R_d$ for which we have to use the procedure GENUSSEARCH to decide whether they are aCM genera; in the last column, the cardinality of $G_d$ and its percentage with respect to $|R_d|$.

| $d$ | Certain genera | Certain gaps | Undecided values | $\|G_d\|$ |
|---|---|---|---|---|
| 25 | 176 (63.77%) | 88 (31.88%) | 13 (4.71%) | 187 (67.75%) |
| 50 | 835 (71.00%) | 289 (24.57%) | 53 (4.51%) | 870 (73.98%) |
| 75 | 2033 (75.27%) | 558 (20.66%) | 111 (4.11%) | 2099 (77.71%) |
| 100 | 3798 (78.29%) | 879 (18.12%) | 175 (3.61%) | 3894 (80.27%) |
| 125 | 6129 (80.37%) | 1244 (16.31%) | 254 (3.33%) | 6261 (82.10%) |
| 150 | 9040 (81.99%) | 1653 (14.99%) | 334 (3.02%) | 9207 (83.50%) |
| 175 | 12 528 (83.24%) | 2094 (13.91%) | 430 (2.86%) | 12 734 (84.61%) |
| 200 | 16 610 (84.31%) | 2574 (13.07%) | 518 (2.63%) | 16 854 (85.55%) |
| 225 | 21 276 (85.19%) | 3084 (12.35%) | 617 (2.47%) | 21 560 (86.32%) |
| 250 | 26 530 (85.92%) | 3623 (11.73%) | 724 (2.34%) | 26 856 (86.98%) |

**Table 2**
In this table, we report the results of a test of Algorithm 2 up to degree 250. The first three columns contain the elapsed time (in milliseconds) for Step 1, Step 2 and Step 3 of Algorithm 2. In the fourth column, there is the total time for the execution (Step 1 + Step 2 + Step 3). The last column contains the time required for determining the set $G_d$ by performing a complete visit of the tree $\mathcal{T}_d$ (even for $d = 75$, we obtain an Out Of Memory Error). The algorithms are implemented in the `Java` language and have been run on a MacBook Pro with an Intel Core 2 Duo 2.4 GHz processor.

| $d$ | Step 1 | Step 2 | Step 3 | Algorithm 2 | Visit $\mathcal{T}_d$ |
|---|---|---|---|---|---|
| 25 | 37.336 ms | 0.164 ms | 38.594 ms | 76.094 ms | 210.769 ms |
| 50 | 82.774 ms | 0.208 ms | 212.868 ms | 295.850 ms | 15 155.87 ms |
| 75 | 21.734 ms | 0.155 ms | 458.117 ms | 480.006 ms | O.O.M. |
| 100 | 47.529 ms | 0.103 ms | 1390.027 ms | 1437.659 ms | O.O.M. |
| 125 | 104.683 ms | 0.279 ms | 4684.598 ms | 4789.56 ms | O.O.M. |
| 150 | 207.936 ms | 0.183 ms | 12 610.461 ms | 12 818.58 ms | O.O.M. |
| 175 | 546.818 ms | 0.227 ms | 37 518.036 ms | 38 065.081 ms | O.O.M. |
| 200 | 665.378 ms | 0.364 ms | 73 552.564 ms | 74 218.306 ms | O.O.M. |
| 225 | 922.599 ms | 0.36 ms | 169 042.878 ms | 169 965.837 ms | O.O.M. |
| 250 | 1395.378 ms | 0.179 ms | 359 836.564 ms | 361 232.121 ms | O.O.M. |

detecting the minimal Castelnuovo–Mumford regularity of a scheme with a given Hilbert polynomial is described in Cioffi et al. (in press).

Denoting by $\rho$ the *regularity* of a Hilbert function, i.e. the minimal degree from which the Hilbert function and the Hilbert polynomial coincide, we can state the following:

**Proposition 6.1.**

$$m^{\mathrm{aCM}}_{d,g} = \min \left\{ \rho \;\middle|\; \begin{array}{l} \rho \textit{ is the regularity of an aCM postulation} \\ \textit{with Hilbert polynomial } dt+1-g \end{array} \right\} + 2$$

**Proof.** Let $f$ be an aCM postulation with Hilbert polynomial $dt + 1 - g$ and regularity $\rho$. Then, the minimal possible Castelnuovo–Mumford regularity of a curve with Hilbert function $f$ is $\rho + 2$. As a matter of fact, by Cioffi and Di Gennaro (2011, Proposition 2.4) this regularity is strictly greater than $\rho + 1$ and if the curve is aCM, it is exactly $\rho + 2$. □

By Proposition 6.1, the value of $m^{\mathrm{aCM}}_{d,g}$ is determined by applying Algorithm 1 in order to find an O-sequence $\mathsf{h}$ of multiplicity $d$ and $g(\mathsf{h}) = g$ with the shortest possible length. Notice that if the length of $\mathsf{h}$ is $s$, then the regularity of $\Sigma^2\mathsf{h}$ is $s - 2$. Thus, we can rewrite the statement in Proposition 6.1 as

$$m_{d,g}^{\mathrm{aCM}} = \min \left\{ s \;\middle|\; \begin{array}{l} s \text{ is the length of an O-sequence } \mathsf{h} \\ \text{with multiplicity } d \text{ and } g(\mathsf{h}) = g \end{array} \right\}.$$

**Example 6.2.** Let us consider the curves of degree $d = 15$ and genus $g = 32$. There are four O-sequences of multiplicity $d$ corresponding to aCM curves of genus $g$:

$$\mathsf{h}_1 = (1,4,3,2,1,1,1,1,1), \quad \mathsf{h}_3 = (1,2,3,4,2,1,1,1),$$
$$\mathsf{h}_2 = (1,3,3,2,2,2,1,1), \quad \mathsf{h}_4 = (1,3,5,1,1,1,1,1,1).$$

Hence, the minimal Castelnuovo–Mumford regularity $m_{d,g}^{\mathrm{aCM}}$ is 8. Applying the results of Cioffi et al. (in press) (see http://www.paololella.it/HSC/Minimal_Hilbert_Functions_and_CM_regularity.html), we notice that the minimal Castelnuovo–Mumford regularity of any projective scheme with Hilbert polynomial $p(t) = 15t - 31$ is 7.

More generally, in the case of an aCM function $f$ with regularity $\rho$ and Hilbert polynomial with *odd* degree, we have that the minimal possible Castelnuovo–Mumford regularity of a scheme $X$ with $H_X = f$ is strictly greater than $\rho + 1$ (see Cioffi and Di Gennaro, 2011, Proposition 2.4). If the degree of the Hilbert polynomial is *even*, an analogous result does not hold, as the following example shows.

**Example 6.3.** The following strongly-stable ideal

$$I = (x_6^2, x_5x_6, x_5^2, x_4x_5, x_3x_5, x_2x_5, x_1x_5, x_4^2x_6, x_3x_4x_6, x_2x_4x_6, x_1x_4x_6, x_3^2x_6, x_2x_3x_6,$$
$$x_1x_3x_6, x_2^3x_6, x_1x_2^2x_6, x_1^2x_2x_6, x_4^4, x_3x_4^3, x_2x_4^3, x_1^4x_6, x_3^3x_4^2, x_3^4x_4, x_3^5) \subset K[x_0,\dots,x_6],$$

where $x_0 < x_1 < \cdots < x_6$, defines a non-aCM surface $X \subset \mathbb{P}^6$ with the aCM postulation $H_X = (1, 7, 21, 44, \dots, 6t^2 - 10t + 21, \dots)$ of regularity $\rho = 4$ and the Castelnuovo–Mumford regularity of $X$ is $5 = \rho + 1$.

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