

# RIGIDITY RESULTS IN DIFFUSION MARKOV TRIPLES

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ABSTRACT. We consider stable solutions of semilinear equations in a very general setting. The equation is set on a Polish topological space endowed with a measure and the linear operator is induced by a carré du champs (equivalently, the equation is set in a diffusion Markov triple).

Under suitable curvature dimension conditions, we establish that stable solutions with integrable carré du champs are necessarily constant (weaker conditions characterize the structure of the carré du champs and carré du champ itéré).

The proofs are based on a geometric Poincaré formula in this setting. From the general theorems established, several previous results are obtained as particular cases and new ones are provided as well.

## 1. INTRODUCTION

A quasilinear equation is an expression of the form

$$(1.1) \quad Lu + F(u) = 0.$$

Here,  $L$  is a linear operator and identity (1.1) often turns out to provide a significant constraint for the solution  $u$ : namely, at any point  $x$ , the operator  $Lu$  at  $x$  has to be perfectly balanced by the nonlinear source  $F(u(x))$  and, as a consequence, the operator is constant along the level sets of the solution.

It is conceivable that this rigid constraint implies suitable classification results: for this, one has typically to consider problems in which (1.1) arises from a variational structure and focus on solutions with “sufficiently small energy”, since high energy solutions may develop some “wild behavior”. To this end, one often considers solutions which are local minimizers of the energy functional. Nevertheless, this minimal property may be uneasy to check in practice and it is therefore customary to look at a more general class of solutions, the so called “stable” solutions, for which the second derivative of the energy functional is nonnegative (in this setting, local minimizers become a special subclass of the stable solutions). We refer to the monograph [19] for a throughout discussion on stable solutions and on several classification results.

After [20, 25], a very useful tool towards the classification of stable solutions has been provided by a series of geometric Poincaré inequalities, originally introduced by Sternberg and Zumbrun in the celebrated articles [47] and [48]. Roughly speaking, in this approach a weighted  $L^2$ -norm of any test function is controlled by a weighted  $L^2$  norm of its gradient. The advantage of this method is that the weights are nonnegative and possess a geometric

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interpretation, hence the possible vanishing of the integral in the Poincaré-type inequalities implies the vanishing of the corresponding geometric weight, which in turn provides a series of useful geometric rigidities. Rigidity results via Poincaré-type inequalities have been recently obtained in different settings, including, among the others, systems of equations [14, 27, 28], manifolds [23, 24, 21, 4, 26, 18], stratified groups [30, 42, 6, 29], equations with drift [22], stratified media [44, 13, 16] and fractional equations [45, 46, 15], and there are also applications for equations in infinite dimensional spaces [10]. The method can be also applied to deduce new weighted Poincaré inequalities from the explicit knowledge of a stable solution, see [31], and it is also flexible enough to deal with Neumann boundary conditions [3, 17].

In this framework, a special role is often played by the geometry of the ambient space. To understand this phenomenon, one can think about the case of one-dimensional solutions in the Euclidean flat space for a bistable nonlinearity, for instance heteroclinic solutions of the mechanical pendulum. If one wants to “bend” these objects to construct solutions on a sphere, a geometric difficulty arises from the fact that the asymptotics “at the point infinity” is not well-determined, thus making it difficult to perform such a bending operation. This very heuristic example suggests that for “curved” manifolds the number and the structure of the stable solutions could be very different from the flat case.

The main objective of this paper is to deal with stable solutions of semilinear equations in the very general setting provided by the Markov triples, see the monograph [2]. Though we have not attempted to reach the widest generality, the setting of Markov triples is an excellent setting that comprises several particular cases at once, by developing an appropriate form of calculus and often providing a general, elegant and unified treatment. Moreover, the setting of Markov triples finds applications in probability and mathematical physics, e.g. to describe quantum ensembles, see e.g. [12] and the references therein.

The environment provided by the Markov triple (described here in details in Section 2) is that of a Polish space endowed by a measure and a carré du champ operator, which provide a variational framework for a general form of equation (1.1). See e.g. [36, 8, 9] for a classical introduction to these kinds of Dirichlet forms and also [1] and the references therein for recent developments.

The main result of this paper (stated in details in Theorem 2.6) is that stable solutions with integral carré du champ in environments with suitable geometric assumptions satisfy additional rigidity properties. Roughly speaking, under strong curvature assumptions, these solutions are necessarily constant and under weak curvature assumptions their level sets need to satisfy suitable geometric identities.

These types of results will be obtained here as a consequence of a very general Poincaré-type inequality (given explicitly in Theorem 2.7).

The rest of this paper is organized as follows. In Section 2, we introduce in details the setting of Markov triples in which we work and we state our main results, the proofs of which are given in Section 3.

Then, in Section 4, we deduce several particular cases from our main results (some of these results were already known in the literature, but follow here as a byproduct of

our general and unified approach; some other results seem to be new to the best of our knowledge).

## 2. FUNCTIONAL SETTING AND MAIN RESULTS

A triple  $(X, \mu, \Gamma)$  is called a *diffusion Markov triple* if it is composed by a Polish topological space  $(X, \tau)$  (i.e. a separable completely metrizable topological space  $X$  with topology  $\tau$ ) endowed with a  $\sigma$ -finite Borel measure  $\mu$  with full support, a class  $\mathcal{A}_0$  of real-valued measurable functions on  $X$  and a symmetric bilinear map (the *carré du champ*) such that  $\Gamma : \mathcal{A}_0 \times \mathcal{A}_0 \rightarrow \mathcal{A}_0$  satisfying the following conditions:

- (1)  $\mathcal{A}_0 \subset L^1(\mu)$  is a vector space, dense in every  $L^p(\mu)$ ,  $1 \leq p < \infty$ , such that

$$\forall f, g \in \mathcal{A}_0 \implies fg \in \mathcal{A}_0$$

and

$$\forall f_1, \dots, f_k \in \mathcal{A}_0, \Psi \in C^\infty(\mathbb{R}^k), \Psi(0) = 0 \implies \Psi(f_1, \dots, f_k) \in \mathcal{A}_0,$$

- (2) The map  $\Gamma$  is bilinear, symmetric and such that, for any  $f \in \mathcal{A}_0$ , it holds that

$$\Gamma(f) := \Gamma(f, f) \geq 0.$$

Moreover, for any  $f_1, \dots, f_k, g \in \mathcal{A}_0$ , and any smooth  $C^\infty$  function  $\Psi : \mathbb{R}^k \rightarrow \mathbb{R}$  such that  $\Psi(0) = 0$ , we have that  $\Psi(f_1, \dots, f_k) \in \mathcal{A}_0$  and

$$\Gamma(\Psi(f_1, \dots, f_k), g) = \sum_{i=1}^k \partial_i \Psi(f_1, \dots, f_k) \Gamma(f_i, g) \quad \mu - \text{a.e.}$$

- (3) For every  $f \in \mathcal{A}_0$ , there is  $C = C(f) > 0$  such that for every  $g \in \mathcal{A}_0$

$$\left| \int_X \Gamma(f, g) d\mu \right| \leq C \|g\|_{L^2(\mu)}.$$

The Dirichlet form  $\mathcal{E}$  is defined for every  $(f, g) \in \mathcal{A}_0 \times \mathcal{A}_0$  by

$$\mathcal{E}(f, g) := \int_X \Gamma(f, g) d\mu.$$

The domain  $\mathcal{D}(\mathcal{E})$  of  $\mathcal{E}$  is the completion of  $\mathcal{A}_0$  with respect to the norm

$$\|f\|_{\mathcal{E}} := (\|f\|_{L^2(\mu)}^2 + \mathcal{E}(f, f))^{1/2}.$$

The Dirichlet form  $\mathcal{E}$  is extended to  $\mathcal{D}(\mathcal{E})$  by continuity together with the map  $\Gamma$ .

- (4)  $L$  is a linear operator on  $\mathcal{A}_0$  defined as

$$\int_X gLf d\mu = - \int_X \Gamma(f, g) d\mu$$

for all  $f, g \in \mathcal{A}_0$ . The domain of the operator  $L$ ,  $\mathcal{D}(L)$ , is defined as the set of  $f \in \mathcal{D}(\mathcal{E})$  for which there exists a constant  $C = C(L) > 0$  such that for any  $g \in \mathcal{D}(\mathcal{E})$

$$|\mathcal{E}(f, g)| \leq C \|g\|_{L^2(\mu)}.$$

On  $\mathcal{D}(L)$ , the operator  $L$  is extended via integration by parts formula for every  $g \in \mathcal{D}(\mathcal{E})$ . The operator  $L$  defined on  $\mathcal{D}(L)$  is always self-adjoint.

- (5) There exists an increasing sequence  $(\xi_k)_{k \geq 1} \subset \mathcal{A}_0$  of functions such that  $\xi_k(x) \in [0, 1]$  for any  $x \in X$ , and

$$\lim_{k \rightarrow +\infty} \xi_k = 1 \quad \mu - \text{a.e. in } X$$

and

$$(2.1) \quad \Gamma(\xi_k) \leq \frac{1}{k} \quad k \geq 1.$$

We also assume the existence of an algebra  $\mathcal{A}_0 \subset \mathcal{A}$  of measurable functions on  $E$ , containing the constant functions and satisfying the following requirements:

- (1') Whenever  $f \in \mathcal{A}$  and  $h \in \mathcal{A}_0$ ,  $hf \in \mathcal{A}_0$ ;  
(2') For any  $f \in \mathcal{A}$ , if

$$\int_E hf d\mu \geq 0 \quad \text{for every positive } h \in \mathcal{A}_0,$$

then  $f \geq 0$ ,

- (3')  $\mathcal{A}$  is stable under composition with smooth  $C^\infty$ -functions  $\Psi : \mathbb{R}^k \rightarrow \mathbb{R}$  with  $\Psi(0) = 0$ , namely if  $f_1, \dots, f_k \in \mathcal{A}$  then  $\Psi(f_1, \dots, f_k) \in \mathcal{A}$ . Furthermore, for all  $f_1, \dots, f_k, g \in \mathcal{A}$ , it holds that

$$(2.2) \quad \Gamma(\Psi(f_1, \dots, f_k), g) = \sum_{i=1}^k \partial_i \Psi(f_1, \dots, f_k) \Gamma(f_i, g) \quad \mu - \text{a.e.}$$

- (4') The operator  $L : \mathcal{A} \rightarrow \mathcal{A}$  is an extension of  $L$  on  $\mathcal{A}_0$ . The carré du champ operator  $\Gamma$  is also defined on  $\mathcal{A} \times \mathcal{A}$  by the formula, for every  $(f, g) \in \mathcal{A} \times \mathcal{A}$ ,

$$(2.3) \quad \Gamma(f, g) = \frac{1}{2} [L(fg) - fLg - gLf] \in \mathcal{A},$$

and for any  $f \in \mathcal{A}$  we set  $\Gamma(f) := \Gamma(f, f)$ ,

- (5') For every  $f \in \mathcal{A}$ ,  $\Gamma(f, f) \geq 0$ ,

- (6') For every  $f \in \mathcal{A}$  and  $g \in \mathcal{A}_0$ , the integration by part formula

$$\int_X \Gamma(f, g) d\mu = - \int_X gLfd\mu = - \int_X fLgd\mu$$

holds true.

- (7') If  $f \in \mathcal{A}$  is such that  $\Gamma(f) = 0$  then  $f$  is constant.

**Remark 2.1.** The assumption on the sign of  $\Gamma(f)$  in (5') and the nondegeneracy condition in (7') are important and nontrivial structural assumptions. Roughly speaking, they reflect the “ellipticity of the operator  $L$ . For instance, if, for  $\mathbb{R}^2 \ni (x, y) \mapsto f(x, y)$ , one considers the d'Alembert operator  $Lf = f_{xx} - f_{yy}$ , one obtains

$$\Gamma(f) = \frac{1}{2} L(f^2) - fLf = ((f_x)^2 + ff_{xx}) - ((f_y)^2 + ff_{yy}) - f(f_{xx} - f_{yy}) = (f_x)^2 - (f_y)^2,$$

which has indefinite sign.

In addition, if, for  $\mathbb{R} \ni x \mapsto f(x)$ , one considers the derivative operator  $Lf = f_x$ , then it follows that

$$\Gamma(f) = \frac{1}{2} L(f^2) - fLf = \frac{1}{2} (f^2)_x - ff_x = 0,$$

therefore there are nontrivial operators producing carré du champs which vanish identically.

It is interesting to point out that operators in “divergence” and “nondivergence” form share the same carré du champs. For instance, if  $a_{ij}$  is a smooth symmetric matrix, and

$$(2.4) \quad L_D f := \sum_{i,j=1}^n (a_{ij} f_i)_j \quad \text{and} \quad L_{ND} f := \sum_{i,j=1}^n a_{ij} f_{ij},$$

a direct computation shows that both  $L_D$  and  $L_{ND}$  satisfy

$$\Gamma(f, g) = \sum_{i,j=1}^n a_{ij} f_i g_j.$$

Remarkably, the difference between the operators  $L_D$  and  $L_{ND}$  is read, in our setting, by condition (6'), which is satisfied by  $L_D$  and not by  $L_{ND}$ . That is, in a sense, while conditions (5') and (7') reflect an elliptic condition into a positive definiteness of an associated quadratic form, condition (6') detects the variational structure of the associated operator.

**Remark 2.2.** Taking  $\Psi(x, y) = xy$  in (2.2) we find that

$$(2.5) \quad \Gamma(f_1 f_2, g) = f_1 \Gamma(f_2, g) + f_2 \Gamma(f_1, g) \quad \mu - \text{a.e.}$$

for any  $f_1, f_2, g \in \mathcal{A}$ .

In addition, from (2.5), we infer that, for any  $f, g \in \mathcal{A}$ ,

$$(2.6) \quad \Gamma(f^2, g^2) = 2f\Gamma(f, g^2) = 2f\Gamma(g^2, f) = 4fg\Gamma(g, f) = 4fg\Gamma(f, g) \quad \mu - \text{a.e.}$$

Moreover, exploiting (2.5) with  $f_1 = f_2 = g = 1$ , we see that

$$\Gamma(1) = \Gamma(1) + \Gamma(1) \quad \mu - \text{a.e.}$$

and therefore

$$\Gamma(1) = 0 \quad \mu - \text{a.e.}$$

Using this identity and formula (2.3) with  $f = g = 1$ , we also infer that

$$0 = \Gamma(1) = \frac{1}{2} [L(1) - L(1) - L(1)] = -\frac{L(1)}{2},$$

and so

$$(2.7) \quad L(1) = 0 \quad \mu - \text{a.e.}$$

To detect the behavior of second derivative operators, it is also classical to introduce the following notation.

**Definition 2.3.** *The carré du champ itéré is the bilinear form  $\Gamma_2 : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  defined as*

$$(2.8) \quad \Gamma_2(f, g) := \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)].$$

We define  $\Gamma_2(f) := \Gamma_2(f, f)$ .

As an example, we point out that when  $L$  is the Laplace operator in  $\mathbb{R}^n$ , then the carré du champ itéré reduces to the square of the norm of the Hessian matrix (see also Appendix C.5 in [2] for more general formulas for Riemannian manifolds).

Now we recall a classical notion of curvature dimension condition in our setting:

**Definition 2.4.** *We say that  $(X, \mu, \Gamma)$  satisfies the  $CD(K, \infty)$  condition, for some  $K \in \mathbb{R}$ , if for any  $f \in \mathcal{A}$*

$$\Gamma_2(f) \geq K\Gamma(f).$$

The following result will be crucial in our setting (see page 9). For the proof of it, see [2, formula (3.3.6)].

**Theorem 2.5.** *Assume that  $(X, \mu, \Gamma)$  satisfies the  $CD(K, \infty)$  condition in Definition 2.4, for some  $K \in \mathbb{R}$ . Then*

$$(2.9) \quad 4\Gamma(f) (\Gamma_2(f) - K\Gamma(f)) \geq \Gamma(\Gamma(f))$$

for every  $f \in \mathcal{A}$ .

In this paper we study the solutions to the following boundary value problem:

$$(2.10) \quad Lu + F(u) = 0 \quad \text{in } X,$$

where  $F \in C^\infty(\mathbb{R})$ . As customary, we say that  $u$  is a weak solution to (2.10) if  $u \in \mathcal{A}$  and

$$(2.11) \quad \int_X \Gamma(u, \varphi) d\mu = \int_X F(u)\varphi d\mu, \quad \text{for any } \varphi \in \mathcal{A}_0.$$

Moreover, we say that a weak solution  $u$  is stable if

$$(2.12) \quad \int_X \Gamma(\varphi) d\mu - \int_X F'(u)\varphi^2 d\mu \geq 0, \quad \text{for any } \varphi \in \mathcal{A}_0.$$

**Theorem 2.6.** *Assume that  $(X, \mu, \Gamma)$  satisfies the curvature dimension condition  $CD(K, \infty)$  for some  $K \geq 0$  and  $\Gamma : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ . For any  $(x, y) \in X \times X$ , let*

$$(2.13) \quad d(x, y) := \text{esssup}\{f(x) - f(y)\}$$

be the distance function in  $(X, \mu, \Gamma)$ , where the essential supremum is computed on bounded functions  $f \in \mathcal{A}$  with  $\Gamma(f) \leq 1$ . Let  $x_0 \in X$  and  $\rho(x) := d(x, x_0)$ , for any  $x \in X$ , and suppose that there exists a sequence of functions  $\rho_k \in \mathcal{A}$  such that

$$(2.14) \quad \rho_k \rightarrow \rho \quad \mu - \text{a.e. in } X \quad \text{and} \quad \|\Gamma(\rho_k)\|_{L^\infty} \leq C_0,$$

for some  $C_0 > 0$ .

Let  $u \in \mathcal{A}$  be a stable solution to (2.10) with

$$(2.15) \quad \int_X \Gamma(u) d\mu < \infty.$$

Then:

$$(2.16) \quad K > 0 \implies \Gamma(u) = 0 \quad \mu - \text{a.e. in } X;$$

$$(2.17) \quad K = 0 \implies \Gamma_2(u) - \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) = 0 \quad \mu - \text{a.e. in } X.$$

In addition, if  $K > 0$  and  $\Gamma(u) \in \mathcal{A} \cap C^0(X)$  then

$$(2.18) \quad u \text{ is constant in } X.$$

The distance function in (2.13), often called intrinsic distance, has been considered also in [2] and it coincides with the Riemannian distance if  $X$  is a Riemannian manifold and with the Carnot-Carathéodory distance if  $X$  is a Carnot-Carathéodory space (see [7] for the definition). In this setting, Assumption (2.14) is related to the fact that  $\rho$  is Lipschitz as defined in [2, Definition 3.3.24]. We recall that the same assumption appears in [49] and it is the analogous to  $|\nabla\rho| \leq 1$  which is satisfied by geodesic distances on any manifold.

The proof of Theorem 2.6 is based on a geometric Poincaré-type inequality, which we state in this setting as follows:

**Theorem 2.7.** *Let  $u \in \mathcal{A}$  be stable weak solution to (2.10). Then,*

$$(2.19) \quad \int_X \left( \Gamma_2(u) - \Gamma \left( \Gamma(u)^{\frac{1}{2}} \right) \right) \varphi^2 d\mu \leq \int_X \Gamma(u) \Gamma(\varphi)$$

for any  $\varphi \in \mathcal{A}_0$ .

### 3. PROOF OF THEOREMS 2.6 AND 2.7

*Proof of Theorem 2.7:* We fix  $\varepsilon > 0$  and take  $\varphi \in \mathcal{A}_0$ . By (4'), we know that  $\Gamma(u) \in \mathcal{A}$ . Since  $\mathcal{A}$  is an algebra (and therefore a vector space) it follows that

$$(3.1) \quad \Gamma(u) + \varepsilon \in \mathcal{A}.$$

Now, we consider a function  $\Psi \in C^\infty(\mathbb{R})$  such that  $\Psi(r) = 0$  for any  $r \leq \varepsilon/4$  and  $\Psi(r) = \sqrt{r}$  for any  $r \geq \varepsilon/2$ . In view of (3') and (3.1), we have that

$$\sqrt{\Gamma(u) + \varepsilon} = \Psi(\Gamma(u) + \varepsilon) \in \mathcal{A}.$$

Consequently, by (1') we conclude that

$$\psi_\varepsilon := \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \in \mathcal{A}_0.$$

Hence, applying (2.12) with  $\varphi$  replaced by  $\psi_\varepsilon$ , we get

$$(3.2) \quad 0 \leq \int_X \Gamma \left( \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \right) d\mu - \int_X F'(u) (\Gamma(u) + \varepsilon) \varphi^2 d\mu.$$

Furthermore, by (2.5), we have that

$$\begin{aligned} & \Gamma \left( \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \right) \\ &= \Gamma \left( \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi, \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \right) \\ &= \sqrt{\Gamma(u) + \varepsilon} \Gamma \left( \varphi, \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \right) + \varphi \Gamma \left( \sqrt{\Gamma(u) + \varepsilon}, \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \right) \\ &= \sqrt{\Gamma(u) + \varepsilon} \Gamma \left( \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi, \varphi \right) + \varphi \Gamma \left( \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi, \sqrt{\Gamma(u) + \varepsilon} \right) \\ &= (\Gamma(u) + \varepsilon) \Gamma(\varphi, \varphi) + \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \Gamma \left( \sqrt{\Gamma(u) + \varepsilon}, \varphi \right) \\ &\quad + \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \Gamma \left( \varphi, \sqrt{\Gamma(u) + \varepsilon} \right) + \varphi^2 \Gamma \left( \sqrt{\Gamma(u) + \varepsilon}, \sqrt{\Gamma(u) + \varepsilon} \right) \\ &= (\Gamma(u) + \varepsilon) \Gamma(\varphi) + 2 \left( \sqrt{\Gamma(u) + \varepsilon} \right) \varphi \Gamma \left( \sqrt{\Gamma(u) + \varepsilon}, \varphi \right) + \varphi^2 \Gamma \left( \sqrt{\Gamma(u) + \varepsilon} \right). \end{aligned}$$

This and (2.6) imply that

$$\Gamma\left(\left(\sqrt{\Gamma(u)+\varepsilon}\right)\varphi\right) = (\Gamma(u)+\varepsilon)\Gamma(\varphi) + \frac{1}{2}\Gamma(\Gamma(u)+\varepsilon, \varphi^2) + \varphi^2\Gamma\left(\sqrt{\Gamma(u)+\varepsilon}\right).$$

Plugging this information into (3.2), we obtain that

$$(3.3) \quad \begin{aligned} & \int_X F'(u)(\Gamma(u)+\varepsilon)\varphi^2 d\mu \\ & \leq \int_X (\Gamma(u)+\varepsilon)\Gamma(\varphi) + \frac{1}{2}\Gamma(\varphi^2, \Gamma(u)+\varepsilon) + \varphi^2\Gamma\left(\sqrt{\Gamma(u)+\varepsilon}\right) d\mu. \end{aligned}$$

Now, we remark that, by Fatou's Lemma,

$$(3.4) \quad \liminf_{\varepsilon \rightarrow 0} \int_X F'(u)(\Gamma(u)+\varepsilon)\varphi^2 d\mu \geq \int_X F'(u)\Gamma(u)\varphi^2 d\mu.$$

Moreover, from (1) we know that  $\Gamma(\varphi)$  is a bounded function and therefore

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0} \int_X (\Gamma(u)+\varepsilon)\Gamma(\varphi) d\mu = \int_X \Gamma(u)\Gamma(\varphi) d\mu + \lim_{\varepsilon \rightarrow 0} \varepsilon \int_X \Gamma(\varphi) d\mu = \int_X \Gamma(u)\Gamma(\varphi) d\mu.$$

We also remark that, for any  $f, g \in \mathcal{A}$ ,

$$(3.6) \quad \begin{aligned} \Gamma(f, g + \varepsilon) &= \frac{1}{2}[L(f(g + \varepsilon)) - fL(g + \varepsilon) - (g + \varepsilon)Lf] \\ &= \frac{1}{2}[L(fg) + \varepsilon Lf - fLg - \varepsilon fL(1) - gLf - \varepsilon Lf] \\ &= \frac{1}{2}[L(fg) - fLg - gLf] \\ &= \Gamma(f, g), \end{aligned}$$

thanks to (2.3) and (2.7). As a consequence, we have that

$$(3.7) \quad \int_X \Gamma(\varphi^2, \Gamma(u) + \varepsilon) d\mu = \int_X \Gamma(\varphi^2, \Gamma(u)) d\mu.$$

Furthermore, we claim that

$$(3.8) \quad \limsup_{\varepsilon \rightarrow 0} \int_X \varphi^2\Gamma\left(\sqrt{\Gamma(u)+\varepsilon}\right) d\mu \leq \int_X \varphi^2\Gamma\left(\sqrt{\Gamma(u)}\right) d\mu.$$

To prove this, we can assume that

$$(3.9) \quad \int_X \varphi^2\Gamma\left(\sqrt{\Gamma(u)}\right) d\mu < +\infty,$$

otherwise (3.8) is true by default. Also, in view of (2.6) (used here with  $f = g = \sqrt{\Gamma(u)+\varepsilon}$ ), we see that

$$\Gamma(\Gamma(u) + \varepsilon) = 4(\Gamma(u) + \varepsilon)\Gamma\left(\sqrt{\Gamma(u) + \varepsilon}\right).$$

From this and (3.6), we obtain that

$$(3.10) \quad \Gamma\left(\sqrt{\Gamma(u) + \varepsilon}\right) = \frac{\Gamma(\Gamma(u) + \varepsilon)}{4(\Gamma(u) + \varepsilon)} = \frac{\Gamma(\Gamma(u))}{4(\Gamma(u) + \varepsilon)}.$$



Similarly, using (2.6) with  $f = g = \sqrt{\Gamma(u)}$ , we see that

$$(3.11) \quad \Gamma(\Gamma(u)) = 4(\Gamma(u)) \Gamma\left(\sqrt{\Gamma(u)}\right).$$

Inserting this into (3.10), we conclude that

$$\varphi^2 \Gamma\left(\sqrt{\Gamma(u) + \varepsilon}\right) = \varphi^2 \frac{\Gamma(u)}{\Gamma(u) + \varepsilon} \Gamma\left(\sqrt{\Gamma(u)}\right) \leq \varphi^2 \Gamma\left(\sqrt{\Gamma(u)}\right),$$

and the latter is a summable function, thanks to (3.9). Therefore,

$$\begin{aligned} \int_X \varphi^2 \Gamma\left(\sqrt{\Gamma(u) + \varepsilon}\right) d\mu &= \int_X \varphi^2 \frac{\Gamma(u)}{\Gamma(u) + \varepsilon} \Gamma\left(\sqrt{\Gamma(u)}\right) d\mu \\ &= \int_X \varphi^2 \left(1 - \frac{\varepsilon}{\Gamma(u) + \varepsilon}\right) \Gamma\left(\sqrt{\Gamma(u)}\right) d\mu \\ &\leq \int_X \varphi^2 \Gamma\left(\sqrt{\Gamma(u)}\right) d\mu, \end{aligned}$$

which in turn implies (3.8).

Therefore, letting  $\varepsilon \rightarrow 0$  in (3.3), and exploiting (3.4), (3.5), (3.7) and (3.8), we conclude that

$$\int_X F'(u) \Gamma(u) \varphi^2 d\mu \leq \int_X \Gamma(u) \Gamma(\varphi) + \frac{1}{2} \Gamma(\varphi^2, \Gamma(u)) + \varphi^2 \Gamma\left(\sqrt{\Gamma(u)}\right) d\mu.$$

As a consequence, by (6'), Definition 2.3 (used here with  $f = g = u$ ) and (2.10), we obtain that

$$(3.12) \quad \begin{aligned} \int_X F'(u) \Gamma(u) \varphi^2 d\mu &\leq \int_X \Gamma(u) \Gamma(\varphi) - \frac{1}{2} \varphi^2 L\Gamma(u) + \varphi^2 \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) d\mu \\ &= \int_X \Gamma(u) \Gamma(\varphi) - \varphi^2 \left(\Gamma_2(u) + \Gamma(u, Lu)\right) + \varphi^2 \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) d\mu \\ &= \int_X \Gamma(u) \Gamma(\varphi) - \varphi^2 \left(\Gamma_2(u) + \Gamma(u, -F(u))\right) + \varphi^2 \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) d\mu. \end{aligned}$$

Besides, using (2.2) with  $\Psi = -F$ , we see that

$$\Gamma(u, -F(u)) \Gamma(-F(u), u) = -F'(u) \Gamma(u) \quad \mu - \text{a.e.}$$

and thus (3.12) becomes

$$\int_X F'(u) \Gamma(u) \varphi^2 d\mu \leq \int_X \Gamma(u) \Gamma(\varphi) - \Gamma_2(u) \varphi^2 + F'(u) \Gamma(u) \varphi^2 + \varphi^2 \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) d\mu.$$

Then, canceling one term, we obtain (2.19), as desired.  $\square$

With this, we are able to prove Theorem 2.6:

*Proof of Theorem 2.6:* Using the identity in (3.11) and Theorem 2.5 we have that

$$(3.13) \quad \begin{aligned} 4\Gamma(u) \left(\Gamma_2(u) - \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right)\right) &= 4\Gamma(u) \Gamma_2(u) - 4\Gamma(u) \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) \\ &= 4\Gamma(u) \Gamma_2(u) - \Gamma(\Gamma(u)) \geq 4K (\Gamma(u))^2, \end{aligned}$$

which is always nonnegative if  $K \geq 0$ . Also, we know that  $\Gamma(0) = \Gamma_2(0) = 0$ , due to (2.3) and (2.8), and therefore

$$\Gamma_2(u) - \Gamma(\Gamma(u)^{\frac{1}{2}}) = 0 \quad \text{in } \{\Gamma(u) = 0\}.$$

Using this and Theorem 2.7 we can write

$$(3.14) \quad \int_{\{\Gamma(u) \neq 0\}} \left( \Gamma_2(u) - \Gamma(\Gamma(u)^{\frac{1}{2}}) \right) \varphi^2 d\mu = \int_X \left( \Gamma_2(u) - \Gamma(\Gamma(u)^{\frac{1}{2}}) \right) \varphi^2 d\mu \\ \leq \int_X \Gamma(u) \Gamma(\varphi) d\mu$$

for every  $\varphi \in \mathcal{A}_0$ .

Now, we fix  $R > 1$  and define  $\Phi = \Phi_R \in C^\infty(\mathbb{R})$ , with

$$(3.15) \quad |\Phi'(t)| \leq 3$$

for any  $|t| \in [R, R+1]$  and

$$(3.16) \quad \Phi(t) := \begin{cases} 1 & \text{if } |t| \leq R, \\ 0 & \text{if } |t| \geq R+1. \end{cases}$$

We also define

$$(3.17) \quad \tilde{\Phi}(t) := \Phi(t) - 1,$$

for any  $t \in \mathbb{R}$ . In this way, we have that  $\tilde{\Phi}(0) = 0$ .

Moreover, we consider  $\xi_k$  as given by (5). We remark that, in light of (2.6) and (2.1),

$$\Gamma(\xi_k^2) = 4\xi_k^2 \Gamma(\xi_k) \leq \frac{4\xi_k^2}{k}.$$

Iterating this, we have that

$$(3.18) \quad \Gamma(\xi_k^4) = 4\xi_k^4 \Gamma(\xi_k^2) \leq \frac{16\xi_k^4}{k}.$$

Therefore, by possibly renaming  $\xi_k$  into  $\xi_k^4$ , we can suppose in view of (5) that  $\xi_k(x) \in [0, 1]$  for any  $x \in X$ , and

$$(3.19) \quad \lim_{k \rightarrow +\infty} \xi_k = 1 \quad \mu - \text{a.e. in } X$$

and

$$(3.20) \quad \Gamma(\xi_k) \leq \frac{16\xi_k^4}{k}.$$

Then, using (1') and the setting in (2.14), we obtain that for every  $k \in \mathbb{N}$  the function  $\rho_k \xi_k$  belongs to  $\mathcal{A}_0$ . Since  $\mathcal{A}_0$  is a vector space we conclude that

$$(3.21) \quad \varphi_k := \tilde{\Phi}(\rho_k \xi_k) + \xi_k \in \mathcal{A}_0.$$

Also, exploiting the bilinearity of  $\Gamma$  we get

$$(3.22) \quad \Gamma(\varphi_k) = \Gamma(\tilde{\Phi}(\rho_k \xi_k) + \xi_k) = \Gamma(\tilde{\Phi}(\rho_k \xi_k)) + 2\Gamma(\tilde{\Phi}(\rho_k \xi_k), \xi_k) + \Gamma(\xi_k).$$

Moreover, using formula (2.2) with  $\Psi := \tilde{\Phi}$ , we obtain that

$$(3.23) \quad \begin{aligned} \Gamma(\tilde{\Phi}(\rho_k \xi_k)) &= (\tilde{\Phi}'(\rho_k \xi_k))^2 \Gamma(\rho_k \xi_k) \\ &= (\Phi'(\rho_k \xi_k))^2 \left( \xi_k^2 \Gamma(\rho_k) + 2\rho_k \xi_k \Gamma(\rho_k, \xi_k) + \rho_k^2 \Gamma(\xi_k) \right) \end{aligned}$$

and

$$(3.24) \quad \begin{aligned} \Gamma(\tilde{\Phi}(\rho_k \xi_k), \xi_k) &= \Phi'(\rho_k \xi_k) \Gamma(\rho_k \xi_k, \xi_k) \\ &= \Phi'(\rho_k \xi_k) \left( \xi_k \Gamma(\rho_k, \xi_k) + \rho_k \Gamma(\xi_k) \right). \end{aligned}$$

Plugging (3.23) and (3.24) into (3.22), we have that

$$(3.25) \quad \begin{aligned} \Gamma(\varphi_k) &= (\Phi'(\rho_k \xi_k))^2 \left( \xi_k^2 \Gamma(\rho_k) + 2\rho_k \xi_k \Gamma(\rho_k, \xi_k) + \rho_k^2 \Gamma(\xi_k) \right) \\ &\quad + 2\Phi'(\rho_k \xi_k) \left( \xi_k \Gamma(\rho_k, \xi_k) + \rho_k \Gamma(\xi_k) \right) + \Gamma(\xi_k). \end{aligned}$$

Now, taking  $\varphi := \varphi_k$  into (3.14) and making use of (3.25), we get that

$$(3.26) \quad \begin{aligned} &\int_{\{\Gamma(u) \neq 0\}} \left( \Gamma_2(u) - \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) \right) \varphi_k^2 d\mu \\ &\leq \int_X \Gamma(u) \left[ (\Phi'(\rho_k \xi_k))^2 \left( \xi_k^2 \Gamma(\rho_k) + 2\rho_k \xi_k \Gamma(\rho_k, \xi_k) + \rho_k^2 \Gamma(\xi_k) \right) \right. \\ &\quad \left. + \Phi'(\rho_k \xi_k) \left( \xi_k \Gamma(\rho_k, \xi_k) + \rho_k \Gamma(\xi_k) \right) + \Gamma(\xi_k) \right] d\mu. \end{aligned}$$

We also point out that, in view of (3.15) and (3.16), and recalling (3.20) and (2.14),

$$(3.27) \quad \begin{aligned} &\int_X \Gamma(u) \left[ (\Phi'(\rho_k \xi_k))^2 \left( \xi_k^2 \Gamma(\rho_k) + 2\rho_k \xi_k \Gamma(\rho_k, \xi_k) + \rho_k^2 \Gamma(\xi_k) \right) \right. \\ &\quad \left. + \Phi'(\rho_k \xi_k) \left( \xi_k \Gamma(\rho_k, \xi_k) + \rho_k \Gamma(\xi_k) \right) + \Gamma(\xi_k) \right] d\mu \\ &\leq \int_{X_{R,k}} \Gamma(u) \left[ 9 \left( C_0 + 2\rho_k \xi_k |\Gamma(\rho_k, \xi_k)| + \frac{16\rho_k^2 \xi_k^2}{k} \right) + 3 \left( |\Gamma(\rho_k, \xi_k)| + \frac{16\rho_k \xi_k}{k} \right) \right] d\mu \\ &\quad + \frac{1}{k} \int_X \Gamma(u) d\mu, \end{aligned}$$

where

$$(3.28) \quad X_{R,k} := \{x \in X \text{ s.t. } \rho_k(x) \xi_k(x) \in [R, R+1]\}.$$

From (3.27) and (3.28), we have

$$\begin{aligned}
(3.29) \quad & \int_X \Gamma(u) \left[ (\Phi'(\rho_k \xi_k))^2 \left( \xi_k^2 \Gamma(\rho_k) + 2\rho_k \xi_k \Gamma(\rho_k, \xi_k) + \rho_k^2 \Gamma(\xi_k) \right) \right. \\
& \quad \left. + \Phi'(\rho_k \xi_k) \left( \xi_k \Gamma(\rho_k, \xi_k) + \rho_k \Gamma(\xi_k) \right) + \Gamma(\xi_k) \right] d\mu \\
\leq & \int_{X_{R,k}} \Gamma(u) \left[ 9 \left( C_0 + 2(R+1) |\Gamma(\rho_k, \xi_k)| + \frac{16(R+1)^2}{k} \right) + 3 \left( |\Gamma(\rho_k, \xi_k)| + \frac{16(R+1)}{k} \right) \right] d\mu \\
& + \frac{1}{k} \int_X \Gamma(u) d\mu.
\end{aligned}$$

Now, we observe that the following Cauchy-Schwarz inequality for  $\Gamma$  holds true: given  $f, g \in \mathcal{A}$  and  $\alpha > 0$ , the fact that  $\Gamma(\cdot, \cdot)$  is a symmetric bilinear map implies that

$$0 \leq \Gamma \left( \alpha f \pm \frac{1}{\alpha} g \right) = \Gamma(\alpha f) + \Gamma \left( \frac{1}{\alpha} g \right) \pm 2\Gamma \left( \alpha f, \frac{1}{\alpha} g \right) = \alpha^2 \Gamma(f) + \frac{1}{\alpha^2} \Gamma(g) \pm 2\Gamma(f, g)$$

and therefore

$$2|\Gamma(f, g)| \leq \alpha^2 \Gamma(f) + \frac{1}{\alpha^2} \Gamma(g).$$

Then, by choosing  $\alpha$  appropriately, we obtain that

$$|\Gamma(f, g)| \leq (\Gamma(f))^{\frac{1}{2}} (\Gamma(g))^{\frac{1}{2}}.$$

As a consequence, we infer from (3.29) that

$$\begin{aligned}
& \int_X \Gamma(u) \left[ (\Phi'(\rho_k \xi_k))^2 \left( \xi_k^2 \Gamma(\rho_k) + 2\rho_k \xi_k \Gamma(\rho_k, \xi_k) + \rho_k^2 \Gamma(\xi_k) \right) \right. \\
& \quad \left. + \Phi'(\rho_k \xi_k) \left( \xi_k \Gamma(\rho_k, \xi_k) + \rho_k \Gamma(\xi_k) \right) + \Gamma(\xi_k) \right] d\mu \\
\leq & \int_{X_{R,k}} \Gamma(u) \left[ 9 \left( C_0 + \frac{32(R+1)\sqrt{C_0}}{\sqrt{k}} + \frac{16(R+1)^2}{k} \right) + 3 \left( \sqrt{\frac{C_0}{k}} + \frac{16(R+1)}{k} \right) \right] d\mu \\
& + \frac{1}{k} \int_X \Gamma(u) d\mu.
\end{aligned}$$

We insert this information into (3.26) and we take the limit as  $k \rightarrow +\infty$ : in this way, we obtain that

$$\begin{aligned}
(3.30) \quad & \lim_{k \rightarrow +\infty} \int_{\{\Gamma(u) \neq 0\}} \left( \Gamma_2(u) - \Gamma \left( \Gamma(u)^{\frac{1}{2}} \right) \right) \varphi_k^2 d\mu \\
\leq & \lim_{k \rightarrow +\infty} \int_{X_{R,k}} \Gamma(u) \left[ 9 \left( C_0 + \frac{32(R+1)\sqrt{C_0}}{\sqrt{k}} + \frac{16(R+1)^2}{k} \right) + 3 \left( \sqrt{\frac{C_0}{k}} + \frac{16(R+1)}{k} \right) \right] d\mu \\
& + \frac{1}{k} \int_X \Gamma(u) d\mu \\
= & 9C_0 \int_{X_R} \Gamma(u),
\end{aligned}$$

where we have used (3.19), (2.15) and (3.28) in the last step, and

$$X_R := \{x \in X \text{ s.t. } \rho(x) \in [R, R+1]\}.$$

We also notice that, by (3.17), (3.19) and (3.21),

$$\lim_{k \rightarrow +\infty} \varphi_k = \tilde{\Phi}(\rho) + 1 = \Phi(\rho),$$

and therefore we deduce from (3.30) that

$$(3.31) \quad \int_{\{\Gamma(u) \neq 0\}} \left( \Gamma_2(u) - \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) \right) (\Phi(\rho))^2 d\mu \leq 9C_0 \int_{X_R} \Gamma(u).$$

Noticing that  $\Phi(\rho(x)) = 1$  for any  $x \in X$  for which  $\rho(x) \leq R$ , thanks to (3.16), and recalling (2.15), we can take the limit as  $R \rightarrow +\infty$  in (3.31), obtaining that

$$(3.32) \quad \int_{\{\Gamma(u) \neq 0\}} \left( \Gamma_2(u) - \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) \right) d\mu \leq 0.$$

Since the integrand in the left hand side of (3.32) is nonnegative (recall (3.13)), this gives (2.17).

Now we suppose that  $K > 0$ . Then, by Definition 2.4, we have that  $(X, \mu, \Gamma)$  also satisfies the  $CD(K, \infty)$  condition with  $K = 0$ . Then, from (2.17) and (3.13), we infer that  $\Gamma(u) = 0$   $\mu$ -a.e. in  $X$ . If in addition  $\Gamma(u) \in C^0(X)$ , we get  $\Gamma(u) = 0$  in  $X$ , which in light of (7') gives (2.18).  $\square$

#### 4. APPLICATIONS TO VECTOR FIELDS SATISFYING THE HÖRMANDER CONDITION

Here we take  $\mathcal{A}_0 := \text{Lip}_c(\mathbb{R}^n)$  and  $\mathcal{A} := \text{Lip}(\mathbb{R}^n)$ . Moreover, we let  $\eta \in \mathcal{A}$  and define

$$d\mu := e^\eta dx.$$

Let  $Z_1, \dots, Z_m$  be smooth vector fields in  $\mathbb{R}^n$  with

$$Z_j = \sum_{i=1}^n Z_i^j \partial_i.$$

We define

$$(4.1) \quad Z_0 f := \sum_{j=1}^m Z_j \eta Z_j f, \quad \text{div} Z_j := \sum_{i=1}^n \partial_i Z_i^j \quad \text{and} \quad \Delta_Z := \sum_{j=1}^m Z_j Z_j.$$

We assume that the family  $(Z_1, \dots, Z_m)$  satisfies the Hörmander condition: at any point  $x \in \mathbb{R}^n$ , consider the vector spaces  $V_p$  generated by the vector fields  $Z_j$  at  $x$ , namely

$$\begin{aligned} V_1 &:= \text{span}\{Z_j \mid 1 \leq j \leq m\}, \\ V_2 &:= \text{span}\{Z_j, [Z_j, Z_k] \mid 0 \leq j, k \leq m\}, \\ &\dots \\ V_d &:= \text{span}\{V_{d-1} \cup \{[Z_j, V] \mid V \in V_{d-1}, 0 \leq j \leq m\}\}, \end{aligned}$$

then there exists  $d \in \mathbb{N}$  such that, for any  $x \in \mathbb{R}^n$ ,  $V_d = \mathbb{R}^n$ .

We recall the following result from [34]:

**Theorem 4.1.** *Let  $Z = (Z_1, \dots, Z_m)$  be a family of smooth vector fields in  $\mathbb{R}^n$  satisfying the Hörmander condition and let  $d$  be the associated Carnot-Carathéodory distance [35], which we assume to be continuous with respect to the Euclidean topology. If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that, for some  $\Lambda \geq 0$ ,*

$$(4.2) \quad |f(x) - f(y)| \leq \Lambda d(x, y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

*then the derivatives  $Z_j f$ , with  $j = 1, \dots, m$ , exist in distributional sense, are measurable functions and  $|(Zf)(x)| \leq \Lambda$  for a.e.  $x \in \mathbb{R}^n$ .*

With the notation introduced in (4.1), we consider the linear operator

$$(4.3) \quad Lg = \sum_{j=1}^m Z_j g \operatorname{div} Z_j + \Delta_Z g + Z_0 g.$$

We remark that

$$(4.4) \quad Z_j(fg) = \sum_{i=1}^m Z_i^j \partial_i(fg) = \sum_{i=1}^m Z_i^j f \partial_i g + \sum_{i=1}^m Z_i^j g \partial_i f = f Z_j g + g Z_j f.$$

Therefore

$$Z_j Z_j(fg) = Z_j(f Z_j g + g Z_j f) = f Z_j Z_j g + g Z_j Z_j f + 2 Z_j f Z_j g,$$

that is, recalling (4.1),

$$(4.5) \quad \Delta_Z(fg) = f \Delta_Z g + g \Delta_Z f + 2 \sum_{j=1}^m Z_j f Z_j g.$$

Moreover, (4.4) implies that

$$(4.6) \quad Z_0(fg) = f Z_0 g + g Z_0 f.$$

In view of (4.3), (4.4), (4.5) and (4.6), and recalling (4'), we define

$$(4.7) \quad \Gamma(f, g) := \sum_{j=1}^m Z_j f Z_j g.$$

Notice that

$$\Gamma(f) = \sum_{j=1}^m (Z_j f)^2 = |Zf|^2.$$

We can now consider the diffusion Markov triple  $(\mathbb{R}^n, \mu, \Gamma)$ . We observe that, with this choice, properties (1),(2) and (3) are satisfied.

We now prove that condition (4) is also satisfied. For this, using an integration by parts, we point out that, for any  $f \in C^\infty(\mathbb{R}^n)$  and any  $\varphi \in C_c^\infty(\mathbb{R}^n)$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} Z_j f \varphi \, d\mu &= \sum_{i=1}^n \int_{\mathbb{R}^n} Z_i^j \partial_i f \varphi e^n \, dx = - \sum_{i=1}^n \int_{\mathbb{R}^n} f \partial_i (Z_i^j \varphi e^n) \, dx \\ &= - \sum_{i=1}^n \int_{\mathbb{R}^n} f (\partial_i Z_i^j \varphi e^n + Z_i^j \partial_i \varphi e^n + Z_i^j \varphi \partial_i \eta e^n) \, dx \\ &= - \int_{\mathbb{R}^n} f (\varphi \operatorname{div} Z_j + Z_j \varphi + Z_j \eta \varphi) \, d\mu. \end{aligned}$$

Taking  $\varphi := Z_j g$ , we thereby conclude that, for every  $f \in C^\infty(\mathbb{R}^n)$  and  $g \in C_c^\infty(\mathbb{R}^n)$  (and, more generally, for every  $f \in \operatorname{Lip}(\mathbb{R}^n)$  and  $g \in \operatorname{Lip}_c(\mathbb{R}^n)$  by a density argument), it holds that

$$\begin{aligned} \int_{\mathbb{R}^n} \Gamma(f, g) \, d\mu &= \sum_{j=1}^m \int_{\mathbb{R}^n} Z_j f Z_j g \, d\mu \\ &= - \int_{\mathbb{R}^n} f \left( \sum_{j=1}^m Z_j g \operatorname{div} Z_j + \Delta_Z g + Z_0 g \right) \, d\mu \\ &= - \int_{\mathbb{R}^n} f Lg \, d\mu, \end{aligned}$$

which is (4) in this setting.

We now prove condition (5). We denote by  $d(x) = d(x, 0)$ , and we consider a function  $\Phi \in C^\infty(\mathbb{R}, [0, 1])$ , with  $|\Phi'(t)| \leq 1$  for any  $|t| \in [1/8, 1/4]$ , and

$$\Phi(t) := \begin{cases} 1 & \text{if } |t| \leq 1/8, \\ 0 & \text{if } |t| \geq 1/4. \end{cases}$$

For every  $k \in \mathbb{N}$ , we define  $\xi_k(x) := \Phi(d(x)^2/k^2) \in \operatorname{Lip}_c(\mathbb{R}^n)$ . Then  $(\xi_k)_{k \in \mathbb{N}}$  is an increasing sequence with  $\xi_k(x) \in [0, 1]$  and  $\xi_k(x) \rightarrow 1$  as  $k \rightarrow +\infty$  for every  $x \in \mathbb{R}^n$ . Moreover, we observe that (4.2) in Theorem 4.1 is satisfied taking  $f := d$  with  $\Lambda = 1$ , and therefore we have that  $|Zd| \leq 1$ . As a consequence

$$|\Gamma(\xi_k)| = |Z\xi_k| = \frac{2|\Phi'(d^2/k^2)|}{k^2} d|Zd| \leq \frac{1}{k},$$

which completes the proof of (5).

Notice that also (1'), (2'), (3'), (4'), (5') and (6') easily follow from the very definition of  $\mathcal{A}$ ,  $L$  and  $\Gamma$ .

We claim that also (7') holds. Indeed, if  $f \in \mathcal{A}$  with  $\Gamma(f) = 0$ , then, by definition,

$$\sum_{j=1}^m (Z_j f)^2 = 0,$$

and so we have that  $Z_j f = 0$  for any  $j = 1, \dots, m$ . Thus, also all iterated derivatives vanish. Then, the conclusion follows, since, by the Hörmander condition, every  $\partial_{x_i} f$  can

be written as a linear combination of iterated derivatives. Therefore,  $f$  is a constant function, and so condition (7') is satisfied.

We now describe some interesting applications of the setting introduced above.

**4.1. Riemannian Manifolds.** Let  $(M, g)$  be a connected Riemannian manifold of dimension  $n$  equipped with the standard Levi-Civita connection  $\nabla$  and let  $G \in C^2(M)$ . As customary in Riemannian geometry, we define the gradient vector of  $f \in C^\infty(M)$  as the vector field whose coordinates are

$$\nabla^i f = g^{ij} \partial_{x_j} f, \quad \text{for any } 1 \leq i \leq n,$$

where  $g^{ij}$  are the coefficients of the inverse matrix  $(g_{ij})_{1 \leq i, j \leq n}$ , and the repeated indices notation has been used. We consider the Markov triple  $(M, \mu, \Gamma)$ , where

$$\Gamma(f, g) := \nabla^i f \partial_{x_i} g.$$

and

$$\mu := e^{-G} dV.$$

Here  $dV$  denotes the Riemannian volume element, namely, in local coordinates,

$$dV = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n,$$

where  $\{dx^1, \dots, dx^n\}$  is the basis of 1-forms dual to the vector basis  $\{\partial_1, \dots, \partial_n\}$ . The Laplace-Beltrami operator  $\Delta_g$  is defined on  $f \in C^\infty(M)$  as

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_{x_i} \left( \sqrt{|g|} g^{ij} \partial_{x_j} f \right),$$

whereas the Hessian matrix  $\nabla^2 f$  of a smooth function  $f$  is defined as the symmetric 2-tensor given in a local patch by

$$(\nabla^2 f)_{ij} = \partial_{ij}^2 f - \Gamma_{ij}^k \partial_k f,$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols, namely

$$\Gamma_{ij}^k = \frac{1}{2} g^{hk} (\partial_i g_{hj} + \partial_j g_{ih} - \partial_h g_{ij}).$$

Given a tensor  $A$ , we also define its norm by  $|A| = \sqrt{AA^*}$ , being  $A^*$  the adjoint of  $A$ .

As proved in [2, 1.11.10] and [2, 1.16.4], we see that

$$(4.8) \quad Lf = \Delta_g f - \langle \nabla G, \nabla f \rangle_g$$

and

$$(4.9) \quad \Gamma_2(f) = |\nabla^2 f| + \text{Ric}(L)(\nabla f, \nabla f),$$

where  $\text{Ric}(L)$  is a symmetric tensor defined from the Ricci tensor  $\text{Ric}_g$  by

$$\text{Ric}(L) = \text{Ric}_g + \nabla^2 G.$$

Observing that

$$\Gamma \left( \Gamma(f)^{\frac{1}{2}} \right) = |\nabla |\nabla f||^2,$$



we use (4.9) and conclude that

$$(4.10) \quad \Gamma_2(u) - \Gamma\left(\Gamma(u)^{\frac{1}{2}}\right) = |\nabla^2 u| + \text{Ric}(L)(\nabla u, \nabla u) - |\nabla|\nabla u||^2.$$

Consequently, for any stable weak solution  $u \in C^\infty(M)$  to

$$(4.11) \quad Lu + F(u) = 0 \quad \text{in } M,$$

the Poincaré inequality in (2.19) of Theorem 2.7 reads as follows:

$$(4.12) \quad \int_M \left( |\nabla^2 u| + \text{Ric}(L)(\nabla u, \nabla u) - |\nabla|\nabla u||^2 \right) \varphi^2 d\mu \leq \int_M |\nabla u|^2 |\nabla \varphi|^2 d\mu,$$

for any  $\varphi \in C_c^\infty(M)$ .

In particular, on  $\mathbb{R}^n$  equipped with the flat Euclidean metric, and for the usual Laplacian  $\Delta$ , inequality (4.12) reads as

$$\int_{\mathbb{R}^n} \left( |\nabla^2 u|^2 + \nabla^2 G(\nabla u, \nabla u) - |\nabla|\nabla u||^2 \right) \varphi^2 d\mu \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 d\mu,$$

for any  $\varphi \in C_c^\infty(\mathbb{R}^n)$ , which is precisely the inequality already proved in [20, 11, 10, 22].

Furthermore, if  $G := 0$  then (4.12) was proved in [23, 24, 21], whereas the general case seems to be new in the literature.

In our setting, we take  $\mathcal{A}_0 := \text{Lip}_c(M)$  and  $\mathcal{A} := \text{Lip}(M)$ , where  $f \in \text{Lip}(M)$  if

$$(4.13) \quad \sup_{x \neq y \in M} \frac{f(x) - f(y)}{d(x, y)} < +\infty,$$

being  $d$  the distance defined in (2.13). Notice that, if we fix  $x_0 \in M$  and we set  $\rho(x) := d(x, x_0)$ , it holds that

$$(4.14) \quad \rho(x) - \rho(y) \leq d(x, y).$$

Indeed, by (2.13), for any  $\epsilon > 0$  there exists  $f_\epsilon$  with  $\Gamma(f_\epsilon) \leq 1$  such that  $\rho(x) \leq \epsilon + f_\epsilon(x) - f_\epsilon(x_0)$ , and therefore

$$\rho(x) - \rho(y) \leq \epsilon + f_\epsilon(x) - f_\epsilon(x_0) - (f_\epsilon(y) - f_\epsilon(x_0)) = \epsilon + f_\epsilon(x) - f_\epsilon(y) \leq \epsilon + d(x, y).$$

From this, taking  $\epsilon$  as small as we wish, we obtain (4.14).

Then, comparing (4.13) and (4.14), we see that  $\rho \in \text{Lip}(M) = \mathcal{A}$ .

We also assume that

$$\lambda \delta^{ij} \leq g^{ij} \leq \frac{\delta^{ij}}{\lambda},$$

for some  $\lambda \in (0, 1]$ . In this way, if  $|\cdot|_E$  is the Euclidean norm of a vector, it holds that

$$\lambda |v|_E^2 = \lambda \delta^{ij} v_i v_j \leq g^{ij} v_i v_j.$$

Then, by (2.13), for any  $\epsilon > 0$  there exists  $\tilde{f}_\epsilon$  with  $\Gamma(\tilde{f}_\epsilon) \leq 1$  such that

$$\begin{aligned}
d(x, y) &\leq \epsilon + \tilde{f}_\epsilon(x) - \tilde{f}_\epsilon(y) \\
&= \epsilon + \int_0^1 \partial_{x_i} \tilde{f}_\epsilon(x + ty) \cdot (x_i - y_i) dt \\
&\leq \epsilon + \frac{1}{\lambda} \int_0^1 \sqrt{g^{ij} \partial_{x_i} \tilde{f}_\epsilon(x + ty) \partial_{x_j} \tilde{f}_\epsilon(x + ty)} |x - y|_E dt \\
&= \epsilon + \frac{1}{\lambda} \int_0^1 \sqrt{\Gamma(\tilde{f}_\epsilon)(x + ty)} |x - y|_E dt \\
&= \epsilon + \frac{|x - y|_E}{\lambda}.
\end{aligned}$$

Hence, taking  $\epsilon$  arbitrary small, it follows that  $d(x, y) \leq \frac{|x-y|_E}{\lambda}$ . Recalling (4.14), we thereby conclude that  $|\nabla\rho|$ , and therefore  $\Gamma(\rho)$ , is bounded by a universal constant depending on  $\lambda$ . Consequently, the assumptions of Theorem 2.6 are satisfied. Then, using Theorem 2.6 and (4.10), we obtain that, in this setting, if the curvature dimension condition  $CD(K, \infty)$  holds true for some  $K \geq 0$ , and  $u$  is a stable solution with  $\int_M |\nabla u|^2 d\mu < +\infty$ , then:

$$(4.15) \quad K > 0 \implies u \text{ is constant in } M;$$

$$(4.16) \quad K = 0 \implies |\nabla^2 u| + \text{Ric}(L)(\nabla u, \nabla u) - |\nabla|\nabla u||^2 = 0 \quad \mu - \text{a.e. in } M.$$

To grasp a geometric flavor of (4.16), one can fix a point  $p \in M$  with  $\nabla u(p) \neq 0$  and consider normal coordinates at  $p$  for which

$$(4.17) \quad g^{ij}(p) = \delta^{ij}, \quad \partial_{x_k} g^{ij}(p) = 0 \quad \text{and} \quad \Gamma_{ij}^k(p) = 0,$$

see e.g. page 55 in [37]. Then, the level set  $S$  of  $u$  passing through  $p$  is locally a submanifold of  $M$  of codimension 1, endowed with a Riemannian structure induced by that of  $M$  (namely if  $v, w \in T_p S \subseteq T_p M$  one can consider  $g(v, w)$  as defining a metric on  $S$ ). Consequently, in view of (4.17), we can reduce the Riemannian term  $|\nabla^2 u| - |\nabla|\nabla u||^2$  to its Euclidean counterpart, which, due to the classical Sternberg-Zumbrun identity (see formula (2.1) of [48]) is larger than  $K^2 |\nabla u|^2$ , being  $K^2$  the sum of the square of the eigenvalues of the second fundamental form of  $S$ , according to the induced Riemannian structure, see Proposition 18 in [21]. Therefore, by (4.16), if the Ricci tensor is nonnegative, it follows that the second fundamental form of  $S$  at  $p$  vanishes, and the Ricci tensor must vanish at  $p$  as well.

Submanifolds with vanishing second fundamental form are called totally geodesic (see e.g. page 104 in [39] or Proposition 1.2 in [43]) and are characterized by the property that any geodesic on the submanifold is also a geodesic on the ambient manifold.

**4.2. Carnot groups.** We recall that a Carnot group  $\mathbb{G}$  is a connected Lie group whose Lie algebra  $\mathcal{G}$  is finite dimensional and stratified of step  $s \in \mathbb{N}$ . Precisely, there exist linear subspaces  $V_1, \dots, V_s$  of  $\mathcal{G}$  such that

$$\mathcal{G} = V_1 \oplus \dots \oplus V_s$$

with

$$[V_1, V_{i-1}] = V_i \quad \text{if } 2 \leq i \leq s \quad \text{and} \quad [V_1, V_s] = \{0\}.$$

Here  $[V_1, V_i] := \text{span}\{[a, b] : a \in V_1, b \in V_i\}$ . Since  $\mathcal{G}$  is stratified, then every element of  $\mathcal{G}$  is the linear combination of commutators of elements of  $V_1$ . We refer to [7] for a complete introduction to the subject.

Let  $\dim(V_1) = m$  and  $Z = (Z_1, \dots, Z_m)$  be a basis of  $V_1$ . The family  $Z$  satisfies the Hörmander condition. Moreover, in this setting,

$$(4.18) \quad \Gamma(f, g) = \sum_{i=1}^m Z_i f Z_i g \quad \text{and} \quad Lf = \Delta_Z f.$$

In order to compute  $\Gamma_2$  we will use the following Bochner-type formula proved in [33, Proposition 3.3] coupled with [33, Lemma 3.1]:

**Theorem 4.2.** *Let  $u$  be a smooth function. Then,*

$$\begin{aligned} & \frac{1}{2} \Delta_Z |Zu|^2 \\ &= \|Z^2 u\|^2 + \sum_{j=1}^m Z_j u Z_j (\Delta_Z u) + 2 \sum_{i,j=1}^m Z_j u [Z_i, Z_j] Z_i u + \sum_{i,j=1}^m Z_j u [Z_i, [Z_i, Z_j]] u, \end{aligned}$$

where  $Z^2 u$  denotes the horizontal Hessian matrix associated to the family  $Z$ , namely the  $m \times m$  matrix whose elements are given by  $u_{ij} := Z_i Z_j u$ , with  $i, j = 1, \dots, m$ .

Now, let  $u \in C^\infty(\mathbb{G})$  be a stable solution to

$$(4.19) \quad \Delta_Z u + F(u) = 0 \quad \text{in } \mathbb{G}.$$

Therefore, recalling (4.18) and using Theorem 4.2,

$$\begin{aligned} \Gamma_2(u) &= \frac{1}{2} L\Gamma(u) - \Gamma(u, Lu) = \frac{1}{2} \Delta_Z |Zu|^2 - \Gamma(u, \Delta_Z u) \\ &= \|Z^2 u\|^2 + \sum_{j=1}^m Z_j u Z_j (\Delta_Z u) + 2 \sum_{i,j=1}^m Z_j u [Z_i, Z_j] Z_i u + \sum_{i,j=1}^m Z_j u [Z_i, [Z_i, Z_j]] u - \Gamma(u, \Delta_Z u) \\ &= \|Z^2 u\|^2 + 2 \sum_{i,j=1}^m Z_j u [Z_i, Z_j] Z_i u + \sum_{i,j=1}^m Z_j u [Z_i, [Z_i, Z_j]] u \\ &= \|Z^2 u\|^2 + \mathcal{R}(u), \end{aligned}$$

where

$$\mathcal{R}(u) := 2 \sum_{i,j=1}^m Z_j u [Z_i, Z_j] Z_i u + \sum_{i,j=1}^m Z_j u [Z_i, [Z_i, Z_j]] u.$$

Therefore, if  $u \in C^\infty(\mathbb{G})$  is a stable solution to (4.19), inequality (2.19) reads as

$$(4.20) \quad \int_{\mathbb{G}} (\|Z^2 u\|^2 - |Z|Zu|^2 + \mathcal{R}(u)) \varphi^2 dx \leq \int_{\mathbb{G}} |Zu|^2 |Z\varphi|^2 dx,$$

for any  $\varphi \in C_c^\infty(\mathbb{G})$ .

Formula (4.20) generalizes to general Carnot groups the Poincaré inequality obtained in [30] in the Heisenberg group and in [42] in the Engel group (we refer the reader to [7]

for the definitions, and we remark that the divergence of the Heisenberg and Engel vector fields vanish in the setting of (4.1)). In particular, in the case of the Heisenberg group, formula (4.20) here reduces to formula (7) in [30], and in the case of the Engel group, formula (4.20) here reduces to the formula in Proposition 3.7 of [42] and  $\mathcal{R}$  here coincides with  $\mathcal{J}$  in Theorem 1.1 of [42]. In its full generality, our formula (4.20) seems to be new in the literature.

In addition, the distance in (2.13) coincides with that of Carnot-Carathéodory in this setting, see [2], and so (2.14) holds true in this case. For completeness, we state (4.20) and we apply Theorem 2.6 to obtain this original result:

**Theorem 4.3.** *Let  $\mathbb{G}$  be a Carnot group whose Lie algebra  $\mathcal{G} = V_1 \oplus \dots \oplus V_s$  is stratified of step  $s$ , with  $V_1$  generated by the basis of vector fields  $(Z_1, \dots, Z_m)$  that satisfy the Hörmander condition. Let  $u \in C^\infty(\mathbb{G})$  be a stable weak solution to  $\Delta_Z u + F(u) = 0$  in  $\mathbb{G}$ . Then,*

$$\int_{\mathbb{G}} (\|Z^2 u\|^2 - |Z|Zu|^2 + \mathcal{R}(u)) \varphi^2 dx \leq \int_{\mathbb{G}} |Zu|^2 |Z\varphi|^2 dx,$$

for any  $\varphi \in C_c^\infty(\mathbb{G})$ .

Assume also that

$$\begin{aligned} \mathcal{R}(u) &\geq 0 \\ \text{and} \quad \int_{\mathbb{G}} |Zu|^2 dx &< \infty. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{R}(u) &= 0 \\ \text{and} \quad \|Z^2 u\|^2 &= |Z|Zu|^2 \text{ a.e. in } \mathbb{G}. \end{aligned}$$

Now, we prove that, for a particular family of Carnot groups, formula (4.20) provides a geometric inequality for every stable solution to (4.19). Model filiform groups are the Carnot groups with the simplest Lie brackets possible while still having arbitrarily large step, see [38]. They have previously been investigated in connection with non-rigidity of Carnot groups [40], quasiconformal mappings between Carnot groups [50], geometric control theory [38] and geometric measure theory [41].

The formal definition is as follows:

**Definition 4.4.** *Let  $n \geq 2$ . The model filiform group of step  $n-1$  is the Carnot group  $\mathbb{E}_n$  whose Lie algebra  $\mathcal{E}_n$  admits a basis  $Z_1, \dots, Z_n$  satisfying  $[Z_i, Z_1] = Z_{i+1}$  for  $1 < i < n$ , with all other Lie brackets among the  $Z_i$  equal to zero.*

The stratification of  $\mathcal{E}_n$  is

$$\mathcal{E}_n = V_1 \oplus \dots \oplus V_{n-1}$$

with  $V_1 = \text{Span}\{Z_1, Z_2\}$  and  $V_i = \text{Span}\{Z_{i-1}\}$  for  $1 < i < n$ .

Proceeding exactly as in [30, formula (19)], we get

$$|Z|Zu|^2 = \frac{1}{|Zu|^2} \langle H_Z u Zu, Zu \rangle \quad \text{in } \{Zu \neq 0\},$$

where

$$H_Z u := Z^2 u (Z^2 u)^T.$$

Whenever  $P \in \{u = k\} \cap \{Zu \neq 0\}$ , we can consider the smooth surface  $\{u = k\}$  and define the intrinsic normal to  $\{u = k\}$  and the intrinsic unit tangent direction to  $\{u = k\}$  as

$$\nu := \frac{Zu(P)}{|Zu(P)|} \quad \text{and} \quad v := \frac{(Z_2u(P), -Z_1u(P))}{|Zu(P)|},$$

respectively. We observe that [30, Lemma 2.1] only depends on the fact that  $\dim V_1 = 2$  and  $\dim V_2 = 1$ , which still hold in every model filiform Carnot group. Therefore, the following result holds:

**Lemma 4.5.** *On  $\{u = k\} \cap \{Zu \neq 0\}$ , it holds that*

$$(4.21) \quad \|Z^2u\|^2 - \langle (H_Zu)\nu, \nu \rangle = |Zu|^2 \left[ h^2 + \left( p + \frac{\langle (Hu)v, \nu \rangle}{|Zu|} \right)^2 \right]$$

where

$$h = \operatorname{div}_Z \nu = Z_1\nu_1 + Z_2\nu_2$$

and

$$p = -\frac{Z_3u}{|\nabla_Z u|}.$$

Plugging (4.21) into (4.20) we get the following geometric Poincaré inequality:

$$\int_{\{Zu \neq 0\}} \left( |Zu|^2 \left[ h^2 + \left( p + \frac{\langle (Hu)v, \nu \rangle}{|Zu|} \right)^2 \right] + \mathcal{R}(u) \right) \varphi^2 dx \leq \int_{\mathbb{E}_n} |Zu|^2 |Z\varphi|^2 dx,$$

for any  $\varphi \in C_c^\infty(\mathbb{E}_n)$ .

We summarize this statement and that of Theorem 2.6 in the following original result:

**Theorem 4.6.** *Let  $\mathbb{E}_n$  be a model filiform group of step  $n - 1$ , as in Definition (4.4). Let  $u \in C^\infty(\mathbb{E}_n)$  be a stable weak solution to  $\Delta_Z u + F(u) = 0$  in  $\mathbb{E}_n$ . Then,*

$$\int_{\{Zu \neq 0\}} \left( |Zu|^2 \left[ h^2 + \left( p + \frac{\langle (Hu)v, \nu \rangle}{|Zu|} \right)^2 \right] + \mathcal{R}(u) \right) \varphi^2 dx \leq \int_{\mathbb{E}_n} |Zu|^2 |Z\varphi|^2 dx,$$

for any  $\varphi \in C_c^\infty(\mathbb{E}_n)$ .

Assume also that

$$\begin{aligned} \mathcal{R}(u) &\geq 0 \\ \text{and} \quad \int_{\mathbb{E}_n} |Zu|^2 dx &< \infty. \end{aligned}$$

Then:

$$(4.22) \quad \left. \begin{aligned} \mathcal{R}(u) &= 0 \quad \text{and} \\ h &= 0 \\ p + \frac{\langle (Hu)v, \nu \rangle}{|Zu|} &= 0 \end{aligned} \right\} \quad \text{a.e. in } \mathbb{E}_n \cap \{Zu \neq 0\}.$$

The two equations in (4.22) on  $\{Zu \neq 0\}$  can be seen as “intrinsic geodesic equations” on the noncritical level sets of the solution  $u$ .

**4.3. Grushin plane.** For a given  $\alpha \in \mathbb{N}$ , the vector fields  $Z_1 = \partial_x$  and  $Z_2 = |x|^\alpha \partial_y$  satisfy the Hörmander condition in  $\mathbb{R}^2$ . We call Grushin plane the metric space  $\mathbb{G}_\alpha = (\mathbb{R}^2, d)$ , where  $d$  is the Carnot-Carathéodory distance induced by  $Z_1$  and  $Z_2$ . Background on the Grushin plane may be found in [5, 38].

Since  $\operatorname{div} Z_1 = \operatorname{div} Z_2 = 0$  and  $Z_0 = 0$ , then, from what we proved for vector fields satisfying the Hörmander condition, we get

$$Lu = \Delta_Z u, \quad \Gamma(f, g) = Z_1 f Z_1 g + Z_2 f Z_2 g \quad \text{and} \quad d\mu = dx.$$

For every solution  $u \in C^\infty(\mathbb{G}_\alpha)$  to  $\Delta_Z u + F(u) = 0$ , we see that, for any  $i \in \{1, 2\}$ ,

$$(4.23) \quad Z_i \Delta_Z u = -F'(u) Z_i u.$$

Let also  $Z_3 := [Z_1, Z_2]$ . We observe that, by [32, Lemma 2.1] and (4.23), we have

$$(4.24) \quad \Delta_Z Z_1 u = Z_1 \Delta_Z u - 2Z_3 Z_2 u = -F'(u) Z_1 u - 2Z_3 Z_2 u$$

$$(4.25) \quad \text{and} \quad \Delta_Z Z_2 u = Z_2 \Delta_Z u + 2Z_3 Z_1 u = -F'(u) Z_2 u + 2Z_3 Z_1 u.$$

As a further consequence of (4.23), we obtain that

$$(4.26) \quad \begin{aligned} \Gamma(u, \Delta_Z u) &= Z_1 u Z_1 \Delta_Z u + Z_2 u Z_2 \Delta_Z u \\ &= -F'(u) (|Z_1 u|^2 + |Z_2 u|^2) = -F'(u) |Zu|^2. \end{aligned}$$

Moreover, direct calculations give

$$\begin{aligned} Z_1 Z_1 (Z_1 u)^2 &= 2(Z_1 Z_1 u)^2 + 2Z_1 u Z_1 Z_1 Z_1 u \\ &= 2(Z_1 Z_1 u)^2 + 2Z_1 u \Delta_Z Z_1 u - 2Z_1 u Z_2 Z_2 Z_1 u, \\ Z_2 Z_2 (Z_1 u)^2 &= 2(Z_2 Z_1 u)^2 + 2Z_1 u Z_2 Z_2 Z_1 u \\ Z_1 Z_1 (Z_2 u)^2 &= 2(Z_1 Z_2 u)^2 + 2Z_2 u Z_1 Z_1 Z_2 u, \\ \text{and} \quad Z_2 Z_2 (Z_2 u)^2 &= 2(Z_2 Z_2 u)^2 + 2Z_2 u Z_2 Z_2 Z_2 u \\ &= 2(Z_2 Z_2 u)^2 + 2Z_2 u \Delta_Z Z_2 u - 2Z_2 u Z_1 Z_1 Z_2 u. \end{aligned}$$

Summing up these equalities and recalling (4.24) and (4.25), we get

$$\begin{aligned} \Delta_Z |Zu|^2 &= 2\|Z^2 u\|^2 + 2Z_1 u \Delta_Z Z_1 u + 2Z_2 u \Delta_Z Z_2 u \\ &= 2\|Z^2 u\|^2 + 2Z_1 u (-F'(u) Z_1 u - 2Z_3 Z_2 u) + 2Z_2 u (-F'(u) Z_2 u + 2Z_3 Z_1 u) \\ &= 2\|Z^2 u\|^2 - 2F'(u) |Zu|^2 - 4Z_1 u Z_3 Z_2 u + 4Z_2 u Z_3 Z_1 u. \end{aligned}$$

Using this and (4.26), we are now in position to compute  $\Gamma_2(u)$  as follows:

$$\begin{aligned} \Gamma_2(u) &= \frac{1}{2} L\Gamma(u) - \Gamma(u, Lu) = \frac{1}{2} \Delta_Z |Zu|^2 - \Gamma(u, \Delta_Z u) \\ &= \|Z^2 u\|^2 - 2Z_1 u Z_3 Z_2 u + 2Z_2 u Z_3 Z_1 u. \end{aligned}$$

Therefore, if  $u \in C^\infty(\mathbb{G}_\alpha)$  is a stable solution to  $\Delta_Z u + F(u)$ , inequality (2.19) reads as

$$\int_{\mathbb{G}_\alpha} \left( \|Z^2 u\|^2 - |Z|Zu|^2 - 2Z_1 u Z_3 Z_2 u + 2Z_2 u Z_3 Z_1 u \right) \varphi^2 dx \leq \int_{\mathbb{G}_\alpha} |Zu|^2 |Z\varphi|^2 dx,$$

for any  $\varphi \in C_c^\infty(\mathbb{G}_\alpha)$ , which coincides with formula (1.10) in Theorem 1.1 of [32].

We conclude pointing out that, since the proof of Theorem 4.6 is based on the commutator relations of the vectors  $Z_1$  and  $Z_2$ , the same result also holds in the Grushin plane (compare with (1.15) and (1.16) of [32]).

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