IL NUOVO CIMENTO **41 C** (2018) 124
DOI 10.1393/ncc/i2018-18124-6

COMMUNICATIONS: SIF Congress 2017

GeoGebra simulations for Feynman's sum over paths approach

M. MALGIERI([1]) on behalf of P. ONORATO([2]) and A. DE AMBROSIS([1])

([1]) *Department of Physics, University of Pavia - Pavia, Italy*
([2]) *Department of Physics, University of Trento - Povo (TN), Italy*

received 1 February 2018

Summary. — We present a collection of interactive simulations meant to complement the educational approach to quantum physics in high school developed since 2013 by the physics education research group at the University of Pavia. The collection, organized as a "book" within the online repository GeoGebra Tube is intended primarily for being used by physics teachers wishing to deepen their understanding of Feynman's sum over paths approach. The simulations represent several problems of physical interest, and aim at clarifying different aspects of Feynman's model. The collection has been used as an educational tool in the context of the IDIFO-6 Master coordinated by the University of Udine, and encouraging indications have been collected. In particular, the use of an open-source software which is very widely known and used in the community of teachers may encourage them to produce their own example simulations, or make easier for them to modify the existing ones according to their preferences and needs.

1. – Introduction

The research group in physics education at the University of Pavia has been working on the teaching of quantum physics in secondary schools since 2013, proposing a reprise and renewal of Feynman's sum over paths approach [1-4] and testing it with students [5]. The didactic use of Feynman's formulation of quantum physics had been proposed by many authors in the past [6-8] and educational advantages have been highlighted both in the previous literature and in our own works. By using the representation of complex numbers as "tiny arrows" already suggested in Feynman's book *QED* [9], the approach requires little advanced mathematics, and uses a very simple formal language, allowing students to concentrate on the conceptual aspects of the theory. Feynman's formulation provides a pictorial representation of the mathematical model underlying the explanation of quantum processes; this aspect is in fact one of the main reasons underlying the scientific fortune of Feynman diagrams, which are a graphical representation of processes involving multiple interacting particles, exchanges between identical quantum objects, creation and destruction processes. The sum over paths approach allows to immediately identify a central difference between classical and quantum physics in the rule for computing the probability of an event which can happen in several alternative, undistinguishable ways. It allows to present to students a unified model for the behaviour of quantum objects which is valid, with minor modification, for both photons and massive particles, thus possibly preventing the difficulty that students interpret photons as wavelets and electrons as classical tiny balls [10]. Finally, it makes the classical limit completely transparent, allowing an easy and convincing derivation of the classical laws from the rules valid for quantum objects.

Like for several other educational proposals based on sum over paths, the Pavia approach adopts computer simulations in order to fully exploit the visualization possibilities offered by Feynman's formulation. Computer simulations have a recognized role in physics education, and they are widely adopted in the teaching of quantum physics, a field in which they allow students to visualize and manipulate experimental situations which are not easily reproducible in high school, and sometimes even undergraduate, laboratories. The number of researchers who focused on developing computer simulations of quantum phenomena for educational use has been steadily growing in the last 20 years. Several online repositories of educational simulations for quantum mechanics exist. To cite only the most famous ones, QUVIS of University of St. Andrews [11]; Physlets of the comPADRE group [12]; PhET of University of Colorado [13]; Visual Quantum Mechanics from Kansas State University [14].

At a very early stage in our research program, we decided to use the software GeoGebra for the design of simulations integrated in our approach. GeoGebra offers the possibility of designing highly interactive simulations through the use of sliding bars ad checkboxes; allows to connect a different representation in adjacent windows of the workspace, and is a very widely known and used software in the community of mathematics and physics teachers. This, as will be discussed in sect. **4**, allows teachers to work autonomously on the simulations we produced, modifying and adapting them to their educational needs.

The general objective of our work is to provide teachers with supporting materials for building teaching-learning sequences with more ambitious goals than simply discussing the initial crisis of classical physics. The common approach adopted in many textbooks puts a great emphasis on the initial stages of development of quantum theory, but omits the discussion of conceptual and foundational issues which have been clarified at a later time. On the other hand, the sum over paths approach allows to build a working model for quantum phenomena which is of great help, even at higher education levels, in interpreting precisely those modern experiments which have shed light on those issues, especially from the field of quantum optics.

Many of the simulations used in our approach were described in ref. [1]. In the present article we will concentrate on discussing the structure of the online repository (sect. **2**), simulations which are new or significantly improved with respect to ref. [1] (sect. **3**) and a simulation produced by student teachers enrolled in the online Master IDIFO-6 (sect. **4**).

2. – The GeoGebra "book"

In this section we introduce the interactive "book", available at the link `https://www.geogebra.org/m/Q6waMV2v` within the online repository GeoGebra Tube. The collection of simulations is intended as an integration of the full lectures for the Pavia proposal, available at the link `http://www-5.unipv.it/didapls/filespdf/dispense2016.pdf`.

The online book is divided in 14 chapters, which closely follow the structure of the Pavia proposal [1, 2, 5]. In turn, each chapter is divided in paragraphs, many of which contain simulations. Those paragraphs which do not contain a simulation are used to summarize part of the content of the lecture notes, in order to make the logical structure of the book transparent to the reader. For paragraphs which do include a simulation, the text, besides discussing the physical phenomenon to which the simulation refers, provides a user guide to the applet, describing the various elements visualized, the parameters which can be varied and their significance and the educational goals of the simulation.

Before proceeding to describe individual simulations, it may be useful to discuss some of our general choices related to their implementation and design. As mentioned in the "Introduction", Feynman's approach provides a support for the visual imagination of students. However, it is necessary that students keep in mind that the visualization refers to the *mathematical model*, which does not necessary bear a one-to-one correspondence to physical reality. Ignoring such cautionary statement can lead to difficulties, such as overly concrete interpretations of the model (*e.g.*, photons imagined as undergoing a physical rotation, and/or as tiny wheels) which have sometimes been reported for Feynman's approach [5, 15]. On a more general level, it is well known that, in some cases, visualizations of quantum processes can become too strong attractors for the mental representations of students, hindering the formation of genuinely quantum models. Finally, research has pointed out that interactivity and the possibility for students of exploring the simulated phenomenon by varying parameters is a necessary condition for simulations to be productive in educational use [16].

According to the above theoretical indications, our approach in the design of simulations can be summarized as follows:

- Simulations strictly adhere to the *source-to-detector* philosophy as described in ref. [8]: all problems, even those concerning confined quantum objects [3] are described as physical situations in which a source, emitting a certain species of quantum objects, is present, and one or more detectors can "click" signalling the presence of the quantum object at a certain point in space.

- Any direct representation of the quantum object itself is avoided; instead, the conceptual machinery of Feynman's approach, consisting of all possible paths and the complex amplitudes (tiny arrows) associated to each path are represented.

- In most simulations, the schematic depiction of the physical setup and the computation of the sum of complex amplitudes and detection probabilities are performed in separate GeoGebra windows. Besides the advantages in clarity of visualization, this may also be useful in preventing students to associate a too concrete interpretation to the amplitude arrows.

- Simulations are designed to allow as much exploration as possible to students; they typically contain several parameters which can be varied, and their meaning is clearly explained in the accompanying instructions.

In fig. 1 we show a snapshot of one of the simplest simulations in our collection, representing the two-slit experiment with pointlike slits, and we highlight the design elements enumerated above.

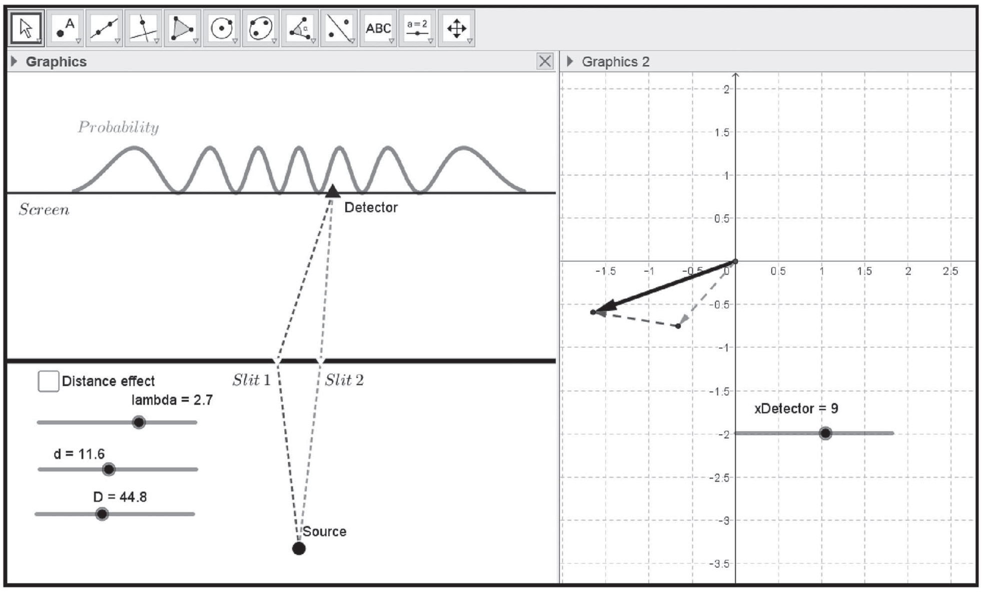


Fig. 1. – GeoGebra simulation of the two-slit experiment with pointlike slits, illustrating the general structure of most simulations in our collection. In the left window we represent the physical setup (source, detector and other elements of physical reality such as, in this case, the slits and screen). We also display all the possible paths for the quantum object which we include in the computation. In the right window, the sum of complex amplitudes for all paths at the detector, providing the final amplitude and probability, is represented. The possibility of varying relevant physical parameters and the detector position is provided by sliders and checkboxes.

3. – New or significantly improved simulations

In this section we describe some of the simulations contained in the online collection which are new or significantly improved with respect to those presented in our previous works [1, 3].

3·1. *Diffraction grating.* – The main objective of this simulation is to provide a solid connection with the experimental part of the Pavia proposal, in which spectra of light coming from different sources are studied using a home-made spectrometer [17, 18]. At this aim, in the current version the default visualization was set so as to provide results comparable with those obtained in the hypothesis of Fraunhofer diffraction. The drawback is that, in this representation, the different paths are no longer immediately discernible, (because they appear as superposed with the units used) and in order to distinguish them, the simulation must be zoomed in. Also, the resulting diffraction figure, the source, and the resulting amplitude arrow were coloured with a variable hue according to the user's choice of the wavelength λ, which is expressed in μm. The formula used, which has been found empirically based on suggestions from several sources, was Hue $= 1.6 - 2.35\lambda$. The simulation is shown in fig. 2.

3·2. *Two-slit interference of massive quantum objects with non-pointlike slits.* – The simulation of the two-slit interference with massive particles and non-pointlike slits was also completely redone from the version presented in [1]. The simulation plays a dual role in the Pavia proposal: first, it serves to show that a model for massive quantum objects

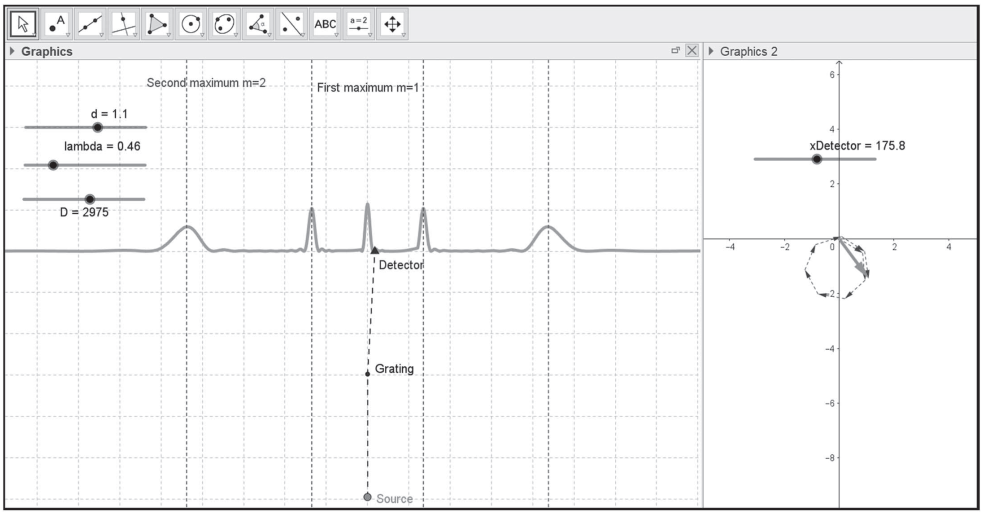


Fig. 2. – Sum over paths simulation of a diffraction grating. In the left window the physical setup, the final probability distribution, and the comparison with the prediction for the positions of the first- and second-order maxima in the hypothesis of Fraunhofer diffraction. In the right window, the construction of the resultant arrow at the detector point, with the phasors corresponding to individual possible paths (small arrows) and the resulting arrow (in a heavier brush stroke). Parameters that can be varied in this case, besides the position of the detector, include the grating pitch d, the light wavelength, the source-detector distance D.

can be constructed which is essentially identical to the one for the photon except for the substitution of the light wavelength with the De Broglie wavelength $\lambda = \frac{h}{p}$. Second, the simulation is useful in explaining the correspondence principle, *i.e.* how classical mechanics emerges as a limit theory for quantum physics. In fig. 3 (top) one can see the expected quantum behaviour, that is an interference pattern with a limited width, which is due to the concurrent effect of diffraction by the non-pointlike slits. However, in fig. 3 (bottom) one can see that, as the mass m of the quantum object is increased (while leaving its energy E constant), the experimental result approaches the expected classical outcome, *i.e.* two heaps of particles, one behind each one of the slits. Furthermore, in the window reserved for the amplitude calculation, one can see that, in the second case, when the detector is placed behind one of the slits, all the paths coming from such slit interfere constructively, while all the paths coming from the other slit interfere destructively. In this sense, one can recover the idea that classically the particle has passed through "only one of the slits".

3˙3. *Resonant tunneling from a double-barrier system.* – At the aim of providing a gradual introduction to the discussion of confined systems, the Pavia proposal introduces a semi-open, resonant setup, in which the detection probability depends very strongly on energy, showing sharp resonance features. The system, in the optical case, consists of a source emitting monochromatic photons towards a detector, with two successive semi-transparent mirrors interposed along the path. In the case of a massive particle the role of semi-transparent mirrors can be played by thin potential barriers of variable height, especially tuned in such a way that for any given energy the square modulus of the transmission coefficient is always $|t|^2 = 0.5$. By varying the energy E of the incoming quantum object, one observes that the detection amplitude goes from a very low minimum, to very sharp maxima, which, as usual, correspond to the case in which all amplitudes associated to two paths differing by a full back and forth reflection in the two-barrier system are in phase.

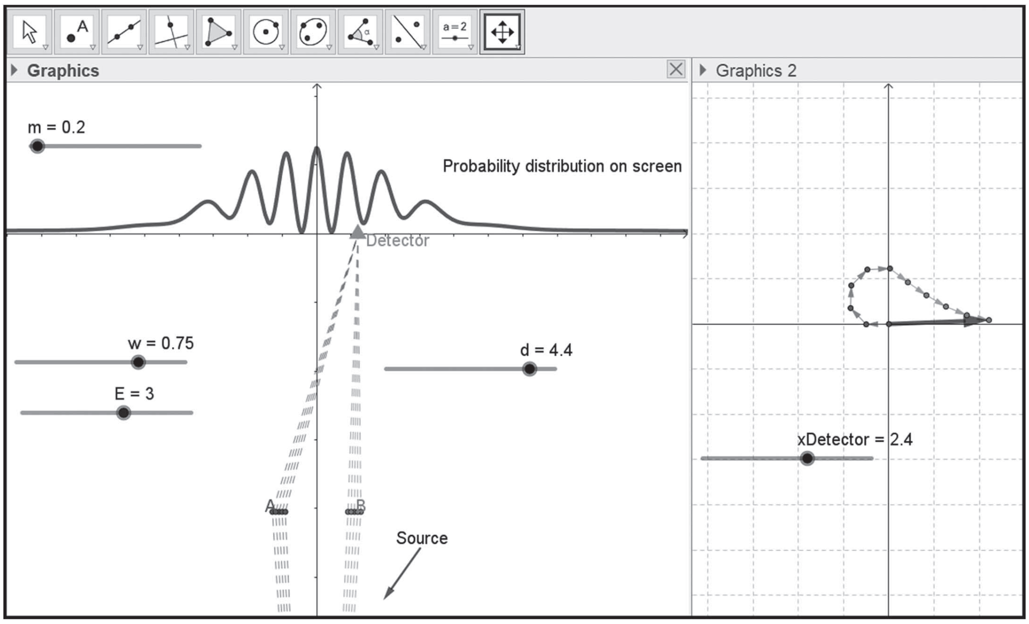
This system does not yet have a discrete set of energy levels; however, it strongly selects some values of energy, for which the probability of detection is much higher. In the simulation, shown in fig. 4, the left window contains the geometrical setting of the problem, the adjustable parameters including the number of paths from source to detector to consider, the graphical representation of the sum of amplitudes at the detector, and an horizontal bar whose length is proportional to the wavelength of the particle. The right window contains the computed probability as a function of energy, which is graphed by a point which moves and is tracked as the energy changes. Resonances, which constitute the main point of presenting this example, are clearly visible.

3˙4. *The Bohr atom.* – In the final part of the Pavia proposal, Bohr's model of the atom as seen from the point of view of the sum over paths approach is introduced, and its limitations are discussed. The model, which is clearly identified as semiclassical, is based on the assumption that the electron is confined to take all possible paths on a circular orbit around the proton, orbits which must be permitted by classical mechanics. Then, the problem becomes one of a quantum object confined to a circumference, subject to a $1/r$ potential, which we deal with using the general method described in ref. [3]. More explicitly, we consider the sum of all paths at fixed energy going from an arbitrary placed "source" to a "detector" on the circular orbit, including those which go through an arbitrary number of full roundabouts. As a result, the Bohr energy levels and allowed orbit radiuses are found by imposing the condition of constructive interference between all paths. The limitations of the models which are discussed are: 1) the model is two dimensional, and 2) it does not take into account all possible paths, but only those confined to a circumference and with a relationship between radius and momentum which is allowed by the rules of classical mechanics. In the Pavia proposal, the orbital model is then discussed qualitatively, showing that the Bohr radius corresponds to the maximum of the $1s$ orbital wave function of hydrogen; this allows to conclude that, in principle, the peaks of such a wave function derive from a constructive interference condition, similar to the one found in the semiclassical model.

A snapshot of the simulation is reported in fig. 5. Note that, of course, in a numerical computation only a finite number of paths can be considered (as defined by the user). As a consequence, the obtained spectrum has lines of finite width, which become narrower as the number of paths is increased.

4. – A GeoGebra simulation produced by teachers

The Pavia proposal, including the simulation repository presented in this article, was used in the course *An approach to quantum physics based on Feynman's sum over paths model* in the context of the online Master for physics teachers IDIFO-6 coordinated by the university of Udine. Some of the teachers who participated in the course produced GeoGebra simulations as part of the assignments required. In fig. 6 we show one of them, which has also been included in the online GeoGebra book. The simulation represents the example of interference in the case of reflection from a thin film, and allows the user to reproduce the formation of the interference pattern found on p. 22 of *QED*, from which the simulation was inspired.



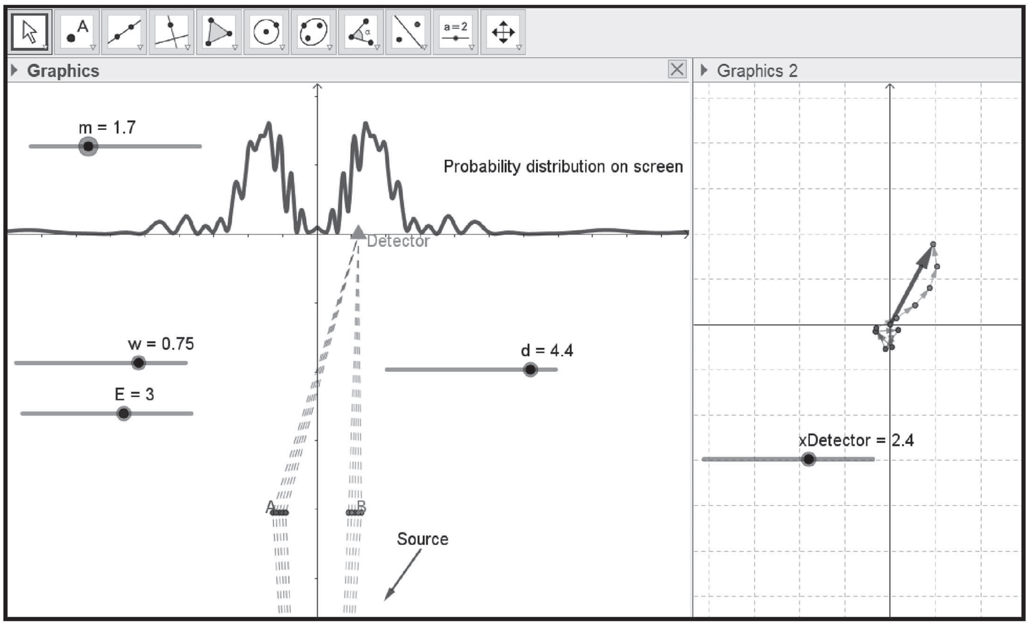


Fig. 3. – Sum over paths simulation of the two-slit interference with massive particles and non-pointlike slits. The simulation is in arbitrary units with $h = 1$. Top: when the De Broglie wavelength of the quantum object is large with respect to the relevant length scales of the setup, an interference pattern modulated by a diffraction figure is found. Bottom: if the De Broglie wavelength becomes small with respect to the slit width and distance, a classical-like result is found. Note that all paths passing through the slit in front of the detector interfere constructively, while all paths passing through the other slit interfere destructively.

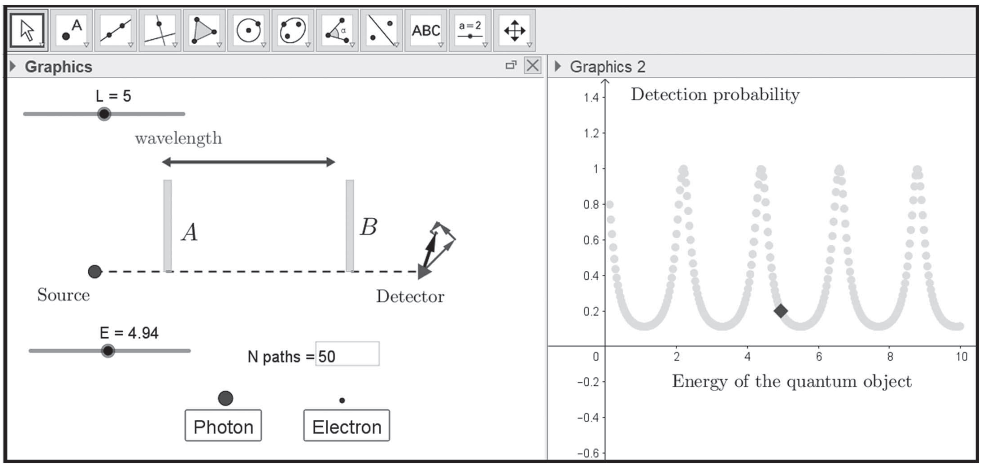


Fig. 4. – GeoGebra simulation of an open system displaying resonant tunneling. In this simulation, which belongs to the final part of the sequence, the computation of the detection probability is performed in the left window, superposed to the detector symbol, in order to leave space in the right window for showing the dependence on energy of the probability itself. Note that the simulation can be performed with both photons and electrons, showing the consequences of the different energy-wavelength relationship.

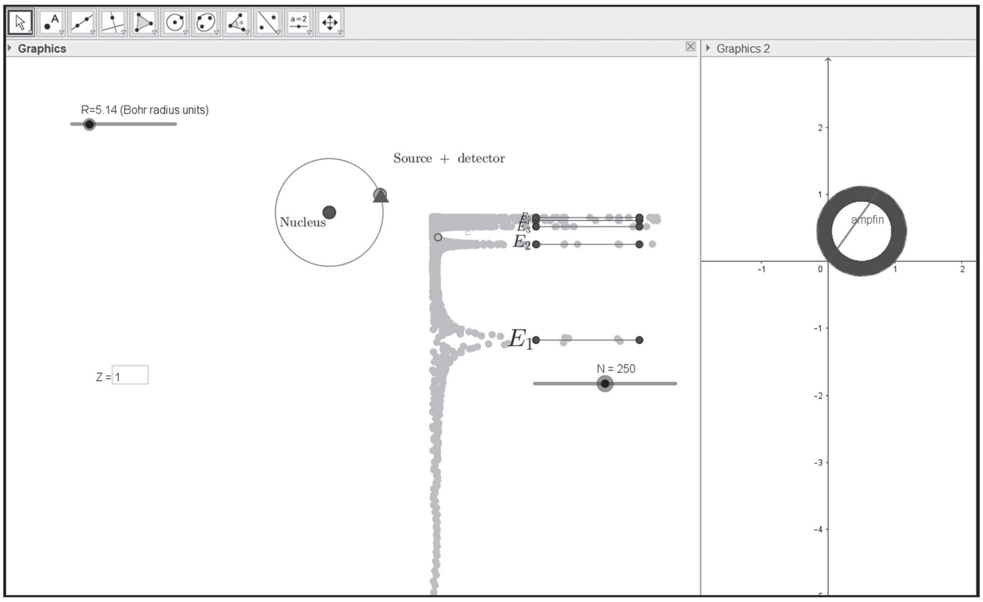


Fig. 5. – GeoGebra simulation of the Bohr model from the point of view of the sum over paths approach. A "source" and a "detector" are placed arbitrarily on the circular orbit; the detection probability has maxima (mathematically, the Green function diverges, see ref. [3]) when paths which differ by an arbitrary number of roundabouts are all in phase. The parameters which can be varied include the number of paths N and the atomic number Z in order to consider hydrogen-like atoms. The results of the simulation are compared to exact predictions for the energy levels.

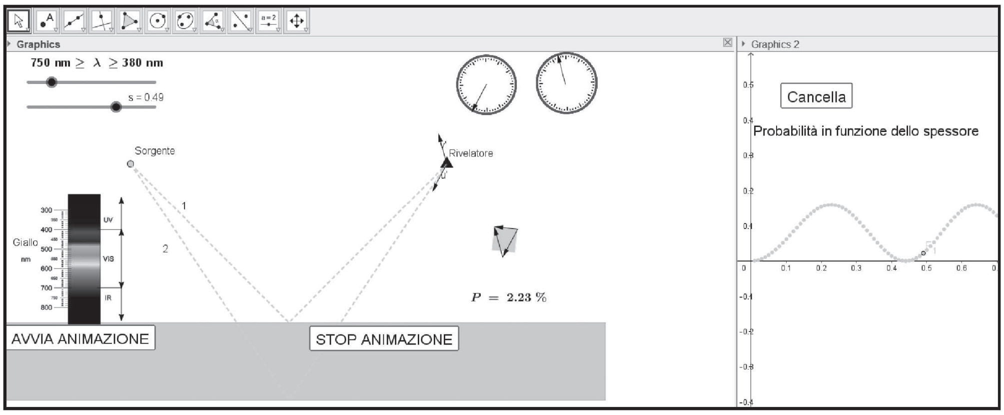


Fig. 6. – Simulation, produced by a teacher, of thin-film reflection as described in the first chapter of *QED*. As can be seen clearly, several techniques and the general philosophy of the simulations of the Pavia group have been reused in this example, which however also shows the personal taste of the teacher. The wavelength of light used can be chosen by the user, and the interference figure is colored accordingly, with the same technique as the simulation in fig. 2. In order to obtain more accurate results, paths including an arbitrary number of "bounces" within the film should be included in the computation.

5. – Conclusions

In this paper we discussed a collection of GeoGebra simulations, comprised within an organic proposal for the teaching of quantum physics at an elementary level using the sum over paths approach, and created within the online repository GeoGebra tube. We believe the collection to constitute a valuable support material for teachers in the construction of educational sequences. Furthermore, having realized the applets with an open-source software, widely known and used in the community of teachers of mathematics and physics, encourages teachers to "reverse-engineer" our examples, adapting them to their own needs and tastes, and reusing the techniques contained in them to produce novel simulations. In the context of the IDIFO-6 Master this fact has proved of some importance for the task of helping those physics teachers of Liceo Scientifico who hold a degree in mathematics (which constitute a majority in the Italian system) to acquire expertise and self-confidence in using the sum over paths model for quantum physics.

The work hereby presented must be considered constantly in progress, as most simulations can be perfected and refined, user instructions and suggestions are continuously improved, and more simulations are in the process of being created and added, as optional material, to the basic structure of the Pavia proposal.

* * *

The authors acknowledge that the simulation shown in fig. 6 was created by Dr. Claudio Sutrini.

REFERENCES

[1] Malgieri M., Onorato P. and De Ambrosis A., *Eur. J. Phys.*, **35** (2014) 055024.
[2] Malgieri M., Onorato P. and De Ambrosis A., *G. Fis.*, **56** (2015) 45.
[3] Malgieri M., Onorato P. and De Ambrosis A., *Am. J. Phys.*, **84** (2015) 678.
[4] Malgieri M., Tenni A., Onorato P. and De Ambrosis A., *Phys. Educ.*, **51** (2016) 055002.
[5] Malgieri M., Onorato P. and De Ambrosis A., *Phys. Rev. Phys. Educ. Res.*, **13** (2017) 010101.
[6] Taylor E. F., Vokos S., O'Meara J. M. and Thornber N. S., *Comput. Phys.*, **12** (1998) 190.
[7] de los Ángeles Fanaro M., Otero M. R. and Arlego M., *Phys. Teach.*, **50** (2012) 156.
[8] Dobson K., Lawrence I. and Britton P., *Phys. Educ.*, **35** (2000) 400.
[9] Feynman R. P., *QED: The Strange Theory of Light and Matter* (Princeton University Press, Princeton, NJ) 1985.
[10] Olsen R. V., *Int. J. Sci. Educ.*, **24** (2002) 565.
[11] Kohnle A., Bozhinova I., Browne D., Everitt M., Fomins A., Kok P. and Swinbank E., *Eur. J. Phys.*, **35** (2014) 015001.
[12] Belloni M., Christian W. and Cox A. J., *Phys. Teach.*, **45** (2007) 488.
[13] McKagan S. B., Handley W., Perkins K. K. and Wieman C. E., *Am. J. Phys.*, **77** (2009) 87.
[14] Zollman D. A., Rebello N. S. and Hogg K., *Am. J. Phys.*, **70** (2002) 252.
[15] Ogborn J., *A First Introduction to Quantum Behavior*, in *Proceedings of the GIREP Conference 2006: Modelling in Physics and Physics Education*, edited by In van den Berg E., Ellermeijer T. and Slooten T. (University of Amsterdam, Alphen aan den Rijn, Netherlands) 2006.
[16] Blake C. and Scanlon E., *J. Comput. Assist. Learn.*, **23** (2007) 491.
[17] Onorato P., Malgieri M. and De Ambrosis A., *Eur. J. Phys.*, **36** (2014) 058001.
[18] Onorato P., Malgieri M. and De Ambrosis A., *Eur. J. Phys.*, **37** (2015) 015301.