Research Article

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Curves on Segre threefolds

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Abstract: We study locally Cohen–Macaulay curves of low degree in the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and investigate the irreducible and connected components, respectively, of the Hilbert scheme of them. We also apply the similar argument to the Segre threefold $\mathbb{P}^2 \times \mathbb{P}^1$.

Keywords: Locally Cohen–Macaulay curve, Hilbert scheme, pure sheaf, moduli space

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1 Introduction

In this paper we study curves in Segre threefolds over the field of complex numbers $\mathbb{C}$. There are three types of Segre threefolds: $\mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In $\mathbb{P}^3$ the structure of the Hilbert scheme of curves has been densely studied by many authors in the last half century. In [11] the connectedness of the Hilbert scheme of curves is proven for the fixed degree and genus of curves, although it is classically known that the locus of smooth curves may not be connected. Recently there have been increased interests on the connectedness of the Hilbert scheme of locally Cohen–Macaulay curves. Up to now, the connectedness has been established only for very small degree [22] or for very large genus [15]. We recommend to see [14] for further results and the state of the art on this problem.

Our main concern is on the connectedness of Hilbert schemes of locally Cohen–Macaulay curves in $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with very small degree. Smooth curves are often the first to be studied and by the Hartshorne–Serre correspondence, globally generated vector bundles on X can have very close relation with smooth curves in X. There is a classification of globally generated vector bundles on X with low first Chern class accomplished by the classification of smooth curves in X with very small degree [3]. One of the advantages in the study of curves in X is that some irreducible components that might appear in the Hilbert scheme in $\mathbb{P}^3$ may disappear in X, so that we can get simpler description of Hilbert schemes; for example, the Hilbert scheme of curves in $\mathbb{P}^3$ of degree three and genus zero has two irreducible components, one with twisted cubics and the other with planar cubics plus extra point, see [24]. The latter case cannot occur in X because X is scheme-theoretically cut out by quadrics.

Our main result is as follows.

Theorem 1.1. *Let* $\mathbf{H}(e_1, e_2, e_3, \chi)_{+,\mathrm{red}}$ *be the reduced Hilbert scheme of locally Cohen–Macaulay curves* C *in* X *with tridegree* (e_1, e_2, e_3) *and* $\chi(\mathcal{O}_C) = \chi$.

(i) $\mathbf{H}(2, 0, 0, a)_{+,\mathrm{red}}$ *is irreducible and rational for* $a \geq 2$.

(ii) $\mathbf{H}(2, 1, 0, a)_{+,\mathrm{red}}$ *has the two irreducible components for* $a \geq 3$.

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(iii) $\mathbf{H}(1,1,1,a)_{+,\mathrm{red}}$ *is irreducible and rational for* $a \in \{1,3\}$, *while* $\mathbf{H}(1,1,1,2)_{+,\mathrm{red}}$ *has the three connected components that are rational.*

(iv) $\mathbf{H}(2,1,1,1)_{+,\mathrm{red}}$ *is irreducible and rational.*

The main ingredient in the study of Hilbert schemes of locally Cohen–Macaulay curves with low degree is a rational ribbon and the Ferrand construction, i.e., a double structure on $\mathbb{P}^1$, see the beginning of Section 3 for a brief summary of its usage in this article. Rational ribbons and their canonical embeddings were studied in [4] and we adopt their results to prove the irreducibility of the Hilbert schemes of double lines in X. Then we investigate the intersecting property of the double lines with other lines in X to investigate irreducible and connected components of the Hilbert schemes, respectively. We recommend to see [19, 20] for studies on the families of double lines in projective spaces. It should be noted that the irreducibility of the space of curves with fixed cohomology in $\mathbb{P}^3$ is investigated in [5]. And the description of the other type of Hilbert scheme is studied in [1].

Let us summarize here the structure of this paper. In Section 2, we introduce the definitions and main properties that will be used throughout the paper, mainly those related to Segre threefold, Hilbert tripolynomial and Hilbert schemes of locally Cohen–Macaulay curves. In Section 3, we pay attention to the Hilbert schemes of curves with tridegree $(2,0,0)$ and conclude their irreducibility using the double structure on $\mathbb{P}^1$. We end the section with the description of the intersection of the double lines with other lines in X, which will be used later on. In Section 4, we move forward to the Hilbert schemes of curves with tridegree $(2,1,0)$, $(1,1,1)$ and $(2,1,1)$, and describe their irreducible and connected components, respectively. In the proof of irreducibility of $\mathbf{H}(2,1,1,1)_{+,\mathrm{red}}$, we use the moduli of stable maps. Finally, in Section 5, we apply our arguments to the case of Segre threefold $\mathbb{P}^2 \times \mathbb{P}^1$.

2 Preliminaries

For three 2-dimensional vector spaces V_1, V_2, V_3 over the field of complex numbers $\mathbb{C}$, let $X \cong \mathbb{P}(V_1)\times\mathbb{P}(V_2)\times\mathbb{P}(V_3)$, which is then embedded into $\mathbb{P}^7 \cong \mathbb{P}(V_0)$ by the Segre map, where $V_0 = V_1 \otimes V_2 \otimes V_3$. It is known that X is the only Del Pezzo with the maximal Picard number $\varrho(X) = 3$. The intersection ring $A(X)$ is isomorphic to $A(\mathbb{P}^1) \otimes A(\mathbb{P}^1) \otimes A(\mathbb{P}^1)$ and so we have

$$A(X) \cong \mathbb{Z}[t_1, t_2, t_3]/(t_1^2, t_2^2, t_3^2).$$

We may identify $A^1(X) \cong \mathbb{Z}^{\oplus 3}$ by $a_1t_1 + a_2t_2 + a_3t_3 \mapsto (a_1, a_2, a_3)$. Similarly, we have $A^2(X) \cong \mathbb{Z}^{\oplus 3}$ by $e_1t_2t_3 + e_2t_3t_1 + e_3t_1t_2 \mapsto (e_1, e_2, e_3)$ and $A^3(X) \cong \mathbb{Z}$ by $ct_1t_2t_3 \mapsto c$.

Let us denote the natural projection of X to the ith factor by $\pi_i\colon X \to \mathbb{P}^1$, and we denote $\pi_1^*\mathcal{O}_{\mathbb{P}^1}(a_1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a_2) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^1}(a_3)$ by $\mathcal{O}_X(a_1, a_2, a_3)$. Then X is embedded into $\mathbb{P}^7$ by the complete linear system $|\mathcal{O}_X(1,1,1)|$ as a subvariety of degree 6, since $(1,1,1)^3 = 6$. We also denote $\mathcal{E}\otimes\mathcal{O}_X(a_1, a_2, a_3)$ by $\mathcal{E}(a_1, a_2, a_3)$ for a coherent sheaf $\mathcal{E}$ on X. We also let $\pi_{ij}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the projection to the (i,j)-factor, i.e., $\pi_{ij}(o_1, o_2, o_3) = (o_i, o_j)$ for $(o_1, o_2, o_3) \in X$.

Proposition 2.1. *For a one-dimensional sheaf* $\mathcal{F}$ *on* X, *there exists a tripolynomial* $\chi_{\mathcal{F}}(x,y,z) \in \mathbb{Q}[x,y,z]$ *of degree one such that*

$$\chi(\mathcal{F}(u,v,w)) = \chi_{\mathcal{F}}(u,v,w)$$

for all $(u,v,w) \in \mathbb{Z}^{\oplus 3}$.

Proof. This follows verbatim from the proof of [2, Proposition 2]. Let $at + b \in \mathbb{Q}[t]$ be the Hilbert polynomial of $\mathcal{F}$ with respect to $\mathcal{O}_X(1,1,1)$. Take a divisor $D_1 \in |\mathcal{O}_X(1,0,0)|$ such that D_1 misses any zero-dimensional components (embedded or isolated) of $\mathrm{Supp}(\mathcal{F})$ and does not contain any component of the one-dimensional reduced scheme associated to $\mathrm{Supp}(\mathcal{F})$. Then this gives us an injective map $j_{D_1}\colon \mathcal{F}(t,t,t) \to \mathcal{F}(t+1,t,t)$ and so we have an exact sequence

$$0 \to \mathcal{F}(t,t,t) \to \mathcal{F}(t+1,t,t) \to \mathcal{F}(t+1,t,t)\otimes\mathcal{O}_{D_1} \to 0. \tag{2.1}$$

Similarly, fix other divisors $D_2 \in |\mathcal{O}_X(0,1,0)|$, $D_3 \in |\mathcal{O}_X(0,0,1)|$ and $D \in |\mathcal{O}_X(1,1,1)|$ to define maps j_{D_2}, j_{D_3} and j_D with the corresponding exact sequences as in (2.1). Set

$$l := h^0(\mathcal{F}(t+1,t,t)\otimes\mathcal{O}_{D_1}), \quad m := h^0(\mathcal{F}(t,t+1,t)\otimes\mathcal{O}_{D_2}), \quad n := h^0(\mathcal{F}(t,t,t+1)\otimes\mathcal{O}_{D_3}),$$

which are independent on t. We claim that $\chi(\mathcal{F}(u,v,w)) = lu + mv + nw + b$ for all $(u,v,w) \in \mathbb{Z}^{\oplus 3}$. From the exact sequence for D, we have $l+m+n = h^0(\mathcal{F}(t+1,t+1,t+1)\otimes\mathcal{O}_D) = a$ and so the claim is true if $u = v = w$. In general, let us assume $u \geq v \geq w$ without loss of generality. Then we get $\chi(\mathcal{F}(u,v,w)) = \chi(\mathcal{F}(w,w,w)) + l(u-w) + m(v-w)$, using the exact sequences for D_1 and D_2 several times. □

Definition 2.2. We call the linear tripolynomial in Proposition 2.1 the *Hilbert tripolynomial* of $\mathcal{F}$ for a purely one-dimensional sheaf $\mathcal{F}$, i.e., $\chi_{\mathcal{F}}(x,y,z) = e_1x + e_2y + e_3z + \chi$ for some $(e_1,e_2,e_3,\chi) \in \mathbb{Z}^{\oplus 4}$. In particular, the Hilbert polynomial of $\mathcal{F}$ with respect to $\mathcal{O}_X(1,1,1)$ is defined to be $\chi_{\mathcal{F}}(t) = \chi_{\mathcal{F}}(t,t,t)$. We also call $\chi_{\mathcal{O}_C}(x,y,z)$ the Hilbert tripolynomial of a curve C.

Let $\mathbf{H}(e_1,e_2,e_3,\chi)$ be the Hilbert scheme of curves in X with the Hilbert tripolynomial $e_1x + e_2y + e_3z + \chi$, and let $\mathbf{H}(e_1,e_2,e_3,\chi)^{\mathrm{sm}}$ be the open locus corresponding to smooth and connected curves.

Definition 2.3. A *locally Cohen–Macaulay* (for short, locally CM) curve in X is a one-dimensional subscheme $C \subset X$ whose irreducible components are all one-dimensional and that has no embedded points. Equivalently, $\mathcal{O}_C$ is purely one-dimensional.

We denote by $\mathbf{H}(e_1,e_2,e_3,\chi)_+$ the subset of $\mathbf{H}(e_1,e_2,e_3,\chi)$ parametrizing the locally CM curves with no isolated point. In particular, we have $\mathbf{H}(e_1,e_2,e_3,\chi)^{\mathrm{sm}} \subset \mathbf{H}(e_1,e_2,e_3,\chi)_+$.

Remark 2.4. Let C be an integral projective curve. By the universal property of fibered product, there is a bijection between the morphisms $u\colon C \to X$ and the triples (u_1,u_2,u_3), with $u_i\colon C \to \mathbb{P}^1$ any morphism. The image $u(C)$ is contained in a two-dimensional factor of X if and only if one of the u_1,u_2,u_3 is constant. We say that a constant map has degree zero. With this convention we may associate to any u a triple $(\deg(u_1),\deg(u_2),\deg(u_3)) \in \mathbb{Z}^{\oplus 3}_{\geq 0}$, and $u(C)$ is a curve if and only if $(\deg(u_1),\deg(u_2),\deg(u_3)) \neq (0,0,0)$. Now assume that u is birational onto its image. With this assumption for all $(a_1,a_2,a_3) \in \mathbb{Z}^{\oplus 3}$, we have

$$u(C)\cdot\mathcal{O}_X(a_1,a_2,a_3) = a_1\deg(u_1) + a_2\deg(u_2) + a_3\deg(u_3).$$

In particular, the degree of the curve $u(C)$ is $\deg(u_1)+\deg(u_2)+\deg(u_3)$.

Lemma 2.5. *Let $C \subset X$ be a locally CM curve with the tridegree (e_1,e_2,e_3). If the tridegree of C_{red} is (b_1,b_2,b_3), with $b_i = 0$ for some i, then we have $e_i = 0$.*

Proof. In general, if $u_i\colon C \to \mathbb{P}^1$ is the ith-projection, with f_i the length of the generic fibre of u_i, then C has tridegree (f_1,f_2,f_3). Now let us assume $i = 3$, i.e., $b_3 = 0$. The restriction of the projection $\pi_{3|C}\colon C \to \mathbb{P}^1$ has degree e_3. Similarly, $\pi_{3|C_{\mathrm{red}}}\colon C_{\mathrm{red}} \to \mathbb{P}^1$ has degree $b_3 = 0$. Thus, $\pi_{3|C_{\mathrm{red}}}$ has finite image and so does $\pi_{3|C}$. In particular, we have $e_3 = 0$. □

Now we end this section by recalling the two major concepts that are used in this article: the ribbons and the Ferrand construction.

For a reduced connected variety Y', a ribbon on Y' is a scheme Z' equipped with an isomorphism $Y' \to Z'_{\mathrm{red}}$ such that the ideal sheaf $\mathcal{L}$ of Y' in Z' is locally free with the condition $\mathcal{L}^{\otimes 2} = 0$ satisfied. A ribbon is simply a scheme Z' which is a ribbon on Z'_{red}, see [4, Section 1].

On the other hand, let X' be a smooth threefold and $Y' \subset X'$ a smooth and irreducible curve. For a fixed positive integer μ, a multiple structure on Y' with multiplicity μ is a locally Cohen–Macaulay scheme $Z' \subset X'$ such that $Z'_{\mathrm{red}} = Y'$ and Z' has multiplicity μ, i.e., for $\mathcal{O}_{X'}(1)\cdot Z' = \mu(\mathcal{O}_{Z'}(1)\cdot Y')$ for one (or for all) ample line bundle(s) $\mathcal{O}_{X'}(1)$ on X'. Throughout this article we only need the case $\mu = 2$, i.e., double structures on the smooth curve Y' and all these double structures are obtained using the following construction due to Ferrand. More generally, let $Y' \subset X'$ be a locally complete intersection of codimension 2. A surjection $\beta\colon N^{\vee}_{Y'|X'} \to \mathcal{L}$ from the conormal bundle to a line bundle $\mathcal{L}$ on Y' defines a closed subscheme Z' for which there exists an

exact sequence

$$0 \to \mathcal{I}_{Z'}/\mathcal{I}_{Y'}^2 \to N_{Y'|X'}^\vee \to \mathcal{L} \to 0.$$

The fact is that Z' is a locally CM curve with $(Z')_{\mathrm{red}} = Y'$ with multiplicity two. We also have $\mathcal{L} \cong \mathcal{I}_{Y',X'}/\mathcal{I}_{Z',X'}$ so that $\mathcal{L}^2 \cong \mathcal{I}_{Y',X'}^2/\mathcal{I}_{Y',X'}\mathcal{I}_{Z',X'}$. This induces an exact sequence of vector bundles on Y':

$$0 \to \mathcal{L}^{\otimes 2} \to (N_{Z'|X'})_{|Y'}^\vee \to \mathcal{I}_{Z'}/\mathcal{I}_{Y'}^2 \to 0,$$

see [6, Section 1]. This construction is called the Ferrand construction. In particular, it is a ribbon in the sense of [4], because the line bundle $\mathcal{L} \cong \mathcal{I}_{Y',X'}/\mathcal{I}_{Z',X'}$ satisfies $\mathcal{L}^{\otimes 2} \cong 0$.

3 Double lines

Notation 3.1. Throughout this article by a line on $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we mean a CM curve on X with tridegree $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$. A double line is by definition a double structure on a line. For each $a \in \mathbb{Z}$, let $\mathbf{D}_a$ be the subset of $\mathbf{H}(2, 0, 0, a)_+$ parametrizing the double lines.

For the moment we take $\mathbf{D}_a$ as a set. In each case it would be clear which scheme-structure is used on it. Since X is a smooth threefold, [6, Remark 1.3] says that each $[B] \in \mathbf{D}_a$ is obtained by the Ferrand construction, and it is a ribbon in the sense of [4] with a line of tridegree $(1, 0, 0)$. Let C_a be the unique split ribbon with $\chi(\mathcal{O}_{C_a}) = a$, and every ribbon is split for $a \geq 1$ by [4, Theorem 1.2]. Each $f \in \mathrm{Aut}(C_a)$ induces an automorphism $\widetilde{f}$ of $\mathbb{P}^1$ and the map $f \mapsto \widetilde{f}$ is surjective. Thus, we get $\dim \mathrm{Aut}(C_a) \geq 3$. Since C_a is equipped with a specification of a normal direction at each point of $\mathbb{P}^1$, we have $\mathrm{Aut}(C_a) \cong \mathrm{Aut}(\mathbb{P}^1)$. In particular, $\dim \mathrm{Aut}(C_a) = 3$.

Remark 3.2. Fix $[A] \in \mathbf{D}_a$ and set $L := A_{\mathrm{red}}$. The curve A is a locally CM curve of degree two with $L \cong \mathbb{P}^1$ as its support (see [6, Remark 1.3]) and so it is a ribbon in the sense of [4], as mentioned in the beginning of this section. The projection π_1 induces a morphism $\pi_{1|A} : A \to \mathbb{P}^1$ whose restriction to $L = A_{\mathrm{red}}$ is the isomorphism $\pi_{1|L} : L \to \mathbb{P}^1$. Thus, A is a split ribbon, see [4, Corollary 1.7]. Hence, $\mathcal{O}_A$ fits into an exact sequence

$$0 \to \mathcal{O}_L(a-2) \to \mathcal{O}_A \to \mathcal{O}_L \to 0, \tag{3.1}$$

which splits as an exact sequence of $\mathcal{O}_L$-modules. Since L is a complete intersection in X of two planes of type $(0, 1, 0)$, $(0, 0, 1) \in \mathrm{Pic}(X)$, the Koszul resolution for $\mathcal{I}_L$ shows that $\mathcal{I}_L/\mathcal{I}_L^2 \cong \mathcal{O}_L^{\oplus 2}$. In particular, the normal bundle $N_{L|X}$ is trivial and double lines supported on L are parametrized by the surjective morphism in $\mathrm{Hom}(\mathcal{O}_L^{\oplus 2}, \mathcal{O}_L(a-2))$, as in [19, Introduction] and [21, Proposition 1.4].

Theorem 3.3. *The description on $\mathbf{D}_a$ is as follows:*

(i) *$\mathbf{D}_a$ is non-empty if and only if $a \geq 2$. It is parametrized by an irreducible and rational variety of dimension $2a - 1$.*

(ii) *We have $\mathbf{D}_a = \mathbf{H}(2, 0, 0, a)_+$ for $a \geq 3$.*

(iii) *$\mathbf{H}(2, 0, 0, 2)_+$ is isomorphic to $\mathbf{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$, the Hilbert scheme of two points in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, it is smooth, irreducible, rational and of dimension four.*

Proof. Part (i) holds by Remark 3.2 and part (ii) follows because two disjoint lines have genus -1. For the proof of part (iii), note that the maps in (3.1) split when $a = 2$, showing that A is a complete intersection. Let $\mathbb{E}$ be the subset of $\mathbf{H}(2, 0, 0, 2)_+$ parametrizing two disjoint lines of tridegree $(1, 0, 0)$. Since $h^1(N_C) = 0$ for every $[C] \in \mathbb{E}$, we have that $\mathbf{H}(2, 0, 0, 1)_+$ is smooth at each point of $\mathbb{E}$ and of dimension $h^0(N_C) = 4$. Obviously, we have

$$\mathbf{H}(2, 0, 0, 2)_{+,\mathrm{red}} = \mathbb{E} \cup \mathbf{D}_2,$$

and $\mathbb{E}$ is irreducible and rational. We have $\dim(\mathbf{D}_2) = 3$ and so it is sufficient to prove that at each $[B] \in \mathbf{D}_2$ the scheme $\mathbf{H}(2, 0, 0, 2)_+$ is smooth and it has dimension four. Fix $[B] \in \mathbf{D}_2$. Since B has only planar singularities, it is locally unobstructed, see [17, 2.12.1]. Hence, $\mathbf{H}(2, 0, 0, 2)_+$ has dimension at least $\chi(N_B)$ at $[B]$ by [17, Theorem 2.15.3]. Since $\mathbf{D}_2$ is irreducible, we get $\chi(N_A) = \chi(N_E)$ for all $[A], [E] \in \mathbf{D}_2$. If B is contained

in a smooth quadric surface $T \in |\mathcal{O}_X(1,1,0)|$, then $N_B \cong \mathcal{O}_B^{\oplus 2}$ and so $\chi(N_B) = 4$. In this case we also have $h^1(N_B) = 0$ by (3.1). Note that there is a connected zero-dimensional subscheme $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree two such that $B \cong L \times Z$: indeed, for each connected zero-dimensional subscheme $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree two, we have $L \times Z \in \mathbf{D}_2$, and the set of all such Z is smooth, irreducible, complete and three-dimensional. There are two types of connected zero-dimensional subscheme $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree two: the ones are contained in a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ and the other ones are the complete intersection of two elements of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$. Thus, our double lines B's are the complete intersection of two elements of $|\mathcal{O}_X(0,1,1)|$. Hence, even for these B's we have $h^1(N_B) = 0$, concluding the proof of the smoothness of $\mathbf{H}(2,0,0,2)_+$.

The same argument works also for the reduced $[C] \in \mathbf{H}(2,0,0,2)_+$ with Z a reduced subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ of degree two. Thus, we get $\mathbf{H}(2,0,0,2)_+ \cong \mathbf{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$, and this is isomorphic to the blow-up of the symmetric product $\mathrm{Sym}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ along the diagonal. □

Remark 3.4. From the proof of Theorem 3.3, each double line in $\mathbf{D}_a$ is associated to the triple (L, f, g) with $L \subset X$ a line of tridegree $(1,0,0)$ and $f, g \in \mathbb{C}[x_0, x_1]_{a-2}$ with no common zero, where x_0 and x_1 are homogeneous linear forms on L.

Remark 3.5. Fix an integer $a \geq 2$ and any $[A] \in \mathbf{D}_a$. Since $h^1(\mathcal{O}_L(a-2)) = 0$, the exponential sequence associated to (3.1)

$$0 \to \mathcal{O}_L(a-2) \to \mathcal{O}_A^* \to \mathcal{O}_L^* \to 1$$

gives that the restriction map $\mathrm{Pic}(A) \to \mathrm{Pic}(L)$ is bijective, as in the proof of [4, Proposition 4.1]. Thus, we have $\mathcal{O}_A(t,u,v) \cong \mathcal{O}_A(t,t,t)$ for all $(t,u,v) \in \mathbb{Z}^{\oplus 3}$ and $\mathcal{O}_A(t,t,t)$ is the only line bundle on A whose restriction to L is $\mathcal{O}_L(t,t,t)$. Hence, the following table computes the cohomology groups of all line bundles on A:

	$h^0(\mathcal{O}_A(t,t,t))$	$h^1(\mathcal{O}_A(t,t,t))$	$h^2(\mathcal{I}_A(t,t,t))$
$-1 \leq t$	$2t+a$	0	0
$-a+1 \leq t \leq -2$	$t+a-1$	$-t-1$	$-t-1$
$t \leq -a$	0	$-2t-a$	$-2t-a$

Indeed, we have $h^2(\mathcal{I}_A(t,t,t)) = h^1(\mathcal{O}_A(t,t,t))$ from $h^1(\mathcal{O}_X(t,t,t)) = h^2(\mathcal{O}_X(t,t,t)) = 0$ for all t. Now the table follows from (3.1) by setting $L := A_{\mathrm{red}}$, because it is a split exact sequence as $\mathcal{O}_L$-sheaves.

Remark 3.6. Take $[A] \in \mathbf{D}_2$. We saw in the proof of part (3) in Theorem 3.3 that A is either the complete intersection of two elements of $|\mathcal{O}_X(0,1,1)|$, the case in which Z is not contained in a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$, or a complete intersection of an element of $|\mathcal{O}_X(0,0,1)|$ (resp. $|\mathcal{O}_X(0,1,0)|$) and an element of $|\mathcal{O}_X(0,2,0)|$ (resp. $|\mathcal{O}_X(0,0,2)|$), the case in which Z is contained in a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. In both cases, we have $h^1(\mathcal{I}_A(t,t,t)) = 0$ for all $t \neq 0$ and $h^1(\mathcal{I}_A) = 1$.

4 Irreducibility of Hilbert schemes

In this section we discuss the topology of $\mathbf{H}(e_1,e_2,e_3,\chi)_{+,\mathrm{red}}$ for small e_i, mainly on the irreducible components. The main strategy is to describe an irreducible sublocus of $\mathbf{H}(e_1,e_2,e_3,\chi)_{+,\mathrm{red}}$ consisting of a special type of curves via double line structures and to determine if its closure is an irreducible component by various methods, including deformation theory. The case $(e_1,e_2,e_3,\chi) = (2,1,1,1)$ is exceptional in a sense that one further needs the irreducibility of a certain moduli space of stable maps and deformation theory on it to prove the irreducibility of $\mathbf{H}(2,1,1,1)_{+,\mathrm{red}}$, see Theorem 4.26.

Lemma 4.1. *Fix $[A] \in \mathbf{D}_a$ with $a \geq 2$ and set $L := A_{\mathrm{red}}$.*

(i) *For a line $L' \subset X$ different from L, we have $\deg(A \cap L') \leq 2$. Moreover, we have $A \cap L' = \emptyset$ if and only if $L \cap L' = \emptyset$.*

(ii) *The following set is a non-empty irreducible and rationally connected variety of dimension $2a+1$:*

$$\{(B, L') \mid [B] \in \mathbf{D}_a,\ L' \text{ is a line of tridegree } (0,1,0) \text{ with } B \cap L' = \emptyset\}.$$

(iii) *The tangent plane T_pA at a point $p \in L$ is a plane containing L and contained in the three-dimensional tangent space T_pX.*

(iv) *$T_pA \cap X$ is the union of three lines L, L_1, L_2 of X through p, with L_1 of tridegree $(0,1,0)$ and L_2 of tridegree $(0,0,1)$. We have $\deg(A \cap L_i) = 2$ if and only if T_pA is the plane spanned by $L \cup L_i$.*

Proof. Since L and L' are different, so $\deg(A \cap L')$ is a well-defined non-negative integer, we have that $\deg(A \cap L') = 0$ if and only if $A \cap L' = \emptyset$, i.e., $L \cap L' = \emptyset$. Since L has tridegree $(1,0,0)$, there is $(o, o') \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $L = \mathbb{P}^1 \times \{(o, o')\}$. The complement of $\mathbb{P}^1 \times \{o'\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ parametrizes the set of all lines T of tridegree $(0,1,0)$ with $T \cap L = \emptyset$. The other assertions are obvious. □

Lemma 4.2. *For $[A] \in \mathbf{H}(2,0,0,2)_+$, there exists a line $R \subset X$ of tridegree $(0,1,0)$ with $\deg(A \cap R) \geq 2$, i.e., $\deg(A \cap R) = 2$ if and only if there exists $Q \in |\mathcal{O}_X(0,0,1)|$ that contains A. In this case, Q is unique, $A \cup R \subset Q$ and there is a one-dimensional family of such lines R.*

Proof. The lemma is obvious if A is a disjoint union of two lines, say $A = \mathbb{P}^1 \times \{(o_2, o_3)\} \cup \mathbb{P}^1 \times \{(p_2, p_3)\}$, because the existence of R is equivalent to $o_3 = p_3$. Now assume $[A] \in \mathbf{D}_2$, say associated to (L, f, g) with $L := A_{\mathrm{red}}$ and $(f, g) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Write $L = \mathbb{P}^1 \times \{(p_2, p_3)\}$. Since $\mathbb{P}^1 \times \mathbb{P}^1 \times \{p_3\}$ is the only element of $|\mathcal{O}_X(0,0,1)|$ containing L, the uniqueness part is obvious. Assume the existence of a line $R \subset X$ of tridegree $(0,1,0)$ with $\deg(A \cap R) \geq 2$. By Lemma 4.1, we have $\deg(A \cap R) = 2$ and $R \cap L$ contains a point, say $p = (p_1, p_2, p_3)$. For each point $q = (q_1, p_2, p_3) \in L$, the pull-backs via the projections π_i, for $i = 2, 3$, of a non-zero tangent vector of $\mathbb{P}^1$ at p_i form a basis of $N_{L,q} \cong \mathbb{C}^2$. Since A has tridegree $(2,0,0)$, the map $\pi_{2|A}$ is induced by an element of $H^0(\mathcal{O}_A)$, i.e., by an element $c \in H^0(\mathcal{O}_L)$, due to (3.1), with $a = 2$ and $\pi_2(L) = \{p_2\}$. The condition $\deg(A \cap R) = 2$ is equivalent to saying that $\pi_{2|A}$ vanishes at p. Since c is a constant, $\pi_{2|A}$ vanishes at all points of L, i.e., $A \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{p_3\}$. □

Theorem 4.3. *We have $\mathbf{H}(2,1,0,1)_+ \cong \mathbb{P}^5 \times \mathbb{P}^1$.*

Proof. It is sufficient to prove that for each $[C] \in \mathbf{H}(2,1,0,1)_+$, there exists $Q \in |\mathcal{O}_X(0,0,1)|$ such that $C \in |\mathcal{O}_Q(1,2)|$, which would give us a morphism from $\mathbf{H}(2,1,0,1)_+$ to $\mathbb{P}^5 \times \mathbb{P}^1$. Its inverse map is obviously defined. If C is reduced, then $\pi_{2|C}$ shows that each irreducible component of C is smooth and rational. Since $\chi(\mathcal{O}_C) = 1$, C is connected. Since C is reduced, connected and of tridegree $(2,1,0)$, the scheme $\pi_3(C)$ is a point and so there is a point $o \in \mathbb{P}^1$ such that $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{o\}$. Now assume that C is not reduced. By Lemma 2.5 and Theorem 3.3, we see that $C = A \cup R$ with $[A] \in \mathbf{D}_a$ for $a \geq 2$ and R a line. Since $\deg(A \cap R) \leq 2$ and $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_A) + 1 - \deg(A \cap R)$, we get $a = 2$ and $\deg(A \cap R) = 2$. We may now apply Lemma 4.2. □

Remark 4.4. The proof of Theorem 4.3 shows that for each $[C] \in \mathbf{H}(2,1,0,1)_+$, there is $Q \in |\mathcal{O}_X(0,0,1)|$ such that $C \in |\mathcal{O}_Q(1,2)|$. This observation gives a way to describe C as follows. We first fix a point $o \in \mathbb{P}^1$ such that $Q = \mathbb{P}^1 \times \mathbb{P}^1 \times \{o\}$ and then we describe C inside Q by an equation of bidegree $(1,2)$. Writing $\mathcal{O}_C(u,v) := \mathcal{O}_Q(u,v)_{|C}$, we have $\mathcal{O}_C(u,v) \cong \mathcal{O}_C(u,v,w)$ for any $(u,v,w) \in \mathbb{Z}^{\oplus 3}$. Note that $\mathcal{I}_{C,Q}(u,v) \cong \mathcal{O}_Q(u,v)(-C) \cong \mathcal{O}_Q(u-1, v-2)$. In particular, for another $[C'] \in \mathbf{H}(2,1,0,1)_+$ and $Q' \in |\mathcal{O}_X(0,0,1)|$ with $C' \subset Q'$, we have $h^i(Q, \mathcal{I}_{C,Q}(u,v)) = h^i(Q', \mathcal{I}_{C',Q'}(u,v))$ for any $(u,v) \in \mathbb{Z}^{\oplus 2}$ and each $i = 0, 1, 2$. On the other hand, we have the exact sequence

$$0 \to \mathcal{O}_Q(u-1, v-2) \to \mathcal{O}_Q(u,v) \to \mathcal{O}_C(u,v) \to 0. \tag{4.1}$$

First assume that either $u \geq 1$, $v \leq -2$ or $u \leq -2$, $v \geq 2$. Then we have $h^0(\mathcal{O}_Q(u,v)) = h^2(\mathcal{O}_Q(u-1, v-2)) = 0$ and the sequence (4.1) gives

$$0 \to H^0(\mathcal{O}_C(u,v)) \to H^1(\mathcal{O}_Q(u-1, v-2)) \to H^1(\mathcal{O}_Q(u,v)) \to H^1(\mathcal{O}_C(u,v)) \to 0.$$

If C is reduced, then it is a connected curve of arithmetic genus zero so that each component of C is smooth and rational: C is either a rational normal curve of degree three or a union of three lines. In the former case ,we have $h^0(\mathcal{O}_C(u,v)) = h^0(\mathcal{O}_{\mathbb{P}^1}(2u+v))$. In the latter case, from the exact sequence

$$0 \to \mathcal{O}_C(u,v) \to \mathcal{O}_L(u,v) \oplus \mathcal{O}_{L_1}(u,v) \oplus \mathcal{O}_{L_2}(u,v) \to \mathcal{O}_{L \cap L_1} \oplus \mathcal{O}_{L \cap L_2} \to 0,$$

with $L \in |\mathcal{O}_Q(1,0)|$ and $L_i \in |\mathcal{O}_Q(0,1)|$ for $i = 1, 2$, we get an exact sequence

$$0 \to H^0(\mathcal{O}_C(u,v)) \to H^0(\mathcal{O}_L(u,v)) \oplus \Big(\bigoplus_{i=1}^{2} H^0(\mathcal{O}_{L_i}(u,v))\Big) \xrightarrow{\eta} \oplus_{i=1}^{2} H^0(\mathcal{O}_{L\cap L_i}),$$

where the rank of the map η is determined only by (u,v); for example, if $(u,v) \in \mathbb{Z}^{\oplus 2}_{>0}$, then η is surjective. If C is not reduced, then we have $C = A \cup R$, with $A \in |\mathcal{O}_Q(0,2)|$ a line with multiplicity two and $R \in |\mathcal{O}_Q(1,0)|$ with $\deg(A \cap R) = 2$. From (3.1), we have $h^0(\mathcal{O}_A(u,v)) = 2u + 2$ (resp. 0) if $u \geq 1$ (resp. $u \leq -2$). Note also that $h^0(\mathcal{O}_R(u,v)) = h^0(\mathcal{O}_{\mathbb{P}^1}(v)) = \max\{0, v+1\}$. From the exact sequence

$$0 \to \mathcal{O}_C(u,v) \to \mathcal{O}_A(u,v) \oplus \mathcal{O}_R(u,v) \to \mathcal{O}_{A\cap R} \to 0,$$

we have an exact sequence

$$0 \to H^0(\mathcal{O}_C(u,v)) \to H^0(\mathcal{O}_A(u,v)) \oplus H^0(\mathcal{O}_R(u,v)) \xrightarrow{\rho} H^0(\mathcal{O}_{A\cap R}) \to \cdots.$$

Here, the map ρ is (resp. not) surjective if $u \geq 1$ (resp. $u \leq -2$). Indeed, in case when $u \leq -2$, ρ is the zero map if and only if $v \leq -1$. In particular, $h^0(\mathcal{O}_C(u,v))$ is determined only by (u,v), and so is $h^1(\mathcal{O}_C(u,v))$. For the other choices for $(u,v) \in \mathbb{Z}^{\oplus 2}$, we have either $h^1(\mathcal{O}_Q(u-1,v-2)) = 0$ or $h^1(\mathcal{O}_Q(u,v)) = 0$. This implies from the sequence (4.1) that $h^i(\mathcal{O}_C(u,v))$ depends only on (u,v) for each $i = 0, 1$. As a summary, we can conclude that $h^i(\mathcal{O}_C(u,v)) = h^i(\mathcal{O}_{C'}(u,v))$ for any $(u,v) \in \mathbb{Z}^{\oplus 2}$ and $i = 0, 1$.

Lemma 4.5. *For a fixed $[A] \in \mathbf{D}_a$ with $a \geq 3$, define $\mathcal{S}$ to be the set of all lines $R \subset X$ with tridegree $(0,0,1)$ and $\deg(A \cap R) \geq 2$. Then $\mathcal{S}$ is a non-empty finite set and we have $\deg(A \cap R) = 2$ for all $R \in \mathcal{S}$.*

Proof. Set $L := A_{\mathrm{red}}$. We have $\chi(\mathcal{O}_B) = 2 < a$ for every $B \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,2)|$ and so $\pi_{12|A} : A \to \mathbb{P}^1 \times \mathbb{P}^1$ is not an embedding by [12, Proposition II.2.3]. Thus, we have $\mathcal{S} \neq \emptyset$. Fix $R \in \mathcal{S}$. Lemma 4.1 gives $\deg(A \cap R) = 2$ and so there is a unique point $o \in L \cap R$. R is the unique line of tridegree $(0,0,1)$ containing o. The condition $\deg(A \cap R) = 2$ is equivalent to the condition that the plane $\langle L \cup R\rangle$ is the tangent plane of A at o. Hence, we have $\deg(A \cap L_1) = 1$ only for the line L_1 of tridegree $(0,1,0)$ containing o. Set $Q := \pi_{12}(L) \times \mathbb{P}^1 \in |\mathcal{O}_X(0,1,0)|$. Assume that $\mathcal{S}$ is infinite. We get that Q contains infinitely many tangent planes of A and so each tangent plane of A is contained in Q. Therefore, we have $\deg(A \cap T) \leq 1$ for all lines $T \subset X$ of tridegree $(0,1,0)$ and so $\pi_{13|A} : A \to \mathbb{P}^1 \times \mathbb{P}^1$ is an embedding by [12, Proposition II.2.3]. We saw that this is false. □

Remark 4.6. Let us fix a double line $[A] \in \mathbf{D}_a$, with $a \geq 3$, that is associated to the triple (L, f, g), with $L \subset X$ a line of tridegree $(1,0,0)$ and $f, g \in \mathbb{C}[x_0, x_1]_{a-2}$ with no common zero. For a fixed point $p = (o_1, o_2, o_3) \in L$ and the line $R \subset X$ of tridegree $(0,1,0)$ passing through p, we have $1 \leq \deg(R \cap A) \leq 2$. Indeed, we have $\deg(R \cap A) = 2$ if and only if f vanishes at p. Since $a > 2$, there exists at least one line R with this property and at most $(a-2)$ such lines exist. If f is general (and in particular if A is general), then f has $(a-2)$ distinct zeros and so there are exactly $(a-2)$ lines R of tridegree $(0,1,0)$ with $\deg(A \cap R) = 2$.

Proposition 4.7. *For each integer $a \geq 4$, we have $\mathbf{H}(2,1,0,a)_{+,\mathrm{red}} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, where we have $[C] \in \mathcal{S}_i$ for each $i = 0, 1, 2$ if and only if $C = A \cup R$, where $[A] \in \mathbf{D}_{a+i-1}$ and R is a line of tridegree $(0,1,0)$ with $\deg(A \cap R) = i$. Furthermore, we get that $\mathcal{S}_i$ is irreducible with $\dim(\mathcal{S}_i) = 2a - 1 + i$ for each $i = 0, 1, 2$.*

Proof. Fix $[C] \in \mathbf{H}(2,1,0,a)_{+,\mathrm{red}}$ and then C is not reduced, because we assumed $a \geq 4$. By Lemma 2.5 and Theorem 3.3, we have $C = A \cup R$, where $[A] \in \mathbf{D}_c$ for some $c \geq 2$ and R is a line of tridegree $(0,1,0)$ with $\deg(A \cap R) = c + 1 - a$. Since $0 \leq \deg(A \cap R) \leq 2$, we have $c \in \{a-1, a, a+1\}$ and so we get a set-theoretic decomposition $\mathbf{H}(2,1,0,1)_{+,\mathrm{red}} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$.

Now we first check that $\mathcal{S}_0$ and $\mathcal{S}_1$ are irreducible. For $[A] \in \mathbf{D}_{a-1}$, the set of all lines R with $R \cap A = \emptyset$ is a non-empty open subset of $\mathbb{P}^1 \times \mathbb{P}^1$. By Theorem 3.3, $\mathcal{S}_0$ is non-empty, irreducible, rational and of dimension $2(a-1) - 1 + 2 = 2a - 1$. For $[A] \in \mathbf{D}_a$, set $L := A_{\mathrm{red}}$ and then a line R of tridegree $(0,1,0)$ satisfies $\deg(A \cap R) > 0$ if and only if $R \cap L \neq \emptyset$. By Lemma 4.5, the set of all such lines R is a non-empty smooth rational curve. Hence, $\mathcal{S}_1$ is rationally connected, irreducible and of dimension $2a$.

For $[A] \in \mathbf{D}_{a+1}$, Remark 4.6 shows that the set of all lines of tridegree $(0,1,0)$ with $\deg(A \cap R) = 2$ is non-empty and finite. So we get $\mathcal{S}_2 \neq \emptyset$ and each irreducible component of $\mathcal{S}_2$ has dimension $2a + 1$. Let $\mathbb{I}$ be the

set of all pairs (A, p) with $[A] \in \mathbf{D}_{a+1}$, $p \in A_{\mathrm{red}}$, and there is a line $R \subset X$ of tridegree $(0, 1, 0)$ with $p \in R$ and $\deg(R \cap A) = 2$. Then it is sufficient to prove that $\mathbb{I}$ is irreducible. For a fixed line $L \subset X$ of tridegree $(1, 0, 0)$ and $g \in \mathbb{C}[x_0, x_1]_{a-1}$ with $g \neq 0$, we define $U(L, g)$ to be the set of all $[A] \in \mathbf{D}_{a+1}$ associated to a triple (L, f, g) for some f, and let $\mathbb{I}_{L,g}$ be the set of all pairs (A, p) with $[A] \in U(L, g)$, $p \in L$ and there is a line $R \subset X$ of tridegree $(0, 1, 0)$ with $\deg(R \cap A) = 2$. The irreducibility of $\mathbb{I}_{L,g}$ is equivalent to the well-known irreducibility of the set of all pairs (f, p) with $p \in \mathbb{P}^1$, $f \in \mathbb{C}[x_0, x_1]_{a-1} \setminus \{0\}$ vanishing at p. Thus, $\mathbb{I}$ is irreducible and so is $\mathcal{S}_2$. □

Theorem 4.8. *For $a \geq 4$, $\overline{\mathcal{S}_0}$ and $\overline{\mathcal{S}_2}$ are the irreducible components of $\mathbf{H}(2, 1, 0, a)_{+,\mathrm{red}}$.*

Proof. Since the support of each element of $\mathcal{S}_1 \cup \mathcal{S}_2$ is connected, we have $\mathcal{S}_0 \notin \overline{\mathcal{S}_1 \cup \mathcal{S}_2}$, and so $\overline{\mathcal{S}_0}$ is an irreducible component of $\mathbf{H}(2, 1, 0, a)_{+,\mathrm{red}}$. Now it remains to prove that $\mathcal{S}_1 \subset \overline{\mathcal{S}_2}$. Fix a general $A \cup R \in \mathcal{S}_1$ with A associated to a triple (L, f, g) and set $\{p\} := R \cap L$. Since $\deg(R \cap A) = 1$, we have $f(p) \neq 0$. For a general $A \cup R$, we may assume that $g(p) \neq 0$. Let $S \subset L$ be a finite set containing all zeros of f and g and the point $x_0 = 0$, but with $p \notin S$. Set $z := x_1/x_0$ and let f_1, g_1 be the elements of $\mathbb{C}[z]$ obtained by dehomogenizing f and g. For a general A, we may assume that $\deg(f_1) = \deg(g_1) = a - 2$. Let Δ denote the diagonal of $(L \setminus S) \times (L \setminus S)$. For all $(u, v) \in ((L \setminus S) \times (L \setminus S)) \setminus \Delta$, set $f_{u,v} := (z - u)f_1$ and $g_{u,v} := (z - v)g_1$. Let $\tilde{f}_{u,v}(x_0, x_1)$ (resp. $\tilde{g}_{u,v}(x_0, x_1)$) be the homogeneous polynomial associated to $f_{u,v}$ (resp. $g_{u,v}$). Let $A_{u,v}$ denote the element of $\mathbf{D}_{a+1}$ associated to $(L, \tilde{f}_{u,v}, \tilde{g}_{u,v})$ and let R_u be the line of tridegree $(0, 1, 0)$ passing through the point of L associated to u. We got a flat family $\{A_{u,v} \cup R_u\}_{u \in \Delta}$ of elements of $\mathcal{S}_2$. As (u, v) tends to (p, p), we get that $A \cup R$ is a flat limit of this family. □

Proposition 4.9. *We have*

$$\mathbf{H}(2, 1, 0, 3)_{+,\mathrm{red}} = \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_2,$$

where each curve $[C] \in \mathcal{T}_i$ for $i \in \{1, 2\}$ is of the form $A \cup R$, where $[A] \in \mathbf{D}_{i+2}$ and R is a line of tridegree $(0, 1, 0)$ with $\deg(A \cap R) = i$. A general element of $\mathcal{T}$ is a disjoint union of three lines. Furthermore, we get that $\mathcal{T}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are all irreducible with dimension 6, 7 and 6, respectively.

Proof. Fix $[C] \in \mathbf{H}(2, 1, 0, 3)_{+,\mathrm{red}}$ and then C is reduced if and only if it is the disjoint union of three lines, two of tridegree $(1, 0, 0)$ and one of tridegree $(0, 1, 0)$. The set $\mathcal{A}$ of all such curves is a non-empty open subset of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and so $\mathcal{A}$ is smooth, irreducible and rational with $\dim(\mathcal{A}) = 6$. Note that $\mathcal{A}$ is not complete since $\mathcal{A} \subsetneq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Now assume that C is not reduced. By Lemma 2.5 and Theorem 3.3, we have $C = A \cup R$, where $[A] \in \mathbf{D}_c$ with $c \geq 2$ and R a line of tridegree $(0, 1, 0)$ with $\deg(A \cap R) = c - 2$. Since $0 \leq \deg(A \cap R) \leq 2$, we have $c \in \{2, 3, 4\}$. Let $\mathcal{T}_i$ be the set of all $C = A \cup R$ with $\deg(A \cap R) = i$. As in the proof of Theorem 4.8, we see that $\mathcal{T}_i \neq \emptyset$ for all i. Set $\mathcal{T} := \mathcal{A} \cup \mathcal{T}_0$ to be the set of all disjoint unions of an element of $\mathbf{H}(2, 0, 0, 2)_{+,\mathrm{red}}$ and a line of tridegree $(0, 1, 0)$. By the case $a = 2$ of Theorem 3.3, $\mathcal{T}_0$ is in the closure of $\mathcal{A}$ and so $\mathcal{T}$ is irreducible. Similarly, as in the proof of Theorem 4.8, we get that $\mathcal{T}_i$ is also irreducible for each $i = 1, 2$. The dimension counting is clear. □

Theorem 4.10. *$\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}_2$ are the irreducible components of $\mathbf{H}(2, 1, 0, 3)_{+,\mathrm{red}}$.*

Proof. As in the proof of Theorem 4.8, we get that $\mathcal{T} \notin \overline{\mathcal{T}_1 \cup \mathcal{T}_2}$ and $\mathcal{T}_1 \subset \overline{\mathcal{T}}_2$, which prove the assertion. □

In $\mathbf{H}(1, 1, 1, 1)$ we have a family of curves formed by three lines through a common point. Denote the locus of such curves by $\mathbf{D}$ and we have $X \cong \mathbf{D} \subset \mathbf{H}(1, 1, 1, 1)$.

Remark 4.11. For $[C] \in \mathbf{D}$, one can check $h^1(\mathcal{O}_C(u, v, w)) = 0$ for all $(u, v, w) \in \mathbb{Z}_{>0}^{\oplus 3}$. Moreover, we have $h^0(\mathcal{I}_C(1, 1, 1)) = 5$, $h^0(\mathcal{I}_C(u, v, w)) = 0$ if $uvw = 0$ and $h^1(\mathcal{I}_C(u, v, w)) = 0$ if $(u, v, w) \in \mathbb{Z}_{>0}^{\oplus 3}$. Note also that for all $[C], [C'] \in \mathbf{D}$, there is $f \in \mathrm{Aut}(X)$ with $f(C') = C$.

Proposition 4.12. *$\mathbf{H}(1, 1, 1, 3)_{+,\mathrm{red}}$ is irreducible, smooth and rational of dimension six.*

Proof. Fix a curve $[C] \in \mathbf{H}(1, 1, 1, 3)_+$ and then C is reduced by Lemma 2.5. We also have $\chi(\mathcal{O}_C) = -2$ and so C has at least three connected components. Thus, C is a disjoint union of three lines, one line for each tridegree $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Hence, set-theoretically, $\mathbf{H}(1, 1, 1, 3)_{+,\mathrm{red}}$ is irreducible, rational and of dimension six. Since $N_C \cong \mathcal{O}_C^{\oplus 2}$, we have $h^1(N_C) = 0$ and so $\mathbf{H}(1, 1, 1, 3)_+$ is smooth. □

Proposition 4.13. $\mathbf{H}(1,1,1,2)_+$ *has three connected components and each of them is smooth and rational of dimension seven.*

Proof. Fix $[C] \in \mathbf{H}(1,1,1,2)_+$ and then C is reduced again by Lemma 2.5. We have $\chi(\mathcal{O}_C) = -1$ and so C has at least two connected components. One of these connected components must be a line. Since $\chi(\mathcal{O}_C) \neq -2$ and $\deg(C) = 3$, C is not the union of three disjoint lines. Hence, C has a unique connected component of degree one. The three connected components of $\mathbf{H}(1,1,1,2)_+$ are distinguished by the tridegree of their degree one component.

Without loss of generality, we may assume that C has a line L of tridegree $(1,0,0)$ as a connected component, say $C = L \sqcup D$ with D of tridegree $(0,1,1)$. We have $D = \{o\} \times D'$ for a point $o \in \mathbb{P}^1$ and a conic $D' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$. Since $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ is irreducible and of dimension three, we get that each connected component of $\mathbf{H}(1,1,1,2)_{+,\mathrm{red}}$ is irreducible, rational and of dimension seven. Since $N_L \cong \mathcal{O}_L^{\oplus 2}$ and $N_D \cong \mathcal{O}_D \oplus \mathcal{O}_D(1,1)$ with $L \cap D = \emptyset$, we get $h^1(N_C) = 0$ and so $\mathbf{H}(1,1,1,2)_+$ is smooth. □

Remark 4.14. As in the proof of Proposition 4.13, set $C = L \sqcup D$ with L of tridegree $(1,0,0)$ and D of tridegree $(0,1,1)$, so that $D = \{o\} \times D'$ for a point $o \in \mathbb{P}^1$ and a conic $D' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$. As an abstract scheme, the isomorphism class of C depends only on the rank of D' as a conic. Take two points o_2 and o_3 on $\mathbb{P}^1$ such that $L = \mathbb{P}^1 \times \{o_2\} \times \{o_3\}$. The assumption $L \cap D = \emptyset$ is equivalent to $(o_2, o_3) \notin D'$. Setting $\nu\colon X \to \mathbb{P}^7$ the Segre embedding, $\nu(L)$ is a line, $\nu(D)$ is a reduced conic and $\nu(\{o\} \times \mathbb{P}^1 \times \mathbb{P}^1)$ is a smooth quadric with $\dim\langle\nu(\{o\} \times \mathbb{P}^1 \times \mathbb{P}^1)\rangle = 3$, where $\langle\ \rangle$ denotes the linear span in $\mathbb{P}^7$. Since $\nu(X)$ is scheme-theoretically cut out by quadrics and X contains no plane, we have $\nu(X) \cap \langle D\rangle = D$. Since $L \cap D = \emptyset$, we get $\nu(L) \cap \langle\nu(D)\rangle = \emptyset$, i.e., $\dim\langle\nu(C)\rangle = 4$, and, in particular, $h^1(\mathcal{I}_C(1,1,1)) = 0$. Now one can check that this implies $h^1(\mathcal{I}_C(u,v,w)) = 0$ for all positive integers u, v, v.

Lemma 4.15. *Each curve in* $\mathbf{H}(1,1,1,1)$ *is connected and reduced.*

Proof. Let us fix $[C] \in \mathbf{H}(1,1,1,1)$. By Lemma 2.5 every one-dimensional component of C is generically reduced, i.e., the purely one-dimensional subscheme E of C_{red} has tridegree $(1,1,1)$. We have $\chi(\mathcal{O}_C) \geq \chi(\mathcal{O}_D)$ for each connected component D of E and equality holds if and only if $D = C$. Since we have $\chi(\mathcal{O}_D) \geq 1$, we get $C = D$, and so C is connected and reduced. □

Lemma 4.16. $\mathbf{H}(1,1,1,1)^{\mathrm{sm}}$ *is irreducible, unirational of dimension six.*

Proof. Let us fix $[C] \in \mathbf{H}(1,1,1,1)$ irreducible. Since $\pi_{1|C}\colon C \to \mathbb{P}^1$ has degree one, C is smooth and rational. In particular, we get $[C] \in \mathbf{H}(1,1,1,1)^{\mathrm{sm}}$. Since $\pi_{i|C}\colon C \to \mathbb{P}^1$, for $i = 1,2,3$, is induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(1)|$, $\mathbf{H}(1,1,1,1)^{\mathrm{sm}}$ is homogeneous for the action of the group

$$\mathrm{Aut}^0(X) = PGL(2) \times PGL(2) \times PGL(2).$$

Thus, the algebraic set $\mathbf{H}(1,1,1,1)^{\mathrm{sm}}_{\mathrm{red}}$ is irreducible and unirational. To show that $\mathbf{H}(1,1,1,1)^{\mathrm{sm}}$ is smooth and of dimension six, it is sufficient to prove that $h^1(N_C) = 0$ and $h^0(N_C) = 6$. Note that we have $\chi(N_C) = 6$. Since X is homogeneous, its tangent bundle TX is globally generated and so is $TX_{|C}$. Since N_C is a quotient of $TX_{|C}$, N_C is also globally generated. Since $C \cong \mathbb{P}^1$, we get $h^1(N_C) = 0$. Indeed, we have $N_C \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. The normal bundle N_C is a direct sum of two line bundles, say of degree $z_1 \geq z_2$ with $z_1 + z_2 = 4$. Since N_C is a quotient of $TX_{|C}$, which is the direct sum of three line bundles of degree two, we get $z_1 = z_2 = 2$. □

Proposition 4.17. $\mathbf{H}(1,1,1,1)$ *is irreducible, unirational of dimension six and smooth outside* $\mathbf{D}$.

Proof. By Lemma 4.16 it is sufficient to prove that $\mathbf{H}(1,1,1,1)$ is smooth at each reducible curve $[C] \notin \mathbf{D}$ and that each reducible element of $\mathbf{H}(1,1,1,1)$ is in the closure of $\mathbf{H}(1,1,1,1)^{\mathrm{sm}}$.

(a) Now assume that C has two irreducible components, say $C = D_1 \cup D_2$ with D_1 a line. Since $\chi(\mathcal{O}_C) = 1$ and the scheme C is reduced with no isolated point and arithmetic genus 0, it follows that C is connected. Since $p_a(C) = 0$, we get $\deg(D_1 \cap D_2) = 1$. In particular, C is nodal and so N_C is locally free. Without loss of generality, we may assume that D_1 has tridegree $(1,0,0)$ and so D_2 is a smooth conic with tridegree $(0,1,1)$. We have $N_{D_1} \cong \mathcal{O}_{D_1}^{\oplus 2}$ and $N_{D_2} \cong \mathcal{O}_{D_2} \oplus \mathcal{O}_{D_2}(0,1,1)$. Since N_C is locally free, we have a Mayer–Vietoris exact

sequence

$$0 \to N_C \to N_{C_{|D_1}} \oplus N_{C_{|D_2}} \to N_{C_{|D_1 \cap D_2}} \to 0. \tag{4.2}$$

Since $\deg(D_1 \cap D_2) = 1$ and C is nodal, the sheaf $N_{C_{|D_1}}$ (resp. $N_{C_{|D_2}}$) is a vector bundle of rank two obtained from N_{D_1} (resp. N_{D_2}) by making one positive elementary transformation at $D_1 \cap D_2$ (see [13, §2], [26, Lemma 5.1] and [25]), i.e., N_{D_i} is a subsheaf of $N_{C_{|D_i}}$ and its quotient $N_{C_{|D_i}}/N_{D_i}$ is a skyscraper sheaf of degree one supported on the point $D_1 \cap D_2$. Since $h^1(D_2, N_{D_2}) = 0$, we get $h^1(N_{C_{|D_2}}) = 0$. We also get $h^1(D_1, N_{C_{|D_1}}) = 0$ and that $N_{C_{|D_1}}$ is spanned. Since $\deg(D_1 \cap D_2) = 1$ and $N_{C_{|D_1}}$ is spanned, the restriction map $H^0(D_1, N_{C_{|D_1}}) \to H^0(D_1 \cap D_2, N_{C_{|D_1 \cap D_2}})$ is surjective. Thus, (4.2) gives $h^1(N_C) = 0$ and so $\mathbf{H}(1, 1, 1, 1)$ is smooth of dimension six at $[C]$. Since the set of all such curves C has dimension five, $[C]$ is in the closure of $\mathbf{H}(1, 1, 1, 1)^{\mathrm{sm}}$.

(b) Now assume that C has at least three components, i.e., $C = D_1 \cup D_2 \cup D_3$ with each D_i a line. First assume that C is nodal. In this case, one of the lines, say D_2, meets the other lines. As in step (a) we first get $h^1(N_{D_2 \cup D_3}) = 0$ and then $h^1(N_C) = 0$. Thus, $\mathbf{H}(1, 1, 1, 1)$ is smooth of dimension six at $[C]$. Since the set of all such C has dimension four, we get that $[C]$ is in the closure of $\mathbf{H}(1, 1, 1, 1)^{\mathrm{sm}}$. Now assume that $[C] \in \mathbf{D}$, say $C = A_1 \cup A_2 \cup A_3$ with a common point p. We can deform A_1 in a family of lines intersecting A_2 at a point different from p and not intersecting A_3. Thus, even in this case $[C]$ is in the closure of $\mathbf{H}(1, 1, 1, 1)^{\mathrm{sm}}$. □

Remark 4.18. By Lemma 4.15 every curve C in $\mathbf{H}(1, 1, 1, 1)_+$ is connected and reducible. If C is smooth, then $\nu(C)$ is a rational normal curve of degree 3 in its linear span and $h^1(\mathcal{I}_C(u, v, w)) = 0$ for all $(u, v, w) \in \mathbb{Z}^{\oplus 3}_{>0}$. The homogeneous ideal of $\nu(C)$ inside $\langle \nu(C) \rangle \cong \mathbb{P}^3$ is generated by three quadrics; one may use the Castelnuovo–Mumford regularity lemma with $h^1(\mathcal{I}_{\nu(C)}(1)) = 0$ and $h^1(\mathcal{O}_{\nu(C)}) = 0$. All these observations are still valid even for C reducible; $\nu(C)$ is either (a) the union of a line and smooth conic or (b) the union of three lines. Thus, in all cases the homogeneous ideal of $\nu(C)$ in $\mathbb{P}^7$ is generated by 4 linear forms and 3 quadratic forms, while none of these 7 equations may be omitted. As a curve in X, the curve C may be cut out by two forms of tridegree $(1, 1, 1)$ and two forms of tridegree $(2, 2, 2)$.

Lemma 4.19. *$\mathbf{H}(2, 1, 1, 1)^{\mathrm{sm}}$ is irreducible, unirational and smooth with dimension eight. In addition, $[C] \in \mathbf{H}(2, 1, 1, 1)^{\mathrm{sm}}$ is a smooth and connected rational curve with tridegree $(2, 1, 1)$.*

Proof. Since no plane cubic curve is contained in X and the intersection of two quadric surfaces in $\mathbb{P}^3$ has $4t$ as its Hilbert polynomial, $[C] \in \mathbf{H}(2, 1, 1, 1)^{\mathrm{sm}}$ is a quartic rational curve. By Remark 2.4, a general element in

$$V = H^0(\mathcal{O}_{\mathbb{P}^1}(2))^{\oplus 2} \times H^0(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2} \times H^0(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2}$$

gives an isomorphism $\alpha\colon \mathbb{P}^1 \to X$ onto its image and so there is an open subset $V_0 \subset V$ with the universal family $\mathcal{V}_0 \subset V_0 \times X$. Since $\mathcal{V}_0$ is flat, it gives a surjection $Y_0 \to \mathbf{H}(2, 1, 1, 1)^{\mathrm{sm}}$ by the universal property of the Hilbert scheme. Since V_0 is rational, $\mathbf{H}(2, 1, 1, 1)^{\mathrm{sm}}$ is unirational. Now fix a curve $[C] \in \mathbf{H}(2, 1, 1, 1)^{\mathrm{sm}}$. Since C is a twisted cubic curve, by adjunction, we have

$$\mathcal{O}_{\mathbb{P}^1}(-2) \cong \omega_C \cong \det(N_C) \otimes \mathcal{O}_X(-2, -2, -2)$$

and so $\det(N_C) \cong \mathcal{O}_{\mathbb{P}^1}(6)$. This implies that $N_C \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $a + b = 6$. Now from the surjection $TX_{|C} \to N_C$, we get that N_C is globally generated and so $a, b \geq 0$. In particular, we have $h^0(N_C) = 8$ and $h^1(N_C) = 0$. Indeed, we have $N_C \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. □

Let us write $\mathbf{H}(2, 1, 1, 1)_{+,\mathrm{red}} = \Gamma_1 \sqcup \Gamma_2$, where Γ_1 consists of the reduced curves and Γ_2 consists of the non-reduced curves.

Lemma 4.20. *Each curve in Γ_1 is connected and its irreducible components are all smooth and rational.*

Proof. For a fixed curve $[C] \in \Gamma_1$, let T be any irreducible component of C. Since either T is a fiber of π_{12} or $\pi_{3|T}$ has degree one, we have that T is smooth and rational. Assume for the moment the existence of a connected curve $C' \subseteq C$ with $p_a(C') > 0$. Since $p_a(C) = 0$, we have $\deg(C') \leq 3$. Since C' is reduced, we get that C' is a plane cubic, contradicting the fact that X contains no plane and it is cut out by quadrics in $\mathbb{P}^7$. The non-existence of C' implies that C is connected. □

Proposition 4.21. *Γ_1 is irreducible with $\mathbf{H}(2,1,1,1)^{\mathrm{sm}}$ as its open subset. In particular, each $[C] \in \Gamma_1$ is connected, and the irreducible components of C are smooth rational curves.*

Proof. Fix a curve $[C] \in \Gamma_1$. By Lemma 4.20, we may let $C_1, \dots, C_h$ be the irreducible components of C with $h \geq 2$ in an ordering so that if $h \geq 3$, then $E_i := C_1 \cup \cdots \cup C_i$ is connected for all $2 \leq i \leq h-1$. Fix an integer i with $1 \leq i \leq h-1$. Since E_i and E_{i+1} are connected with arithmetic genus zero, we have $\deg(C_{i+1} \cap E_i) = 1$. Since $TX \cong \mathcal{O}_X(2,0,0) \oplus \mathcal{O}_X(0,2,0) \oplus \mathcal{O}_X(0,0,2)$, the Mayer–Vietoris exact sequence

$$0 \to TX_{|E_{i+1}} \to TX_{|E_i} \oplus TX_{|C_{i+1}} \to TX_{|E_i \cap C_{i+1}} \to 0$$

and induction on i give $h^1(TX_{|C}) = 0$. Since the natural map $TX_{|C} \to N_C$ has cokernel supported on the finite set $\mathrm{Sing}(C)$, we have $h^1(N_C) = 0$ and so $\mathbf{H}(2,1,1,1)_+$ is smooth at $[C]$. If C is nodal, by induction on i, we get that each E_i is smoothable and, in particular, C is smoothable in X, i.e., $[C]$ is contained in the closure of $\mathbf{H}(2,1,1,1)^{\mathrm{sm}}$ in $\mathbf{H}(2,1,1,1)$.

Now assume that C is not nodal and then we get $3 \leq h \leq 4$. If $h = 3$, we may find an ordering so that $\deg(C_1) = 2$, $\deg(C_2) = \deg(C_3) = 1$ and C_1, C_2, C_3 contain a common point, say p, and neither C_2 nor C_3 is the tangent line of C_1 at p. Since $p \in C_2 \cap C_3$, the lines C_2 and C_3 have different tridegree and so we may fix $C_1 \cup C_2$ and move C_3 in the family of all lines meeting C_2 and with the tridegree of C_3. Thus, we may deform C to a nodal curve and hence again we get that $[C]$ is contained in the closure of $\mathbf{H}(2,1,1,1)^{\mathrm{sm}}$ in $\mathbf{H}(2,1,1,1)$. If $h = 4$, then each irreducible component of C is a line. Since two of these components have the same tridegree, C has a unique triple point and we may use the argument above for the case $h = 3$. □

Proposition 4.22. *For $[C] \in \Gamma_1$, we have $h^1(\mathcal{I}_C(t)) = 0$ for all $t \in \mathbb{Z}$.*

Proof. The Mayer–Vietoris exact sequence gives $h^0(\mathcal{O}_{E_i}(1)) = \deg(E_i) + 1$ for all i even in the non-nodal case. In particular, we get $h^0(\mathcal{O}_C(1)) = 5$. Let $M \subset \mathbb{P}^7$ be the linear span of C. Since $h^0(\mathcal{O}_C(1)) = 5$, we have $\dim(M) \leq 4$. Let $H \subset M$ be a general hyperplane. Assume for the moment $\dim(M) = 4$. In this case, C is linearly normal in M. The scheme $H \cap C$ is the union of four points. Since C is connected and linearly normal in M, we have $h^1(M, \mathcal{I}_C(t)) = 0$ for all $t \leq 1$. The case $t = 1$ of the exact sequence

$$0 \to \mathcal{I}_{C,M}(t-1) \to \mathcal{I}_{C,M}(t) \to \mathcal{I}_{C\cap H,H}(t) \to 0 \tag{4.3}$$

gives that $C \cap H$ is formed by four points of H spanning H. This implies that $h^1(H, \mathcal{I}_{C\cap H,H}(t)) = 0$ for all $t > 0$. By induction on t, (4.3) gives $h^1(M, \mathcal{I}_{C,M}(t)) = 0$ for all $t \geq 2$. To conclude we only need to exclude that $\dim(M) < 4$. We have $\dim(M) > 2$, because X is cut out by quadrics and contains no plane. Now assume $\dim(M) = 3$. Since X contains no plane and no quadric surface, $X \cap M$ is an algebraic set cut out by quadrics and with connected components of dimension at most one. $X \cap M$ is not the complete intersection of two quadrics of M, because $X \cap M$ contains C of degree four and $p_a(C) = 0$. Since $h^0(M, \mathcal{O}_C) = 1$, the case $t = 1$ of (4.3) gives that $C \cap H$ spans the plane H. Hence, $h^1(H, \mathcal{I}_{C\cap H,H}(2)) = 0$. Since $h^1(M, \mathcal{I}_{C,M}(1)) = 1$ and $H \cap C$ is formed by four points spanning the plane H, the case $t = 2$ of (4.3) gives $h^1(M, \mathcal{I}_{C,M}(2)) \leq 1$ and hence $h^0(M, \mathcal{I}_{C,M}(2)) \leq 2$, a contradiction. □

Remark 4.23. Fix $[C] \in \Gamma_2$, i.e., C is not reduced. By Lemma 2.5, C_{red} has tridegree $(1,1,1)$ and the nilradical of $\mathcal{O}_C$ is supported by a line L of tridegree $(1,0,0)$. There is a unique reduced curve $E \subset C$ with E of tridegree $(0,1,1)$; E is either a disjoint union of two lines or a reduced conic. Set $\mathcal{J} := \mathrm{Ann}_{\mathcal{O}_C}(\mathcal{I}_{E,C})$. The $\mathcal{O}_C$-sheaf $\mathcal{J}$ is the ideal sheaf of a degree two structure supported by L, possibly with embedded components. Let C' be the curve with $\mathcal{J}$ as its ideal sheaf and A the maximal locally CM subcurve of C', which is obtained by taking as its ideal sheaf in C the intersection of the non-embedded components of a primary decomposition of $\mathcal{I}_{C'}$. The curve A is a locally CM curve of degree two with $L \coloneqq \mathbb{P}^1$ as its support, i.e., $[A] \in \mathbf{D}_a$ for some $a \geq 2$. We have $\chi(\mathcal{O}_A) = a$.

Lemma 4.24. *If $C = A \cup E \in \Gamma_2$ with $[A] \in \mathbf{D}_a$ for some $a \geq 2$ and E is a reduced conic of tridegree $(0,1,1)$, then it is contained in $\overline{\Gamma_1}$.*

Proof. Let $\langle E \rangle$ be the plane spanned by E. Since X contains no plane cubic, we have $\deg(L \cap \langle E \rangle) \leq 1$. Considering a general hyperplane $H \subset \mathbb{P}^7$ containing $\langle E \rangle$ with $L \notin H$, we get $\deg(A \cap \langle E \rangle) \leq 2$ and so $\deg(A \cap E) \leq 2$. Thus, we have $\chi(\mathcal{O}_A) \geq a - 1$ and so $a = 2 = \deg(A \cap E)$. This implies that $\deg(L \cap E) = 1$.

Set $Z := A \cap E$. Since E is connected, there is $o \in \mathbb{P}^1$ such that $E \subset \{o\} \times \mathbb{P}^1 \times \mathbb{P}^1$ and so $E = \{o\} \times E'$ with $E' \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$. Since $Z \subset E \subset \{o\} \times \mathbb{P}^1 \times \mathbb{P}^1$, there is a zero-dimensional subscheme $Z' \subset E$ of degree two such that $Z = \{o\} \times Z'$. Since $\deg(A) = 2$, Z is the scheme-theoretic intersection of A and $\{o\} \times \mathbb{P}^1 \times \mathbb{P}^1$. By Theorem 3.3, A is smoothable, i.e., there are an integral curve Δ with $o \in \Delta$ and a flat family $\{A_t\}_{t \in \Delta}$ with $A = A_o$ and A_t a disjoint union of two lines for all $t \in \Delta \setminus \{o\}$. Set $Z_t := A_t \cap \{o\} \times \mathbb{P}^1 \times \mathbb{P}^1$. We have $Z_t = \{o\} \times Z'_t$ for a zero-dimensional subscheme $Z'_t \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree two with $Z'_o = Z'$ and Z'_t reduced for all $t \in \Delta \setminus \{o\}$. Fix a general $q \in E'$. Decreasing Δ if necessary, we may assume $q \notin Z'_t$ for any t and so $|\mathcal{O}_{Z'_t \cup \{q\}}(1,1)|$ contains a unique curve, say E'_t. Since $q \in E'$, we have $E'_o = E'$. Set $E_t = \{o\} \times E'_t$. The algebraic family $\{A_t \cup E_t\}_{t \in \Delta}$ is a flat family. Since $[A_t \cup E_t] \in \Gamma_1$ for $t \neq o$, we have $[A_o \cup E_o] \in \overline{\Gamma_1}$. □

Now we are ready to prove our main result on the irreducibility of $\mathbf{H}(2,1,1,1)_{+,\mathrm{red}}$ in Theorem 4.26. The main technical ingredient is to use the deformation theory of maps and elementray modification of sheaves. In general, let Y be a smooth projective variety. A map $f\colon C \to Y$ is called *stable* if C has at worst nodal singularities and $|\mathrm{Aut}(f)| < \infty$. Let $\mathbf{M}(Y,\beta)$ be the moduli space of isomorphism classes of stable maps $f\colon C \to X$ with genus $g(C) = 0$ and $f_*[C] = \beta \in H_2(X,\mathbb{Z})$. Note that if Y is a projective homogeneous variety, then the space $\mathbf{M}(Y,\beta)$ is irreducible, see [16]. The local structure of the space $\mathbf{M}(Y,\beta)$ was well-studied in [18, Proposition 1.4, 1.5].

Proposition 4.25. *Fix $[f\colon C \to Y] \in \mathbf{M}(Y,\beta)$. Then the tangent space (resp. the obstruction space) of $\mathbf{M}(Y,\beta)$ at $[f]$ is given by*

$$\mathrm{Ext}^1([f^*\Omega^1_Y \to \Omega^1_C], \mathcal{O}_C) \qquad (\textit{resp.}\ \mathrm{Ext}^2([f^*\Omega^1_Y \to \Omega^1_C], \mathcal{O}_C)),$$

where $[f^*\Omega^1_Y \to \Omega^1_C]$ is thought of as a complex of sheaves of the degrees -1 and 0.

Let $\mathcal{E}$ be a flat family of sheaves on Y parameterized by a smooth variety S. Let Z be a smooth divisor of S such that $\mathcal{E}_{|Z}$ has a flat family $\mathcal{A}$ of quotient sheaves. Then the kernel $\ker\{\mathcal{E} \twoheadrightarrow \mathcal{E}|_{Z\times Y} \twoheadrightarrow \mathcal{A}\}$ of the composition map is called an *elementary modification of the sheaf* $\mathcal{E}$ along Z. The effect of elementary modification at Z is the interchange of the sub and quotient sheaves (cf. [8, Example 2.9]).

Theorem 4.26. $\mathbf{H}(2,1,1,1)_{+,\mathrm{red}}$ *is irreducible.*

Proof. By Proposition 4.21, it is enough to show that $\Gamma_2 \subset \overline{\Gamma_1}$. By Remark 4.23 and Lemma 4.24, we may assume that $C = A \cup E \in \Gamma_2$ with $[A] \in \mathbf{D}_a$ for some $a \geq 2$ and E a disjoint union of two lines, say L_1 of tridegree $(0,1,0)$ and L_2 of tridegree $(0,0,1)$. Let us show that the locus Γ_{2b} of these types of curves, is contained in $\overline{\Gamma_1}$ (cf. [7, Proposition 5.10]). Note that the space Γ_{2b} is a $\mathbb{P}^1$ (or its open subset)-bundle over $(\mathbb{P}^1 \times \mathbb{P}^1) \times ((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \mathrm{D})$, where D is the diagonal. Here, the first $\mathbb{P}^1 \times \mathbb{P}^1$ parameterizes the supporting lines of the double lines A and the second $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \mathrm{D}$ parameterizes the ordered pairs (L_1, L_2) of two lines. Also the fiber $\mathbb{P}^1 \cong \mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}_{C_0}, \mathcal{O}_L(-1))$ parameterizes the non-split extensions:

$$0 \to \mathcal{O}_L(-1) \to \mathcal{F} \to \mathcal{O}_{C_0} \to 0, \tag{4.4}$$

where $C_0 = L \cup L_1 \cup L_2$.

Consider the moduli space $\mathbf{M}(X,\beta)$ of stable maps $f\colon C'_0 \to X$ of genus zero and $f_*[D] = \beta \in H_2(X,\mathbb{Z})$ of tridegree $(2,1,1)$. Let Θ_{2b} be the locus of stable maps

$$f\colon C'_0 = L' \cup L'_1 \cup L'_2 \to X$$

with $f(C'_0) = C_0 = L \cup L_1 \cup L_2$ such that $\deg(f_{|L'}) = (2,0,0)$, $\deg(f_{|L'_1}) = (0,1,0)$ and $\deg(f|_{L'_2}) = (0,0,1)$. Then one can easily see that Θ_{2b} is a $\mathbb{P}^2$-bundle over $(\mathbb{P}^1 \times \mathbb{P}^1) \times ((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \mathrm{D})$, where $\mathbb{P}^2$ parameterizes the stable maps of degree two on L. To apply the modification method as in [9], we need to choose a smooth chart of $\mathbf{M}(X,\beta)$ at $[f]$. In fact, from [23, Theorem 0.1], the space of maps in $\mathbf{M}(X,\beta)$ around $[f]$ can be obtained as the SL(2)-quotient

$$\mathbf{M}(X,\beta) \cong \mathbf{M}(\mathbb{P}^1 \times X, (1,\beta))/\,\mathrm{Aut}(\mathbb{P}^1)$$

of the moduli space $\mathbf{M}(\mathbb{P}^1 \times X, (1,\beta))$ of stable maps in $\mathbb{P}^1 \times X$ of bidegree $(1,\beta)$, where $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{SL}(2)$ canonically acts on $\mathbf{M}(\mathbb{P}^1 \times X, (1,\beta))$, see [8, §3.1]. Among the fiber over $[f]$ along the GIT-quotient map, if we choose a graph map f' such that the restriction on L' is of bidegree $(1,(2,0,0))$ which doubly covers $\mathbb{P}^1 \times L \subset \mathbb{P}^1 \times X$, then f' has the trivial automorphism. Hence, around $[f]$, the space $\mathbf{M}(\mathbb{P}^1 \times X, (1,\beta))$ is

a smooth chart, compatible with the SL(2)-action. Thus, the argument in [9, Lemma 4.6] about the construction of the Kodaira-Spencer map of the space of maps can be naturally applied in our situation.

Now let us compute the normal space of Θ_{2b}. Consider the long exact sequence:

$$0 \to \mathrm{Ext}^0(\Omega^1_{C_0}, \mathcal{O}_{C_0}) \to \mathrm{Ext}^0(\Omega^1_X, \mathcal{O}_{C_0}) \to \mathrm{Ext}^0(N^\vee_{C_0|X}, \mathcal{O}_{C_0}) \xrightarrow{\psi} \mathrm{Ext}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0}) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_{C_0}) = 0.$$

The last term is zero by the convexity of X and $\mathrm{Ext}^0(N^\vee_{C_0|X}, \mathcal{O}_{C_0}) \cong H^0(N_{C_0|X}) \cong \mathbb{C}^6$ because of the smoothness of $\mathbf{H}(1,1,1,1)$. Since C_0 has two node points, we get $\mathrm{Ext}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0}) = \mathbb{C}^2$. Therefore, $\ker(\psi)$ in the above means the deformation of C_0, while keeping the two node points. That is, this is the deformation space of the base space of Θ_{2b}. On the other hand, as a similar computation did in the proof of [9, Lemma 4.10], we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\psi) & \longrightarrow & \mathrm{Ext}^0(N^\vee_{C_0|X}, \mathcal{O}_{C_0}) & \longrightarrow & \mathrm{Ext}^1(\Omega^1_{C_0}, \mathcal{O}_{C_0}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \frac{T_{[f]}\mathbf{M}(X,\beta)}{T_{[f]}\mathbf{M}(C_0,\beta)} & \longrightarrow & \mathrm{Ext}^0(f^*N^\vee_{C_0|X}, \mathcal{O}_{C_0'}) & \longrightarrow & \mathrm{Ext}^2([f^*\Omega^1_{C_0} \to \Omega^1_D], \mathcal{O}_{C_0'}) & \longrightarrow & 0.
\end{array}$$

Hence, the normal space of Θ_{2b} is $\mathrm{coker}(\zeta)$, which is isomorphic to $\mathrm{Ext}^0(N^\vee_{C_0|X}, \mathcal{O}_L(-1))$ obtained from the exact sequence

$$0 \to \mathcal{O}_{C_0} \to f_*\mathcal{O}_{C_0'} \to \mathcal{O}_L(-1) \to 0.$$

Moreover, the Kodaira–Spencer map $T_{[f]}\mathbf{M}(\mathbb{P}^1 \times X, (1,\beta)) \to \mathrm{Ext}^1_X(f_*\mathcal{O}_{C_0'}, f_*\mathcal{O}_{C_0'})$ in [9, Equation (4.11)] descends to the normal space, which is compatible with the map

$$N_{\Theta_{2b}|\mathbf{M}(X,\beta),[f]} = \mathrm{Ext}^0(N^\vee_{C_0|X}, \mathcal{O}_L(-1)) \cong \mathrm{Ext}^0(\mathcal{I}_{C_0}, \mathcal{O}_L(-1)) \cong \mathrm{Ext}^1(\mathcal{O}_{C_0}, \mathcal{O}_L(-1)).$$

This implies that if we do the modification of $f_*\mathcal{O}_{C_0'}$ along the normal direction, the modified sheaf must lie in $\mathrm{Ext}^1(\mathcal{O}_{C_0}, \mathcal{O}_L(-1))$ bijectively, see[9, Lemma 4.6]. Since $\mathbf{M}(X,\beta)$ is irreducible by [16] and Γ_1 can be regarded as an open subset of $\mathbf{M}(X,\beta)$ due to Lemma 4.21 and [10, Theorem 2], we get that $\Gamma_{2b} \subset \overline{\Gamma}_1$. □

5 Segre threefold $\mathbb{P}^2 \times \mathbb{P}^1$

In this section we take $X := \mathbb{P}^2 \times \mathbb{P}^1$ and set $\nu\colon X \to \mathbb{P}^5$ to be the Segre embedding; in most cases we adopt the same notations as in the case of $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For a locally CM curve $C \subset X$ with pure dimension one, the *bidegree* $(e_1, e_2) \in \mathbb{Z}^{\oplus 2}$ is defined to be the pair (e_1, e_2) of integers $e_1 := \deg(\mathcal{O}_C(1,0))$ and $e_2 := \deg(\mathcal{O}_C(0,1))$, where the degree is computed using the Hilbert function of the $\mathcal{O}_X$-sheaves $\mathcal{O}_C(1,0)$ and $\mathcal{O}_C(0,1)$ with respect to the ample line bundle $\mathcal{O}_X(1,1)$. We also say that $C = \emptyset$ has bidegree $(0,0)$. Since $\mathcal{O}_C(1,0)$ and $\mathcal{O}_C(0,1)$ are spanned, we have $e_1 \geq 0$ and $e_2 \geq 0$, i.e., $(e_1, e_2) \in \mathbb{N}^{\oplus 2}$. We have $\deg(C) = \deg(\mathcal{O}_C(1,1)) = e_1 + e_2$. Note that the bipolynomial of $\mathcal{O}_C$ is of the form $e_1x + e_2y + \chi$ for some $\chi \in \mathbb{Z}$ and $\chi = \chi(\mathcal{O}_C)$.

As in the proof of Lemma 2.5 we get the following.

Lemma 5.1. *Let $C \subset X$ be a locally CM curve with the bidegree (e_1, e_2). If the bidegree of C_{red} is (b_1, b_2) with $b_i = 0$ for some i, then we have $e_i = 0$.*

Proposition 5.2. $\mathbf{H}(1,1,1)$ *is smooth and irreducible of dimension five, and all its elements are reduced.*

Proof. Let us fix $[C] \in \mathbf{H}(1,1,1)$. By Lemma 5.1, every one-dimensional component of C is generically reduced, i.e., the purely one-dimensional subscheme E of C_{red} has bidegree $(1,1)$. We have $\chi(\mathcal{O}_C) \geq \chi(\mathcal{O}_D)$ for each connected component D of E and equality holds if and only if $D = C$. Since we have $\chi(\mathcal{O}_D) \geq 1$, we get $C = D$ and that C is connected. If C is irreducible, then it is a smooth conic. Since N_C is a quotient of $TX_{|C}$, we get $h^1(N_C) = 0$. This implies that $\mathbf{H}(1,1,1)$ is smooth at $[C]$ and of dimension $h^0(N_C) = \deg(N_C) + 2 = \deg(TX_{|C}) = 5$. Indeed, we have $N_C \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Now assume that C is reducible, say the union of a line D_1 of bidegree $(1, 0)$ and a line D_2 of bidegree $(0, 1)$. Since $\deg(D_1 \cap D_2) \leq 1$ and $[C] \in \mathbf{H}(1, 1, 1)_+$, we get $\deg(D_1 \cap D_2) = 1$ and that C is nodal. Since $h^1(TX_{|C}) = 0$ and the natural map $TX_{|C} \to N_C$ is supported at the point $D_1 \cap D_2$, we have $H^1(N_C) = 0$. Hence, $\mathbf{H}(1, 1, 1)$ is again smooth at $[C]$ and of dimension $\deg(N_C) + 2 = 5$. Since the set of all such reducible curves is four-dimensional, each such curve is in the closure of the open subset of $\mathbf{H}(1, 1, 1)$ parametrizing the smooth curves. □

Remark 5.3. Any curve in $\mathbf{H}(1, 1, 1)$ is a reduced conic and so it is arithmetically Cohen–Macaulay. Note also that C as a curve in X is cut out by two forms of bidegrees $(1, 0)$ and $(1, 1)$, respectively. Thus, one can compute the cohomology of the twists of $\mathcal{I}_C$ from the exact sequence

$$0 \to \mathcal{O}_X(-2, -1) \to \mathcal{O}_X(-1, 0) \oplus \mathcal{O}_X(-1, -1) \to \mathcal{I}_C \to 0;$$

for example, $h^0(\mathcal{I}_C(1, 0)) = 1$, $h^0(\mathcal{I}_C(0, 1)) = 0$ and $h^1(\mathcal{I}_C(u, v)) = 0$ for all $(u, v) \in \mathbb{Z}^{\oplus 2}_{\geq 0}$.

Proposition 5.4. $\mathbf{H}(1, 1, 2)$ *is smooth and irreducible of dimension five. It parametrizes the disjoint unions of two lines, one of bidegree* $(1, 0)$ *and the other of bidegree* $(0, 1)$.

Proof. By Lemma 5.1, any curve $[C] \in \mathbf{H}(1, 1, 2)_+$ is reduced. If C is irreducible, we get $\chi(\mathcal{O}_C) = 1$, a contradiction. If $C = D_1 \cup D_2$ with lines D_1 of bidegree $(1, 0)$ and D_2 of bidegree $(0, 1)$, we get $D_1 \cap D_2 = \emptyset$. Then we have $h^1(N_C) = 0$ and

$$h^0(N_C) = h^0(N_{D_1}) + h^0(N_{D_2}) = \deg(TX_{|D_1}) + \deg(TX_{|D_2}) = 5.$$ □

Remark 5.5. For a curve $[C] \in \mathbf{H}(1, 1, 2)$, the homogeneous ideal of $\nu(C)$ in $\mathbb{P}^5$ is generated by 2 forms of degree one (corresponding to elements of $|\mathcal{O}_X(1, 1, 1)|$) and 4 forms of degree two (corresponding to elements of $|\mathcal{O}_X(2, 2, 2)|$). We also have $h^0(\mathcal{I}_C(1, 0)) = h^0(\mathcal{I}_C(0, 1)) = 0$, $h^1(\mathcal{I}_C) = 1$ and $h^1(\mathcal{I}_C(u, v)) = 0$ for all $(u, v) \in \mathbb{Z}^{\oplus 2}_{>0}$,

Remark 5.6. Using the argument in the proof of Proposition 5.2, we get that $\mathbf{H}(1, 1, \chi)_+ = \emptyset$ if either $\chi \leq 0$ or $\chi \geq 3$.

In the case of Segre threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the main ingredient is the knowledge on the Hilbert scheme of double lines. So we suggest the following results for the Segre threefold $\mathbb{P}^2 \times \mathbb{P}^1$, as in Theorem 3.3. As in the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, let $\mathbf{D}_a$ be the subset of $\mathbf{H}(0, 2, a)_+$ parametrizing the double lines whose reduction is a line of bidegree $(0, 1)$ in $X = \mathbb{P}^2 \times \mathbb{P}^1$ for each $a \in \mathbb{Z}$. For the moment, we take $\mathbf{D}_a$ as a set and it would be clear in each case which scheme-structure is used on it. Since X is a smooth threefold, [6, Remark 1.3] says that each $[B] \in \mathbf{D}_a$ is obtained by Ferrand's construction and, in particular, it is a ribbon in the sense of [4] with a line of bidegree $(1, 0)$ as its support. Let $\mathbf{R}_a$ be the subset of $\mathbf{H}(2, 0, a)_+$ parametrizing the double structures on lines of bidegree $(1, 0)$.

Proposition 5.7. *The description on* $\mathbf{R}_a$ *is as follows:*

(i) $\mathbf{R}_a$ *is non-empty if and only if* $a \geq 2$. *It is parametrized by an irreducible and rational variety of dimension* $2a - 1$.

(ii) *We have* $\mathbf{R}_a = \mathbf{H}(2, 0, a)_+$ *for* $a \geq 3$.

(iii) $\mathbf{H}(2, 0, 2)_+$ *is smooth, irreducible, rational and of dimension four.*

Proof. Each element of $\mathbf{R}_a$ is a ribbon in the sense of [4]. For any line $L \subset X$ of bidegree $(1, 0)$, let $\mathbf{R}_a(L)$ denote the set of all $[A] \in \mathbf{R}_a$ such that $A_{\mathrm{red}} = L$. The set of all lines of X with bidegree $(1, 0)$ is isomorphic to $\mathbb{P}^1$. Any line $L \subset X$ of bidegree $(1, 0)$ has trivial normal bundle and so $\mathbf{R}_a(L)$ is parametrized by the pairs (f, g) with $f \in H^0(\mathcal{O}_L(a - 2))$ and $g \in H^0(\mathcal{O}_L(a - 2))$ with no common zero. Here we have the convention that (L, f_1, g_1) and (L, f_2, g_2) give the same element of $\mathbf{R}_a(L)$ if and only if there is $t \in \mathbb{C}^\times$ with $f_1 = tg_1$ and $f_2 = tg_2$. Hence, we get parts (1) and (2) of Proposition 5.7 for $a \geq 3$, and for any $a \geq 2$, each element of $\mathbf{R}_a$ is a split ribbon.

The set $\mathbf{H}(2, 0, 2)_+$ is the disjoint union of $\mathbf{R}_2$ and the set $\mathcal{T}$ of all disjoint unions of two different lines of bidegree $(1, 0)$. $\mathcal{T}$ is isomorphic to the symmetric product of two copies of $\mathbb{P}^2$ and so it is smooth and rational with $\dim(\mathcal{T}) = 4$. Fix a line $L \subset X$ of bidegree $(1, 0)$ such that $L = \mathbb{P}^1 \times \{o\}$ with $o \in \mathbb{P}^2$, and $[A] \in \mathbf{R}_2(L)$

determined by $(f, g) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, up to a non-zero scalar. The pair (f, g) defines a zero-dimensional scheme $v \subset \mathbb{P}^2$ of degree two with $v_{\mathrm{red}} = \{o\}$. Let $R \subset \mathbb{P}^2$ be the line spanned by v and then A is contained in $L \times R$ as a curve of bidegree $(2, 0)$, and hence it is a flat deformation of a family of pairs of disjoint lines of $L \times R$ and so of X. We also get that the normal sheaf N_A of A in X is isomorphic to $\mathcal{O}_L \oplus \mathcal{O}_L(1)$. Hence, we get $h^1(N_A) = 0$ and so $\mathbf{H}(2, 0, 2)_+$ is smooth at $[A]$. □

Remark 5.8. For a fixed $[C] \in \mathbf{H}(2, 0, 2)_+$, we see that the linear span $\langle v(C)\rangle$ is isomorphic to $\mathbb{P}^3$ and the homogeneous ideal of $v(C)$ in $\mathbb{P}^5$ is generated by 2 linear forms and 4 quadratic forms. One can also compute $h^1(\mathcal{I}_C) = 1$ and $h^1(\mathcal{I}_C(u, u)) = 0$ for all $u > 0$. Now we have $C \subset W$ with $W \in |\mathcal{O}_X(0, 2)|$. If $C \notin \mathbf{R}_2$, then $v(W)$ is embedded as a disjoint union of two planes and so it spans $\mathbb{P}^5$; in this case, we have $v(C) = v(W) \cap H$, where H is one hyperplane in the pencils of hyperplanes containing the two lines $v(C)$. Indeed, any hyperplane in the pencil cut out C inside W.

Below we give a description on $\mathbf{D}_a$ as in Theorem 3.3 and Proposition 5.7, which can be proven by the same way.

Proposition 5.9. *The description on $\mathbf{D}_a$ is as follows:*
(i) *$\mathbf{D}_a$ is non-empty if and only if $a \geq 2$. It is parametrized by an irreducible variety of dimension $2a - 1$.*
(ii) *We have $\mathbf{D}_a = \mathbf{H}(0, 2, a)_+$ for $a \geq 3$.*
(iii) *$\mathbf{H}(0, 2, 2)_+$ is smooth, irreducible, rational and of dimension four.*

Remark 5.10. Similarly, as in $\mathbf{H}(2, 0, 2)_+$, any curve C in $\mathbf{H}(0, 2, 2)_+$ is a union of two disjoint lines. Then the homogeneous ideal of $v(C)$ in $\mathbb{P}^5$ is generated by 2 linear forms and 4 quadratic forms. One can also compute $h^1(\mathcal{I}_C) = 1$ and $h^1(\mathcal{I}_C(u, v)) = 0$ for all $(u, v) \in \mathbb{Z}^{\oplus 2}_{>0}$.

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